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Thesis for the Degree of Master of Science

Regularity for inhomogeneous evolution equations in ς-convex space

by

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The Graduate School

Pukyong National University

February 2010

Regularity for inhomogeneous evolution equations in ζ-convex space
(ζ-convex space에서의 비동차 발전 방정식에 대한 정칙성)

Advisor: Prof. Jin Mun Jeong

by

Eun Young Ju

A thesis submitted in partial fulfillment of the requirements
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Regularity for inhomogeneous evolution equations in ζ-convex space

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February 25, 2010

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ζ-convex space에서의 비동차 발전 방정식에 대한 정칙성

주 은 영

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요　약

본 논문은 $\zeta - convex$공간상에서 미분연산자로 구성된 방물형의 방정식의 해에 대한 정칙성 문제를 다루고 있다. 주 작용소의 정의구역과 본래의 전공간사이의 보간들을 주 작용소에 의해 생성된 해석적 반군으로 수식화하고 공간이론을 이용하여 $H_{p,q} = (W_0^{1,p}, W^{-1,p})_{1/q,q}$(1<p<$\infty$, $\frac{1}{p} + \frac{1}{q} = 1$)의 관계식을 보이고 주 작용소의 해석적 반군을 생성하는 성질을 이용하여 주 결과를 얻어내었다.

$W^{1,p'}(\Omega)$ $(p' = p/(p-1))$가 조밀한 공간으로서 그의 공액공간을 $W^{-1,p}(\Omega)$로 했을때 $H_{p,q}$, $W_0^{1,p}$가 $\zeta - convex$공간임을 고려하여 다음과 같은 $\zeta - convex$공간 상에서 A를 포함하는 초기치 문제:

$$\begin{cases} \frac{d}{dt}u(t) = Au(t) + f(t) \\ u(0) = u_0 \end{cases}$$

에서, $f = L^q(0, T; W^{-1,p}(\Omega))$ 그리고 $u_0 \in H_{p,q}$로 주어지면 위의 초기치 문제의 해는 유일하게 존재하며, 아울러

$$u \in L^q(0, T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0, T; W^{-1,p}(\Omega)) \subset C([0, T]; H_{p,q}).$$

임을 증명하였다.

공간적 이론을 토대로 주 작용소의 기본 성질을 다루어 기존의 결과들을 보다 일반적이고 응용 가능한 결과를 얻었다는데 본 논문의 의미가 있다고 하겠다.

1 Introduction

This paper is concerned with regularity of solutions for an abstract parabolic type equation;

$$\begin{cases} \frac{du(t)}{dt} = Au(t) + f(t), \quad t \in (0,T], \\ u(0) = u_0 \end{cases} \tag{1.1}$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order as follows:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \Omega$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$.

If we put that $Au = -\mathcal{A}(x, D_x)u$ then it is known that A generates an analytic semigroup in $W^{-1,p}(\Omega)$ where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p'}(\Omega), p' = p/(p-1)$ as is seen in [1]. Therefore, from the interpolation theory. First, we will prove that the operator A generates an analytic semigroup in $H_{p,q} = (W_0^{1,p}, W^{-1,p})_{1/q,q}$.

If $-A$ is the infinitesimal generator of an analytic semigroup in a complex Banach space X, we find that in general it is false that problem (1.1) has a solution $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ in case $f \in L^p(0,T;X)$. As in Da Prato and Grisvard [6](also see [18, 4]), we can obtain L^2- regularity for the strong solutions, while in the Hilbert space setting. Moreover as the better result in [5], if X is ζ-convex, we also obtain $L^p(p > 1)$-regularity results for solution of (1.1) mentioned above.

Concerning ζ-convex Banach space, we recall that every Hilbert space is ζ-convex. Cartesian products and quotients of ζ-convex spaces are ζ-convex. By proving that A is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, and $W^{-1,p}(\Omega)$ are ζ-convex spaces, it is easily seen that $H_{p,q}$ is also ζ-convex.

