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Semi Hausdorffness on Generalized Intuitionistic Fuzzy Filters

by

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Graduate School of Education

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Semi Hausdorffness on Generalized Intuitionistic Fuzzy Filters
(일반화된 직관적 퍼지 필터 상에서의 반-하우스도르프성)

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A Dissertation

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요 약

본 논문에서는 일반화된 직관적 퍼지 필터(generalized intuitionistic fuzzy filter, GIFF)상의 반-하우스도르프(semi Hausdorff)의 개념을 소개하고 그에 대한 몇 가지 성질들을 소개하였다. 그리고 일반화된 직관적 퍼지 필터의 수열적 수렴성과 수열적 컴팩트의 개념을 소개하고, 그 기본적인 성질을 밝혔다. 또한, 일반화된 직관적 퍼지 위상공간에서 일반화된 직관적 퍼지 필터의 수렴성 및 일반화된 직관적 퍼지적인 쌓인점(generalized intuitionistic fuzzily cluster points)을 정의하여 일반화된 직관적 퍼지 위상공간이 일반화된 직관적 퍼지적인 컴팩트(generalized intuitionistic fuzzily compact)이기위한 필요충분조건임을 조사하였다.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [19]. Atanassov [1] generalized this idea to intuitionistic fuzzy sets, and later there has been much progress in the study of intuitionistic fuzzy sets by many authors [1-5,9,10,13]. On the other hand, Lowen [14] introduced the concept of fuzzy filter and defined convergence in a fuzzy topological space which enables us to characterize fuzzy compactness. Many results on fuzzy filter are obtained by De Prada and Saralegui [11,12] and Ramakrishnan and Nayagam [16]. More recently, Mondal and Samanta [13] introduced definitions of GIF sets, generalized intuitionistic fuzzy relations and generalized intuitionistic fuzzy topology and studied some of their properties. The notion of Hausdorff GIF filter was intoduced and studied by Park et al. [20].

In this thesis, a new notion of semi Hausdorffness, which can not be defined in usual theory of filters is introduced and studied to some extend in chapter 3. Moreover, we prove that an equivalent condition of generalized intuitoinistic fuzzily compactness through GIF filter convergence has been studied in chapter 4.

2 Preliminaries

Definition 2.1 [1] Let X be a nonempty fixed set. An intuitionistic fuzzy set A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$.

Obviously, every fuzzy set $\{\langle \mu_A(x), x\rangle : x \in X\}$ on X is an intuitionistic fuzzy set of the form $A = \{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle : x \in X\}$. For an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$, it is observed that $\mu_A(x) + \gamma_A(x) \leq 1$ for each $x \in X$ and hence $\mu_A(x) \wedge \gamma_A(x) \not> \frac{1}{2}$ for each $x \in X$. For examples, the attributes (i) 'beauty' and 'fatty' (ii) 'attentiveness'and 'dullness' (iii) 'interior' and 'frontier' etc. are such that both attributes are not significant simultaneously, but sum of their degrees may exceed 1. Having motivated from the observation, Montal and Samanta [13] defined a generalized intuitionistic fuzzy set as follows:

Definition 2.2 [13] Let X be a nonempty fixed set. A generalized intuitionistic fuzzy set (GIF set for short) A in X is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each $x \in X$ to the set A, respectively, and $\mu_A(x) \wedge \gamma_A(x) \leq \frac{1}{2}$ for each $x \in X$. Thus every intuitionistic fuzzy set is GIF but not conversely.

A GIF set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ in X can identified to an ordered pair $\langle \mu_A(x), \gamma_A(x)\rangle$ in $[0,1]^X \times [0,1]^X$ or to element in $([0,1] \times [0,1])^X$. For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \gamma_A\rangle$ for the GIF set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$.

Definition 2.3 [13] Let A and B be GIF sets in X in the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then

(a) $A \subset B$ iff $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$;

(b) $A = B$ iff $A \subset B$ and $B \subset A$;

(c) $A^c = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$;

(d) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$;

(e) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$;

(f) If $\{A_i : i \in J\}$ is an arbitrary family of GIF sets in X, then

$$\bigcap A_i = \{\langle x, \bigwedge \mu_{A_i}(x), \bigvee \gamma_{A_i}(x)\rangle : x \in X\},$$
$$\bigcup A_i = \{\langle x, \bigvee \mu_{A_i}(x), \bigwedge \gamma_{A_i}(x)\rangle : x \in X\}.$$

(g) $0_\sim = \{\langle x, 0, 1\rangle : x \in X\}$ and $1_\sim = \{\langle x, 1, 0\rangle : x \in X\}$.

Remark 2.4 (a) $A = A^c$ iff $\mu_A(x) = \gamma_A(x)$ for all $x \in X$. Thus degree of membership equals nonmembership for all $x \in X$. That is A is maximal fuzzy.

(b) $0_\sim = {1_\sim}^c$ and $1_\sim = {0_\sim}^c$.

Definition 2.5 [13] Let X and Y be two nonempty sets and $f : X \to Y$ be a function. Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ be a GIF set in X and $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ be a GIF set in Y.

(a) The inverse image $f^{-1}(B)$ of B under f is the GIF set in X defined by

$$f^{-1}(B) = \{\langle x, \mu_{f^{-1}(B)}(x), \gamma_{f^{-1}(B)}(x)\rangle : x \in X\},$$

where $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$ and $\gamma_{f^{-1}(B)}(x) = \gamma_B(f(x))$ for each $x \in X$.

(b) The image $f(A)$ of A under f is the GIF in Y defined by

$$f(A) = \{\langle y, \mu_{f(A)}(y), \gamma_{f(A)}(y)\rangle : y \in Y\}$$

where

$$\mu_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise,} \end{cases}$$
$$\gamma_{f(A)}(y) = \begin{cases} \bigwedge_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.6 [20] Let $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in Y\}$ be GIF set in X and Y, respectively. We define the cartesian product of A and B as GIF set in $X \times Y$ defined as follows:

$$A \times B = \{\langle (x, y), \mu_A(x) \wedge \mu_B(y), \gamma_A(x) \vee \gamma_B(y)\rangle : x \in X,\ y \in Y\}.$$

Definition 2.7 [20] A nonempty family $\mathcal{F}$ of GIF sets on X is said to be a fuzzy filter of GIF sets or a GIF filter if

(a) $0_\sim \notin \mathcal{F}$,

(b) If A, $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$,

(c) If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$.

Definition 2.8 [20] A nonempty family $\mathcal{B}$ of GIF sets is called a GIF filter base if $\mathcal{B}$ does not contain $0_\sim$ and the intersection of any two element of $\mathcal{B}$ contains an element of $\mathcal{B}$. A family $\mathcal{S}$ is called a subbase of a GIF filter base if it is nonempty and the intersection of any finite number of elements of $\mathcal{S}$ is not $0_\sim$.

Remark 2.9 [20] If $\mathcal{S}$ is a subbase of a GIF filter, then the family $\mathcal{B}(\mathcal{S})$ consisting of all finite intersections of elements of $\mathcal{S}$ is a GIF filter base. If $\mathcal{B}$ is a GIF filter base, then the family $\mathcal{F}(\mathcal{B})$, consisting of all GIF sets A such that $A \supseteq B$ for some $B \in \mathcal{B}$, is a GIF filter. Furthermore, $\mathcal{B}(\mathcal{S})$ and $\mathcal{F}(\mathcal{B})$ are uniquely determined by $\mathcal{S}$ and $\mathcal{B}$, respectively.

Definition 2.10 [20] Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be GIF filters. A function $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is called GIF filter continuous with respect to $(\mathcal{F}_1, \mathcal{F}_2)$ if for every $F \in \mathcal{F}_2$, $f^{-1}(F) \in \mathcal{F}_1$.

Definition 2.11 [20] Let $f : X \to Y$ be a function. A GIF set A in X is said to be f-invariant if $f(x) = f(y)$ implies $\mu_A(x) = \mu_A(y)$ and $\gamma_A(x) = \gamma_A(y)$.

