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Strong Differential Subordination and Superordination for Certain Meromorphic Functions

by

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Strong Differential Subordination and Superordination for Certain Meromorphic Functions
(유리형 다엽함수들에 대한 강 미분 종속과 초종속)

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요　약

기하 함수 이론은 지금까지 많은 학자들에 의하여 연구되어 왔다. 특히, Miller와 Mocanu는 미분종속 이론을 소개하고 해석함수들의 종속문제와 그 쌍대개념인 초 종속 문제를 연구하여 다양한 기하학적 성질들을 조사하였다 (cf. [10, 11]). 그리고 Antonino [2, 3]는 강 미분 종속과 소개하여 미분종속이론을 확장·발전 시켰다. 최근, G. I. Oros와 G. Oros [13]는 강 미분 초 종속 개념을 소개하여 여러 기하학적 성질들을 조사하였다.

본 연구에서는 Antonino와 G. I. Oros와 G. Oros의 강 미분 종속 및 초 종속 이론을 응용하여 적당한 admissible 함수들의 족들을 도입하여 Liu-Srivastava 연산자와 관련된 유리형 다엽함수들의 강 미분 종속과 그 쌍대문제인 강 미분 초 종속 보존 성질들을 연구하였다. 또한, 이 연산자에 대하여 sandwich 형태의 결과들을 조사하였다.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

and let $\mathcal{H}_0 \equiv \mathcal{H}[0,1]$ and $\mathcal{H}_1 \equiv \mathcal{H}[1,1]$.

Let $H(z,\zeta)$ be analytic in $\mathbb{U} \times \overline{\mathbb{U}}$ and let $f(z)$ be analytic and univalent in $\mathbb{U}$. Then the function $H(z,\zeta)$ is said to be strongly subordinate to $f(z)$, or $f(z)$ is said to be strongly superordinate to $H(z,\zeta)$, written as $H(z,\zeta) \prec\prec f(z)$, if for $\zeta \in \overline{\mathbb{U}}$, the function of z, $H(z,\zeta)$ is subordinate to $f(z)$. We note that $H(z,\zeta) \prec\prec f(z)$ if and only if $H(0,\zeta) = f(0)$ and $H(\mathbb{U} \times \overline{\mathbb{U}}) \subset f(\mathbb{U})$ (cf. [2,3,12]).

Let Σ_p denote the class of all analytic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (n \in \mathbb{N}; z \in \mathbb{D} := \mathbb{U}\backslash\{0\}). \tag{1.1}$$

For two functions f(z) given by (1.1) and $g(z)$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k \quad (n \in \mathbb{N}; z \in \mathbb{D}),$$

the Hadamard product (or convolution) of f and g is defined (as usual) by

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z) \quad (n \in \mathbb{N}; z \in \mathbb{D}).$$

For complex parameters $\alpha_1, \cdots, \alpha_l$ and $\beta_1, \cdots, \beta_m$ ($\beta_j \neq 0, -1, -2, \cdots; j = 1, \cdots, m$), we now define the generalized hypergeometric function ${}_lF_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$ as follows [14,15]:

$$ {}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_l)_k}{(\beta_1)_k\cdots(\beta_m)_k}\frac{z^k}{k!} $$

$$ (l \le m+1;\ l,m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\};\ z\in\mathbb{U}), $$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$ (\nu)_k := \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & \text{if } k=0 \text{ and } \nu\in\mathbb{C}\backslash\{0\}, \\ \nu(\nu+1)\cdots(\nu+k-1) & \text{if } k\in\mathbb{N} \text{ and } \nu\in\mathbb{C}. \end{cases} $$

Corresponding to a function $\mathcal{F}_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z)$ defined by

$$ \mathcal{F}_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z) := z^{-p}{}_lF_m(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z), $$

the Liu-Srivastaba operator $H_p^{l,m}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m) : \Sigma_p \longrightarrow \Sigma_p$ is defined by the following Hadamard product(or convolution):

$$ H_p^{l,m}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m)f(z) := \mathcal{F}_p(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m;z) * f(z). \tag{1.2} $$

so that, for a function f of form (1.1), we have

$$ H_p^{l,m}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m)f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_l)_k}{(\beta_1)_k\cdots(\beta_m)_k}\frac{a_k}{(k+p)!}z^k. $$

If for convenience, we write

$$ H_p^{l,m}(\alpha_1) := H_p^{l,m}(\alpha_1,\cdots,\alpha_l;\beta_1,\cdots,\beta_m). \tag{1.3} $$

Then one can easily verify from definition (1.2) that

$$z(H_p^{l,m}(\alpha_1))' = \alpha_1 H_p^{l,m}(\alpha_1+1)f(z) - (\alpha_1+p)H_p^{l,m}(\alpha_1)f(z). \tag{1.4}$$

We note that the definition of the linear operator $H_p^{l,m}(\alpha_1)$ was introduced and studied by Liu and Srivastava [9]. This operator $H_p^{l,m}(\alpha_1)$ was motivated essentially by Dziok and Srivastava [4]. Some interesting developments associated with the generalized hypergeometric function were considered recently by (for example) Dziok and Srivastava [5,6] and Liu and Srivastava [7,8].