In view of Sobolev's embedding theorem, we remark that $L^1(\Omega) \subset W^{-1,p}(\Omega)$ if $1 < p < n/(n-1)$ as is seen in [5]. Hence, we can investigate the system (1.1) in the space $W^{-1,p}(\Omega)$. Furthermore, it is known that $W^{-1,p}(\Omega)$ is ζ-convex and the initial value problem (1.1) has a unique solution $u \in L^q(0,T; W_0^{1,p}(\Omega) \cap W^{1,q}(0,T; W^{-1,p}(\Omega))$ for any $u_0 \in H_{p,q}$ and $f \in L^q(0,T; W^{-1,p}(\Omega))$ (see Theorem 3.1 in [5]). Thereafter, we can apply the method of Dore and Venni [5] to the system (1.1) to show the existence and uniqueness of the solution

$$u \in L^q(0,T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T; W^{-1,p}(\Omega)) \subset C([0,T]; H_{p,q}).$$

2 Notations

For an integer $m \geq 0$, $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω. For $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p,} = (\sum_{\alpha \leq m} ||D^\alpha f||_p^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad ||f||_{m,\infty} = \max_{\alpha \leq m} ||D^\alpha u||_\infty,$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $|| \cdot ||_p$. Let $p^{'} = p/(p-1)$, $1 < p < \infty$. $W^{-1,p}(\Omega)$ stands for the dual space $W_0^{1,p^{'}}(\Omega)^*$ of $W_0^{1,p^{'}}(\Omega)$ whose norm is denoted by $|| \cdot ||_{-1,p}$.

For a closed linear operator of A in some Banach space, $\rho(A)$ denotes the resolvent set of A. If X is a Banach space and the notation $(\cdot, \cdot)_{X^*,X}$ is the duality pairing between X^* and X.

$L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable. $C^m([0,T];X)$ will denote the set of all m-times continuously differentiable functions from $[0,T]$ into X.

If X and Y are two Banach spaces, $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$. For an interpolation couple of Banach spaces X_0 and X_1, $(X_0,X_1)_{\theta,p}$ and $[X_0,X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively.

3 Elliptic boundary value problem in $W^{-1,p}(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the following elliptic differential operator of second order with real and smooth coefficients:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \overline{\Omega}$. The operator

$$\mathcal{A}'(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\cdot) + c(x)$$

is the formal adjoint of $\mathcal{A}$.

For $1 < p < \infty$, we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \tag{3.1}$$

$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

For $p' = p/(p-1)$, we can also define the realization $\mathcal{A}'$ in $L^{p'}(\Omega)$ under Dirichlet boundary condition by $A'_{p'}$:

$$D(A'_{p'}) = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega),$$

$$A'_{p'} v = \mathcal{A}' v \quad \text{for} \quad v \in D(A'_{p'}).$$

It is known that the adjoint operator of A_p considered as a closed linear operator in $L^p(\Omega)$ coinside with $A'_{p'}$:

$$A_p^* = A'_{p'}$$

and $-A_p$ and $-A'_{p'}$ generate analytic semigroups in $L^p(\Omega)$ and $L^{p'}(\Omega)$, respectively[[19], section 7.3]. For the sake of simplicity we assume that the closed half plane $\{\lambda : \mathrm{Re}\lambda \le 0\}$ is contained in $\rho(A_p) \cap \rho(A'_{p'})$, hence in particular $0 \in \rho(A_p) \cap \rho(A'_{p'})$, by adding some positive constant to $\mathcal{A}$ if necessary.

In what follows we make $D(A_p)$ and $D(A'_{p'})$ Banach space endowing them with graph norm of A_p and $A'_{p'}$, respectively, Since $D(A'_{p'})$ and $W_0^{1,p'}(\Omega)$ are dense subspaces of $W_0^{1,p'}(\Omega)$ and $L^{p'}(\Omega)$, respectively, we may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A'_{p'})^*.$$

Lemma 3.1 *Let $(A'_{p'})'$ be the adjoint operator $A'_{p'}$. Then $(A'_{p'})'$ is an isomorphism from $L^p(\Omega)$ to $D(A'_{p'})^*$ and the restriction of $(A'_{p'})'$ to $D(A_p)$ coincides with A_p.*

Proof. For any $f \in L^p(\Omega)$ and $v \in D(A'_{p'})$, we have

$$((A'_{p'})'f, v)_{D(A'_{p'})^*, D(A'_{p'})} = (f, A'_{p'}v)_{L^p(\Omega), L^{p'}(\Omega)}.$$

