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On πgp-locally closed sets and πgp-locally continuous functions

by

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Graduate School of Education

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(πgp-국소폐집합과 πgp-국소연속함수에 관하여)

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CONTENTS

Abstract(Korean) i

1. Introduction 1

2. Preliminaries 2

3. πgp-locally closed sets 4

4. Properties of πgp-lc sets 9

5. πgp-locally continuous and πgp-locally irresolute functions .. 13

References 17

πgp-국소 폐집합과 πgp-국소 연속 함수에 관하여

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요약

본 논문에서는 πgp-개집합과 πgp-폐집합의 정의와 이를 이용한 g-lc 집합, p-lc 집합를 포함하는 πgp-lc 집합, πgp-lc^* 집합, πgp-lc^{**} 집합이라 불리는 세 가지 새로운 집합을 소개하고 그 집합들의 모임인 πGp-$LC(X,\tau)$, πGp-$LC^*(X,\tau)$ 및 πGp-$LC^{**}(X,\tau)$이 갖는 성질에 대하여 알아보았다.

또한, 그러한 집합들을 이용하여 πgp-lc 연속 함수, πgp-lc^* 연속 함수, πgp-lc^{**} 연속 함수와 πgp-lc irresolute 함수, πgp-lc^* irresolute 함수, πgp-lc^{**} irresolute 함수를 정의하고 이들의 관계를 살펴보았다.

1 Introduction

Levine [10] initiated the investigation of so-called g-closed sets in topological spaces, since then many modifications of g-closed sets were defined and investigated by many authors. Zaitsev [14] introduced the concept of π-closed sets and a class of topological spaces called quasi-normal spaces. Recently, Dontchev and Noiri [6] defined the concept of πg-closed sets as a weak form of g-closed sets and used this notion to obtain a characterization and some preservation theorems for quasi-normal spaces. More recently, Park et al. [13] introduced and studied the notion of πgp-closed sets which is implied by that of πg-closed sets. The notions of πgp-open sets, πgp-$T_{1/2}$ spaces, πgp-continuity and πgp-irresoluteness are also introduced by Park et al. [13]. The notion of a locally closed set in topological space was implicitly introduced by Kuratowski and Sierpienski [9]. According to Bourbaki [4] a subset of a topological space X is locally closed in X. In 1989, Ganster and Reilly [8] continued the study of locally closed set and also introduced the concept of LC-continuous functions to find a decomposition of continuous functions. Balachandran et al. [3] introduced the concept of generalized continuity. Arockia Rani et al. [1] introduced regular generalized locally closed sets and obtained six more new classes of generalized continuity using the concept of regular generalized closed sets [12]. They also introduced $\pi G\alpha$-LC sets, $\pi G\alpha$-LC^* sets, $\pi G\alpha^{**}$-LC sets and different classes of continuous and irresolute functions [2].

The purpose of this paper is to introduce three new classes of sets called πgp-lc sets, πgp-lc^* sets, πgp-lc^{**} sets which contain the class of glc-sets and p-lc sets by using the notion of πgp-open sets and πgp-closed sets. Also we introduce some different classes of continuity and irresoluteness and study some of their properties.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. We denote the closure and the interior of a set A by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = \mathrm{int}(\mathrm{cl}(A))$ (resp. $A = \mathrm{cl}(\mathrm{int}(A))$). The finite union of regular open sets is said to be π-open [14]. The complement of a π-open set is said to be π-closed [14]. A subset A is said to be α-open [11] (resp. *pre*-open) if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ (resp. $A \subset \mathrm{int}(\mathrm{cl}(A))$). The complement of an α-open (resp. *pre*-open) set is said to be α-closed (resp. *pre*-closed). The preclosure is denoted by $\mathrm{pcl}(A)$.

We recall the following definitions used in sequel.

Definition 2.1 A subset A of a space (X, τ) is called

(a) g-closed [6] if $\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is open in X;

(b) $\pi g\alpha$-closed [3] if $\alpha\mathrm{cl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X;

(c) πgp-closed [13] if $\mathrm{pcl}(A) \subset U$ whenever $A \subset U$ and U is π-open in X.

Definition 2.2 A subset A of (X, τ) is called

(a) locally closed set [7] (briefly, lc set) if $A = G \cap F$ where G is open and F is closed;

(b) $\pi g\alpha$-lc set [2] if $S = A \cap B$ where A is $\pi g\alpha$-open in X and B is $\pi g\alpha$-closed in X;

(c) $\pi g\alpha$-lc^* set [2] if there exist a $\pi g\alpha$-open set A and a closed set B such that $S = A \cap B$;

(d) $\pi g\alpha$-lc^{**} set [2] if there exist an open set A and a $\pi g\alpha$-closed set B such that $S = A \cap B$.

