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Optimality Conditions and Duality for Nondifferentiable Multiobjective Programming Problems

by

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미분불가능한 다목적 계획문제의 최적조건과 쌍대성

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by
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요 약

본 논문에서는 등호제약식과 부등호제약식을 가지고 목적함수에 지지함수를 포함하는 미분불가능한 다목적 계획문제에 대하여 일반화된 볼록조건 아래에서 약 유효해를 이용하여 최적조건과 쌍대관계를 정립하였다.

먼저, 약 유효해를 이용한 Fritz John과 Kuhn-Tucker 최적조건을 제시하고, 일반화된 볼록 조건 아래에서 약 유효해에 관한 약 쌍대정리 및 강 쌍대정리를 증명하였다. 마지막으로 약 벡터안장점과 약 유효해와의 상등관계를 밝혔다.

1 Introduction and Preliminaries

The role of optimality criteria in mathematical programming is important from both theoretical and computational points of view. Perhaps the best known conditions for optimality are the Fritz John and the Kuhn-Tucker conditions. And there have been much increasing interests in developing optimality and duality relations. Therefore, some researchers have made the study of optimality conditions and they have established duality theorems.

In [2], Jeyakumar introduced ρ-invexity for nonsmooth scalar-valued functions and studied duality theorems for nonsmooth optimization problems. Moreover, it is shown that the equivalence between saddle-points and optima holds for a much large class of non-differentiable non-convex problems. Jeyakumar and Mond [3] defined generalized V-invexity for differentiable multiobjective programming problems. Also, they established the sufficient optimality conditions and duality results as in the scalar case.

Further developments in this direction are founded in Mishra and Mukherjee [11] and Kuk et al. [5]. Mishra and Mukherjee [11] extended the results of Jeyakumar and Mond [3] to multiobjective nonsmooth programming. And Kuk et al. [5] defined the concept of (V, ρ)-invexity for vector-valued functions, which is a generalization of the concept of V-invexity concept([3, 11]). Moreover, they proved the generalized Karush-Kuhn-Tucker sufficient optimality theorems as well as weak and strong duality for nonsmooth multiobjective programs under the (V, ρ)-invexity assumptions. Duality theorems for nondifferentiable programming problem with a square root term were obtained by Lal et al. [7].

In 1996, Mond and Schechter [12] studied duality and optimality for nondifferentiable multiobjective programming problems in which each component of the objective function contains the support function of a compact convex set. In nondifferentiable multiobjective programs involving the support function, further developments for duality relations were found in Kim et al. [4] and Liang et al. [6].

Liang et al. [6] introduced the concept of (F, α, ρ, d)-convexity and obtained some corresponding optimality conditions and duality results based on the properties of sublinear fractional and generalized convex functions. Also, Kim et al. [4] established necessary and sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems under (V, ρ)-invexity assumptions introduced in Kuk et al. [5]. In order to establish sufficient optimality conditions and duality relations, we present the concept of generalized (F, α, ρ, d)-convexity which is related to various generalized convexity by several authors([2, 3, 5, 6, 7, 11, 13, 14]).

Recently, based on Mond and Schechter [12], Yang et al. [14] introduced a class of nondifferentiable multiojbective programming problems involving the support function of a compact convex set. They constructed a general dual model for a class of nondifferentiable multiobjective programs and established only weak duality theorems for efficient solutions under the generalized (F, ρ)-convexity assumptions. Subsequently, Kim et al. [8] established generalized second order symmetric duality in nondifferentiable multiobjective programming problems.

In this paper, we present the concept of generalized (F, α, ρ, d)-convexity and formulate a class of nondifferentiable multiobjective programming problems involving the support function of a compact convex set and linear functions. And we obtain the necessary and sufficient optimality theorems and generalized duality theorems for weakly efficient solutions under generalized (F, α, ρ, d)-convexity assumptions. Moreover, we get the equivalence of saddle points and weakly efficient solution.

In Section 2, we obtain the necessary and sufficient optimality conditions under generalized (F, α, ρ, d)-convexity assumptions. And in Section 3 we give the weak and strong duality theorems for weakly efficient solutions. Both weak duality theorems and strong duality theorems are established by using necessary and sufficient optimality conditions. Finally, we derive the weak vector saddle point theorems for multiobjective programming problems under generalized (F, α, ρ, d)-convexity assumptions in Section 4.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its nonnegative orthant.

The following notation will be used for vectors in $\mathbb{R}^n$:

$$x < y \iff x_i < y_i,\ i = 1, 2, \cdots, n;$$

$$x \leqq y \iff x_i \leqq y_i,\ i = 1, 2, \cdots, n;$$

$$x \leq y \iff x_i \leqq y_i,\ i = 1, 2, \cdots, n \text{ but } x \neq y;$$

$x \not< y$is the negation of$x < y$;

$x \not\leq$ y is the negation of$x \leq y$.

For $x, u \in \mathbb{R}$, $x \leqq u$ and $x < u$ have the usual meaning.

We consider the following nondifferentiable multiobjective programming problem:

$$\text{(MPE)} \quad \text{Minimize} \quad (f_1(x) + s(x|D_1), \cdots, f_p(x) + s(x|D_p))$$

$$\text{subject to} \quad g(x) \leqq 0, \quad Bx = c,$$

where f and g are differentiable functions from $\mathbb{R}^n \to \mathbb{R}^p$ and $\mathbb{R}^n \to \mathbb{R}^m$, respectively; B is a $q \times n$ matrix, $c \in R^q$, and D_i, for each $i \in P = \{1, 2, \cdots, p\}$, is a compact convex set of $\mathbb{R}^n$.

Further let, $S := \{x \in \mathbb{R}^n \mid g_i(x) \leqq 0,\ B_k x = c_k,\ i = 1, \cdots, m,\ k = 1, \cdots, q\}$ and $I(x) := \{i \mid g_i(x) = 0\}$ for any $x \in \mathbb{R}^n$.

Definition 1.1 *A feasible solution $\bar{x}$ is a weakly efficient solution of (MPE) if there exists no other $x \in S$ such that $f(x) < f(\bar{x})$.*

Definition 1.2 *[12] Let D be a compact convex set in $\mathbb{R}^n$. The support function $s(x|D)$ is defined by*

$$s(x|D) := max\{x^T y : y \in D\}.$$

The support function $s(x|D)$, being convex and everywhere finite, has a subdifferential, that is, there exists z such that

$$s(y|D) \geq s(x|D) + z^T(y - x) \text{ for all } y \in D.$$

Equivalently,

$$z^T x = s(x|D).$$

The subdifferential of $s(x|D)$ is given by

$$\partial s(x|D) := \{z \in D : z^T x = s(x|D)\}.$$

Definition 1.3 *A functional $F : X \times X \times R^n \to \mathbb{R}$ is sublinear in its third component, if for all $x, u \in X$,*

(i) $F(x, u; a_1 + a_2) \leqq F(x, u; a_1) + F(x, u; a_2)$ for all $a_1, a_2 \in \mathbb{R}^n$; and

(ii) $F(x, u; \alpha a) = \alpha F(x, u; a)$ for all $\alpha \in \mathbb{R}_+$, and for all $a \in \mathbb{R}^n$.