Definition 2.12 [20] Let $\mathcal{F}$ and $\mathcal{G}$ be GIF filters on X and Y, respectively. Then a function $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is called GIF filter open if for every $F \in \mathcal{F}$, $f(F) \in \mathcal{G}$. In addition, if f GIF filter continuous and bijective, then f is called a GIF filter homeomorphism.

Definition 2.13 [20] Two GIF sets A and B in X are said to intersect at $x \in X$ if $\mu_A(x) + (1 - \gamma_B(x)) > 1$ or $\mu_B(x) + (1 - \gamma_A(x)) > 1$. Otherwise A and B do not intersect at x. A and B are said to be disjoint if these sets do not intersect anywhere.

Definition 2.14 [20] A GIF filter $(X, \mathcal{F})$ is called Hausdorff if for any $x,\ y \in X$ with $x \neq y$, there exist $F_1,\ F_2 \in \mathcal{F}$ such that $\gamma_{F_1}(x) < \frac{1}{2}$, $\gamma_{F_2}(y) < \frac{1}{2}$ and $\mu_{F_1}(z) + (1 - \gamma_{F_2}(z)) \leq 1$ and $\mu_{F_2}(z) + (1 - \gamma_{F_1}(z)) \leq 1$ for any $z \in X$.

Definition 2.15 [20] Let $(X, \mathcal{F})$ be a GIF filter. A sequence $\{x_n\}$ of X is said to converge GIF filterly to x (denoted by $\{x_n\} \longrightarrow_{gif} x$), and x is called a GIF limit of $\{x_n\}$, if for every $F \in \mathcal{F}$ such that $\gamma_F(x) < \frac{1}{2}$, there exists $n_0 \in \mathbf{N}$ such that $\mu_F(x_n) > \frac{1}{2}$ for all $n \geq n_0$, equivalently $1 - \gamma_{F^c}(x_n) < \frac{1}{2}$ for all $n \geq n_0$.

3 Semi Hausdorff GIF filters

Remark 3.1 From the Theorem 4.6 of [20], in a Hausdorff GIF filter $(X, \mathcal{F})$, every GIF filterly convergent sequence of points of X has exactly one GIF limit.

But the converse need not be true as is seen in the following example.

Example 3.2 Let X be an uncountable set and $\mathcal{F} = \{F \in (I \times I)^X \mid F^c$ has countable support$\}$ where support$(X) = \{x \in X \mid \mu(x) > 0\}$. Clearly $(X, \mathcal{F})$ is a GIF filter. Now every sequence $\{x_n\}$ of points of X converges GIF filterly uniquely if it converges. In fact here every sequence does not converge. For, let $x \in X$. Consider $F \in \mathcal{F}$ such that

$$\mu_F(z) = \begin{cases} 1, & z \neq x_n,\ x \text{ or } z = y, \\ \frac{1}{4}, & z = x_n, \\ \frac{3}{4}, & z = x, \end{cases}$$

$$\gamma_F(z) = \begin{cases} 0, & z \neq x_n,\ x \text{ or } z = y, \\ \frac{3}{4}, & z = x_n, \\ \frac{1}{4}, & z = x. \end{cases}$$

Clearly $\gamma_F(x) < \frac{1}{2}$. But $\mu_F(x_n) < \frac{1}{2}$ for all $n \in Z_+$ and hence $\{x_n\}$ does not converge GIF filterly to x. But $(X, \mathcal{F})$ is not a Hausdorff GIF filter. For, let $x, y \in X$ such that $x \neq y$. Suppose there exists $F_1, F_2 \in \mathcal{F}$ such that $\gamma_{F_1}(x) < \frac{1}{2}, \gamma_{F_2}(y) < \frac{1}{2}$ and $\mu_{F_1}(z) + (1 - \gamma_{F_2}(z)) \leq 1$ and $\mu_{F_2}(z) + (1 - \gamma_{F_1}(z)) \leq 1$ for all $z \in X$. Since $F_1, F_2 \in \mathcal{F}$, ${F_1}^c$ and ${F_2}^c$ have countable support say $\{x_n\}_{n \in Z_+}$ and $\{y_m\}_{m \in Z_+}$ respectively. Hence F_1 and F_2 have value 1 on $X - \{x_n, y_m\}_{n,m \in Z_+}$. Since X is uncountable, there exists $z \in X - \{x_n, y_m\}_{n,m \in Z_+}$ such that $\mu_{F_1}(z) = 1$ and $\mu_{F_2}(z) = 1$, which is a contradiction to the fact that $\mu_{F_1}(z) + (1 - \gamma_{F_2}(z)) \leq 1$ and $\mu_{F_2}(z) + (1 - \gamma_{F_1}(z)) \leq 1$ for all $z \in X$.

Definition 3.3 A GIF filter $(X, \mathcal{F})$ is called semi Hausdorff if every sequence of points converges GIF filterly to at most one point.

Definition 3.4 Let $(X, \mathcal{F})$ be a GIF filter. Then $(X, \mathcal{F})$ is called nearly T_1 if for every $x, y \in X$ with $x \neq y$, there exist $F_1, F_2 \in \mathcal{F}$ such that $\gamma_{F_1}(x) < \frac{1}{2},\ \gamma_{F_2}(y) < \frac{1}{2}$ and $\mu_{F_1}(y) \leq \frac{1}{2},\ \mu_{F_2}(x) \leq \frac{1}{2}$.

Theorem 3.5 *Every semi Hausdorff GIF filter is nearly T_1 GIF filter.*

Proof Let $(X, \mathcal{F})$ be a semi Hausdorff GIF filter. Let $x, y \in X$ such that $x \neq y$. Assuming $x_n = x$, for every $n \in Z_+$, we have $\{x_n\} \to_{gif} x$. Since $(X, \mathcal{F})$ is semi Hausdorff, $\{x_n\} \not\to_{gif} y$. Therefore there exists $F_1 \in \mathcal{F}$ such that $\gamma_{F_1}(y) < \frac{1}{2}$, $\mu_{F_1}(x) \leq \frac{1}{2}$ for all $n \in Z_+$. Similarly, by taking $y_n = y$ for every $n \in Z_+$, we have $F_2 \in \mathcal{F}$ such that $\gamma_{F_2}(x) < \frac{1}{2}$, $\mu_{F_2}(y) \leq \frac{1}{2}$. Hence $(X, \mathcal{F})$ is nearly T_1. □

The converse need not be true as is seen in the following example.

Example 3.6 Let X be an infinite set and $\mathcal{F} = \{F \in (I \times I)^X |\ F^c$ has finite support$\}$ where support$(X) = \{x \in X |\ \mu(x) > 0\}$. Clearly $(X, \mathcal{F})$ is a GIF filter. First, to prove $(X, \mathcal{F})$ is nearly T_1, consider $x, y \in X$ with $x \neq y$. Let $F_1, F_2 \in \mathcal{F}$ such that

$$\gamma_{F_1}(z) = \begin{cases} \frac{1}{4}, & z = x, \\ 1, & z \neq x, \end{cases} \qquad \mu_{F_1}(z) = \begin{cases} \frac{1}{4}, & z = y, \\ 1, & z \neq y, \end{cases}$$

$$\gamma_{F_2}(z) = \begin{cases} \frac{1}{4}, & z = y, \\ 1, & z \neq y, \end{cases} \qquad \mu_{F_2}(z) = \begin{cases} \frac{1}{4}, & z = x, \\ 1, & z \neq x. \end{cases}$$

Then support of ${F_1}^c$ and ${F_2}^c$ are $\{y\}$ and $\{x\}$, respectively. Therefore $\gamma_{F_1}(x) < \frac{1}{2}$, $\gamma_{F_2}(y) < \frac{1}{2}$ and $\mu_{F_1}(y) \leq \frac{1}{2}$, $\mu_{F_2}(x) \leq \frac{1}{2}$. Next, to prove that $(X, \mathcal{F})$ is not semi Hausdorff, consider $\{x_n\}$ of X such that $x_i \neq x_j$ for $i \neq j$. Now $\{x_n\}$ converges GIF filterly to all points of X. For, let $x \in X$ and $F \in \mathcal{F}$ such that $\gamma_F(x) < \frac{1}{2}$. Since F^c has finite support and $\{x_n\}$ is an infinite sequence of distinct points, $\mu_F(x_n) = 1$ for all but finite number of points of $\{x_n\}$. Therefore $\{x_n\}$ converges GIF filterly to x and hence to all points of X. Hence $(X, \mathcal{F})$ is not semi Hausdorff.