The collection of all lc sets, (resp. $\pi g\alpha$-lc sets, $\pi g\alpha$-lc^* sets, $\pi g\alpha$-lc^{**} sets) of (X, τ) will be denoted by $LC(X, \tau)$, (resp. $\pi G\alpha$-$LC(X, \tau)$, $\pi G\alpha$-$LC^*(X, \tau)$, $\pi G\alpha$-$LC^{**}(X, \tau)$).

Definition 2.3 A function $f : (X, \tau) \to (Y, \sigma)$ is called

(a) lc continuous [7] if $f^{-1}(V) \in LC(X, \tau)$ for each open set V of (Y, σ);

(b) lc irresolute [7] if $f^{-1}(V) \in LC(X, \tau)$ for each open set V of $LC(Y, \sigma)$;

(c) Sub-lc continuous [7] if there is a sub-base B for (Y, σ) such that $f^{-1}(V) \in LC(X, \tau)$ for each $V \in B$.

Definition 2.4 A space (X, τ) is called

(a) submaximal space [5] if every dense subset of X is open;

(b) door space [4] if every subset of X is either open or closed in X;

(c) $\pi g\alpha$-$T_{1/2}$ space if every $\pi g\alpha$-closed set is α-closed.

Theorem 2.5 [13] If A is π-open and πgp-closed in (X, τ), then A is pre-closed and hence clopen.

Lemma 2.6 [13] If $A \subset X \subset Y$ and Y is open in X, then $\mathrm{pcl}_Y(A) = \mathrm{pcl}_X(A) \cap Y$.

Lemma 2.7 [13] Let Y is open in X. Then

(a) If A is π-open in Y, then there exists a π-open set B in X such that $A = B \cap Y$.

(b) If A is π-open in X, then $A \cap B$ is π-open in Y.

Theorem 2.8 [13] Let $A \subset Y \subset X$. Then

(a) If Y is open in X and A is πgp-closed in X, then A is πgp-closed in Y.

(b) If Y is πgp-closed and regular open in X and A is πgp-closed in Y, then A is πgp-closed in X.

3 πgp-locally closed sets

In this section we define πgp-locally closed sets which are weaker forms of locally closed sets and compare it with existing weaker forms of sets.

Definition 3.1 A subset S of a space (X, τ) is called

(a) p-lc set if $S = G \cap F$ where G is pre-closed and F is pre-open;

(b) πgp-lc set if $S = A \cap B$ where A is πgp-open in X and B is πgp-closed in X;

(c) πgp-lc^* if there exist a πgp-open set A and a closed set B such that $S = A \cap B$;

(d) πgp-lc^{**} if there exist an open set A and a πgp-closed set B such that $S = A \cap B$.

The collection of all p-lc sets, (resp. πgp-lc sets, πgp-lc^* sets, πgp-lc^{**} sets) of (X, τ) will be denoted by p-$LC(X, \tau)$, (resp. πGp-$LC(X, \tau)$, πGp-$LC^*(X, \tau)$, πGp-$LC^{**}(X, \tau)$).

Remark 3.2 From Definition 3.1, we have the following diagram of implications.

closed set
↓
locally closed set
↙ ↘
$\pi g\alpha$-lc^{**} set $\pi g\alpha$-lc^* set
| ↘ ↙ |
↓ $\pi g\alpha$-lc set ↓
πgp-lc^{**} set | πgp-lc^* set
↘ ↓ ↙
πgp-lc set

In the above remark the relationship cannot be reversible as the following example illustrates.

Example 3.3 (a) Let $X = \{a, b, c, d, e\}$ and $\tau = \{X, \emptyset, \{a\}, \{e\}, \{a, e\}, \{c, d\}, \{a, c, d\}, \{c, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$, then $\{b, c\} \in \pi Gp\text{-}LC^*(X, \tau)$ but $\{a, c\} \notin \pi G\alpha\text{-}LC^*(X, \tau)$.

(b) Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$, then

(i) $\{b, c, d\} \in \pi Gp\text{-}LC(X, \tau)$ but $\{b, c, d\} \notin \pi Gp\text{-}LC^*(X, \tau)$.

(ii) $\{a, c\} \in \pi Gp\text{-}LC(X, \tau)$ but $\{a, c\} \notin \pi Gp\text{-}LC^{**}(X, \tau)$.

Remark 3.4 Every πgp-closed (resp. πgp-open) set is πgp-lc set and every locally closed set is πgp-closed but not conversely.

Example 3.5 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$. Then $\{a, c\}$ is a πgp-closed set but $\{a, c\}$ is not a locally closed set.

Remark 3.6 If $A \in LC(X, \tau)$, then $A \in \pi Gp\text{-}LC^*(X, \tau)$ and $\pi Gp\text{-}LC^{**}(X, \tau)$. The converse is not true as seen in the following example. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$. Then $LC(X, \tau) = \{X, \emptyset, \{c\}, \{a, b\}\}$ and $\{a\} \in \pi Gp\text{-}LC^*(X, \tau)$ and $\pi Gp\text{-}LC^*(X, \tau) = \pi Gp\text{-}LC^{**}(X, \tau) = P(X)$. Hence $\{a\} \in \pi Gp\text{-}LC^{**}(X, \tau)$ but $\{a\} \notin LC(X, \tau)$.