To prove our results, we need the following definitions and theorems.

Definition 1.1 ([12], cf. [10]). Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the (second order) strong differential subordination

$$\phi(p(z), zp'(z), zp''(z); z, \zeta) \prec\prec h(z), \tag{1.6}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the strong differential subordination, or more simply a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.6). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.6) is said to be the best dominant.

Recently, Oros [13] introduced the following strong differential superordinations, as the dual concept of strong differential subordinations.

Definition 1.2 ([13], cf. [11]). Let $\varphi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If $p(z)$ and $\varphi(p(z), zp'(z), zp''(z); z, \zeta)$ are univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$ and satisfy the (second order) strong differential superordination

$$h(z) \prec\prec \varphi(p(z), zp'(z), zp''(z); z, \zeta), \tag{1.7}$$

then $p(z)$ is called a solution of the strong differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the strong differential superordination, or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.7). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.7) is said to be the best subordinant.

Denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty \right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \backslash E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.3 [12]. Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; z, \zeta) \notin \Omega$$

whenever $r = q(\xi)$, $s = k\xi q'(\xi)$ and

$$\mathrm{Re}\left\{ \frac{t}{s} + 1 \right\} \geq k\mathrm{Re}\left\{ \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right\}$$

for $z \in \mathbb{U}, \xi \in \partial\mathbb{U} \backslash E(q)$, $\zeta \in \overline{\mathbb{U}}$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Definition 1.4 [13]. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \xi, \zeta) \in \Omega$ whenever $r = q(z)$, $s = zq'(z)/m$ for $z \in \mathbb{U}$ and

$$\text{Re}\left\{\frac{t}{s}+1\right\} \leq \frac{1}{m}\text{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\}$$

for $z \in \mathbb{U}, \xi \in \partial\mathbb{U}, \zeta \in \overline{\mathbb{U}}$ and $m \geq n \geq 1$. We write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

For the above two classes of admissible functions, G. I. Oros and G. Oros proved the following theorems.

Theorem 1.1 [12]. *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega,$$

then $p(z) \prec q(z)$.

Theorem 1.2 [13]. *Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{Q}(a)$ *and*

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta)$$

is univalent in $\mathbb{U}$ *for* $\zeta \in \overline{\mathbb{U}}$, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\}$$

implies $q(z) \prec p(z)$.

In the present paper, making use of the differential subordination and superordination results of G. I. Oros and G. Oros [12,13], we determine certain classes of admissible functions and obtain some subordination and superordination implications of meromorphic multivalent functions associated with the Liu Srivastava operator $H_p^{l,m}(\alpha_1)$ defined by (1.3). Additionally, new differential sandwich-type theorems are obtained. We remark in passing that some results on differential subordination and superordination for the operator $H_p^{l,m}(\alpha_1)$ were obtained by Ali *et al.* [1].

2. Subordination Results

Firstly, we begin by proving the subordination theorem involving the integral operator $H_p^{l,m}(\alpha_1)$ defined by (1.3). For this purpose, we need the following class of admissible functions.

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$ and $\alpha_1 \in \mathbb{C}$ with $\alpha_1 \neq 0, -1$. The class of admissible functions $\Phi_{\mathcal{I},1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z, \zeta) \notin \Omega$ whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi) + \alpha_1 q(\xi)}{\alpha_1},$$

and

$$\mathrm{Re}\left\{\frac{(\alpha_1+1)(w-u)}{v-u} - (2\alpha_1+1)\right\} \geq k\mathrm{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\}$$

for $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U} \backslash E(q)$, $\zeta \in \overline{\mathbb{U}}$ and $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_{\mathcal{I},1}[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

$$\left\{\phi(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1+1) f(z), z^p H_p^{l,m}(\alpha_1+2) f(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\} \subset \Omega, \tag{2.1}$$

then

$$z^p H_p^{l,m}(\alpha_1) f(z) \prec q(z).$$

Proof. Define the function $p(z)$ in $\mathbb{U}$ by

$$p(z) := z^p H_p^{l,m}(\alpha_1) f(z). \tag{2.2}$$

From (2.2) with the relation (1.4), we get

$$z^p H_p^{l,m}(\alpha_1 + 1) f(z) = \frac{zp'(z) + \alpha_1 p(z)}{\alpha_1}. \tag{2.3}$$