The proof of the following theorem is immediate.

Theorem 3.7 *Let $(X, \mathcal{F})$ be a semi Hausdorff GIF filter. Let $Y \subseteq X$ is also a semi Hausdorff GIF filter if no element of $\mathcal{F}$ vanishes on Y.*

Theorem 3.8 *A semi Hausdorff GIF filter is invarient under every bijective GIF filter open map.*

Proof Let $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ be a GIF filter open map and $(X, \mathcal{F}_1)$ be a semi Hausdorff GIF filter. Suppose that $(Y, \mathcal{F}_2)$ is not a semi Hausdorff GIF filter, then there exists $y_n \in Y$ such that $\{y_n\}$ converges GIF filterly to distinct point y and y'. Since f is bijective, $\{f^{-1}(y_n)\}$ is sequence of points of X. Let $f^{-1}(y_n) = x_n$. Let $f^{-1}(y) = a$ and $f^{-1}(y') = b$. Since f is bijective, $a \neq b$. To prove that $f^{-1}(y_n) = x_n \in X$ converges to a and b, let $F \in \mathcal{F}_1$ be a GIF filter open set such that $\gamma_F(a) < \frac{1}{2}$. Now $f(F) \in \mathcal{F}_2$ and $\gamma_{f(F)}(y) = \gamma_F(f^{-1}(y)) = \gamma_F(a) < \frac{1}{2}$. Therefore $\mu_{f(F)}(y_n) > \frac{1}{2}$ for all but finite number of n's. Since f is $1-1$, f is invarient and hence $\mu_F(x_n) = \mu_{f^{-1}(f(F))}(x_n) = \mu_{f(F)}(y_n) > \frac{1}{2}$ for all but finite number of n's. Hence $\{x_n\} \to_{gif} a$. Similarly $\{x_n\} \to_{gif} b$, which is a contradiction to the fact that $(X, \mathcal{F}_1)$ is semi Hausdorff. □

Lemma 3.9 [20] *Let $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ be a GIF filter continuous function and let $\{x_n\} \to_{gif} x$. Then $\{f(x_n)\} \to_{gif} f(x)$.*

Proof Let $G \in \mathcal{F}_2$ such that $\gamma_G(f(x)) < \frac{1}{2}$. Since f is GIF filter continuous, $f^{-1}(G) \in \mathcal{F}_1$ and $\gamma_{f^{-1}(G)}(x) < \frac{1}{2}$. Since $\{x_n\} \to_{gif} x$, there exists $n_0 \in \mathbf{N}$ such that $\mu_{f^{-1}(G)}(x_n) > \frac{1}{2}$ for all $n \geq n_0$. Then $\mu_G(f(x_n)) = \mu_{f^{-1}(G)}(x_n) > \frac{1}{2}$ for all $n \geq n_0$. Hence $\{f(x_n)\} \to_{gif} f(x)$. □

Lemma 3.10 *If the inverse image of subbasic GIF filter open set is GIF filter open, then the function is GIF filter continuous.*

Proof Let $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ be a function. Let $\mathcal{B}_1$ be a base for $\mathcal{F}_1$ and $\mathcal{B}_2$ be a base for $\mathcal{F}_2$ and $F \in \mathcal{F}_2$. Since $F \in \mathcal{F}_2$, there exists $G \in \mathcal{B}_2$ such that $G \subseteq F$. Also $G = \bigcap_{n=1}^{m} G_n$ where G_n are subbasic GIF filter open sets. Therefore $f^{-1}(G_n) \in \mathcal{F}_1$ for $n = 1, 2, 3, \cdots$. Hence $f^{-1}(G) = f^{-1}(\bigcap_{n=1}^{m} G_n) = \bigcap_{n=1}^{m} f^{-1}(G_n) \in \mathcal{F}_1$. Since $f^{-1}(G) \subseteq f^{-1}(F)$, $f^{-1}(F) \in \mathcal{F}_1$. Therefore f is GIF filter continuous. □

The proof of the following corollary is immediate.

Corollary 3.11 *A function is GIF filter continuous if and only if the inverse image of subbasic GIF filter open set is GIF filter open.*

Lemma 3.12 *Let* $\{(X_\alpha, \mathcal{F}_\alpha)\}$ *be an indexed family of GIF filters and* $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$ *be the product GIF filter on* $\prod X_\alpha$. *Then* $(X_\alpha, \mathcal{F}_\alpha)$ *is GIF filter homeomorphic to a subspace of* $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$, *where every* X_α *is a nonempty set.*

Proof Since X_α is nonempty, we can fix $x_\beta \in X_\beta$ for all $\beta \neq \alpha$. Define $f : X_\alpha \to \prod X_\alpha$ such that $f(x_\alpha) = (x_j)$ where $x_j = x_\alpha,\ j = \alpha$ and $x_j = x_\beta,\ j = \beta \neq \alpha$. Then f is well defined and 1-1. To prove that f is GIF filter continuous, consider a subbasic GIF filter open set $p_\alpha^{-1}(F_\alpha) \in \prod \mathcal{F}_\alpha$ where F_α is GIF filter open in X_α and let $p_\alpha : \prod X_\alpha \to X_\alpha$ be the projection map. Also since $\mu_{f^{-1}(p_\alpha^{-1}(F_\alpha))}(x_\alpha) = \mu_{(p_\alpha^{-1}(F_\alpha))}(f(x_\alpha)) = \mu_{F_\alpha}(x_\alpha)$ and $\gamma_{f^{-1}(p_\alpha^{-1}(F_\alpha))}(x_\alpha) = \gamma_{(p_\alpha^{-1}(F_\alpha))}(f(x_\alpha)) = \gamma_{F_\alpha}(x_\alpha)$, $f^{-1}(p_\alpha^{-1}(F_\alpha))$ is GIF filter open in X_α. By Lemma 3.10, f is GIF filter continuous.

Consider a GIF filter open set F_α in $(X_\alpha, \mathcal{F}_\alpha)$. Let $S = \{(x_j) |\ x_j = x_\alpha,\ j = \alpha$ and $x_j = x_\beta$ for all $j = \beta \neq \alpha\}$. Now, by the definition of f,

$$f^{-1}(x) = \begin{cases} x_\alpha, & x \in S, \\ 0, & x \notin S. \end{cases}$$

Clearly $p_\alpha^{-1}(F_\alpha)|_S = f(F_\alpha)$. Hence $f(F_\alpha)$ is GIF filter open in S as a subspace of $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$. Therefore $f^{-1} : S \to (X_\alpha, \mathcal{F}_\alpha)$ is GIF filter continuous and $(X_\alpha, \mathcal{F}_\alpha)$ is GIF filter homeomorphic to a subspace of $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$. □

Theorem 3.13 *Let* $\{(X_\alpha, \mathcal{F}_\alpha)\}$ *be an indexed family of GIF filters. Then* $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$ *is a semi Hausdorff GIF filter if and only if each* $(X_\alpha, \mathcal{F}_\alpha)$ *is a semi Hausdorff GIF filter.*

Proof Let $\{(X_\alpha, \mathcal{F}_\alpha)\}$ be a family of semi Hausdorff GIF filters. Suppose that $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$ is not a semi Hausdorff GIF filter, there exists a sequence $\{x_n\}$ of points of $\prod X_\alpha$ which converges GIF filterly to distinct point x and y. Since

$x \neq y$, there exists an index β such that $x_\beta \neq y_\beta$. Since the projection map $p_\beta : \prod X_\alpha \to X_\beta$ is GIF filter continuous in product GIF filter, by Lemma 3.9, $\{p_\beta(x_n)\}$ converges to x_β and y_β GIF filterly with $x_\beta \neq y_\beta$, which is a contradiction to semi Hausdorff GIF filterness of $(X_\beta, \mathcal{F}_\beta)$. Hence $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$ is semi Hausdorff.

Now we prove the converse part.