We introduce the following definitions due to the concept of (F, α, ρ, d)-convexity defined by Liang et al. [6].

Let $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a sublinear functional; let the function $\phi : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $u \in \mathbb{R}^n, \rho \in \mathbb{R}$, and $d(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Definition 1.4 *The function ϕ is said to be (F, α, ρ, d)-convex at u if*

$$\phi(x) - \phi(u) \geqq F(x, u; \alpha(x, u)\nabla\phi(u)) + \rho d^2(x, u), \quad \forall x \in \mathbb{R}^n.$$

Definition 1.5 *The function ϕ is (F, α, ρ, d)-quasiconvex at u if*

$$\phi(x) \leqq \phi(u) \Rightarrow F(x, u; \alpha(x, u)\nabla\phi(u)) \leqq -\rho d^2(x, u), \quad \forall x \in \mathbb{R}^n.$$

Definition 1.6 *The function ϕ is (F, α, ρ, d)-pseudoconvex at u if*

$$F(x, u; \alpha(x, u)\nabla\phi(u)) \geqq -\rho d^2(x, u) \Rightarrow \phi(x) \geqq \phi(u), \quad \forall x \in \mathbb{R}^n.$$

Definition 1.7 *The function ϕ is strictly (F, α, ρ, d)-pseudoconvex at u if for all $x \in \mathbb{R}^n$, $x \neq u$ such that*

$$F(x, u; \alpha(x, u)\nabla\phi(u)) \geqq -\rho d^2(x, u) \Rightarrow \phi(x) > \phi(u), \quad \forall x \in \mathbb{R}^n.$$

Remark 1.1 *(i) When* $\alpha(x,u) = 1$*, the concept of* (F,α,ρ,d)*-convexity is the same as that of* (F,ρ)*-convexity in [13].*

(ii) When $F(x,u;\alpha(x,u)\nabla\phi(u)) = \alpha(x,u)\nabla\phi(u)\eta(x,u)$*, for a certain function* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*, the concept of* (F,α,ρ,d)*-convexity is the same as that of* (V,ρ)*-invexity in [6].*

We give a generalization of Gordan's theorem for the convex and linear functions.

Theorem 1.1 [9] *Let* F *be an* m*-dimensional convex vector function on the convex set* $\mathbb{R}^n$*. Let* B *be a given* $q \times n$ *matrix with linearly independent rows, and let* c *be a given* q*-dimensional vector. Then either*

I. $F(x) < 0,\ Bx = c$ *has a solution* $x \in \mathbb{R}^n$

or

II. $\langle (p,q),(F(x), Bx - c)\rangle \geqq 0$*, for all* $x \in \mathbb{R}^n$*, for some* $p \geq 0,\ \ p \in \mathbb{R}^m$*,* $q \in \mathbb{R}^q$*,*

but never both.

2 Optimality Conditions

In this section, we establish Fritz John and Kuhn-Tucker necessary and sufficient conditions for weakly efficient solutions of (MPE).

Theorem 2.1 (Fritz John Necessary Optimality Conditions) *Suppose that* $f_i, g_j : \mathbb{R}^n \to \mathbb{R},\ i = 1, \cdots, p,\ j = 1, \cdots, m,$ *are differentiable and the vectors* $B_k,\ k = 1, \cdots, q,$ *are linearly independent. If* $\bar{x} \in S$ *is a weakly efficient solution of (MPE), then there exist* $\lambda_i,\ i = 1, \cdots, p,\ \mu_j,\ j = 1, \cdots, m,\ \nu_k,\ k = 1, \cdots, q,$ *and* $w_i \in D_i,\ i = 1, \cdots, p$ *such that*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq 0.$$

Proof. Let $h_i(x) = s(x|D_i),\ i = 1, \cdots, p.$ Since D_i is convex and compact, $h_i : \mathbb{R}^n \to \mathbb{R}$ is a convex function and hence $\forall d \in \mathbb{R}^n$,

$$h_i'(\bar{x}; d) = \lim_{\lambda \to 0+} \frac{h_i(\bar{x} + \lambda d) - h_i(\bar{x})}{\lambda}$$

is finite. Also, $\forall d \in \mathbb{R}^n$,

$$\begin{aligned}(f_i + h_i)'(\bar{x}; d) &= \lim_{\lambda \to 0+} \frac{f_i(\bar{x} + \lambda d) + h_i(\bar{x} + \lambda d) - f_i(\bar{x}) - h_i(\bar{x})}{\lambda} \\ &= \lim_{\lambda \to 0+} \frac{f_i(\bar{x} + \lambda d) - f_i(\bar{x})}{\lambda} + \lim_{\lambda \to 0+} \frac{h_i(\bar{x} + \lambda d) - h_i(\bar{x})}{\lambda}\end{aligned}$$

$$
\begin{aligned}
&= f_i'(\bar{x}; d) + h_i'(\bar{x}; d) \\
&= \langle \nabla f_i(\bar{x}), d \rangle + h_i'(\bar{x}; d).
\end{aligned}
$$

Since $\bar{x}$ is a weakly efficient solution of (MPE),

$$
\begin{cases}
\langle \nabla f_i(\bar{x}), d \rangle + h_i'(\bar{x}; d) < 0, \; i = 1, \cdots, p \\
\langle \nabla g_j(\bar{x}), d \rangle < 0, \; j \in I(\bar{x}) \\
\langle B_k, d \rangle = 0, \; k = 1, \cdots, q
\end{cases}
$$

has no solution $d \in \mathbb{R}^n$. By Gordan theorem for convex functions, there exist $\lambda_i \geqq 0$, $i = 1, \cdots, p$, $\mu_j \geqq 0$, $j \in I(\bar{x})$, and ν_k, $k = 1, \cdots, q$, not all zero, such that for any $d \in \mathbb{R}^n$,

$$
\begin{aligned}
&\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d \rangle + \sum_{i=1}^{p} \lambda_i h_i'(\bar{x}; d) \\
&\qquad + \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d \rangle + \sum_{k=1}^{q} \nu_k \langle B_k, d \rangle \geqq 0. \qquad (2.1)
\end{aligned}
$$