Definition 3.7 A space is a πgp-space if every πgp-open set is open in X.

Theorem 3.8 Let (X, τ) be a πgp-space, then

(a) $\pi Gp\text{-}LC^{**}(X, \tau) = LC(X, \tau)$.

(b) $\pi Gp\text{-}LC^{**}(X, \tau) \subset GLC(X, \tau)$.

(c) $\pi Gp\text{-}LC^{**}(X, \tau) \subset \alpha LC(X, \tau)$.

Proof Obvious.

Definition 3.9 A space is (X, τ) is πgp-$T_{1/2}$ spaces if every πgp-closed set is *pre*-closed.

Theorem 3.10 If X is a πgp-$T_{1/2}$ space, then $\pi Gp\text{-}LC(X, \tau) = p\text{-}LC(X, \tau)$.

Proof It follows from Definition 3.9.

The converse of the above theorem need not hold.

Example 3.11 Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\ \{a, c, d\}, \{a, b, d\}\}$, then $\pi Gp\text{-}LC(X, \tau) = p\text{-}LC(X, \tau) = P(X)$. But $\{a\}$ is πgp-closed set not *pre*-closed.

Theorem 3.12 If X is a πgp-space, then

$$\pi Gp\text{-}LC(X, \tau) = \pi Gp\text{-}LC^{*}(X, \tau) = \pi Gp\text{-}LC^{**}(X, \tau).$$

Proof Straightforward.

The hypothesis in Theorem 3.12 can be weakened as follows.

Theorem 3.13 If $\pi GpO(X, \tau) \subset LC(X, \tau)$ and suppose that collections of all πgp-closed (or πgp-open) sets are closed under finite intersection, then

$$\pi Gp\text{-}LC(X, \tau) = \pi Gp\text{-}LC^{*}(X, \tau) = \pi Gp\text{-}LC^{**}(X, \tau).$$

Proof Let $A \in \pi Gp\text{-}LC(X)$. Then $A = P \cap Q$ where P is πgp-open and Q is πgp-closed. Since $\pi GpO(X, \tau) \subset LC(X, \tau)$ implies $\pi GpC(X, \tau) \subset LC(X, \tau)$, we have Q is locally closed. Let $Q = M \cap N$ where M is open and N is closed. Hence $A = (P \cap M) \cap N$ where $(P \cap M)$ is πgp-open and N is closed. Hence $A \in \pi Gp\text{-}LC^{*}(X)$. For any space X, $\pi Gp\text{-}LC^{*}(X) \subset \pi Gp\text{-}LC(X)$. Thus $\pi Gp\text{-}LC(X) = \pi Gp\text{-}LC^{*}(X)$. Let $B \in \pi Gp\text{-}LC(X)$. Then $B = P \cap Q$ where P is locally closed, we have $P = M \cap N$ where M is open and N is closed. Hence $A = M \cap (N \cap Q)$ where M is open and $N \cap Q$ is πgp-closed. For any space X, $\pi Gp\text{-}LC^{**}(X) \subset \pi Gp\text{-}LC(X)$. Thus $\pi Gp\text{-}LC(X) = \pi Gp\text{-}LC^{**}(X)$.

Now, we obtain a characterization for $\pi gp\text{-}lc^{*}$ sets as follows.

Theorem 3.14 For a subset S of (X, τ) the following are equivalent:

(a) $S \in \pi Gp\text{-}LC^{**}(X, \tau)$.
(b) $S = P \cap \mathrm{cl}(S)$ for some πgp-open set P.
(c) $\mathrm{cl}(S) \setminus S$ is πgp-closed.
(d) $S \cup (X \setminus \mathrm{cl}(S))$ is πgp-open.

Proof $(a) \Rightarrow (b)$: Let $S \in \pi Gp\text{-}LC^*(X,\tau)$. Then there exist a πgp-open set P and a closed set F in (X,τ) such that $S = P \cap F$. Since $S \subset P$ and $S \subset \mathrm{cl}(S)$, we have $S \subset P \cap \mathrm{cl}(S)$. Conversely, $P \cap \mathrm{cl}(S) \subset P \cap F = S$ since $\mathrm{cl}(S) \subset F$. Hence $S = P \cap \mathrm{cl}(S)$.

$(b) \Rightarrow (a)$: Since P is πgp-open and $\mathrm{cl}(S)$ is closed, $S = P \cap \mathrm{cl}(S) \in \pi Gp\text{-}LC^*(X,\tau)$.