Let $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$ be a semi Hausdorff GIF filter. By Lemma 3.12, each X_α is GIF filter homeomorphic to a subspace of $(\prod X_\alpha, \prod \mathcal{F}_\alpha)$. By Theorems 3.7 and 3.8, $(X_\alpha, \mathcal{F}_\alpha)$ is a semi Hausdorff GIF filter. □

Note 3.14 Let $(Y, \mathcal{F})$ be a GIF filter. For defining the pointwise convergence filter on Y^X, if we take $S(x, F) = \{f \in Y^X |\ \mu_F(f(x)) > \frac{1}{2},\ \mu_F(f(x)) + \gamma_F(f(x)) \leq 1\}$ or $S(p, F) = \{f \in Y^X |\ f(p) \in F\}$ where $x \in X$, F is a GIF filter open set in Y(i.e., $F \in \mathcal{F}$) and p is a GIF point, analogous to the crisp theory, the collection $S(x, F)$ and $S(p, F)$ need not be a subbasis for a filter as seen in the following example.

Example 3.15 Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$. Let $\mathcal{B} = \{B_1, B_2, B_3\}$, $B_i \in \mathcal{B}, i = 1, 2, 3$ such that

$$\mu_{B_1}(z) = \begin{cases} l, & z = y_1, \\ m, & z = y_2, \\ n, & z = y_3, \end{cases} \qquad \gamma_{B_1}(z) = \begin{cases} l_0, & z = y_1, \\ m_0, & z = y_2, \\ n_0, & z = y_3, \end{cases}$$

$$\mu_{B_2}(z) = \begin{cases} l_0, & z = y_1, \\ m_0, & z = y_2, \\ n_0, & z = y_3, \end{cases} \qquad \gamma_{B_2}(z) = \begin{cases} l, & z = y_1, \\ m, & z = y_2, \\ n, & z = y_3, \end{cases}$$

$$\mu_{B_3}(z) = \begin{cases} l_0, & z = y_1, \\ m, & z = y_2, \\ n, & z = y_3, \end{cases} \qquad \gamma_{B_3}(z) = \begin{cases} l, & z = y_1, \\ m_0, & z = y_2, \\ n_0, & z = y_3, \end{cases}$$

where $m, n \leq \frac{1}{2} < l$ and $l + l_0 \leq 1, m + m_0 \leq 1, n + n_0 \leq 1$. Now $(Y, \mathcal{F})$ is a GIF filter generated by $\mathcal{B}$. Let $S(x, F) = \{f \in Y^X |\ \mu_F(f(x)) > \frac{1}{2},\ \mu_F(f(x)) +$

$\gamma_F(f(x)) \leq 1\}$. Now, since $S(x_1, F_1) \cap S(x_1, F_2) = \emptyset$, the collection $S(x, F)$ is not a subbasis for a filter. Similarly, the collection $S(p, F) = \{f \in Y^X | \ f(p) \in F\}$ fails to be a subbasis for a filter.

Note 3.16 If $S(x, F) = \{f \in Y^X | \ \mu_F(f(x)) > 0, \ \mu_F(f(x)) + \gamma_F(f(x)) \leq 1\}$, this collection is a subbasis for the filter. For, consider some $S(x_1, F_1)$ and $S(x_2, F_2)$. Now we prove that $\mu_{F_1}(f(x_1)) > 0$ and $\mu_{F_2}(f(x_2)) > 0$ for some function $f \in Y^X$. Since $\mu_{F_1} \wedge \mu_{F_2} \neq 0$, there exists $y \in Y$ such that $(\mu_{F_1} \wedge \mu_{F_2})(y) > 0$. Let $f : X \to Y$ be defined by $f(x_1) = f(x_2) = y$. Therefore $\mu_{F_1}(f(x_1)) = \mu_{F_1}(y) > 0$, $\mu_{F_2}(f(x_2)) = \mu_{F_2}(y) > 0$. Hence $f \in S(x_1, F_1) \cap S(x_2, F_2)$.

Also since $\mu_{F_1} \wedge \mu_{F_2} \wedge \cdots \wedge \mu_{F_n} \neq 0$, $(\mu_{F_1} \wedge \mu_{F_2} \wedge \cdots \wedge \mu_{F_n})(y) \neq 0$ for some y. Consider the function $f : X \to Y$ such that $f(x_i) = y$ for all i. Now clearly $\mu_{F_1}(f(x_1)) = \mu_{F_1}(y) > 0$. Similarly, $\mu_{F_i}(f(x_i)) > 0$ for all i. Therefore $f \in S(x_1, F_1) \cap S(x_2, F_2) \cap \cdots \cap S(x_n, F_n)$.

Note 3.17 But here we do not arrive at some basic results. So to generalize GIF pointwise convergence filter we need the following definition.

Definition 3.18 A GIF filter $(X, \mathcal{F})$ is called strong if $F_1, F_2 \in \mathcal{F}$, then $(\mu_{F_1} \wedge \mu_{F_2})(x) > \frac{1}{2}$ for some $x \in X$.

Note 3.19 Let $(Y, \mathcal{F})$ be a strong GIF filter. Then we have $S(x, F) = \{f \in Y^X | \ \mu_F(f(x)) > \frac{1}{2}, \ \mu_F(f(x)) + \gamma_F(f(x)) \leq 1\}$. Now $S(x, F)$ is a subbasis for a filter. For, if $F_1, F_2 \in \mathcal{F}$, $(\mu_{F_1} \wedge \mu_{F_2})(y) > \frac{1}{2}$ for some y. Now there exists a function f such that $f(x_1) = y$ and $f(x_2) = y$. Therefore $\mu_{F_1}(f(x_1)) > \frac{1}{2}$ and $\mu_{F_2}(f(x_2)) > \frac{1}{2}$. Hence $f \in S(x_1, F_1) \cap S(x_2, F_2)$.

Definition 3.20 The filter $\mathbf{F}$ generated by the subbasis $\{S(x, F)\}_{x \in X, F \in \mathcal{F}}$ is called the pointwise convergence filter on Y^X with respect to the strong GIF filter $(Y, \mathcal{F})$.

Theorem 3.21 In the above pointwise convergence filter $\mathbf{F}$ on Y^X with respect to a strong GIF filter $(Y, \mathcal{F})$, $f_n \to f$ if and only if $f_n(x) \to f(x)$ GIF filterly for every $x \in X$.

Proof Assume $f_n \to f$ in the above filter. To prove $f_n(x) \to f(x)$ GIF filterly for every $x \in X$, consider $x \in X$ and a GIF filter open set $F \in \mathcal{F}$ such that $\mu_F(f(x)) > \frac{1}{2}$, $\mu_F(f(x)) + \gamma_F(f(x)) \leq 1$. Hence $f \in S(x, F)$. Since $f_n \to f$ and $f \in S(x, F)$, there exists $n_0 \in \mathbf{N}$ such that $f_n \in S(x, F)$ for all $n \geq n_0$. Hence $\mu_F(f_n(x)) > \frac{1}{2}$, $\mu_F(f_n(x)) + \gamma_F(f_n(x)) \leq 1$ for all $n \geq n_0$. Hence $f_n(x) \to f(x)$ GIF filterly for every $x \in X$.

Conversely, suppose $f_n(x) \to f(x)$ GIF filterly for every $x \in X$. To prove $f_n \to f$ in the strong GIF filter, let $S(x, F)$ be a subbasis open set containing f. Then $\mu_F(f(x)) > \frac{1}{2}$. Since $f_n(x) \to_{gif} f(x)$, $\mu_F(f_n(x)) > \frac{1}{2}$ for all $n \geq n_0$ for some $n_0 \in \mathbf{N}$. Hence $f_n \in S(x, F)$ for all $n \geq n_0$. Hence $f_n \to f$. □

Definition 3.22 A filter $(X, \mathbf{F})$ is said to be semi Hausdorff filter (semi T_2 filter) if and only if every sequence in X has at most one limit.