Let $A = \{\sum_{i=1}^{p} \lambda_i [\nabla f_i(\bar{x}) + \xi_i] + \sum_{j \in I(\bar{x})} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k \mid \xi_i \in \partial h_i(\bar{x}), \; i = 1, \cdots, p\}$. Then $0 \in A$. Assume to the contrary that $0 \notin A$. By separation theorem, there exists $d^* \in \mathbb{R}^n$, $d^* \neq (0, \cdots, 0)$ such that $\forall a \in A$, $\langle a, d^* \rangle < 0$, that is, $\forall \xi_i \in \partial h_i(\bar{x})$

$$
\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d^* \rangle + \sum_{i=1}^{p} \lambda_i \langle \xi_i, d^* \rangle + \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d^* \rangle + \sum_{k=1}^{q} \nu_k \langle B_k, d^* \rangle < 0.
$$

Hence, we have

$$\sum_{i=1}^{p} \lambda_i \langle \nabla f_i(\bar{x}), d^* \rangle + \sum_{i=1}^{p} \lambda_i h_i'(\bar{x}; d^*) + \sum_{j \in I(\bar{x})} \mu_j \langle \nabla g_j(\bar{x}), d^* \rangle + \sum_{k=1}^{q} \nu_k \langle B_k, d^* \rangle < 0,$$

which contradicts (2.1).

Letting $\mu_j = 0, \ \forall j \notin I(\bar{x})$, we get

$$0 \in \sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i \partial h_i(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k$$

and

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \neq 0.$$

Since $\partial h_i(\bar{x}) = \{w_i \in D_i \mid \langle w_i, \bar{x} \rangle = s(\bar{x}|D_i)\}$, we obtain the desired result.□

Theorem 2.2 (Kuhn-Tucker Necessary Optimality Conditions) *Suppose that $f_i, g_j : \mathbb{R}^n \to \mathbb{R}, \ i = 1, \cdots, p, \ j = 1, \cdots, m$, are differentiable and the vectors $B_k, \ k = 1, \cdots, q$, are linearly independent. Assume that there exist $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle < 0, \ j \in I(\bar{x})$ and $\langle B_k, z^* \rangle = 0, \ k = 1, \cdots, q$. If $\bar{x} \in S$ is a weakly efficient solution of (MPE), then there exist $\lambda_i, \ i = 1, \cdots, p, \ \mu_j, \ j = 1, \cdots, m, \ \nu_k, \ k = 1, \cdots, q$, and $w_i \in D_i, \ i = 1, \cdots, p$ such that*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i), \ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p) \geq 0,$$

$$(\mu_1, \cdots, \mu_m) \geq 0.$$

Proof. Since $\bar{x}$ is a weakly efficient solution of (MPE), by Theorem 2.1, there exist $\lambda_i, \ i = 1, \cdots, p, \ \mu_j, \ j = 1, \cdots, m, \ \nu_k, \ k = 1, \cdots, q,$ and $w_i \in D_i, \ i = 1, \cdots, p$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i), \ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq 0.$$

Assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle < 0, \ \forall j \in I(\bar{x})$ and $\langle B_k, z^* \rangle = 0, \ k = 1, \cdots, q$. Then $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. Assume to the contrary that $(\lambda_1, \cdots, \lambda_p) = (0, \cdots, 0)$. Then $(\mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq$

$(0, \cdots, 0)$. If $\mu = 0$, then $\nu \neq 0$. Since B_k is linearly independent, $\nu_1 B_1 + \cdots + \nu_q B_q = 0$ has a trivial solution $\nu = 0$, this contradicts to the fact that $\nu \neq 0$. So $\mu \geq 0$. Define $\mu_{j \in I_{(x)}} > 0$, $\mu_{j \notin I_{(x)}} = 0$. Since $\langle \nabla g_j(\bar{x}), z^* \rangle < 0$, $j \in I(\bar{x})$, we have $\sum_{j=1}^m \mu_j \langle \nabla g_j(\bar{x}), z^* \rangle < 0$ and so $\sum_{j=1}^m \mu_j \langle \nabla g_j(\bar{x}), z^* \rangle + \sum_{k=1}^q \nu_k \langle B_k, z^* \rangle < 0$. This is a contradiction. Hence $(\lambda_1, \cdots, \lambda_p) \neq (0, \cdots, 0)$. $\square$

Theorem 2.3 (Fritz John Sufficient Optimality Conditions) *Let* $(\bar{x}, \lambda, w, \mu, \nu)$ *satisfy the Fritz John optimality conditions as follows:*

$$\sum_{i=1}^p \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^p \lambda_i w_i + \sum_{j=1}^m \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^q \nu_k B_k = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i), \ i = 1, \cdots, p,$$

$$\sum_{j=1}^m \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m, \nu_1, \cdots, \nu_q) \neq 0.$$

Assume that one of the following conditions hold:

(a) $f_i(\cdot) + (\cdot)^T w_i$ *is* (F, α, ρ_i, d)*-pseudoconvex at* $\bar{x}$ *and* $\sum_{j=1}^m \mu_j g_j(\cdot) + \sum_{k=1}^q \nu_k (B_k(\cdot) - c_k)$ *is strictly* (F, α, β, d)*-pseudoconvex at* $\bar{x}$ *with* $\beta + \sum_{i=1}^p \lambda_i \rho_i \geqq 0$*; or*

(b) $\sum_{i=1}^p \lambda_i (f_i(\cdot) + (\cdot)^T w_i)$ *is* (F, α, ρ, d)*-quasiconvex at* $\bar{x}$ *and* $\sum_{j=1}^m \mu_j g_j(\cdot) + \sum_{k=1}^q \nu_k (B_k(\cdot) - c_k)$ *is strictly* (F, α, β, d)*-pseudoconvex at* $\bar{x}$ *with* $\beta + \rho \geqq 0$.

Then $\bar{x}$ is a weakly efficient solution of (MPE).

Proof. (a) Suppose that $\bar{x}$ is not a weakly efficient solution of (MPE). Then there exists $x^* \in S$ such that $f_i(x^*) + s(x^*|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i)$. Since $\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p$,

$$\begin{aligned} f_i(x^*) + x^{*T} w_i &= f_i(x^*) + s(x^*|D_i) \\ &< f_i(\bar{x}) + s(\bar{x}|D_i) \\ &= f_i(\bar{x}) + \bar{x}^T w_i. \end{aligned}$$

By the (F, α, ρ_i, d)-pseudoconvexity of $f_i(\cdot) + (\cdot)^T w_i$ at $\bar{x}$, we obtain

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\nabla f_i(\bar{x}) + w_i)) < -\rho_i d^2(x^*, \bar{x}).$$

By the sublinearity of F,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i(\nabla f_i(\bar{x}) + w_i)) \leqq -\sum_{i=1}^{p} \lambda_i \rho_i d^2(x^*, \bar{x}).$$

With $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$, we have

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k)) \geqq -\beta d^2(x^*, \bar{x}).$$

Since $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k)$ is strictly (F, α, β, d)-pseudoconvex,

$$\sum_{j=1}^{m} \mu_j g_j(x^*) + \sum_{k=1}^{q} \nu_k (B_k x^* - c_k) > \sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k).$$

By $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0$ and $\sum_{k=1}^{q} \nu_k (B_k x^* - c_k) = \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k) = 0$, we get

$$\sum_{j=1}^{m} \mu_j g_j(x^*) > 0,$$

which contradicts the condition that $\sum_{j=1}^{m} \mu_j g_j(x*) \leqq 0$.