$(c) \Rightarrow (d)$: Let $F = \mathrm{cl}(S) \setminus S$. Then F is πgp-closed by assumption. $X \setminus F = X \cap (\mathrm{cl}(S) \setminus S)^c = S \cup (X \setminus \mathrm{cl}(S))$. Since $X \setminus F$ is πgp-open, we have that $S \cup (X \setminus \mathrm{cl}(S))$ is πgp-open.

$(d) \Rightarrow (c)$: Let $U = S \cup (X \setminus \mathrm{cl}(S))$. Then U is πgp-open. This implies $X \setminus U = X \setminus (S \cup (X \setminus \mathrm{cl}(S))) = (X \setminus S) \cap \mathrm{cl}(S) = \mathrm{cl}(S) \setminus S$ is πgp-closed.

$(b) \Rightarrow (d)$: Let $S = P \cap \mathrm{cl}(S)$ for some πgp-open set P. $S \cup (X \setminus \mathrm{cl}(S)) = P \cap (\mathrm{cl}(S) \cup X \setminus \mathrm{cl}(S)) = P \cap X = P$ which is πgp-open.

$(d) \Rightarrow (b)$: Let $U = S \cup (X - \mathrm{cl}(S))$. Then U is πgp-open. Now, $U \cap \mathrm{cl}(S) = (S \cup (X \setminus \mathrm{cl}(S))) \cap \mathrm{cl}(S) = (S \cap \mathrm{cl}(S)) \cup (X \setminus \mathrm{cl}(S) \cap \mathrm{cl}(S)) = S \cup \emptyset = S$ for some πgp-open set U.

Remark 3.15 It is not true that $S \in \pi Gp\text{-}LC^*(X,\tau)$ iff $S \subset \mathrm{int}(S \cup (X \setminus \mathrm{cl}(S)))$. Let $S = \{b, d\}$ be a subset of the topological space (X,τ) given in Example 3.3 (b). Then $S \notin \mathrm{int}(S \cup (X \setminus \mathrm{cl}(S)))$ but $S \in \pi Gp\text{-}LC^*(X,\tau)$.

Definition 3.16 A topological space (X,τ) is called πgp-submaximal if every dense subset in it is πgp-open.

Theorem 3.17 Every submaximal space is πgp-submaximal.

Proof It follows from Definition 3.16.

Converse of the above is not true as seen in the following example.

Example 3.18 Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Let $A = \{b, d\}$. Then A is dense in X such that A is πgp-open but not open.

Theorem 3.19 A topological space (X, τ) is πgp-submaximal if and only if $\pi Gp\text{-}LC^{*}(X, \tau) = P(X)$.

Proof *Necessity* : Let $S \in P(X)$. Let $U = S \cup (X \setminus \mathrm{cl}(S))$. Then $\mathrm{cl}(U) = X$. U is dense in X and X is πgp-submaximal implies U is πgp-open. By Theorem 3.14, $S \in \pi Gp\text{-}LC^{*}(X, \tau)$.

Sufficiency : Let S be a dense subset of (X, τ). Then $S = (X \setminus \mathrm{cl}(S)) = S$ is πgp-open. Hence (X, τ) is πgp-submaximal.

Definition 3.20 The intersection of all πgp-closed sets of X containing A is the πgp-closure of A is denoted by $\pi gp\mathrm{cl}(A)$.

Theorem 3.21 Let S be any subset of (X, τ). If $S \in \pi Gp\text{-}LC^{**}(X, \tau)$, then there exist an open set P such that $S = P \cap \pi gp\mathrm{cl}(S)$.

Proof Let $S \in \pi Gp\text{-}LC^{**}(X, \tau)$. Then there exist an open set P and πgp-closed set F of (X, τ) such that $S = P \cap F$. Since $S \subset P$ and $S \subset \pi gp\mathrm{cl}(S)$, we have $S \subset P \cap \pi gp\mathrm{cl}(S)$. Since $\pi gp\mathrm{cl}(S) \subset F$, we have $P \cap \pi gp\mathrm{cl}(S) \subset P \cap F \subset S$. Thus $S = P \cap \pi gp\mathrm{cl}(S)$.

4 Properties of πgp-lc sets

Theorem 4.1 Let A and B be any two subsets of (X, τ). Suppose that the collection of πgp-closed set of (X, τ) is closed under finite intersection, then the following are true.

(a) If $A \in \pi Gp\text{-}LC(X, \tau)$ and B is πgp-open (or πgp-closed), then

$$A \cap B \in \pi Gp\text{-}LC(X, \tau).$$

(b) If $A \in \pi Gp\text{-}LC^*(X, \tau)$ and $B \in \pi Gp\text{-}LC^*(X, \tau)$, then

$$A \cap B \in \pi Gp\text{-}LC^*(X, \tau).$$

Proof (a) $A \in \pi Gp\text{-}LC(X, \tau)$ implies $A \cap B = (G \cap F) \cap B$ for some πgp-open set G and πgp-closed set F. If B is πgp-open, then $A \cap B = (G \cap B) \cap F \in \pi Gp\text{-}LC(X, \tau)$. If B is πgp-closed, then $A \cap B = G \cap (F \cap B) \in \pi Gp\text{-}LC(X, \tau)$, since $F \cap B$ is πgp-closed.