Corollary 3.23 *The above filter* $\mathbf{F}$ *on* Y^X *is a semi* T_2 *filter if* $(Y, \mathcal{F})$ *is a semi Hausdorff GIF filter.*

Proof Let $(Y, \mathcal{F})$ be a semi Hausdorff GIF filter. Suppose that the above filter $\mathbf{F}$ on Y^X is not semi T_2, then there exists $f_n \in Y^X$ such that $f_n \to f$ and $f_n \to g$ with $f \neq g$. By the above Theorem 3.21, $f_n(x) \to_{gif} f(x)$ for all $x \in X$ and $f_n(x) \to_{gif} g(x)$ for all $x \in X$. Since $(Y, \mathcal{F})$ is semi Hausdorff, $f(x) = g(x)$ for all $x \in X$, which contradicts the fact that $f \neq g$. Hence the above filter $\mathbf{F}$ on Y^X is semi T_2. □

Theorem 3.24 *Let* $(Y, \mathcal{F})$ *be a strong GIF filter such that filter* $\mathbf{F}$ *on* Y^X *is semi* T_2 *for every indexing set* X *in the pointwise convergence filter with respect* $(Y, \mathcal{F})$. *Then* $(Y, \mathcal{F})$ *is semi Hausdorff.*

Proof Suppose that $(Y, \mathcal{F})$ is not a semi Hausdorff GIF filter, then there exists $y_n \in Y$ such that $y_n \to_{gif} x$ and $y_n \to_{gif} y$ with $x \neq y$. Define $f_n, f_x, f_y : X \to Y$ by $f_n(z) = y_n$, $f_x(z) = x$ and $f_y(z) = y$ for all $z \in X$. Then clearly f_n, f_x and f_y are elements of Y^X. Now we claim that $f_n \to f_x$ and $f_n \to f_y$. For, consider a subbasic filter open set $S(t, F)$ containing f_x where $t \in X$ and $F \in \mathcal{F}$. Hence

$\mu_F(f_x(t)) > \frac{1}{2}$, $\mu_F(f_x(t)) + \gamma_F(f_x(t)) \leq 1$. So $\mu_F(x) > \frac{1}{2}$, $\gamma_F(x) < \frac{1}{2}$. Since $y_n \to_{gif} x$ and $\gamma_F(x) < \frac{1}{2}$, we have $\mu_F(y_n) > \frac{1}{2}$ for every $n \geq n_0$ for some $n_0 \in \mathbf{N}$. Therefore $\mu_F(f_n(t)) = \mu_F(y_n) > \frac{1}{2}$ for every $n \geq n_0$. Hence $f_n \in S(t, F)$ for every $n \geq n_0$. So $f_n \to f_x$. Similarly, $f_n \to f_y$. So we get a contradiction to semi T_2-ness of Y^X. Hence $(Y, \mathcal{F})$ is semi Hausdorff. □

Definition 3.25 Let $(X, \mathcal{F})$ be a GIF filter. A subset S of X is called sequentially GIF filterly compact if every sequence in S has subsequence converging GIF filterly to a point in S.

Definition 3.26 Let $(X, \mathcal{F})$ be a GIF filter. A subset S of X is called sequentially GIF filterly closed if no sequence in S converges GIF filterly to a point in the complement of S.

Theorem 3.27 *In a semi Hausdorff GIF filter $(X, \mathcal{F})$, every set which is sequentially GIF filterly compact is sequentially GIF filterly closed.*

Proof Let S be a sequentially GIF filterly compact subset of $(X, \mathcal{F})$. Suppose that S is not sequentially GIF filterly closed, then there is a sequence $\{x_n\}$ in S such that $\{x_n\} \to_{gif} x$ and $x \notin S$. Since S is sequentially GIF filterly compact, there is a subsequence $\{x_{n_k}\}$ converging GIF filterly to a point $y \in S$. But as a subsequence of a GIF filterly convergent sequence converging to x, $\{x_{n_k}\} \to_{gif} x$. So we have $\{x_{n_k}\} \to_{gif} x$ and $\{x_{n_k}\} \to_{gif} y$ with $x \neq y$, which is a contradiction to the fact that $(S, \mathcal{F}_S)$ is a semi Hausdorff GIF filter, being a subspace of a semi Hausdorff GIF filter. Hence S is sequentially GIF filterly closed. □

Theorem 3.28 *A GIF filter $(X, \mathcal{F})$ is semi Hausdorff if and only if the diagonal set $\Delta = \{(x, x) | \ x \in X\}$ is sequentially GIF filterly closed.*

Proof If $(X, \mathcal{F})$ is a semi Hausdorff GIF filter and suppose that there is a sequence $(x_n, x_n) \in \Delta$ converging GIF filterly to $(x, y) \notin \Delta$, then $\{x_n\} \to_{gif} x$ and $\{x_n\} \to_{gif} y$. For, take $F \in \mathcal{F}$ such that $\gamma_F(x) < \frac{1}{2}$. Let $p_i : X \times X \to X, \ i = 1, 2$ be the projection maps on the i^{th} coordinate. We have $\gamma_{p_1^{-1}(F)}(x, y) =$

$\gamma_F(p_1(x,y)) = \gamma_F(x) < \frac{1}{2}$. Since p_1 is GIF filter continuous in product filter, $p_1^{-1}(F)$ is a GIF filter open set such that $\gamma_{p_1^{-1}(F)}(x,y) = \gamma_F(x) < \frac{1}{2}$ and hence $\mu_{p_1^{-1}(F)}(x_n, x_n) > \frac{1}{2}$ for all but finite number of n's. Therefore we got $\mu_F(x_n) = \mu_F(p_1(x_n, x_n)) = \mu_{p_1^{-1}(F)}(x_n, x_n) > \frac{1}{2}$ for all but finite number of n's. So $\{x_n\} \rightarrow_{gif} x$. Similarly $\{x_n\} \rightarrow_{gif} y$. Hence we have $\{x_n\} \rightarrow_{gif} x$ and $\{x_n\} \rightarrow_{gif} y$ with $x \neq y$ which contradicts semi Hausdorffness of X. Therefore Δ is sequentially closed.

Conversely, let Δ be sequentially GIF filterly closed and suppose that X is not a semi Hausdorff GIF filter, we have a sequence $\{x_n\}$ of X such that $\{x_n\} \rightarrow_{gif} x$ and $\{x_n\} \rightarrow_{gif} y$ with $x \neq y$. Now we claim that $(x_n, x_n) \rightarrow_{gif} (x,y)$. For, let F be a GIF filter open set in the product GIF filter $\mathcal{F}$ such that $\gamma_F(x,y) < \frac{1}{2}$. Hence we have a GIF filter open set $F_1 \times F_2$ such that $\gamma_{F_1 \times F_2}(x,y) = \gamma_{F_1}(x) \vee \gamma_{F_2}(y) < \frac{1}{2}$ and $F_1 \times F_2 \subseteq F$. Since $\{x_n\} \rightarrow_{gif} x$ and $\{x_n\} \rightarrow_{gif} y$ and $\gamma_{F_1}(x) < \frac{1}{2}$ and $\gamma_{F_2}(y) < \frac{1}{2}$, we get $\mu_{F_1}(x_n) > \frac{1}{2}$ for all but finite number of n's. Similarly, $\mu_{F_2}(x_n) > \frac{1}{2}$ for all but finite number of n's. Hence $\mu_F(x_n, x_n) \geq \mu_{F_1 \times F_2}(x_n, x_n) = \mu_{F_1}(x_n) \wedge \mu_{F_2}(x_n) > \frac{1}{2}$ for all but finite number of n's and hence $(x_n, x_n) \rightarrow_{gif} (x,y)$ with $x \neq y$, which is a contradiction to the fact that Δ is sequentially GIF filterly closed. □

Definition 3.29 A function $f : (X, \mathcal{F}_X) \rightarrow (Y, \mathcal{F}_Y)$ is called sequentially GIF filterly continuous if $\{x_n\} \rightarrow_{gif} x \Rightarrow \{f(x_n)\} \rightarrow_{gif} f(x)$.