(b) Suppose that $\bar{x}$ is not a weakly efficient solution of (MPE). Then there exists $x^* \in S$ such that $f_i(x^*) + s(x^*|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i)$. Since $\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,$

$$f_i(x^*) + x^{*T} w_i < f_i(\bar{x}) + \bar{x}^T w_i.$$

Using $\lambda_i \geqq 0$, we have

$$\sum_{i=1}^{p} \lambda_i (f_i(x^*) + x^{*T} w_i) \leqq \sum_{i=1}^{p} \lambda_i (f_i(\bar{x}) + \bar{x}^T w_i).$$

By the (F, α, ρ, d)-quasiconvexity of $\sum_{i=1}^{p} \lambda_i (f_i(\cdot) + (\cdot)^T w_i)$ at $\bar{x}$, we obtain

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i (\nabla f_i(\bar{x}) + w_i)) \leqq -\rho d^2(x^*, \bar{x}).$$

With $\beta + \rho \geqq 0$, we have

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) (\sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k)) \geqq -\beta d^2(x^*, \bar{x}).$$

Since $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k)$ is strictly (F, α, β, d)-pseudoconvex,

$$\sum_{j=1}^{m} \mu_j g_j(x^*) + \sum_{k=1}^{q} \nu_k (B_k x^* - c_k) > \sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k).$$

By $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0$ and $\sum_{k=1}^{q} \nu_k(B_k x^* - c_k) = \sum_{k=1}^{q} \nu_k(B_k \bar{x} - c_k) = 0$, we get

$$\sum_{j=1}^{m} \mu_j g_j(x^*) > 0,$$

which contradicts the condition that $\sum_{j=1}^{m} \mu_j g_j(x*) \leqq 0$. $\square$

Theorem 2.4 (Kuhn-Tucker Sufficient Optimality Conditions) *Let $(\bar{x}, \lambda, w, \mu, \nu)$ satisfy the Kuhn-Tucker optimality conditions as follows:*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{m} \mu_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k = 0,$$

$$\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i), \; i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p) \geq 0,$$

$$(\mu_1, \cdots, \mu_m) \geq 0.$$

Assume that one of the following conditions hold:

(a) $f_i(\cdot) + (\cdot)^T w_i$ is (F, α, ρ_i, d)-pseudoconvex at $\bar{x}$ and $\sum_{j=1}^{m} \mu_j g_j(\cdot) + \sum_{k=1}^{q} \nu_k(B_k(\cdot) - c_k)$ is (F, α, β, d)-quasiconvex at $\bar{x}$ with $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$; or

(b) $\sum_{i=1}^{p} \lambda_i(f_i(\cdot) + (\cdot)^T w_i)$ is (F, α, ρ, d)-pseudoconvex at $\bar{x}$ and $\sum_{j=1}^{m} \mu_j g_j(\cdot) + \sum_{k=1}^{q} \nu_k(B_k(\cdot) - c_k)$ is (F, α, β, d)-quasiconvex at $\bar{x}$ with $\beta + \rho \geqq 0$. Then $\bar{x}$ is a weakly efficient solution of (MPE).

Proof. (a) Suppose that $\bar{x}$ is not a weakly efficient solution of (MPE). Then there exists $x^* \in S$ such that $f_i(x^*) + s(x^*|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i)$. Since $\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,$

$$\begin{aligned} f_i(x^*) + x^{*T} w_i &= f_i(x^*) + s(x^*|D_i) \\ &< f_i(\bar{x}) + s(\bar{x}|D_i) \\ &= f_i(\bar{x}) + \bar{x}^T w_i. \end{aligned}$$

By the (F, α, ρ_i, d)-pseudoconvexity of $f_i(\cdot) + (\cdot)^T w_i$ at $\bar{x}$, we obtain

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\nabla f_i(\bar{x}) + w_i)) < -\rho_i d^2(x^*, \bar{x}).$$

By the sublinearity of F,

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i (\nabla f_i(\bar{x}) + w_i)) < -\sum_{i=1}^{p} \lambda_i \rho_i d^2(x^*, \bar{x}).$$

With $\beta + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$, we have

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k)) > -\beta d^2(x^*, \bar{x}).$$

Since $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k)$ is (F, α, β, d)-quasiconvex,

$$\sum_{j=1}^{m} \mu_j g_j(x^*) + \sum_{k=1}^{q} \nu_k (B_k x^* - c_k) > \sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k (B_k \bar{x} - c_k).$$

By $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0$ and $\sum_{k=1}^{q} \nu_k(B_k x^* - c_k) = \sum_{k=1}^{q} \nu_k(B_k \bar{x} - c_k) = 0$, we get

$$\sum_{j=1}^{m} \mu_j g_j(x^*) > 0,$$

which contradicts the condition that $\sum_{j=1}^{m} \mu_j g_j(x*) \leqq 0$.

(b) Suppose that $\bar{x}$ is not a weakly efficient solution of (MPE). Then there exists $x^* \in S$ such that $f_i(x^*) + s(x^*|D_i) < f_i(\bar{x}) + s(\bar{x}|D_i)$. Since $\langle w_i, \bar{x} \rangle = s(\bar{x}|D_i),\ i = 1, \cdots, p,$

$$f_i(x^*) + x^{*T} w_i < f_i(\bar{x}) + \bar{x}^T w_i.$$

Using $\lambda_i \geq 0$, we have

$$\sum_{i=1}^{p} \lambda_i(f_i(x^*) + x^{*T} w_i) < \sum_{i=1}^{p} \lambda_i(f_i(\bar{x}) + \bar{x}^T w_i).$$

By the (F, α, ρ, d)-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(\cdot) + (\cdot)^T w_i)$ at $\bar{x}$, we obtain

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x}) \sum_{i=1}^{p} \lambda_i(\nabla f_i(\bar{x}) + w_i)) < -\rho d^2(x^*, \bar{x}).$$

With $\beta + \rho \geqq 0$, we have

$$F(x^*, \bar{x}; \alpha(x^*, \bar{x})(\sum_{j=1}^{m} \mu_i \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k B_k)) > -\beta d^2(x^*, \bar{x}).$$