(b) If $A, B \in \pi Gp\text{-}LC^*(X, \tau)$, then by Theorem 3.14, there exist πgp-open sets P and Q such that $A = P \cap \mathrm{cl}(A)$ and $B = Q \cap \mathrm{cl}(B)$. Since $P \cap Q$ is also πgp-open, then $A \cap B = (P \cap Q) \cap (\mathrm{cl}(A) \cap \mathrm{cl}(B)) \in \pi Gp\text{-}LC^*(X, \tau)$.

Theorem 4.2 Let A and B be any two subsets of (X, τ). Suppose that the collection of πgp-closed set (X, τ) is closed under finite intersection. If $A \in \pi Gp\text{-}LC^{**}(X, \tau)$ and B is closed or open, then $A \cap B \in \pi Gp\text{-}LC^{**}(X, \tau)$.

Proof If $A \in \pi Gp\text{-}LC^{**}(X, \tau)$, then there exist an open set G in (X, τ) and a πgp-closed set F in (X, τ) such that $A \cap B = (G \cap F) \cap B$. If B is open, then $A \cap B = (G \cap B) \cap F \in \pi Gp\text{-}LC^{**}(X, \tau)$. If B is closed, then $A \cap B = G \cap (B \cap F) \in \pi Gp\text{-}LC^{**}(X, \tau)$.

Theorem 4.3 Let A be any subset of (X, τ) and let $A \subset Z$ which Z is πgp-open in (X, τ) and regular closed. Suppose that the collection of all πgp-open sets of (X, τ) is closed under finite intersection. If $A \in \pi Gp\text{-}LC^*(Z, \tau/Z)$, then $A \in \pi Gp\text{-}LC^*(X, \tau)$.

Proof If $A \in \pi Gp\text{-}LC^*(Z, \tau/Z)$, there is a πgp-open set G in $(Z, \tau/Z)$ such that $A = G \cap \mathrm{cl}_Z(A)$ where $\mathrm{cl}_Z(A) = Z \cap \mathrm{cl}(A)$. Since G and Z are πgp-open, $G \cap Z$ is also πgp-open. This implies that $A = (G \cap Z) \cap \mathrm{cl}(A) \in \pi Gp\text{-}LC^*(X, \tau)$.

Remark 4.4 The following example shows that one of the assumption in the above theorem (i.e., Z is πgp-open in (X, τ)) cannot be removed.

Example 4.5 Let $X = \{a, b, c, d\}$ and $\tau = \{X, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$. Let $Z = A = \{a, c\}$. Then Z is not πgp-open in X and $\tau/Z = \{Z, \emptyset, \{c\}\}$. Hence $A \in \pi Gp\text{-}LC^*(Z, \tau/Z)$ but $A \notin \pi Gp\text{-}LC^*(X, \tau)$.

Lemma 4.6 Let Z be regular open and πgp-closed in (X, τ) and $F \subset Z$. If F is πgp-closed in $(Z, \tau/Z)$, then F is πgp-closed in (X, τ).

Theorem 4.7 Suppose that the collection of all πgp-closed sets of (X, τ) is closed under finite intersection. If Z is πgp-closed and regular open in (X, τ) and $A \in \pi Gp\text{-}LC^*(Z, \tau/Z)$, then $A \in \pi Gp\text{-}LC(X, \tau)$.

Proof Let $A \in \pi Gp\text{-}LC^*(Z, \tau/Z)$. Then $A = G \cap F$ for some πgp-open set G in $(Z, \tau/Z)$ and some closed set F in $(Z, \tau/Z)$. F is closed in Z. By Lemma 4.6, Z is πgp-closed and regular open in X implies F is πgp-closed in (X, τ). Hence $A = G \cap F \in \pi Gp\text{-}LC(X, \tau)$.

Theorem 4.8 If Z is closed and open in (X, τ) and $A \in \pi Gp\text{-}LC(Z, \tau/Z)$, then $A \in \pi Gp\text{-}LC(X, \tau)$.

Proof Let $A \in \pi Gp\text{-}LC(Z, \tau/Z)$. Then $A = G \cap F$ where G is πgp-open in $(Z, \tau/Z)$ and F is πgp-closed in $(Z, \tau/Z)$. Since Z is closed and open in (X, τ) by Lemma 4.6, G and F are πgp-open and πgp-closed respectively in (X, τ). Therefore $A \in \pi Gp\text{-}LC(X, \tau)$.

Theorem 4.9 If $A \in \pi Gp\text{-}LC^{**}(Z, \tau/Z)$, where Z is πgp-closed and regular open in (X, τ), then $A \in \pi Gp\text{-}LC^{**}(X, \tau)$.