Theorem 3.30 *Let $f : (X, \mathcal{F}_X) \rightarrow (Y, \mathcal{F}_Y)$ and $g : (X, \mathcal{F}_X) \rightarrow (Y, \mathcal{F}_Y)$ be sequentially GIF filterly continuous functions. Then $(Y, \mathcal{F}_Y)$ is a semi Hausdorff GIF filter if and only if the set $A = \{x \in X | \ f(x) = g(x)\}$ is sequentially GIF filterly closed.*

Proof Let $(Y, \mathcal{F}_Y)$ be a semi Hausdorff GIF filter. Suppose that A is not sequentially GIF filterly closed, then there exists $x_n \in A$ such that $\{x_n\} \rightarrow_{gif} x$ with $x \notin A$. Hence $f(x_n) = g(x_n)$ and $f(x) \neq g(x)$. Since $\{x_n\} \rightarrow_{gif} x$, f and g are sequentially GIF filterly continuous, we have $\{f(x_n)\} \rightarrow_{gif} f(x)$ and $\{g(x_n)\} \rightarrow_{gif} g(x)$. Since $f(x_n) = g(x_n)$, $\{f(x_n)\} \rightarrow_{gif} f(x)$ and $\{f(x_n)\} \rightarrow_{gif}$

$g(x)$ with $f(x) \neq g(x)$, we have a contradiction to the fact that $(Y, \mathcal{F}_Y)$ is semi Hausdorff.

Now the converse follows from the previous theorem by taking $X = Y \times Y$, f and g to be projections which are sequentially GIF filterly continuous, then $\Delta = A$. $\square$

4 Convergence in GIF topological spaces

In this chapter, new notions of GIF filter convergence and GIF cluster points and introduced and some GIF topological properties are studied through those notions.

Definition 4.1 Let X be a nonempty set. A nonvoid family $\mathcal{T}$ of GIF sets on X is said to be a GIF topology if

(a) $1_{\sim},\ 0_{\sim} \in \mathcal{T}$

(b) The union of every family of GIF sets in $\mathcal{T}$ is a GIF set in $\mathcal{T}$

(c) The intersection of every finite family of GIF sets in $\mathcal{T}$ is a GIF set in $\mathcal{T}$

Definition 4.2 Let $(X, \mathcal{T})$ be a GIF topological space and let $(X, \mathcal{F})$ be a GIF filter. A point $x \in X$ is called a generalized intuitionistic fuzzily cluster point of $(X, \mathcal{F})$ ($(X, \mathcal{F})$ accumulates x) if for every $U \in \mathcal{T}$ with $\mu_U(x) \geq \frac{1}{2}, \gamma_U(x) \leq \frac{1}{2}$, there exists $z \in X$ such that $\mu_U(z) + (1 - \gamma_F(z)) > 1$ or $\mu_F(z) + (1 - \gamma_U(z)) > 1, \forall F \in \mathcal{F}$.

Definition 4.3 Let $(X, \mathcal{T})$ be a GIF topological space. A GIF filter $(X, \mathcal{F})$ is said to converge to a point $x \in X$ if for every $U \in \mathcal{T}$ with $\mu_U(x) \leq \frac{1}{2}$, there exists $F \in \mathcal{F}$ such that $F \subseteq U$.

The prove of the following note is immediate from definitions.

Note 4.4 If a strong GIF filter $(X, \mathcal{F})$ converges to a point $x \in X$, then $(X, \mathcal{F})$ accumulates x.

Theorem 4.5 *Let $(X, \mathcal{T})$ be a GIF topological space and let $(X, \mathcal{F})$ be a GIF filter. A point $x \in X$ is a generalized intuitionistic fuzzily cluster point of $(X, \mathcal{F})$ if and only if $\mu_{\bar{F}}(x) > \frac{1}{2}, \forall F \in \mathcal{F}$.*

Proof Let $x \in X$ be a generalized intuitionistic fuzzily cluster point. Suppose that there exists $F \in \mathcal{F}$ such that $\mu_{\bar{F}}(x) \leq \frac{1}{2},\ \mu_{\bar{F}^c}(x) \geq \frac{1}{2}$ and $\bar{F}^c \in \mathcal{T}$. Clearly $\mu_{\bar{F}^c}(z) + (1 - \gamma_{\bar{F}}(z)) \leq 1,\ \forall z \in X$ and hence $\mu_{\bar{F}^c}(z) + (1 - \gamma_F(z)) \leq 1,\ \forall z \in X$.

So there exists $F \in \mathcal{F}$, $\bar{F}^c \in \mathcal{T}$ such that $\mu_{\bar{F}^c}(z) + (1 - \gamma_F(z)) \leq 1$, $\forall z \in X$. This contradicts the fact that $x \in X$ is a generalized intuitionistic fuzzily cluster point of $(X, \mathcal{F})$.

Now we prove the converse part. Suppose $\mu_{\bar{F}}(x) > \frac{1}{2}$, $\forall F \in \mathcal{F}$, we have to prove that x is a generalized intuitionistic fuzzily cluster point of $(X, \mathcal{F})$. By assuming the contrary, we have $F \in \mathcal{F}$ and $U \in \mathcal{T}$ such that $\mu_U(x) \geq \frac{1}{2}$ and $\mu_F(z) + (1 - \gamma_U(z)) \leq 1$, $\forall z \in X$. So $\mu_F(z) \leq \gamma_U(z) = \mu_{U^c}(z)$, $\forall z \in X$ and hence U^c is a GIF closed set containing F. Hence $\mu_F(z) \leq \mu_{U^c}(z) \leq 1 - \gamma_{U^c}(z) = 1 - \mu_U(z)$, $\forall z \in X$. So we have $\mu_F(z) \leq \frac{1}{2}$, which contradicts to our hypothesis. □

Definition 4.6 A GIF topological space $(X, \mathcal{T})$ is called nearly Hausdorff if for every pair of elements $x \neq y$ of X, there exist GIF open sets $F_1, F_2 \in \mathcal{T}$ such that $\mu_{F_1}(x) > \frac{1}{2}$ and $\mu_{F_2}(y) > \frac{1}{2}$, $\mu_{F_1}(z) + (1 - \gamma_{F_2}(z)) \leq 1$ and $\mu_{F_2}(z) + (1 - \gamma_{F_1}(z)) \leq 1$, $\forall z \in X$.

Theorem 4.7 *Let $(X, \mathcal{T})$ be a nearly Hausdorff GIF topological space. Then every convergent strong GIF filter in X converges uniquely.*

Proof Let $(X, \mathcal{T})$ be a nearly Hausdorff GIF space and let $\mathcal{F}$ be any strong GIF filter on X. Suppose that $\mathcal{F}$ converges to two distinct points x and y, by nearly Hausdorffness of $(X, \mathcal{T})$, there exist $F_1, F_2 \in \mathcal{T}$ with $\mu_{F_1}(x) > \frac{1}{2}$, $\mu_{F_2}(x) > \frac{1}{2}$ and $\mu_{F_1}(z) + (1 - \gamma_{F_2}(z)) \leq 1$ and $\mu_{F_2}(z) + (1 - \gamma_{F_1}(z)) \leq 1$, $\forall z \in X$. Since $\mathcal{F}$ converges to x, $F_1 \in \mathcal{F}$. Similarly, $F_2 \in \mathcal{F}$. So by strong GIF filterness of $\mathcal{F}$, $(\mu_{F_1} \wedge \mu_{F_2})(z) > \frac{1}{2}$ for some $z \in X$. So $\mu_{F_1}(z) + \mu_{F_2}(z) > 1$, which is a contradiction. □

Definition 4.8 Let $(X, \mathcal{T})$ be a GIF topological space. Let $A \subseteq X$. A point $x \in X$ is called a generalized intuitionistic fuzzily limit point of A if for every GIF open set $U \in \mathcal{T}$ such that $\mu_U(x) \geq \frac{1}{2}$, then $\mu_U(z) \geq \frac{1}{2}$, $\mu_U(z) + \gamma_U(z) \leq 1$ for some $z \in A - \{x\}$.

Definition 4.9 Let $(X, \mathcal{T})$ be GIF topological spaces. A subset C of X is called generalized intuitionistic fuzzily closed if it contains all its generalized intuitionistic fuzzily limit points.

Definition 4.10 Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be GIF topological spaces. A function $f : X \to Y$ is called nearly GIF continuous if $f^{-1}(A)$ is generalized intuitionistic fuzzily closed in $(X, \mathcal{T}_1)$ for every generalized intuitionistic fuzzily closed set A in $(Y, \mathcal{T}_2)$.