Since $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k(B_k \bar{x} - c_k)$ is (F, α, β, d)-quasiconvex,

$$\sum_{j=1}^{m} \mu_j g_j(x^*) + \sum_{k=1}^{q} \nu_k(B_k x^* - c_k) > \sum_{j=1}^{m} \mu_j g_j(\bar{x}) + \sum_{k=1}^{q} \nu_k(B_k \bar{x} - c_k).$$

By $\sum_{j=1}^{m} \mu_j g_j(\bar{x}) = 0$ and $\sum_{k=1}^{q} \nu_k(B_k x^* - c_k) = \sum_{k=1}^{q} \nu_k(B_k \bar{x} - c_k) = 0$, we get

$$\sum_{j=1}^{m} \mu_j g_j(x^*) > 0,$$

which contradicts the condition that $\sum_{j=1}^{m} \mu_j g_j(x*) \leqq 0$. □

3 Duality Theorems

In this section, we introduce a generalized dual programming problem and establish weak and strong duality theorems under generalized (F, α, ρ, d)-convexity assumptions. Now we propose the following general dual (MDE) to (MPE):

(MDE) Maximize

$$(f_1(u) + u^T w_1 + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j(B_j u - c_j),$$

$$\cdots, f_p(u) + u^T w_p + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j(B_j u - c_j))$$

subject to

$$\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) + y^T \nabla g(u) + \sum_{k=1}^{q} z_k B_k = 0, \quad (3.1)$$

$$\sum_{i \in I_\alpha} y_i g_i(u) + \sum_{j \in J_\alpha} z_j(B_j u - c_j) \geqq 0, \ \alpha = 1, \cdots, r, \quad (3.2)$$

$$y \geqq 0, \quad w_i \in D_i, \ i = 1, \cdots, p,$$

$$\lambda = (\lambda_1, \cdots, \lambda_p) \in \Lambda^+,$$

where $I_\alpha \subset M = \{1, \cdots, m\},\ \alpha = 0, 1, \cdots, r$ with $\cup_{\alpha=0}^{r} I_\alpha = M$ and $I_\alpha \cap I_\beta = \emptyset$ if $\alpha \neq \beta,\ J_\alpha \subset Q = \{1, \cdots, q\},\ \alpha = 0, 1, \cdots, r$ with $\cup_{\alpha=0}^{r} J_\alpha = Q$ and $J_\alpha \cap J_\beta = \emptyset$ if $\alpha \neq \beta$.

Let $\Lambda^+ = \{\lambda \in \mathbb{R}^p :\ \lambda \geqq 0,\ \lambda^T e = 1,\ e = (1, \cdots, 1)^T \in \mathbb{R}^p\}$.

Theorem 3.1 (Weak Duality) *Assume that for all feasible x of (MPE) and all feasible (u, λ, w, y, z) of (MDE), if $\sum_{i \in I_\alpha} y_i g_i(\cdot) + \sum_{j \in J_\alpha} z_j (B_j(\cdot) - c_j)$ $(\alpha = 1, \cdots, r)$ is $(F, \alpha, \beta_\alpha, \rho)$-quasiconvex at u and assuming that one of the following conditions hold:*

(a) $f_i(\cdot) + (\cdot)^T w_i + \sum_{i \in I_0} y_i g_i(\cdot) + \sum_{j \in J_0} z_j (B_j(\cdot) - c_j)$ is (F, α, ρ_i, d)-pseudoconvex at u with $\sum_{\alpha=1}^{r} \beta_\alpha + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$; or

(b) $\sum_{i=1}^{p} \lambda_i (f_i(\cdot) + (\cdot)^T w_i) + \sum_{i \in I_0} y_i g_i(\cdot) + \sum_{j \in J_0} z_j (B_j(\cdot) - c_j)$ is (F, α, ρ, d)-pseudoconvex at u with $\sum_{\alpha=1}^{r} \beta_\alpha + \rho \geqq 0$.

Then the following cannot hold:

$$f(x) + s(x|C) < f(u) + u^T w + \sum_{i \in I_0} y_i g_i(u) e + \sum_{j \in J_0} z_j (B_j u - c_j) e. \tag{3.3}$$

Proof. Since x is feasible for (MPE) and (u, λ, w, y, z) is feasible for (MDE), we have

$$\sum_{i \in I_\alpha} y_i g_i(x) + \sum_{j \in J_\alpha} z_j (B_j x - c_j) \leqq 0 \leqq \sum_{i \in I_\alpha} y_i g_i(u) + \sum_{j \in J_\alpha} z_j (B_j u - c_j),$$
$$\alpha = 1, \cdots, r.$$

By the $(F, \alpha, \beta_\alpha, d)$-quasiconvexity of $\sum_{i \in I_\alpha} y_i g_i(u) + \sum_{j \in J_\alpha} z_j (B_j u - c_j),\ \alpha = 1, \cdots, r$, it follows that

$$F(x,u;\alpha(x,u)(\sum_{i\in I_\alpha} y_i \nabla g_i(u) + \sum_{j\in J_\alpha} z_j B_j)) \leqq -\beta_\alpha d^2(x,u),\ \ \alpha = 1,\cdots,r. \quad (3.4)$$

On the other hand, by (3.1) and the sublinearity of F, we have

$$F(x,u;\alpha(x,u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) + \sum_{i\in I_0} y_i \nabla g_i(u) + \sum_{j\in J_0} z_j B_j))$$

$$+ \sum_{\alpha=1}^{r} F(x,u;\alpha(x,u)(\sum_{i\in I_\alpha} y_i \nabla g_i(u) + \sum_{j\in J_\alpha} z_j B_j))$$

$$\geqq F(x,u;\alpha(x,u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) + y^T \nabla g(u) + \sum_{k=1}^{q} z_k B_k))$$

$$= 0. \quad (3.5)$$

Combination (3.4) and (3.5) gives

$$F(x,u;\alpha(x,u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) + \sum_{i\in I_0} y_i \nabla g_i(u) + \sum_{j\in J_0} z_j B_j))$$

$$\geqq (\sum_{\alpha=1}^{r} \beta_\alpha) d^2(x,u). \quad (3.6)$$

Now suppose, contrary to the result, that (3.3) holds. Since $x^T w_i \leqq s(x|D_i)$, we have for all $i \in \{1,\cdots,p\}$

$$f_i(x) + x^T w_i + \sum_{i\in I_0} y_i g_i(x) + \sum_{j\in J_0} z_j (B_j x - c_j)$$

$$\leqq f_i(x) + x^T w_i$$

$$\leqq f_i(x) + s(x|D_i)$$

$$< f_i(u) + u^T w_i + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j (B_j u - c_j). \tag{3.7}$$