So, due to $0 \in \rho(A'_{p'})$, we have that $(A'_{p'})'$ is an isomorphism from $L^p(\Omega)$ to $D(A'_{p'})^*$. If $f \in L^p(\Omega)$ and $v \in D(A'_{p'})$, then

$$((A'_{p'})'u, v)_{D(A'_{p'})^*, D(A'_{p'})} = (u, A'_{p'}v)_{L^p(\Omega), L^{p'}(\Omega)} = (A_p u, v)_{D(A'_{p'})^*, D(A'_{p'})}.$$

This implies that the restriction of $(A'_{p'})'$ to $D(A_p)$ coincides with A_p. $\square$

Lemma 3.2 *Let $\widetilde{A}$ be the restriction of $(A'_{p'})'$ to $W_0^{1,p}(\Omega)$. Then the operator $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction $\widetilde{A'}$ of $(A_p)' \in B(L^{p'}(\Omega), D(A_p)^*)$ to $W_0^{1,p'}(\Omega)$ is an isomorphism from $W_0^{1,p'}(\Omega)$ to $W^{-1,p'}(\Omega)$.*

Proof. From the result of Seeley [11] (see also Triebel [[15], p. 321], [3]) we obtain that

$$[D(A_p), L^p(\Omega)]_{1/2} = W_0^{1,p}(\Omega), \tag{3.2}$$

$$[D(A'_{p'}), L^{p'}(\Omega)]_{1/2} = W_0^{1,p'}(\Omega). \tag{3.3}$$

Regarding the dual spaces, from (3.3) it follows that

$$[L^p(\Omega), D(A'_{p'})^*]_{1/2} = [D(A'_{p'}), L^{p'}(\Omega)]^*_{1/2} = W^{-1,p}(\Omega).$$

This, together with $0 \in \rho(A'_{p'})$, implies that the operator $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$ by the interpolation theory. $\square$

It is not difficult to see that, for $u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,p'}(\Omega)$, $\widetilde{A}u = \mathcal{A}u$ and $\widetilde{A'}v = \mathcal{A}'v$, both understood in the distribution sense, and

$$(\widetilde{A}u, v) = a(u, v) = (u, \widetilde{A'}v) \tag{3.4}$$

where $a(u, v)$ is the associated sesquilinear form:

$$a(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} (a_{i,j}(x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \overline{v} + c(x) u \overline{v} \right\} dx.$$

The following results are from Section 3 in Jeong [8].

Lemma 3.3 *The operators $-\widetilde{A}$ and $-\widetilde{A'}$ generate analytic semigroups in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively. Furthermore, the inequality*

$$||(\widetilde{A})^{is}||_{B(W^{-1,p}(\Omega))} \le Ce^{\gamma|s|}, \quad -\infty < s < \infty, \tag{3.5}$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$.

For any $q \in (1, \infty)$, we set

$$Z_{p,q} = (D(A_p), L^p(\Omega))_{1/q,q}, \quad H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q}. \tag{3.6}$$

Remark 3.1 *Concerning ζ-convex Banach space, we recall that every Hilbert space is ζ-convex. Cartesian products and quotients of ζ-convex spaces are ζ-convex. If (X, Y) is an interpolation couple spaces of ζ-convex spaces,*

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$(X,Y)_{\theta,p}$ with $1 < p < \infty$ and $[X,Y]_\theta$ are ζ-convex. Moreover, if X is ζ-convex and $1 < p < \infty$ then every L^p space of X-valued functions is ζ-convex(see [19, 17] and the bibliography therein). Since $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ are ζ-convex spaces. From the interpolation theory and definitions of the operator $\widetilde{A}$, it is easily seen that $H_{p,q}$ and $Z_{p,q}$ are also ζ-convex.

Proposition 3.1 *The operators $-\widetilde{A}$ and $-\widetilde{A'}$ generate analytic semigroups in $H_{p,q}$ and $H_{p',q'}$, respectively.*

Proof. By lemma 3.3, since $-A_p$ and $-\widetilde{A}$ generate analytic semigroup in $L^p(\Omega)$ and $W^{-1,p}(\Omega)$, respectively, there exists an angle $\gamma \in (0, \frac{\pi}{2})$ such that

$$\Sigma = \{\lambda : \gamma \le \arg\lambda \le 2\pi - \gamma\} \subset \rho(A_p) \cap \rho(\widetilde{A}), \tag{3.7}$$

$$||(\lambda - A_p)^{-1}||_{B(L^p(\Omega))} \le C/|\lambda|, \quad \lambda \in \Sigma, \tag{3.8}$$

$$||(\lambda - \widetilde{A})^{-1}||_{B(W^{-1,p}(\Omega))} \le C/|\lambda|, \quad \lambda \in \Sigma. \tag{3.9}$$