Theorem 4.11 *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be GIF topological spaces. and $f : X \to Y$ be any map.*

(a) *If f is GIF filter continuous, then $\mathcal{F} \to x$ implies $f(\mathcal{F}) \to f(x)$.*

(b) *If $\mathcal{F} \to x$ implies $f(\mathcal{F}) \to f(x)$ for every GIF filter $\mathcal{F}$ on X, then f is nearly GIF continuous.*

Proof (a) Assume that f is GIF continuous and $\mathcal{F} \to x$. To prove that $f(\mathcal{F}) \to f(x)$, let $U \in \mathcal{T}_2$ such that $\mu_U(f(x)) \geq \frac{1}{2}$. Since f is GIF filter continuous, $f^{-1}(U) \in \mathcal{T}_1$ and clearly $\mu_{f^{-1}(U)}(x) = \mu_U(f(x)) \geq \frac{1}{2}$. Since $\mathcal{F} \to x$, $f^{-1}(U) \in \mathcal{F}$. Hence $f(f^{-1}(U)) \in f(\mathcal{F})$. Since $f(f^{-1}(U)) \subseteq U$, $U \in f(\mathcal{F})$. Hence $f(\mathcal{F}) \to f(x)$.

(b) If $\mathcal{F} \to x$ implies $f(\mathcal{F}) \to f(x)$ for every GIF filter $\mathcal{F}$ on X, we have to prove that $f : X \to Y$ is nearly GIF continuous. Let A be a generalized intuitionistic fuzzily closed set in Y. Now we prove that $f^{-1}(A)$ is generalized intuitionistic fuzzily closed in X. If $f^{-1}(A) = X$, then it is generalized intuitionistic fuzzily closed. Suppose $f^{-1}(A) \neq X$, let $x \notin f^{-1}(A)$. Clearly $f(x) \notin A$. Since A is generalized intuitionistic fuzzily closed, there exists $U \in \mathcal{T}_2$ such that $\mu_U(f(x)) \geq \frac{1}{2}$ and $\mu_U^{-1}[\frac{1}{2}, 1] \cap A = \emptyset$. Now let $\mathcal{F}$ be a GIF filter generated by $\mathcal{B} = \{G \in \mathcal{T}_1 | \ \mu_G(x) \geq \frac{1}{2}\}$. Clearly $\mathcal{F} \to x$. So by hypothesis, $f(\mathcal{F}) \to f(x)$. Hence $U \in f(\mathcal{F})$. Clearly $f^{-1}(U) \in \mathcal{F}$ and $\mu_{f^{-1}(U)}(x) \geq \frac{1}{2}$. So we have $G \in \mathcal{B} \subseteq f^{-1}(U)$. Hence we have $G \in \mathcal{T}_1$ and $\mu_G(x) \geq \frac{1}{2}$. Now we claim that $\mu_G^{-1}[\frac{1}{2}, 1] \cap f^{-1}(A) = \emptyset$. If not, $z \in \mu_G^{-1}[\frac{1}{2}, 1] \cap f^{-1}(A)$, $f(z) \in A$ and $\mu_G(z) \geq \frac{1}{2}$. Hence we have $\mu_{f^{-1}(U)}(z) \geq \frac{1}{2}$ and $f(z) \in A$. So we have $f(z) \in \mu_U^{-1}[\frac{1}{2}, 1] \cap A$, which is a contradiction. So, x is not a generalized intuitionistic fuzzily limit point of $f^{-1}(A)$ and hence $f^{-1}(A)$ is generalized intuitionistic fuzzily closed. □

Definition 4.12 Let $\alpha \in [0,1]$. For a GIF subset F of X, $\tilde{F}_\alpha = \{x \in X | \ \mu_F(x) < \alpha, \gamma_F(x) \geq 1-\alpha\}$ is called a α-strictly lowerlevel set of F.

Definition 4.13 Let $(X, \mathcal{T})$ be a GIF topological space. A collection δ of GIF open sets is called a generalized intuitionistic fuzzily open cover for $A \subseteq X$ if for every $z \in A$, there exists $F \in \delta$ such that $\mu_F(z) \geq \frac{1}{2}$, $\mu_F(z) + \gamma_F(z) \leq 1$.

Definition 4.14 A subset $A \subseteq X$ of a GIF topological space $(X, \mathcal{T})$ is called a generalized intuitionistic fuzzily compact if for every GIF open cover δ for A, there exists a finite subcollection δ_0 of δ such that for every $z \in A$, there exists $F \in \delta_0$ with $\mu_F(z) \geq \frac{1}{2}$, $\mu_F(z) + \gamma_F(z) \leq 1$.

Theorem 4.15 *Let $(X, \mathcal{T})$ be a GIF topological space. Then X is generalized intuitionistic fuzzily compact if and only if for every collection $\mathcal{C}$ of generalized intuitionistic fuzzily closed sets in X satisfying the finite intersection property has nonempty intersection.*

Proof Let $\mathcal{F}$ be a collection of generalized intuitionistic fuzzily closed sets satisfying the finite intersection property. We have to prove the $\cap_{C \in \mathcal{C}} C \neq \emptyset$. Suppose that $\cap_{C \in \mathcal{C}} C = \emptyset$. So $(\cap_{C \in \mathcal{C}} C)^c = X$. By demorgan's law, $\cup_{C \in \mathcal{C}} C^c = X$. Let $x \in X$, then $x \in {C_x}^c$ for some $C_x \in \mathcal{C}$. Since C_x is generalized intuitionistic fuzzily closed and $x \notin C_x$, we have $U_x \in \mathcal{T}$ such that $\mu_{U_x}(x) \geq \frac{1}{2}$ and $\mu_{U_x}(z) < \frac{1}{2}$ for every $z \in C_x$. Clearly $\sigma = \{U_x | \ x \in X\}$ is a generalized intuitionistic fuzzily open covering of X. Since X is generalized intuitionistic fuzzily compact, we have a finite subcollection $\{U_{x_1}, U_{x_2}, \ldots, U_{x_n}\}$ of σ such that for every $z \in X$, we have $\mu_{U_{x_i}}(z) \geq \frac{1}{2}$ for some i. Hence $X = \bigcup_{i=1}^n \mu_{U_{x_i}}^{-1}[\frac{1}{2}, 1]$ and hence $\bigcap_{i=1}^n (\mu_{U_{x_i}}^{-1}[\frac{1}{2}, 1])^c = \emptyset$. Now we claim that $C_{x_i} \subseteq (\mu_{U_{x_i}}^{-1}[\frac{1}{2}, 1])^c$. For, to each $z \in C_{x_i}$, $\mu_{U_{x_i}}(z) < \frac{1}{2}$ and hence $z \notin \mu_{U_{x_i}}^{-1}[\frac{1}{2}, 1]$ and hence $z \in (\mu_{U_{x_i}}^{-1}[\frac{1}{2}, 1])^c$. Hence $\bigcap_{i=1}^n C_{x_i} = \emptyset$, which contradicts the finite intersection property of $\mathcal{C}$.

