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Subordination and Superordination Preserving Properties for Certain Multivalent Functions

by

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(다엽함수들에 대한 종속과 초 종속 보존 성질들)

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다엽함수들에 대한 미분종속과 초 종속 보존 성질들

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요 약

기하 함수 이론은 지금까지 많은 학자들에 의하여 연구되어 왔다. 특히, Miller와 Mocanu는 미분종속 이론을 소개하고 해석함수들의 종속문제와 그 쌍대개념인 초 종속 문제를 연구하여 다양한 기하학적 성질들을 조사하였다 (cf. [4, 5]).

본 연구에서는 Miller와 Mocanu의 미분종속 및 초 종속 이론을 응용하여 적당한 admissible 함수들의 족들을 도입하여 fractional differintegral 연산자와 관련된 다엽함수들의 미분종속과 그 쌍대문제인 미분 초 종속 보존 성질들을 연구하였다. 또한, 이 연산자에 대하여 sandwich 형태의 결과들을 조사하였다.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\},$$

and let $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H}_1 \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{U}) \tag{1.1}$$

and let $\mathcal{A}_1 \equiv \mathcal{A}$. Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [4, 11]).

With a view to introducing a fractional differintegral operator, we begin by recalling the following definitions of fractional calculus (that is, fractional intgral and fractional derivative of an arbitrary order) considered by Owa [6] (see also [7], [10] and [11]).

Definition 1.1 The fractional integral of order $\lambda (\lambda > 0)$ is defined, for a function f, analytic in a simply-connected region of the complex plane containing the origin by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z - \zeta)^{1-\lambda}} d\zeta,$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 1.2. Under the Definition 1.1, the fractional derivative of f of order $\lambda(\lambda \geq 0)$ is defined by

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta & (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \leq \lambda < n+1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{cases}$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed as in Definition 1.1.

We observe that, for a function f, given by (1.1), we have

$$D_z^{\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} z^{p-\lambda} + \sum_{n=1}^{\infty} \frac{\Gamma(n+p+1)}{\Gamma(n+p+1-\lambda)} a_{p+n} z^{n+p-\lambda}, \tag{1.2}$$

provided that $z \in \widetilde{\mathbb{U}}$, where $\widetilde{\mathbb{U}} = \mathbb{U}$ if $-\infty < \lambda \leq p$ and $\widetilde{\mathbb{U}} = \mathbb{U} \backslash \{0\}$ if $p < \lambda < p+1$, and $D_z^{\lambda} f(z)$ is, respectively, the fractional integral of f of order $-\lambda$ when $-\infty < \lambda < 0$ and the fractional derivative of f of order λ when $0 \leq \lambda < p+1$.

In view of (1.2), we now define the fractional differintegral operator $\Omega_z^{\lambda,p} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ for a function f of the form (1.1) and for a real number $\lambda(-\infty < \lambda < p+1)$ by

$$\begin{aligned} \Omega_z^{\lambda,p} f(z) &= \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z) \\ &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)} a_{k+p} z^{k+p}. \end{aligned} \tag{1.3}$$

It is easily seen from (1.3) that

$$z(\Omega_z^{\lambda,p} f(z))' = (p-\lambda)\Omega_z^{\lambda+1,p} f(z) + \lambda \Omega_z^{\lambda,p} f(z) \ \ (-\infty < \lambda < p;\ z \in \mathbb{U}). \tag{1.4}$$

We also note that

$$\Omega_z^{0,p} f(z) = f(z), \quad \Omega_z^{1,p} f(z) = \frac{zf'(z)}{p}.$$

The fractional differential operator $\Omega_z^{\lambda,p}$ with $0 \leq \lambda < 1$ was investigated by Srivastava and Aouf [8]. More recently, Srivastava and Mishra [9] obtained several interesting properties and characteristics for certain subclasses of p-valent analytic functions involving the differintegral operator $\Omega_z^{\lambda,p}$ when $-\infty < \lambda < 1$. We further observe that $\Omega_z^{\lambda,1}$ is the operator introduced by Owa and Srivastava [7].

Denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \backslash E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.3 [4]. Let

$$\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$$

and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \tag{1.4}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.4). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.4) is said to be the best dominant.

Definition 1.4 [5]. Let

$$\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$$

and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.5}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.5). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.5) is said to be the best subordinant.

Definition 1.5 [4]. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissiblity condition $\psi(r, s; z) \notin \Omega$ whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$ and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}$$

for $z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}} \backslash E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Definition 1.6 [5]. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissiblity condition $\psi(r, s; \zeta) \notin \Omega$ whenever $r = q(z)$, $s = zq'(z)/m$ for $z \in \mathbb{U}$ and

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \leq k\mathrm{Re}\left\{\frac{\zeta q''(z)}{q'(z)} + 1\right\}$$

for $z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}$ and $m \geq n \geq 1$. We write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

For the above two classes of admissible functions, Miller and Mocanu proved the following theorems.