Proof Let $A \in \pi Gp\text{-}LC^{**}(Z, \tau/Z)$. Then $A = G \cap F$ where G is πgp-open in $(Z, \tau/Z)$ and F is πgp-closed in $(Z, \tau/Z)$. Since Z is πgp-closed and regular open, G is open set in (X, τ) and F is open set and πgp-closed set in (X, τ). Then $A \in \pi Gp\text{-}LC^{**}(X, \tau)$.

Definition 4.10 Let $A, B \in X$. Then A and B are said to be *separated* if $A \cap \mathrm{cl}(B) = \emptyset$ and $B \cap \mathrm{cl}(A) = \emptyset$.

Theorem 4.11 Suppose the collection of all πgp-open sets of (X, τ) are closed under finite unions. Let $A, B \in \pi Gp\text{-}LC^{*}(X, \tau)$. If A and B are separated in (X, τ), then $A \cup B \in \pi Gp\text{-}LC^{*}(X, \tau)$.

Proof Since $A, B \in \pi Gp\text{-}LC^{*}(X, \tau)$ by Theorem 3.14, there exist πgp-open sets P and Q of (X, τ) such that $A = P \cap \mathrm{cl}(A)$ and $B = Q \cap \mathrm{cl}(B)$. Put $U = P \cap (X \setminus \mathrm{cl}(B))$ and $V = Q \cap (X \setminus \mathrm{cl}(A))$. Then U and V are πgp-open subsets of (X, τ). Thus $A = U \cap \mathrm{cl}(A)$, $B = V \cap \mathrm{cl}(B)$ and $U \cap \mathrm{cl}(B) = \emptyset, V \cap \mathrm{cl}(A) = \emptyset$. Consequently $A \cup B = (U \cap V) \cap (\mathrm{cl}(A \cup B))$ shows that $A \cup B \in \pi Gp\text{-}LC^{*}(X, \tau)$.

Remark 4.12 The following example shows that one of assumption of Theorem 4.11 (i.e., A and B are separated) cannot be removed.

In Example 3.3 (b), $\{b\} \in \pi Gp\text{-}LC^{*}(X, \tau), \{c, d\} \in \pi Gp\text{-}LC^{*}(X, \tau)$. However $\{b\}$ and $\{c, d\}$ are not separated and $\{b, c, d\} \notin \pi Gp\text{-}LC^{*}(X, \tau)$.

Theorem 4.13 Let $\{Z_i : i \in \Gamma\}$ be a finite πgp-closed cover of (X, τ) and let A be a subset of (X, τ). If $A \cap Z_i \in \pi Gp\text{-}LC^{**}(Z_i, \tau/Z_i)$ for each $i \in \Gamma$, then $A \in \pi Gp\text{-}LC^{**}(X, \tau)$.

Proof For each $i \in \Gamma$, there exist an open set $U_i \in \tau/Z_i$ and πgp-closed set F_i of $(Z_i, \tau/Z_i)$ such that $A \cap Z_i = (U_i \cap F_i) \cap Z_i = U_i \cap (F_i \cap Z_i)$. Then $A = \cup\{A \cap Z_i : i \in \Gamma\} = \cup\{U_i : i \in \Gamma\} \cap [\cup\{F_i \cap Z_i : i \in \Gamma\}]$ and hence $A \in \pi Gp\text{-}LC^{**}(X, \tau)$.

Theorem 4.14 Let (X, τ) and (Y, σ) be any two topological spaces. Then

(a) If $A \in \pi Gp\text{-}LC(X, \tau)$ and $B \in \pi Gp\text{-}LC(Y, \sigma)$, then

$$A \times B \in \pi Gp\text{-}LC(X \times Y, \tau \times \sigma).$$

(b) If $A \in \pi Gp\text{-}LC^*(X, \tau)$ and $B \in \pi Gp\text{-}LC^*(Y, \sigma)$, then

$$A \times B \in \pi Gp\text{-}LC^*(X \times Y, \tau \times \sigma).$$

(c) If $A \in \pi Gp\text{-}LC^{**}(X, \tau)$ and $B \in \pi Gp\text{-}LC^{**}(Y, \sigma)$, then

$$A \times B \in \pi Gp\text{-}LC^{**}(X \times Y, \tau \times \sigma).$$

Proof Let $A \in \pi Gp\text{-}LC(X, \tau)$ and $B \in \pi Gp\text{-}LC(Y, \sigma)$. Then there exist πgp-open sets V and V_1 of (X, τ) and πgp-closed sets W and W_1 of (Y, σ) respectively such that $A = V \cap W$ and $B = V_1 \cap W_1$. Then $A \times B = (V \cap W) \times (V_1 \cap W_1) = (V \times V_1) \cap (W \times W_1)$ hold and hence $A \times B \in \pi Gp\text{-}LC(X \times Y, \tau \times \sigma)$.