Conversely, let σ be a generalized intuitionistic fuzzily open covering. We claim that the $\frac{1}{2}$-strictly lowerlever set $\tilde{U}_{\frac{1}{2}}$ is generalized intuitionistic fuzzily closed for every $U \in \sigma$. For, let $y \notin \tilde{U}_{\frac{1}{2}}$. Hence $\mu_U(y) \geq \frac{1}{2}$, $\gamma_U(y) < \frac{1}{2}$. So we have a GIF open set $F \in \mathcal{T}$ such that $\mu_F(y) \geq \frac{1}{2}$ and $\mu_F(z) < \frac{1}{2}$ for every $z \in \tilde{U}_{\frac{1}{2}}$. Consider $\mathcal{C} = \{\tilde{U}_{\frac{1}{2}} | U \in \sigma\}$.Cleary $\mathcal{C}$ is a collection of generalized intuitionistic fuzzily closed sets. Since σ is a generalized intuitionistic fuzzily open cover of X, for every $x \in X$, there exists $G \in \sigma$ such that $\mu_G(x) \geq \frac{1}{2}$. Hence for every $x \in X$, $x \notin \tilde{U}_{\frac{1}{2}}$ for some $G \in \sigma$. So $\bigcap_{U \in \sigma} \tilde{U}_{\frac{1}{2}} = \emptyset$. Hence by hypothesis, $\mathcal{C}$ does not satisfy finite intersection property. Hence there exists a finite subcollection $\sigma_0 \subseteq \sigma$ such that $\bigcap_{U \in \sigma_0} \tilde{U}_{\frac{1}{2}} = \emptyset$. So for every $x \in X$, there exists $U \in \sigma_0$ such that $x \notin \tilde{U}_{\frac{1}{2}}$ and hence $\mu_U(x) \geq \frac{1}{2}$. Hence X is generalized intuitionistic fuzzily compact. □

Theorem 4.16 *A GIF topological space $(X, \mathcal{T})$ is generalized intuitionistic fuzzily compact if and only if each strong GIF filter in X has at least one generalized intuitionistic fuzzily cluster point.*

Proof By Theorem 4.15, it is enough to prove that every collection of generalized intuitionistic fuzzily closed sets with finite intersection property has nonempty intersection if and only if every strong GIF filter in X has at least one generalized intuitionistic fuzzily cluster point.

Assume the hypothesis, let $\mathcal{F}$ be a strong GIF filter in X. It is enough to prove that $\mathcal{F}$ has at least one generalized intuitionistic fuzzily cluster point. Now for every $F \in \mathcal{F}$, we have $\mu_{\bar{F}}{}^{-1}(\frac{1}{2}, 1]$ is generalized intuitionistic fuzzily closed. Since $\mathcal{F}$ is a strong GIF filter, we have $(\mu_{F_1} \wedge \mu_{F_2})(x) > \frac{1}{2}$ for every pair of $F_1, F_2 \in \mathcal{F}$ for some $x \in X$. So $\Omega = \{\mu_{\bar{F}}^{-1}(\frac{1}{2}, 1] |\ F \in \mathcal{F}\}$ is clearly a collection of generalized intuitionistic fuzzily closed sets with finite intersection property and hence by hypothesis, we have $\bigcap_{F \in \mathcal{F}} \mu_{\bar{F}}{}^{-1}(\frac{1}{2}, 1] \neq \emptyset$. Therefore we have $z \in X$ such that $\mu_{\bar{F}}(z) > \frac{1}{2}, \forall F \in \mathcal{F}$. By Theorem 4.5, z is a generalized intuitionistic fuzzily cluster point of $\mathcal{F}$.

Conversely, we assume the hypothesis. Let Ω be a collection of generalized intuitionistic fuzzily closed sets that satisfies finite intersection property. Let $A \in \Omega$ and if $z \notin A$, by the definition of generalized intuitionistic fuzzily closed set, then z is not a generalized intuitionistic fuzzily limit point of A and hence there exists $U \in \mathcal{T}$ such that $\mu_U(z) \geq \frac{1}{2}$ and $\mu_U(y) < \frac{1}{2}$, $\forall y \in A$. So we have a GIF closed set U^c with $\mu_{U^c}(z) \leq \frac{1}{2}$ and $\mu_{U^c}(y) > \frac{1}{2}, \forall y \in A$. So for every $A \in \Omega$

and for each $z \notin A$, there exists a GIF closed set $F_{(z,A)}$ with $\mu_{F_{(z,A)}}(y) > \frac{1}{2}$, $\forall y \in A$ and $\mu_{F_{(z,A)}}(z) \leq \frac{1}{2}$.

Now consider $\mathcal{S} = \{F_{(z,A)} | \ A \in \Omega$ and $x \notin A\}$. By finite intersection property of Ω, for any finite subcollection $\mathcal{S}_0$ of $\mathcal{S}$, we have $\bigwedge_{F \in \mathcal{S}_0} \mu_F(t) > \frac{1}{2}$ for some $t \in X$. So the GIF filter $\mathcal{F}$ generated by $\mathcal{S}$ is clearly a strong GIF filter. Now by hypothesis, this strong GIF filter $\mathcal{F}$ has at least one generalized intuitionistic fuzzily cluster point. Let it be x. By Theorem 4.5, $\mu_{\bar{F}}(x) > \frac{1}{2}$, $\forall F \in \mathcal{F}$. Hence $\mu_{\bar{F}_{(z,A)}}(x) > \frac{1}{2}$, $\forall F_{(z,A)} \in \mathcal{S}$. Since $F_{(z,A)}$ is GIF closed, $\mu_{F_{(z,A)}}(x) > \frac{1}{2}$, $\forall z \notin A$ and $A \in \Omega$. Clearly $x \in A$, $\forall A \in \Omega$. For if $x \notin B$, for some $B \in \Omega$, we have $F_{(x,B)} \in \mathcal{S}$ such that $\mu_{F_{(x,B)}}(y) > \frac{1}{2}$, $\forall y \in B$ and $\mu_{F_{(x,B)}}(x) \leq \frac{1}{2}$, which is a contradiction. Hence $\bigcap_{A \in \Omega} A \neq \emptyset$. $\square$

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, in: V. Sgurev, Ed., VII ITKR's Session, Sofia (June 1983 Central Sci. and Techn. Library, Bulg. Academy of Sciences, 1984).

[2] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87–96.

[3] K. Atanassov, *Review and new results on intuitionistic fuzzy sets*, Preprint IM-MFAIS-1-88. Sofia, 1988.

[4] K. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems **61** (1994), 137–142.

[5] K. Atanassov and S. Stoeva, *Intuitionistic fuzzy sets*, in: Polish Symp. on Interval & Fuzzy Mathematics, Poznan (August 1983), 23–26.

[6] Shi-Zhong Bai, *PS-convergence theory of fuzzy nets and its application*, Inform. Sci. **153** (2003), 237–245.

[7] Shi-Zhong Bai, *Near PS-compact L-subsets*, Inform. Sci. **155** (2003), 111–118.

[8] Shi-Zhong Bai, *The semi-precompactness axioms*, Inform. Sci. **165** (2004), 139–148.

[9] D. Çoker, *On fuzzy inclusion in intuitionistic sense*, J. Fuzzy Math. **4** (1996), 701–714.

[10] D. Çoker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems **88** (1997), 81–89.

[11] M. A. de Prada Vincente and M. Saralegui Aranguren, *Fuzzy filters*, J. Math. Anal. Appl. **129** (1988), 560–568.

[12] M. A. de Prada Vincente and M. Saralegui Aranguren, *t-Prefilter theory*, Fuzzy Sets and Systems **38** (1990), 115–124.

[13] T. K. Mondal and S. K. Samanta, *Generalized intuitionistic fuzzy sets*, J. Fuzzy Math. **10** (2002), 839–861.

[14] R. Lowen, *Convergence in fuzzy topological spaces*, General Topology and its applns. **10** (1979), 147-160.

[15] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems **48** (1992), 371–375.

[16] P. V. Ramakrishnan and V. Lakshmana Gomathi Nayagam, *Hausdorff interval valued fuzzy filters*, J. Korean Math. Soc. **39** (2002), 137–148.

[17] S. A. El-Sheikh, *A new approach to fuzzy bitopological spaces*, Inform. Sci. **137** (2001), 283–301.

[18] A. P. Šostak, *On fuzzy topological structure*, Supp. Rend. Circ. Mat. Palermo Ser. II **14** (1985), 89–103.

[19] L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.

[20] J. H. Park and J. K. Park, *Hausdorffness on generalized intuitionistic fuzzy filters*, Inform. Sci. **168** (2004), 95–110.

[21] Geetha Sivaraman and V. Lakshmana Gomathi Nayagam, *Semi Hausdorff fuzzy filters*, Int. J. Math. Math. Sci. **2007** (2007), ID 92423.

[22] P. V. Ramakrishnan and V. Lakshmana Gomathi Nayagam, *Nearly fuzzy husdorff spaces*, Indian J. Pure and Applied Math. **31** (2000), 695–712.

[23] P. V. Ramakrishnan and V. Lakshmana Gomathi Nayagam, *Fuzzily compact spaces*, J. Fuzzy Math. **10** (2002), 615–625.