Theorem 1.1 [4]. *Let* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If the analytic function* $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then $p \prec q$.

Theorem 1.2 [5]. *Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{Q}(a)$ *and* $\psi(p(z), zp'(z); z)$ *is univalent in* $\mathbb{U}$, *then*

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathbb{U}\}$$

implies $q \prec p$.

In the present paper, making use of the differential subordination and superordination results of Miller and Mocanu [4, 5], we determine certain classes of admissible functions and obtain some subordination and superordination implications of multivalent functions associated with the fractional differintegral operator $\Omega_z^{\lambda,p}$ defined by (1.3). Additionally, new differential sandwich-type theorems are obtained. We remark in passing that some similar problems for analytic and meromorphic functions associated with linear operators were considered by Ali *et al.* [1, 2, 3].

2. Subordination Results

Firstly, we begin by proving the subordination theorem involving the integral operator $\Omega_z^{\lambda,p}$ defined by (1.3). For this purpose, we need the following class of admissible functions.

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$ and $\lambda < p-2$. The class of admissible functions $\Phi_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever

$$u = q(\zeta), \ v = \frac{k\zeta q'(\zeta) + (p-\lambda-1)q(\zeta)}{p-\lambda}$$

and

$$\mathrm{Re}\left\{\frac{(p-\lambda)(p-\lambda-1)w - (p-\lambda-2)(p-\lambda-1)u}{(p-\lambda)v-(p-\lambda-1)u} - 2(p-\lambda)+3\right\} \geq k\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}$$

for $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_{I,1}[\Omega, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(\frac{\Omega_z^{\lambda,p}f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p}f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p}f(z)}{z^{p-1}}; z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{2.1}$$

then

$$\frac{\Omega_z^{\lambda,p}f(z)}{z^{p-1}} \prec q(z).$$

Proof. Define the function p in $\mathbb{U}$ by

$$p(z) := \frac{\Omega_z^{\lambda,p}f(z)}{z^{p-1}}. \tag{2.2}$$

By making use of (1.4) and (2.2), we get

$$\frac{\Omega_z^{\lambda+1,p}f(z)}{z^{p-1}} = \frac{zp'(z)+(p-\lambda-1)p(z)}{p-\lambda}. \tag{2.3}$$

Further computations show that

$$\frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}} = \frac{z^2 p''(z) + 2(p-\lambda-1)zp'(z) + (p-\lambda-2)(p-\lambda-1)p(z)}{(p-\lambda-1)(p-\lambda)}. \tag{2.4}$$

Define the transformation from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, v = \frac{s + (p-\lambda-1)r}{p-\lambda} \text{ and } w = \frac{t + 2(p-\lambda-1)s + (p-\lambda-2)(p-\lambda-1)r}{(p-\lambda-1)(p-\lambda)}. \tag{2.5}$$

Let

$$\begin{aligned} &\psi(r,s,t;z) = \phi(u,v,w;z) \\ &= \phi\left(r, \frac{s+(p-\lambda-1)r}{p-\lambda}, \frac{t+2(p-\lambda-1)s+(p-\lambda-2)(p-\lambda-1)r}{(p-\lambda-1)(p-\lambda)}; z\right). \end{aligned} \tag{2.6}$$

Using equations (2.2), (2.3) and (2.4), from (2.6), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}; z\right) \tag{2.7}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

Note that

$$\frac{t}{s} + 1 = \frac{(p-\lambda-1)(p-\lambda)w - (p-\lambda-2)(p-\lambda-1)u}{(p-\lambda)v - (p-\lambda-1)u} - 2(p-\lambda) + 3.$$

and so the admissibility condition for $\phi \in \Phi_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi \in \Psi[\Omega, q]$. Therefore by Theorem 1.1, $p \prec q$ or

$$\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec q(z),$$

which evidently completes the proof of Theorem 2.1.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{I,1}[h(\mathbb{U}), q]$ is written as $\Phi_{I,1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 2.5.

Theorem 2.2. *Let $\phi \in \Phi_{I,1}[h, q]$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right) \prec h(z), \tag{2.8}$$

then

$$\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec q(z).$$

Our next result is an extension of Theorem 2.2 to the case where the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 2.1. *Let $\Omega \subset \mathbb{C}$ and q be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\phi \in \Phi_I[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right) \in \Omega,$$

then

$$\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec q(z).$$

Proof. Theorem 2.1 yields $\Omega_z^{\lambda,p} f(z)/z^{p-1} \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

Theorem 2.3. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfy one of the following conditions*:

(1) *$\phi \in \Phi_I[h, q_\rho]$ for some $\rho \in (0, 1)$, or*

(2) *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}_p$ satisfies (2.8), *then*

$$\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec q(z).$$

Proof. The proof is similar to that [16, Theorem 2.3d] and so is omitted.