In view of (3.8)

$$||A_p(\lambda - A_p)^{-1}u||_p = ||(\lambda - A_p)^{-1}A_p u||_p \le \frac{C}{|\lambda|}||A_p u||_p,$$

for any $u \in D(A_p)$, we have

$$||(\lambda - A_p)^{-1}||_{B(D(A_p))} \le \frac{C}{|\lambda|}. \tag{3.10}$$

From (3.8) and (3.10) it follows that

$$||(\lambda - \widetilde{A})^{-1}||_{B(W_0^{1,p}(\Omega))} \le \frac{C}{|\lambda|} \tag{3.11}$$

and, hence from (3.10), (3.11) and the definition of the space $H_{p,q}$ we have that

$$||(\lambda - \widetilde{A})^{-1}||_{B(H_{p,q})} \le \frac{C}{|\lambda|}.$$

Therefore we have shown that $-\widetilde{A}$ generates an analytic semigroup in $H_{p,q}$. □

Proposition 3.2 *There exists a constant $C > 0$ such that*

$$||\widetilde{A}^{is}||_{B(H_{p,q})} \le Ce^{\gamma|\lambda|}, s \in \mathbb{R},$$

where γ is the constant in (3.7).

Proof. From Theorem 1 of Seeley [10] and Proposition 3.2 of Jeong [8] there exists a constant $C > 0$ such that

$$||(A_p)^{\epsilon+is}||_{B(L^p(\Omega))} \le Ce^{\gamma|s|}, \tag{3.12}$$

$$||\widetilde{A}^{\epsilon+is}||_{B(W^{-1,p}(\Omega))} \le Ce^{\gamma|s|}, \tag{3.13}$$

for any $s \in \mathbb{R}$ and $\epsilon > 0$. From (3.12) it follows

$$||(A_p)^{\epsilon+is}||_{B(D(A_p))} \le Ce^{\gamma|s|}, \tag{3.14}$$

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and hence, from (3.12) and (3.14) we obtain

$$||\widetilde{A}^{\epsilon+is}||_{B(W_0^{1,p}(\Omega))} \le Ce^{\gamma|s|}. \tag{3.15}$$

Hence from (3.5), (3.14) and (3.15) we have shown that

$$||\widetilde{A}^{\epsilon+is}||_{B(H_{p,q})} \le Ce^{\gamma|s|}.$$

So the proof is complete. □

Remark 3.2 *Propositions 3.1, 3.2 say that* $-\widetilde{A}$ *generates analytic semigroup* $\{e^{t\widetilde{A}} : t \ge 0\}$ *in* $H_{p,q}$ *as well as in* $W^{-1,p}(\Omega)$. *Hence we may assume that there is a constant* $M_0 > 0$ *such that*

$$||e^{t\widetilde{A}}||_{B(L^p(\Omega))} \le M_0, \quad ||e^{t\widetilde{A}}||_{B(H_{p,q})} \le M_0, \quad ||e^{t\widetilde{A}}||_{B(W^{-1,p}(\Omega))} \le M_0.$$

From now on, in virtue of Proposition 3.1, 3.2, we study such a simple initial value problem in $W^{-1,p}(\Omega)$ or in $H_{p,q}$ as

$$\begin{cases} u'(t) + \widetilde{A}u(t) = f(t), \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{LE}$$

Remark 3.3 *If* $-A$ *is the infinitesimal generator of an analytic semigroup in a complex Banach space* X*, we find that in general it is false that problem (LE) has a solution* $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ *in case* $f \in L^p(0,T;X)$.

As in Da Prato and Grisvard [19](also see [18, 4], section 5.5 of [12]), we can obtain L^2- regularity for the strong solutions, while in the Hilbert space setting. Moreover as the better result in [5], if X is ζ-convex, we also obtain $L^p(p > 1)$-regularity results for solution of (LE) mentioned above.

From Theorem 3.5.3 of Butzer and Berens [2] we obtain the following result.