If (a) holds, then we get

$$F(x, u; \alpha(x, u)(\nabla f_i(u) + w_i + \sum_{i \in I_0} y_i \nabla g_i(u) + \sum_{j \in J_0} z_j B_j))$$

$$< -\rho_i d^2(x, u), \quad \forall i \in \{1, \cdots p\}. \tag{3.8}$$

From $\lambda \in \Lambda^+$, (3.8) and the sublinearity of F, we have

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) + \sum_{i \in I_0} y_i \nabla g_i(u) + \sum_{j \in J_0} z_j B_j))$$

$$< (-\sum_{i=1}^{p} \lambda_i \rho_i) d^2(x, u). \tag{3.9}$$

Since $\sum_{\alpha=1}^{r} \beta_\alpha + \sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$, it follows from (3.9) that

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i (\nabla f_i(u) + w_i) + \sum_{i \in I_0} y_i \nabla g_i(u) + \sum_{j \in J_0} z_j B_j))$$

$$< (\sum_{\alpha=1}^{r} \beta_\alpha) d^2(x, u),$$

which contradicts (3.6). Hence (3.3) cannot hold.

If (b) holds, then from $\lambda \in \Lambda^+$ and (3.7), it follows that

$$\sum_{i=1}^{p} \lambda_i (f_i(x) + x^T w_i) + \sum_{i \in I_0} y_i g_i(x) + \sum_{j \in J_0} z_j (B_j x - c_j)$$

$$< \sum_{i=1}^{p} \lambda_i(f_i(u) + u^T w_i) + \sum_{i\in I_0} y_i g_i(u) + \sum_{j\in J_0} z_j(B_j u - c_j).$$

Then, by the (F, α, ρ, d)-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(\cdot)+(\cdot)^T w_i)+\sum_{i\in I_0} y_i$ $g_i(\cdot) + \sum_{j\in J_0} z_j(B_j(\cdot) - c_j)$ at u,

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) + \sum_{i\in I_0} y_i \nabla g_i(u) + \sum_{j\in J_0} z_j B_j))$$
$$< -\rho d^2(x, u). \tag{3.10}$$

Since $\sum_{\alpha=1}^{r} \beta_\alpha + \rho \geqq 0$, it follows from (3.10) that

$$F(x, u; \alpha(x, u)(\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) + w_i) + \sum_{i\in I_0} y_i \nabla g_i(u) + \sum_{j\in J_0} z_j B_j))$$
$$< (\sum_{\alpha=1}^{r} \beta_\alpha) d^2(x, u),$$

which contradicts (3.6). Hence (3.3) cannot hold. □

Theorem 3.2 (Strong Duality) *If $\bar{x} \in S$ is a weakly efficient solution of (MPE), and assume that there exists $z^* \in \mathbb{R}^n$ such that $\langle \nabla g_j(\bar{x}), z^* \rangle < 0,\ \forall j \in I(\bar{x}),\ \langle B_k, z^* \rangle = 0,\ k = 1, \cdots, q$, and the vectors $B_k,\ k = 1, \cdots, q$, are linearly independent, then there exist $\bar{\lambda} \in \mathbb{R}^p,\ \bar{w}_i \in D_i,\ i = 1, \cdots, p,\ \bar{y} \in \mathbb{R}^m$, and $\bar{z} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is feasible for (MDE) and $\bar{x}^T \bar{w}_i = s(\bar{x}|D_i),\ i = 1, \cdots, p$. Moreover, if the assumptions of weak duality are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is a weakly efficient solution of (MDE).*

Proof. By Theorem 2.2, there exist $\bar{\lambda} \in \mathbb{R}^p$, $\bar{y} \in \mathbb{R}^m$, $\bar{z} \in \mathbb{R}^q$, and $\bar{w}_i \in D_i$, $i = 1, \cdots, p$ such that

$$\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i \bar{w}_i + \sum_{j=1}^{m} \bar{y}_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} z_k B_k = 0,$$

$$< \bar{w}_i, \bar{x} >= s(\bar{x}|D_i), \ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \bar{y}_j g_j(\bar{x}) = 0,$$

$$(\lambda_1, \cdots, \lambda_p) \geq 0,$$

$$(\mu_1, \cdots, \mu_m) \geq 0.$$

Thus $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is a feasible for (MDE) and $\bar{x}^T \bar{w}_i = s(\bar{x}|D_i)$, $i = 1, \cdots, p$. Notice that

$$\begin{aligned} & f_i(\bar{x}) + s(\bar{x}|D_i) \\ = & f_i(\bar{x}) + \bar{x}^T \bar{w}_i \\ = & f_i(\bar{x}) + \bar{x}^T \bar{w}_i + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) + \sum_{j \in J_0} \bar{z}_j (B_j \bar{x} - c_j). \end{aligned}$$

By Theorem 3.1, we obtain that the following cannot hold:

$$\begin{aligned} & (f_1(\bar{x}) + s(\bar{x}|D_1), \cdots, f_p(\bar{x}) + s(\bar{x}|D_p)) \\ & \quad < (f_1(u) + u^T w_1 + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j (B_j u - c_j) \\ & \qquad , \cdots, f_p(u) + u^T w_p + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j (B_j u - c_j)), \end{aligned}$$

where (u, λ, w, y, z) is any feasible solution of (MDE). Since $\bar{x}^T \bar{w}_i = s(\bar{x}|D_i)$, we have that the following cannot hold:

$$(f_1(\bar{x}) + \bar{x}^T \bar{w}_1 + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) + \sum_{j \in J_0} \bar{z}_j (B_j \bar{x} - c_j)$$

$$, \cdots, f_p(\bar{x}) + \bar{x}^T \bar{w}_p + \sum_{i \in I_0} \bar{y}_i g_i(\bar{x}) + \sum_{j \in J_0} \bar{z}_j (B_j \bar{x} - c_j))$$

$$< (f_1(u) + u^T w_1 + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j (B_j u - c_j)$$

$$, \cdots, f_p(u) + u^T w_p + \sum_{i \in I_0} y_i g_i(u) + \sum_{j \in J_0} z_j (B_j u - c_j)).$$

Since $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is a feasible solution for (MDE), $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{y}, \bar{z})$ is a weakly efficient solution of (MDE). Hence the result holds. □

Remark 3.1 *If we replace the conditions of Theorem 3.1 and Theorem 3.2 by the ones of Theorem 2.1 in [14], we can establish our weak and strong duality theorems for efficient solutions.*

Remark 3.2 *If $B = 0$ and $d = 0$, the primal problem (MPE) and the dual problem (MDE) become the primal problem (VP) and the dual problem (VD) considered in Yang et al. [14] respectively. So our weak duality Theorem 3.1 extends and improves Theorem 2.1 in Yang et al. [14].*

Remark 3.3 *Let $D_i = \{B_i w : w^T B_i w \leqq 1\}$. Then $s(x|D_i) = (x^T B_i x)^{1/2}$ and the sets D_i, $i = 1, \cdots, p$, are compact and convex.*

(i) $B = 0$, $d = 0$, $I_0 = M$ and $I_\alpha = \emptyset$, $\alpha = 1, \cdots, r$, then (MPE) and (MDE) reduce to (VP) and $(VDP)_1$ in Lal et al. [7], respectively.