Further computations show that

$$z^p H_p^{l,m}(\alpha_1 + 2) f(z) = \frac{z^2 p''(z) + 2(\alpha_1 + 1) z p'(z) + \alpha_1(\alpha_1 + 1) p(z)}{\alpha_1(\alpha_1 + 1)}. \tag{2.4}$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s + \alpha_1 r}{\alpha_1},\ w = \frac{t + 2(\alpha_1 + 1)s + \alpha_1(\alpha_1 + 1) r}{\alpha_1(\alpha_1 + 1)}. \tag{2.5}$$

Let

$$\begin{aligned} \psi(r, s, t; z) &= \phi(u, v, w; z) \\ &= \phi\left(r, \frac{s + \alpha_1 r}{\alpha_1}, \frac{t + 2(\alpha_1 + 1)s + \alpha_1(\alpha_1 + 1) r}{\alpha_1(\alpha_1 + 1)}; z, \zeta\right). \end{aligned} \tag{2.6}$$

Using equations (2.2), (2.3) and (2.4), from (2.6), we obtain

$$\begin{aligned} &\psi(p(z), zp'(z), z^2 p''(z); z) \\ &= \phi\left(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1 + 1) f(z), z^p H_p^{l,m}(\alpha_1 + 2) f(z); z, \zeta\right). \end{aligned} \tag{2.7}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z, \zeta) \in \Omega.$$

Note that

$$\frac{t}{s} + 1 = \frac{(\alpha_1 + 1)(w - u)}{v - u} - (2\alpha_1 + 1)$$

and so the admissibility condition for $\phi \in \Phi_{\mathcal{I},1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi \in \Psi_p[\Omega, q]$. Therefore by Theorem 1.1, $p \prec q$ or

$$z^p H_p^{l,m}(\alpha_1) f(z) \prec q(z),$$

which evidently completes the proof of Theorem 2.1.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{I},1}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathcal{I},1}[h, q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi \in \Phi_{\mathcal{I},1}[h, q]$. If $f \in \Sigma_p$ satisfies*

$$\phi(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1 + 1) f(z), z^p H_p^{l,m}(\alpha_1 + 2) f(z); z, \zeta) \prec\prec h(z), \qquad (2.8)$$

then

$$z^p H_p^{l,m}(\alpha_1) f(z) \prec q(z).$$

Our next result is an extension of Theorem 2.1 to the case where the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q(0) = 1$. Let $\phi \in \Phi_{\mathcal{I},1}[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1 + 1) f(z), z^p H_p^{l,m}(\alpha_1 + 2) f(z); z, \zeta\right) \in \Omega,$$

then

$$z^p H_p^{l,m}(\alpha_1) f(z) \prec q(z).$$

Proof. Theorem 2.1 yields $z^p H_p^{l,m}(\alpha_1) f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 2.3. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_{\mathcal{I},1}[h, q_\rho]$ for some $\rho \in (0, 1)$, or*

(2) *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathcal{I},1}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.8), then

$$z^p H_p^{l,m}(\alpha_1) f(z) \prec q(z).$$

Proof. The proof is similar to that [10, Theorem 2.3d] and so is omitted.

The next theorem yields the best dominant of the differential subordination (2.8).

Theorem 2.4. *Let h be univalent in $\mathbb{U}$. Let $\phi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + \alpha_1 q(z)}{\alpha_1}, \frac{z^2 q''(z) + 2(\alpha_1 + 1) z q'(z) + \alpha_1(\alpha_1 + 1) q(z)}{\alpha_1(\alpha_1 + 1)}; z, \zeta\right) = h(z) \tag{2.9}$$

has a solution q with $q(0) = 1$ and satisfy one of the following conditions:

(1) *$q \in \mathcal{Q}_1$ and $\phi \in \Phi_{\mathcal{I},1}[h, q]$,*

(2) *$q(z)$ is univalent in $\mathbb{U}$ and $\phi \in \Phi_{\mathcal{I},1}[h, q_\rho]$ for some $\rho \in (0,1)$, or*

(3) *$q(z)$ is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_{\mathcal{I},1}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.8) and

$$\phi\left(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1+1) f(z), z^p H_p^{l,m}(\alpha_1+2) f(z); z, \zeta\right)$$

is analytic in $\mathbb{U}$, then

$$z^p H_p^{l,m}(\alpha_1) f(z) \prec q(z).$$

and $q(z)$ is the best dominant.