Lemma 3.4 *For any $1 < p$ and $q \in (0, \infty)$, we have*

$$Z_{p,q} = (D(A_p), L^p(\Omega))_{1/q,q} = \{x \in L^p(\Omega) : \int_0^T ||\widetilde{A}e^{t\widetilde{A}}x||_p^q dt < \infty\},$$

and

$$H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q} = \{x \in W^{-1,p}(\Omega) : \int_0^T ||\widetilde{A}e^{t\widetilde{A}}x||_{-1,p}^q dt < \infty\}.$$

In order to prove the solvability of the initial equation (LE), we establish necessary estimates applying the result of [5] to (LE) considered as an equation in $H_{p,q}$ as well as in $W^{-1,p}(\Omega)$.

Proposition 3.3 *Suppose that $\widetilde{A}$ is defined as in Lemma 3.2. Then the following results hold:*
1) Let $1 < p,\ q < \infty$, Then for any $u_0 \in H_{p,q}$ and $f \in L^q(0, T; W^{-1,p}(\Omega))$, there exists a unique solution u of (LE) belonging to

$$\mathcal{W} \equiv L^q(0, T; W_0^{1,p}(\Omega)) \bigcap W^{1,q}(0, T; W^{-1,p}(\Omega)) \subset C([0, T]; H_{p,q}) \quad (3.16)$$

and satisfying

$$||u||_{\mathcal{W}} \le C_1(||u_0||_{p,q} + ||f||_{L^q(0,T;W^{-1,p}(\Omega))}), \tag{3.17}$$

where C_1 is a constant depending on T.

2) Let $u_0 \equiv 0$ and $f \in L^q(0,T;H_{p,q})$, $T > 0$. Then there exists a unique solution u of (LE) belonging to

$$\mathcal{W}_0 \equiv L^q(0,T;W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega))\bigcap W^{1,q}(0,T;H_{p,q})$$

and satisfying

$$||u||_{\mathcal{W}_0} \le C_1||f||_{L^q(0,T;H_{p,q})},$$

where C_1 is a constant depending on T.

Proof. In virtue of Remark 3.2 the mild solution of (LE) is represented by

$$u(t) = e^{-t\widetilde{A}}u_0 + \int_0^t e^{-(t-s)\widetilde{A}}f(s)ds, \quad t \ge 0.$$

If $t \mapsto f(t)$ belongs to $L^q(0,T;X)$ we set $||f(t)||_{L_t^q(0,T;X)} = ||f||_{L^q(0,T;X)}$, Analogous notations we are used when $L^q(0,T;X)$ is replaced by another Banach space of functions. For the sake of simplicity, we may consider

$$||v||_{-1,p} \le ||v||_{p,q}, \quad v \in H_{p,q}.$$

Now, by Lemma 3.4, Remark 3.2, and the isomorphism of $\widetilde{A}$, we have that

$$||e^{-t\widetilde{A}}u_0||_{L^q_t(0,T;W^{1,p}_0(\Omega))} \leq \mathrm{const} \cdot \left(\int_0^T ||\widetilde{A}e^{t\widetilde{A}}u_0||^q_{-1,p}dt \right)^{1/q}$$

$$\leq \mathrm{const} \cdot ||e^{t\widetilde{A}}u_0||_{W^{1,q}_t(0,T;W^{-1,p}(\Omega))} \leq c_0||u_0||_{p,q}.$$

For any $f \in L^q(0,T;W^{-1,p}(\Omega))$, set

$$(e^{-\widetilde{A}} * f)(t) = \int_0^t e^{(t-s)\widetilde{A}}f(s)ds, \quad 0 \leq t \leq T.$$

Since $-\widetilde{A}$ generates an analytic semigroup $\{e^{-t\widetilde{A}} : 0 \leq t < \infty\}$ in $W^{-1,p}(\Omega)$ and applying Theorem 3.2 of [5] to the equation (LE), we have (3.17)(see Theorem 2.3 of [4]) and

$$e^{-\widetilde{A}} * f \in L^q(0,T;W^{1,p}_0(\Omega)) \bigcap W^{1,q}(0,T;W^{-1,p}(\Omega)).$$

The last inclusion relation of (3.16) is well known and is an easy consequence of the definition of real interpolation space by the trace method.