Proofs of (b) and (c) are similar to (a).

5 πgp-locally continuous and πgp-locally irresolute functions

In this section we use πgp-lc sets, πgp-lc^* sets, πgp-lc^{**} sets to generalize πgp-lc continuous function and πgp-lc irresolute function.

Definition 5.1 A function $f : (X, \tau) \to (Y, \sigma)$ is called

(a) πgp-lc continuous if $f^{-1}(V) \in \pi Gp\text{-}LC(X, \tau)$ for every $V \in \sigma$;
(b) πgp-lc^* continuous if $f^{-1}(V) \in \pi Gp\text{-}LC^*(X, \tau)$ for every $V \in \sigma$;
(c) πgp-lc^{**} continuous if $f^{-1}(V) \in \pi Gp\text{-}LC^{**}(X, \tau)$ for every $V \in \sigma$;
(d) πgp-lc irresolute if $f^{-1}(V) \in \pi Gp\text{-}LC(X, \tau)$ for every $V \in \pi Gp\text{-}LC(Y, \sigma)$;
(e) πgp-lc^* irresolute if $f^{-1}(V) \in \pi Gp\text{-}LC^*\ (X, \tau)$ for every $V \in \pi Gp\text{-}LC^*(Y, \sigma)$;
(f) πgp-lc^{**} irresolute if $f^{-1}(V) \in \pi Gp\text{-}LC^{**}(X, \tau)$ for every $V \in \pi Gp\text{-}LC^{**}(Y, \sigma)$.

Proposition 5.2 If f is πgp-lc irresolute, then it is πgp-lc continuous.

Proof It follows from Definition 5.1.

Remark 5.3 From Definitions 2.3 and 5.1, we have the following diagram of implications.

$$
\begin{array}{ccccc}
 & & lc \text{ continuous} & & \\
 & \swarrow & & \searrow & \\
\pi g\alpha\text{-}lc^{**} \text{ continuous} & & & & \pi g\alpha\text{-}lc^{*} \text{ continuous} \\
| & \searrow & & \swarrow & | \\
\downarrow & & \pi g\alpha\text{-}lc \text{ continuous} & & \downarrow \\
\pi gp\text{-}lc^{**} \text{ continuous} & & | & & \pi gp\text{-}lc^{*} \text{ continuous} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \pi gp\text{-}lc \text{ continuous} & &
\end{array}
$$

In the above remark the relationship cannot be reversible as the following example illustrates.

Example 5.4 (a) Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b\}, \{c\}$ $, \{a, b\}, \{b, c\}, \{a, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is πgp-lc^* continuous and πgp-lc^{**} continuous but not lc continuous.

(b) Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is πgp-lc continuous but not πgp-lc^* continuous since $\{a, c\} \in \sigma$ but $f^{-1}(\{a, c\}) \notin \pi Gp$-$LC^*(X, \tau)$.

(c) Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is πgp-lc^* continuous but not $\pi g\alpha$-lc^* continuous since $\{a, b, d\} \in \sigma$ but $f^{-1}(\{a, b, d\}) \notin \pi G\alpha$-$LC^*(X, \tau)$.

Lemma 5.5 $f : (X, \tau) \to (Y, \sigma)$ be a function.

(a) If f is πgp-lc^* irresolute, then f is πgp-lc^* continuous.

(b) If f is πgp-lc^{**} irresolute, then f is πgp-lc^{**} continuous.

The converse of the above need not be true in general as can be seen in the following example.

Example 5.6 Let $X = \{a, b, c, d\}, \tau = \{X, \emptyset, \{b\}, \{c, d\}, \{b, c, d\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{a, c\}, \{a, b, c\}, \{b, c, d\}\}$. Let $f : (X, \tau) \to (X, \sigma)$ be the identity mapping. Then f is πgp-lc^* continuous but not πgp-lc^* irresolute since $\{b, c, d\} \in \pi Gp$-$LC^*(Y, \sigma)$ but $f^{-1}(\{b, c, d\}) \notin \pi Gp$-$LC^*(X, \tau)$.

Proposition 5.7 Any map defined on a door space is πgp-lc irresolute.

Proof Let (X, τ) be a door space and (Y, σ) be any space. Define a map $f : (X, \tau) \to (Y, \sigma)$ and let $A \in \pi Gp$-$LC(Y, \sigma)$. Then $f^{-1}(A)$ is either open or closed in (X, τ). In both cases $f^{-1}(A) \in \pi Gp$-$LC(X, \tau)$. Hence f is πgp-lc irresolute.

Theorem 5.8 A topological space (X, τ) is πgp-submaximal iff every function having (X, τ) as it domain is πgp-lc^* continuous.