Proof. Following the same arguments in [10, Theorem 2.3e], we deduce that $q(z)$ is a dominant from Theorem 2.2 and Theorem 2.3. Since $q(z)$ satisfies (2.9), it is also a solution of (2.8) and therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant.

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_{\mathcal{I},1}[\Omega, q]$, denoted by $\Phi_{\mathcal{I},1}[\Omega, M]$, is described below.

Definition 2.2. Let Ω be a set in $\mathbb{C}$, $\alpha_1 \in \mathbb{C}$ with $\alpha_1 \neq 0, -1$ and $M > 0$. The class of admissible functions $\Phi_{\mathcal{I},1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{(k+\alpha_1)Me^{i\theta}}{\alpha_1}, 1 + \frac{L + (2k+\alpha_1)(\alpha_1+1)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z, \zeta\right) \notin \Omega \qquad (2.9)$$

whenever $z \in \mathbb{U}$, $\mathrm{Re}\left\{Le^{-i\theta}\right\} \geq (k-1)kM$, $\theta \in \mathbb{R}$, $\zeta \in \overline{\mathbb{U}}$ and $k \geq 1$.

Corollary 2.2. *Let* $\phi \in \Phi_{\mathcal{I},1}[\Omega, M]$. *If* $f \in \Sigma_p$ *satisfies*

$$\phi\left(z^p H_p^{l,m}(\alpha_1)f(z), z^p H_p^{l,m}(\alpha_1+1)f(z), z^p H_p^{l,m}(\alpha_1+2)f(z); z, \zeta\right) \in \Omega,$$

then

$$z^p H_p^{l,m}(\alpha_1)f(z) \prec Mz.$$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{\mathcal{I},1}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{I},1}[M]$.

Corollary 2.3. *Let* $\phi \in \Phi_{\mathcal{I},1}[M]$. *If* $f \in \Sigma_p$ *satisfies*

$$\left|\phi(z^p H_p^{l,m}(\alpha_1)f(z), z^p H_p^{l,m}(\alpha_1+1)f(z), z^p H_p^{l,m}(\alpha_1+2)f(z); z, \zeta)\right| < M,$$

then

$$|z^p H_p^{l,m}(\alpha_1)f(z)| < M.$$

Corollary 2.4. *Let* $\operatorname{Re}\{\alpha_1\} > 0$, $M > 0$ *and let* $C(\zeta)$ *be analytic function in* $\overline{U}$ *with* $\operatorname{Re}\{\xi D(\zeta)\} \geq 0$ *for* $\xi \in \partial\mathbb{U}$. *If* $f \in \Sigma_p$ *satisfies*

$$\left|z^p H_p^{l,m}(\alpha_1+1)f(z) - z^p H_p^{l,m}(\alpha_1)f(z) + D(\zeta)\right| < \frac{\operatorname{Re}\{\alpha_1\}}{|\alpha_1|^2}M,$$

then

$$\left| z^p H_p^{l,m}(\alpha_1) f(z) z^p \right| < M.$$

Proof. This follows from Corollary 2.2 by taking $\phi(u, v, w; z, \zeta) = v - u + D(\zeta)$ and $\Omega = h(\mathbb{U})$, where $h(z) = \frac{\text{Re}\{\alpha_1\}}{|\alpha_1|^2} Mz$. To use Corollary 2.2, we need to show that $\phi \in \Phi_{\mathcal{I},1}[\Omega, M]$, that is, the admissible condition (2.9) is satisfied. This follows since

$$\begin{aligned}
&\left| \phi\left(1 + Me^{i\theta}, 1 + \frac{(k+\alpha_1)Me^{i\theta}}{\alpha_1}, 1 + \frac{L + (2k+\alpha_1)(\alpha_1+1)Me^{i\theta}}{\alpha_1(\alpha_1+1)}; z, \zeta\right) \right| \\
&= \left| \frac{k}{\alpha_1} Me^{i\theta} + D(\zeta) \right| \\
&\geq \frac{\text{Re}\{\alpha_1\}}{|\alpha_1|^2} M
\end{aligned}$$

for $z \in \mathbb{U}$, $\text{Re}\left\{Le^{-i\theta}\right\} \geq (k-1)kM$, $\theta \in \mathbb{R}$ and $k \geq 1$. Hence by Corollary 2.2, we deduce the required result.