The proof of 2) is obtained by applying the argument of 1) term by term to the equation (LE) due to (3.1) in the space $H_{p,q}$. □

Remark 3.4 *By terms of Proposition 3.3, the result of [[5], Theorem 2.1] implies that if* $u_0 \in (D(A), L^p(\Omega))_{1/q,q} \equiv Z_{p,q}$ *and* $f \in L^q(0,T;L^p(\Omega))$, *then there exists a unique solution* u *of (LE) belonging to*

$$\mathcal{W}_1 \equiv L^q(0,T;D(A)) \bigcap W^{1,q}(0,T;L^p(\Omega)) \subset C([0,T];Z_{p,q}) \qquad (3.18)$$

and satisfying

$$||u||_{\mathcal{W}_1} \leq C_1(||u_0||_{Z_{p,q}} + ||f||_{L^q(0,T;L^p(\Omega))}), \tag{3.19}$$

where C_1 is a constant depending on T.

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4 Properties of the principal operator

This section is to investigate the regularity of solutions for an abstract parabolic type equation (1.1) in the strong sense in case for any $u_0 \in H_{p,q}(1 < p, q < \infty)$ and $f \in L^q(0,T; W^{-1,p}(\Omega))$. Now, we put that

$$Au = -\mathcal{A}(x, D_x)u \quad \text{i.e.,} \quad A = \widetilde{A} \tag{4.1}$$

which was defined in the previous section, and $\mathcal{A}(x, D_x)$ is restriction to $W_0^{1,p}(\Omega)$ with real coefficients:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$ and $\{a_{ij}(x)\}$ is positive definite uniformly in Ω, i.e., there exists a positive number c_1 such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq c_1|\xi|^2 \tag{4.2}$$

for all $x \in \bar{\Omega}$ and all real vectors ξ, $b_i \in C^1(\Omega)$, and $c \in L^\infty(\Omega)$. On the other hand, by this hypothesis, there exists a certain K such that $|b_i(x)| \leq K$ and $|c(x)| \leq K$ hold almost everywhere.

We denote the pairings between $L^{p'}(\Omega)$ and $L^p(\Omega)$, $W^{-1,p}(\Omega)$ and $W_0^{1,p'}(\Omega)$, and $D(A'_{p'})^*$ and $D(A'_{p'})$ all by $(\cdot,\cdot)$ with no fear of confusion.

Theorem 4.1 *Let the operator A be defined by (4.1). Then A is symmetric bounded operator from $W_0^{1,p}(\Omega)$ into $W^{-1,p}(\Omega)$ and there exist constants $\omega_1 > 0,\ \omega_2 \geq 0$ such that for any $u \in W_0^{1,p}(\Omega)$*

$$||Au||_{-1} \geq \omega_1 ||u||_{1,2}^2 - \omega_2 ||u||_P^2. \tag{4.3}$$

Proof. For each $u, v \in H_1(\Omega)$, we put

$$a(u,v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} \bar{v} + cu\bar{v} \right\} dx. \tag{4.4}$$

Since $\{a_{ij}\}$ is real symmetric, by (4.2) the inequality

$$\sum_{i,j=1}^n a_{ij}(x) \zeta_i \bar{\zeta_j} \geq c_0 |\zeta|^2 \tag{4.5}$$

holds for all complex vectors $\zeta = (\zeta_1, \cdots, \zeta_n)$. Hence, by (4.3),(4.4) we have

$$\begin{aligned} \operatorname{Re} a(u,u) &\geq \int_\Omega c_0 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx - K \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right| |u| dx - K \int_\Omega |u|^2 dx \\ &\geq c_0 \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dx - K \int_\Omega \sum_{i=1}^n \left(\frac{\varepsilon}{2} \left| \frac{\partial u}{\partial x_i} \right|^2 + \frac{1}{2\epsilon} |u|^2 \right) dx \\ &\quad - K \int_\Omega |u|^2 dx \\ &= \left(c_0 - \frac{\epsilon}{2} K \right) \sum_{i=1}^n \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 dx - \left(\frac{nK}{2\varepsilon} + K \right) \int_\Omega |u|^2 dx. \end{aligned}$$

By choosing $\epsilon = c_0K^{-1}$, we obtain

$$\mathrm{Re\,a(u,u)} \geq \frac{c_0}{2}\sum_{i=1}^{n}\int_{\Omega}\left|\frac{\partial u}{\partial x_i}\right|^2 dx - \left(\frac{nK^2}{2c_0} + K\right)\int_{\Omega}|u|^2 dx$$

$$= \frac{c_0}{2}\|u\|_1^2 - \left(\frac{nK^2}{2c_0} + K + \frac{c_0}{2}\right)\|u\|^2.$$

Therefore, regarding as (3.3), the proof is completed. □

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