(ii) $B = 0$, $d = 0$, $I_0 = \emptyset$, $I_1 = M$ and $I_\alpha = \emptyset$, $\alpha = 2, \cdots, r$, then (MPE) and (MDE) reduce to (VP) and $(VDP)_2$ in Lal et al. [7], respectively.

4 Weak Vector Saddle Point Theorems

In this section, we prove weak vector saddle point theorems for the multi-objective program (MPE).

For the problem (MPE), a point (x, λ, μ, ν) is said to be a critical point if x is a feasible point for (MPE), and

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \lambda_i w_i + \sum_{j=1}^{m} \mu_j \nabla g_j(x) + \sum_{k=1}^{q} \nu_k B_k = 0,$$

$$\langle w_i, x \rangle = s(x|D_i),\ i = 1, \cdots, p,$$

$$\sum_{j=1}^{m} \mu_j g_j(x) + \sum_{k=1}^{q} \nu_k (B_k x - c_k) = 0,$$

$$(\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_m) \geqq 0,$$

$$\lambda^T e = 1.$$

Let $L(x, \mu, \nu) = f(x) + s(x|D) + \mu^T g(x) e + \nu^T (Bx - c) e$, where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, and $\nu \in \mathbb{R}^q$. Then, a point $(\bar{x}, \bar{\mu}, \bar{\nu}) \in \mathbb{R}^n \times \mathbb{R}^m_+ \times \mathbb{R}^q$ is said to be a weak vector saddle point if

$$L(\bar{x},\mu,\nu) \not\geqslant L(\bar{x},\bar{\mu},\bar{\nu}) \not\geqslant L(x,\bar{\mu},\bar{\nu})$$

for all $x \in \mathbb{R}^n, \mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^q$.

Theorem 4.1 *Let $(\bar{x},\bar{\lambda},\bar{\mu},\bar{\nu})$ be a critical point of (MPE). Assume that $f_i(\cdot)+(\cdot)^T w_i$ is (F,α,ρ_i,d)-convex at $\bar{x}$ and $\bar{\mu}^T g(\cdot)+\bar{\nu}^T(B(\cdot)-c)$ is (F,α,β,d)-convex at $\bar{x}$ with $\sum_{i=1}^p \bar{\lambda}_i\rho_i+\beta \geqq 0$. Then $(\bar{x},\bar{\mu},\bar{\nu})$ is a weak vector saddle point of (MPE).*

Proof. Since $f_i(\cdot)+(\cdot)^T w_i$ is $(F,\alpha,\rho_i,d)-$convex at $\bar{x}$, we obtain

$$\begin{aligned}&[f_i(x)+x^T w_i]-[f_i(\bar{x})+\bar{x}^T w_i]\\ &\qquad \geqq F(x,\bar{x};\alpha(x,\bar{x})(\nabla f_i(\bar{x})+w_i))+\rho_i d^2(x,\bar{x}),\ i=1,\cdots,p,\end{aligned}$$

and the sublinearity of F,

$$\begin{aligned}&\sum_{i=1}^p \bar{\lambda}_i[f_i(x)+x^T w_i]-\sum_{i=1}^p \bar{\lambda}_i[f_i(\bar{x})+\bar{x}^T w_i]\\ &\qquad \geqq F(x,\bar{x};\alpha(x,\bar{x})(\sum_{i=1}^p \bar{\lambda}_i[\nabla f_i(\bar{x})+w_i]))+(\sum_{i=1}^p \bar{\lambda}_i\rho_i)d^2(x,\bar{x}). \quad (4.1)\end{aligned}$$

Since $\bar{\mu}^T g(\cdot)+\bar{\nu}^T(B(\cdot)-c)$ is $(F,\alpha,\beta,d)-$convex at $\bar{x}$, we have

$$\begin{aligned}&[\bar{\mu}^T g(x)+\bar{\nu}^T(Bx-c)]-[\bar{\mu}^T g(\bar{x})+\bar{\nu}^T(B\bar{x}-c)]\\ &\qquad \geqq F(x,\bar{x};\alpha(x,\bar{x})(\bar{\mu}^T\nabla g(\bar{x})+\bar{\nu}^T B_k))+\beta d^2(x,\bar{x}). \qquad (4.2)\end{aligned}$$

Combination (4.1) and (4.2), we get

$$\sum_{i=1}^{p} \bar{\lambda}_i [f_i(x) + x^T w_i] - \sum_{i=1}^{p} \bar{\lambda}_i [f_i(\bar{x}) + \bar{x}^T w_i]$$

$$+[\bar{\mu}^T g(x) + \bar{\nu}^T (Bx - c)] - [\bar{\mu}^T g(\bar{x}) + \bar{\nu}^T (B\bar{x} - c)]$$

$$\geqq F(x, \bar{x}; \alpha(x, \bar{x})(\sum_{i=1}^{p} \bar{\lambda}_i [\nabla f_i(\bar{x}) + w_i]))$$

$$+F(x, \bar{x}; \alpha(x, \bar{x})(\bar{\mu}^T \nabla g(\bar{x}) + \bar{\nu}^T B_k)) + (\sum_{i=1}^{p} \bar{\lambda}_i \rho_i) d^2(x, \bar{x}) + \beta d^2(x, \bar{x}).$$

Since $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\nu})$ is a critical point for (MPE), then there exists

$$\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{i=1}^{p} \bar{\lambda}_i w_i + \sum_{j=1}^{m} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{k=1}^{q} \bar{\nu}_k B_k = 0.$$