Proof Suppose that $f:(X,\tau)\to(Y,\sigma)$ is a function. By Theorem 3.19, we have that $f^{-1}(V)\in P(X)=\pi Gp\text{-}LC^*(X,\tau)$ for each open set V of (Y,σ). Therefore f is πgp-lc^* continuous. Conversely, let every map having (X,τ) as domain be πgp-lc^* continuous. Let $Y=\{0,1\}$ be the the Sierpinski space with topology $\sigma=\{Y,\emptyset,\{0\}\}$. Let V be a subset of (X,τ) and $f:(X,\tau)\to(Y,\sigma)$ be a function defined by $f(x)=0$ for every $x\in V$ and $f(x)=1$ for every $x\notin V$. By assumption, f is πgp-lc^* continuous and hence $f^{-1}(0)=V\in\pi Gp\text{-}LC^*(X,\tau)$. Therefore we have $P(X)=\pi Gp\text{-}LC^*(X,\tau)$ and by Theorem 3.19, (X,τ) is πgp-submaximal.

Proposition 5.9 If $f:(X,\tau)\to(Y,\sigma)$ is πgp-lc^{**} continuous and subset B is open in (X,τ), then the restriction of f to B, say $f/B:(B,\tau/B)\to(Y,\sigma)$, is πgp-lc^{**} continuous.

Proof Let V be an open set of (Y,σ). Then $f^{-1}(V)=G\cap F$ for some open set G and πgp-closed set F of (X,τ). Now $G\cap B\in\tau/B$ and F is πgp-closed subset of $(B,\tau/B)$. But $(f/B)^{-1}(V)=(G\cap B)\cap F$. Hence $(f/B)^{-1}(V)\in\pi Gp\text{-}LC^{**}(B,\tau/B)$. This implies that f/B is πgp-lc^{**} continuous.

We recall the definition of the combination of two function: Let $X=A\cup B$ and $f:A\to Y$ and $h:B\to Y$ be two functions. We say that f and h are *compatible* if $f/A\cap B=h/A\cap B$. If $f:A\to Y$ and $h:B\to Y$ are compatible, then the function $f\bigtriangledown h:X\to Y$ defined as

$$(f\bigtriangledown h)(x)=\begin{cases} f(x), & \text{for every } x\in A\\ h(x), & \text{for every } x\in B\end{cases}$$

is called the combination of f and h.

Next we introduce pasting lemma for πgp-lc^{**} continuous (resp. πgp-lc^{**} irresolute) function.

Theorem 5.10 Let $X = A \cup B$ where A and B are πgp-closed subsets of (X, τ) and $f : (A, \tau/A) \to (Y, \sigma)$ and $h : (B, \tau/B) \to (Y, \sigma)$ be compatible functions.

(a) If f and h are πgp-lc^{**} continuous, then $f \bigtriangledown h : (X, \tau) \to (Y, \sigma)$ is πgp-lc^{**} continuous.

(b) If f and h are πgp-lc^{**} irresolute, then $f \bigtriangledown h : (X, \tau) \to (Y, \sigma)$ is πgp-lc^{**} irresolute.

Proof (a) Let $V \in \sigma$. Then $(f \bigtriangledown h)^{-1}(X) \cap A = f^{-1}(X)$ and $(f \bigtriangledown h)^{-1}(X) \cap B = h^{-1}(V)$. By assumption $(f \bigtriangledown h)^{-1}(X) \cap A \in \pi Gp\text{-}LC^{**}(A, \tau/A)$ and $(f \bigtriangledown h)^{-1}(X) \cap B \in \pi Gp\text{-}LC^{**}(B, \tau/B)$. Therefore by Theorem 4.13,

$$(f \bigtriangledown h)^{-1}(V) \in \pi Gp\text{-}LC^{**}(X, \tau)$$

and hence $f \bigtriangledown h$ is πgp-lc^{**} continuous.

(b) Proof is similar to (a).

In the end of this section we have the theorem concerning the composition of function.

Theorem 5.11 Let $f : (A, \tau) \to (Y, \sigma)$ and $g = (Y, \sigma) \to (Z, \eta)$ be the function. Then

(a) $g \circ f$ is πgp-lc irresolute (resp. πgp-lc^{*} irresolute, πgp-lc^{**} irresolute) if f and g are πgp-lc irresolute (resp. πgp-lc^{*} irresolute, πgp-lc^{**} irresolute).

(b) $g \circ f$ is πgp-lc continuous if f is πgp-lc irresolute and g is πgp-lc continuous.

(c) $g \circ f$ is πgp-lc^{*} continuous if f is πgp-lc^{*} continuous and g is continuous.

(d) $g \circ f$ is πgp-lc continuous if f is πgp-lc continuous and g is continuous.

(e) $g \circ f$ is πgp-lc^{*} continuous if f is πgp-lc^{*} irresolute and g is πgp-lc^{*} continuous.

(f) $g \circ f$ is πgp-lc^{**} continuous if f is πgp-lc^{**} irresolute and g is πgp-lc^{**} continuous.

Proof It follows from Definition 5.1 and Proposition 5.2.

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