3. Superordination and Sandwich-type Results

The dual problem of differential subordination, that is, differential superordination of the fractional differintegral operator $z^p H_p^{l,m}(\alpha_1)$ defined by (1.2) is investigated in this section. For this purpose, the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}_1$ with $zq'(z) \neq 0$ and $\alpha_1 \in \mathbb{C}$ with $\alpha_1 \neq 0, -1$. The class of admissible functions $\Phi'_{\mathcal{I},1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; \xi, \zeta) \in \Omega$ whenever

$$u = q(z), \quad v = \frac{zq'(z)/m + \alpha_1 q(z)}{\alpha_1},$$

and

$$\operatorname{Re}\left\{\frac{(\alpha_1+1)(w-u)}{v-u}-(2\alpha_1+1)\right\} \leq \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)}+1\right\}$$

for $z \in \mathbb{U}$, $\xi \in \partial\mathbb{U}$, $\zeta \in \overline{\mathbb{U}}$ and $m \geq 1$.

Theorem 3.1. *Let $\phi \in \Phi'_{\mathcal{I},1}[\Omega, q]$. If $f \in \Sigma_p$, $z^p H_p^{l,m}(\alpha_1) f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1+1) f(z), z^p H_p^{l,m}(\alpha_1+2) f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1+1) f(z), z^p H_p^{l,m}(\alpha_1+2) f(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\} \tag{3.1}$$

implies

$$q(z) \prec z^p H_p^{l,m}(\alpha_1) f(z).$$

Proof. From (2.7) and (3.1), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2 p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_{\mathcal{I},1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q \prec p$ or

$$q(z) \prec z^p H_p^{l,m}(\alpha_1) f(z),$$

which evidently completes the proof of Theorem 3.1.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi'_{\mathcal{I},1}[h(\mathbb{U}), q]$ is written as $\Phi'_{\mathcal{I},1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $q \in \mathcal{H}_1$, h be analytic in $\mathbb{U}$ and $\phi \in \Phi'_{\mathcal{I},1}[h, q]$. If $f(z) \in \Sigma_p$, $z^p H_p^{l,m}(\alpha_1) f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1 + 1) f(z), z^p H_p^{l,m}(\alpha_1 + 2) f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi(z^p H_p^{l,m}(\alpha_1) f(z), z^p H_p^{l,m}(\alpha_1 + 1) f(z), z^p H_p^{l,m}(\alpha_1 + 2) f(z); z, \zeta)$$

implies

$$q(z) \prec z^p H_p^{l,m}(\alpha_1) f(z).$$

Theorem 3.1 and Theorem 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain ϕ.

Theorem 3.3. *Let h be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^2 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + \alpha_1 q(z)}{\alpha_1}, \frac{z^2 q''(z) + 2(\alpha_1 + 1) z q'(z) + \alpha_1(\alpha_1 + 1) q(z)}{\alpha_1(\alpha_1 + 1)}; z, \zeta\right) = h(z)$$

has a solution $q \in \mathcal{Q}_1$. *If* $\phi \in \Phi'_{\mathcal{I},1}[h,q]$, $f \in \Sigma_p$, $z^p H_p^{l,m}(\alpha_1)f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(z^p H_p^{l,m}(\alpha_1)f(z), z^p H_p^{l,m}(\alpha_1+1)f(z), z^p H_p^{l,m}(\alpha_1+2)f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec\prec \phi\left(z^p H_p^{l,m}(\alpha_1)f(z), z^p H_p^{l,m}(\alpha_1+1)f(z), z^p H_p^{l,m}(\alpha_1+2)f(z); z, \zeta\right)$$

implies

$$q(z) \prec z^p H_p^{l,m}(\alpha_1)f(z),$$

and $q(z)$ *is the best subordinant.*

Combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich-type theorem.

Theorem 3.4. *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{U}$, h_2 *be univalent function in* $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \Phi_{\mathcal{I},1}[h_2, q_2] \cap \Phi'_{\mathcal{I},1}[h_1, q_1]$. *If* $f \in \Sigma_p$, $H_p^{l,m}(\alpha_1)f(z) \in \mathcal{H}_1 \cap \mathcal{Q}_1$ *and*

$$\phi\left(z^p H_p^{l,m}(\alpha_1)f(z), z^p H_p^{l,m}(\alpha_1+1)f(z), z^p H_p^{l,m}(\alpha_1+2)f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec\prec \phi(z^p H_p^{l,m}(\alpha_1)f(z), z^p H_p^{l,m}(\alpha_1+1)f(z), z^p H_p^{l,m}(\alpha_1+2)f(z); z, \zeta) \prec\prec h_2(z)$$

implies

$$q_1(z) \prec z^p H_p^{l,m}(\alpha_1)f(z) \prec q_2(z).$$

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