By the sublinearity of F and $\sum_{i=1}^{p} \bar{\lambda}_i \rho_i + \beta \geqq 0$, we have

$$F(x, \bar{x}; \alpha(x, \bar{x})(\sum_{i=1}^{p} \bar{\lambda}_i [\nabla f_i(\bar{x}) + w_i]))$$

$$+F(x, \bar{x}; \alpha(x, \bar{x})(\bar{\mu}^T \nabla g(\bar{x}) + \bar{\nu}^T B_k)) + (\sum_{i=1}^{p} \bar{\lambda}_i \rho_i) d^2(x, \bar{x}) + \beta d^2(x, \bar{x})$$

$$\geqq F(x, \bar{x}; \alpha(x, \bar{x})(\sum_{i=1}^{p} \bar{\lambda}_i [\nabla f_i(\bar{x}) + w_i] + \bar{\mu}^T \nabla g(\bar{x}) + \bar{\mu}^T B_k))$$

$$+(\sum_{i=1}^{p} \bar{\lambda}_i \rho_i) d^2(x, \bar{x}) + \beta d^2(x, \bar{x})$$

$$\geqq (\sum_{i=1}^{p} \bar{\lambda}_i \rho_i) d^2(x, \bar{x}) + \beta d^2(x, \bar{x})$$

$$\geqq 0.$$

Hence, we get

$$\sum_{i=1}^{p} \bar{\lambda}_i [f_i(x) + x^T w_i] + \bar{\mu}^T g(x) + \bar{\nu}^T (Bx - c)$$

$$\geqq \sum_{i=1}^{p} \bar{\lambda}_i [f_i(\bar{x}) + \bar{x}^T w_i] + \bar{\mu}^T g(\bar{x}) + \bar{\nu}^T (B\bar{x} - c).$$

From $s(x|D_i) \geqq x^T w_i,\ i = 1, \cdots, p$, we have

$$\sum_{i=1}^{p} \bar{\lambda}_i [f_i(x) + s(x|D_i)] + \bar{\mu}^T g(x) + \bar{\nu}^T (Bx - c)$$

$$\geqq \sum_{i=1}^{p} \bar{\lambda}_i [f_i(\bar{x}) + s(\bar{x}|D_i)] + \bar{\mu}^T g(\bar{x}) + \bar{\nu}^T (B\bar{x} - c).$$

Using $\bar{\lambda}_i \geqq 0$, and $\bar{\lambda}^T e = 1$,

$$f(x) + s(x|D) + \bar{\mu}^T g(x) e + \bar{\nu}^T (Bx - c) e$$

$$\nless f(\bar{x}) + s(\bar{x}|D) + \bar{\mu}^T g(\bar{x}) e + \bar{\nu}^T (B\bar{x} - c) e,\ \ for\ any\ x \in \mathbb{R}^n.$$

Now, since $\mu^T g(\bar{x}) + \nu^T (B\bar{x} - c) \leqq 0$, we have

$$[\bar{\mu}^T g(\bar{x}) + \bar{\nu}^T (B\bar{x} - c)] - [\mu^T g(\bar{x}) + \nu^T (B\bar{x} - c)] \geqq 0,$$

for any $\mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^q$. Thus,

$$\sum_{i=1}^{p} \bar{\lambda}_i[f_i(\bar{x}) + s(\bar{x}|D_i)] + \mu^T g(\bar{x}) + \nu^T(B\bar{x} - c)$$

$$\leqq \sum_{i=1}^{p} \bar{\lambda}_i[f_i(\bar{x}) + s(\bar{x}|D_i)] + \bar{\mu}^T g(\bar{x}) + \bar{\nu}^T(B\bar{x} - c).$$

Using $\bar{\lambda}_i \geqq 0$, and $\bar{\lambda}^T e = 1$,

$$f(\bar{x}) + s(\bar{x}|D) + \mu^T g(\bar{x})e + \nu^T(B\bar{x} - c)e$$

$$\not> f(\bar{x}) + s(\bar{x}|D) + \bar{\mu}^T g(\bar{x})e + \bar{\nu}^T(B\bar{x} - c)e,$$

for any $\mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^q$.

Therefore, $(\bar{x}, \bar{\mu}, \bar{\nu})$ is a weak vector saddle point of (MPE). □

Remark 4.1 *If we replace the (F, α, ρ_i, d)-convexity of $f_i(\cdot) + (\cdot)^T w_i$ and $\bar{\mu}^T g(\cdot) + \bar{\nu}^T(B(\cdot) - c)$ with $\sum_{i=1}^{p} \bar{\lambda}_i \rho_i + \beta \geqq 0$ by $(F, \alpha, \rho, d)-$convexity of $f(\cdot) + (\cdot)^T w + \bar{\mu}^T g(\cdot)e + \bar{\nu}^T(B(\cdot) - c)e$ at $\bar{x}$ with $\sum_{i=1}^{p} \bar{\lambda}_i \rho_i \geqq 0$ in Theorem 4.1, then this theorem is also valid.*

Theorem 4.2 *If there exists $(\bar{\mu}, \bar{\nu}) \in \mathbb{R}^m_+ \times \mathbb{R}^q$ such that $(\bar{x}, \bar{\mu}, \bar{\nu})$ is a weak vector saddle point, then $\bar{x}$ is a weakly efficient solution of (MPE).*

Proof. Assume that $(\bar{x}, \bar{\mu}, \bar{\nu})$ is a weak vector saddle-point. From the left inequality of saddle-point conditions,

$$f(\bar{x}) + s(\bar{x}|D) + \mu^T g(\bar{x})e + \nu^T(B\bar{x} - c)e$$
$$\not\geq f(\bar{x}) + s(\bar{x}|D) + \bar{\mu}^T g(\bar{x})e + \bar{\nu}^T(B\bar{x} - c)e,$$

for any $\mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^q$, and hence we have

$$\mu^T g(\bar{x}) + \nu^T(B\bar{x} - c) \leqq \bar{\mu}^T g(\bar{x}) + \bar{\nu}^T(B\bar{x} - c), \tag{4.3}$$

for any $\mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^q$.

Since μ can be taken arbitrary large and $B\bar{x} = c$, we have

$$g(\bar{x}) \leqq 0.$$

Hence,

$$\bar{\mu}^T g(\bar{x}) \leqq 0.$$

Letting $\mu = 0$ in (4.3) and $B\bar{x} = c$, we have

$$\bar{\mu}^T g(\bar{x}) \geqq 0.$$

Therefore,

$$\bar{\mu}^T g(\bar{x}) = 0.$$

By $B\bar{x} = c$, we obtain

$$\bar{\mu}^T g(\bar{x}) + \bar{\nu}^T(B\bar{x} - c) = 0. \tag{4.4}$$

Now, from the right inequality of saddle point conditions and (4.4), we have for any feasible x for (MPE)

$$f(\bar{x}) + s(\bar{x}|D) + \bar{\mu}^T g(\bar{x})e + \bar{\nu}^T(B\bar{x} - c)e$$
$$\not\geq f(x) + s(x|D) + \bar{\mu}^T g(x)e + \bar{\nu}^T(Bx - c)e,$$

i.e.

$$f(\bar{x}) + s(\bar{x}|D) \not> f(x) + s(x|D).$$

Hence, $\bar{x}$ is a weakly efficient solution of (MPE). □

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