The next theorem yields the best dominant of the differential subordination (2.7).

Theroem 2.4. *Let h be univalent in $\mathbb{U}$. Let $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + (p-\lambda-1)q(z)}{p-\lambda}, \frac{z^2q''(z) + 2(p-\lambda-1)zq'(z) + (p-\lambda-2)(p-\lambda-1)q(z)}{(p-\lambda-1)(p-\lambda)}; z\right) = h(z) \qquad (2.9)$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

(1) *$q \in \mathcal{Q}_0$ and $\phi \in \Phi_I[h, q]$,*

(2) *q is univalent in $\mathbb{U}$ and $\phi \in \Phi_I[h, q_\rho]$ for some $\rho \in (0, 1)$, or*

(3) *q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_I[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}_p$ *satisfies* (2.8)*, then*

$$\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec q(z).$$

and q *is the best dominant.*

Proof. Following the same arguments in [4, Theorem 2.3e], we deduce that q is a dominant from Theorem 2.2 and Theorem 2.3. Since q satisfies (2.9), it is also a solution of (2.8) and therefore q will be dominated by all dominants. Hence q is the best dominant.

In the particular case $q(z) = Mz,\ M > 0$, the class $\Phi_{I,1}[\Omega, q]$ of admissible functions becomes the class $\Phi_{I,1}[\Omega, M]$.

Definition 2.2. Let Ω be a set in $\mathbb{C}$, $\lambda < p-2$ and $M > 0$. The class of admissible functions $\Phi_{I,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that

$$\phi\left(Me^{i\theta}, \frac{(k+p-\lambda-1)Me^{i\theta}}{p-\lambda}, \frac{L+[2k(p-\lambda-1)+(p-\lambda-2)(p-\lambda-1)]Me^{i\theta}}{(p-\lambda-1)(p-\lambda)}\right) \notin \Omega \tag{2.10}$$

whenever $z \in \mathbb{U}$, $\mathrm{Re}\left\{Le^{-i\theta}\right\} \geq (k-1)kM$, $\theta \in \mathbb{R}$ and $k \geq 1$.

Corollary 2.2. *Let* $\phi \in \Phi_{I,1}[\Omega, M]$*. If* $f \in \mathcal{A}_p$ *satisfies*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}; z\right) \in \Omega,$$

then

$$\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec Mz.$$

When $\Omega = \{w : |w| < M\} = q(\mathbb{U})$, the class $\Phi_{I,1}[\Omega, M]$ is simply denoted by $\Phi_{I,1}[M]$.

Cor0llary 2.3. *Let $\phi \in \Phi_{I,1}[M]$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)\right| < M,$$

then

$$\left|\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}\right| < M.$$

Cor0llary 2.4. *Let $\lambda < p - 1$ and $M > 0$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}\right| < M,$$

then

$$\left|\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}\right| < M.$$

Proof. This follows from Corollary 2.6 by taking $\phi(u, v; z) = v$.

Cor0llary 2.5. *Let $\lambda < p - 2$ and $M > 0$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|(p-\lambda-1)(p-\lambda)\frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}} + (p-\lambda)\frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}\right.$$
$$\left.-(p-\lambda-2)(p-\lambda-1)\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}\right| < [3(p-\lambda)-2]M,$$

then

$$\left|\frac{\Omega_z^{\lambda,p}f(z)}{z^{p-1}}\right| < M.$$

Proof. This follows from Corollary 2.2 by taking $\phi(u,v,w;z) = (p-\lambda-1)(p-\lambda)w+(p-\lambda)v-(p-\lambda-2)(p-\lambda-1)u$ and $\Omega = h(\mathbb{U})$, where $h(z) = [3(p-\lambda)-2]Mz$. To use Corollary 2.2, we need to show that $\phi \in \Phi_{I,1}[\Omega, M]$, that is, the admissible condition (2.10) is satisfied. This follows since

$$\begin{aligned}
&\left|\phi\left(Me^{i\theta}, \frac{(k+p-\lambda-1)Me^{i\theta}}{p-\lambda}, \frac{L+[2k(p-\lambda-1)+(p-\lambda-2)(p-\lambda-1)]Me^{i\theta}}{(p-\lambda-1)(p-\lambda)}\right)\right| \\
&= \left|L-[2k(p-\lambda-1)+(p-\lambda-2)(p-\lambda-1)Me^{i\theta}+(k+p-\lambda-1)Me^{i\theta}\right. \\
&\qquad\qquad \left.-(p-\lambda-2)(p-\lambda-1)Me^{i\theta}\right| \\
&= \left|L+[2k(p-\lambda-1)+(k+p-\lambda-1)]Me^{i\theta}\right| \\
&\geq \operatorname{Re}\left\{Le^{-i\theta}\right\}+[2k(p-\lambda-1)+(k+p-\lambda-1)]M \\
&\geq k(k-1)M+[2k(p-\lambda-1)+(k+p-\lambda-1)]M \\
&= [3(p-\lambda)-2]M
\end{aligned}$$

for $z \in \mathbb{U}$, $\operatorname{Re}\left\{Le^{-i\theta}\right\} \geq (k-1)kM$, $\theta \in \mathbb{R}$ and $k \geq 1$. Hence by Corollary 2.5, we deduce the required result.

3. Superordination and Sandwich-type Results

The dual problem of differential subordination, that is, differential superordination of the fractional differintegral operator defined by (1.3) is investigated in this section. For this purpose, the class of admissible functions is given in the following definition.

Definition 3.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}_0$ and $\lambda < p-2$. The class of admissible functions $\Phi'_{I,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissiblity condition $\phi(u, v, w; \zeta) \in \Omega$ whenever

$$u = q(z), \;\; v = \frac{zq'(z)/m + (p-\lambda-1)q(z)}{p-\lambda}$$

and

$$\mathrm{Re}\left\{\frac{(p-\lambda)(p-\lambda-1)w - (p-\lambda-2)(p-\lambda-1)u}{(p-\lambda)v - (p-\lambda-1)u} - 2(p-\lambda) + 3\right\} \leq \frac{1}{m}\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}$$

for $z \in \mathbb{U}$, $\zeta \in \overline{\mathbb{U}}$ and $m \geq 1$.

Theorem 3.1. *Let $\phi \in \Phi'_{I,1}[\Omega, q]$. If $f \in \mathcal{A}_p$, $\Omega_z^{\lambda,p} f(z)/z^{p-1} \in \mathcal{Q}_0$ and*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right) : z \in \mathbb{U}\right\} \tag{3.1}$$

implies

$$q(z) \prec \frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}.$$

Proof. From (2.7) and (3.1), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z); z), z^2 p''(z); z) : z \in \mathbb{U}\right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_{I,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 1.2, $q \prec p$ or

$$q(z) \prec \frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi'_{I,1}[h(\mathbb{U}), q]$ is written as $\Phi'_{I,1}[h, q]$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $q \in \mathcal{H}_0$, h is analytic in $\mathbb{U}$ and $\phi \in \Phi'_{I,1}[h, q]$. If $f \in \mathcal{A}_p$, $\Omega_z^{\lambda,p} f(z)/z^{p-1} \in \mathcal{Q}_0$ and*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)$$

implies

$$q(z) \prec \frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}.$$

Theorem 3.1 and Theorem 3.2 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.2). The following theorem proves the existence of the best subordinant of (3.2) for certain ϕ.

Theorem 3.3. *Let h be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)+(p-\lambda-1)q(z)}{p-\lambda}, \frac{z^2q''(z)+2(p-\lambda-1)zq'(z)+(p-\lambda-2)(p-\lambda-1)q(z)}{(p-\lambda-1)(p-\lambda)}; z\right) = h(z)$$

has a solution $q \in \mathcal{Q}_0$. *If* $\phi \in \Phi'_I[h, q]$, $f \in \mathcal{A}_p$, $\Omega_z^{\lambda,p} f(z) \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$h(z) \prec \phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)$$

implies

$$q(z) \prec \frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}},$$

and q *is the best subordinant.*

Proof. The proof is similar to that of Theorem 2.4 and so is omitted.

Combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich-type theorem.

Theroem 3.4. *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{U}$, h_2 *be univalent function in* $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$ *and* $\phi \in \Phi_{I,1}[h_2, q_2] \cap \Phi'_{I,1}[h_1, q_1]$. *If* $f \in \mathcal{A}_p$, $\Omega_z^{\lambda,p} f(z)/z^{p-1} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ *and*

$$\phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{U}$, *then*

$$h_1(z) \prec \phi\left(\frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+1,p} f(z)}{z^{p-1}}, \frac{\Omega_z^{\lambda+2,p} f(z)}{z^{p-1}}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{\Omega_z^{\lambda,p} f(z)}{z^{p-1}} \prec q_2(z).$$

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