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Some Inclusion Properties for Certain Classes of Meromorphic Function

by

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Some Inclusion Properties for Certain Classes of Meromorphic Function
(유리형 함수족들에 대한 포함성질들)

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by

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유리형함수들의 족들에 대한 포함성질들

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요 약

유리형 함수들의 다양한 기하학적 성질들에 관한 연구는 지금까지 많은 학자들에 의하여 연구되고 있다. 최근, Noor[8,11,12]는 개단위원 내부에서 정의된 해석함수들의 적분연산자를 소개하고 여러 함수족들을 정의하여 그들사이의 포함관계들을 연구하였다.

본 논문에서는 Noor에 의하여 소개된 적분연산자의 개념을 유리 형 함수들에도 확장하고, 이 새로운 연산자를 이용하여 새로운 유리 형 함수들의 족들을 소개하였다. 또한 Miiler 와 Mocano [3,10]의 결과들을 응용하여 함수 족들에 대한 여러 가지 포함성질들을 조사하였으며 유리형 함수들의 적분 보존성질을 조사하였다.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k$$

which are analytic in the punctured open unit disk $\mathcal{D}=\{z\in\mathbb{C}:0<|z|<1\}$. If f and g are analytic in $\mathcal{U}=\mathcal{D}\cup\{0\}$, we say that f is subordinate to g, written $f\prec g$ or $f(z)\prec g(z)$, if there exists a Schwarz function w in $\mathcal{U}$ such that $f(z)=g(w(z))$. For $0\leq\eta,\beta<1$, we denote by $\Sigma^*(\eta)$ and $\Sigma_k(\eta)$ and $\Sigma_c(\eta,\beta)$ the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order η and convex of order η and colse-to-convex of order β and type η in $\mathcal{U}$ (for details, see, e.g. [5]).

Let $\mathcal{M}$ be the class of analytic functions ϕ in $\mathcal{U}$ normalized by $\phi(0)=1$, and let $\mathcal{N}$ be the subclass of $\mathcal{M}$ consisting of those functions ϕ which are univalent in $\mathcal{U}$ and for which $\phi(\mathcal{U})$ is convex and $\mathrm{Re}\{\phi(z)\}>0$ $(z\in\mathcal{U})$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\Sigma^*(\eta,\phi)$, $\Sigma_k(\eta,\phi)$ and $\Sigma_c(\eta,\beta;\ \phi,\psi)$ of the class Σ for $0\leq\eta,\beta<1$ and $\phi,\psi\in\mathcal{N}$, which are defined by

$$\Sigma^*(\eta;\ \phi):=\left\{f\in\Sigma:\ \frac{1}{1-\eta}\left(-\frac{zf'(z)}{f(z)}-\eta\right)\prec\phi(z)\text{ in }\mathcal{U}\right\},$$

$$\Sigma_k(\eta;\ \phi):=\left\{f\in\Sigma:\ \frac{1}{1-\eta}\left(-\left\{1+\frac{zf''(z)}{f'(z)}\right\}-\eta\right)\prec\phi(z)\text{ in }\mathcal{U}\right\},$$

and

$$\Sigma_c(\eta,\beta;\ \phi,\psi):=\left\{f\in\Sigma:\exists g\in\Sigma^*(\eta;\ \phi)\ \text{ s.t.}\frac{1}{1-\beta}\left(-\frac{zf'(z)}{g(z)}-\beta\right)\prec\psi(z)\text{ in }\mathcal{U}\right\}.$$

We note that the classes mentioned above is motivated essentially by the familiar classes which have been used widely on the space of analytic and univalent functions in $\mathcal{U}$(see, for details, [2,6,9]) and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of Σ [1,4,5].

Let

$$f_\lambda(z) = \frac{1}{z(1-z)^{\lambda+1}} \ \ (\lambda > -1;\ z \in \mathcal{D})$$

and let $f_{\lambda,\mu}$ be defined such that

$$f_\lambda(z) * f_{\lambda,\mu}(z) = \frac{1}{z(1-z)^{\mu}} \ \ (\lambda > -1;\ \mu > 0;\ z \in \mathcal{D}), \tag{1.1}$$

where the symbol $(*)$ stands for the Hadamard product(or convolution). Then we define the operator $I_{\lambda,\mu} : \Sigma \to \Sigma$ as follows:

$$I_{\lambda,\mu} f(z) = (f_{\lambda,\mu} * f)(z) \ \ (f \in \Sigma; \lambda > -1;\ \mu > 0). \tag{1.2}$$

In particular, we note that $I_{0,2}f(z) = zf'(z) + 2f(z)$ and $I_{1,2}f(z) = f(z)$. In view of (1.1) and (1.2), we obtain the useful identities as follows:

$$z\left(I_{\lambda+1,\mu}f(z)\right)' = (\lambda+1)I_{\lambda,\mu}f(z) - (\lambda+2)I_{\lambda+1,\mu}f(z). \tag{1.3}$$

and

$$z\left(I_{\lambda,\mu}f(z)\right)' = \mu I_{\lambda,\mu+1}f(z) - (\mu+1)I_{\lambda,\mu}f(z). \tag{1.4}$$

The operator $I_{\lambda,\mu}$ is closely related to the Choi-Saigo-Srivastava operator for analytic and univalent functions [2], which extends the Noor integral operator studied by Liu [7](also, see [8,11,12]).

Next, by using the operator $I_{\lambda,\mu}$, we introduce the following classes of meromorphic functions for $\phi, \psi \in \mathcal{N}$, $\lambda > -1$, $\mu > 0$ and $0 \le \eta, \beta < 1$:

$$\Sigma^*(\lambda, \mu;\ \eta;\ \phi) := \{f \in \Sigma : I_{\lambda,\mu}f \in \Sigma^*(\eta; \phi)\},$$

$$\Sigma_k(\lambda,\mu;\ \eta;\ \phi) := \{f \in \Sigma : I_{\lambda,\mu}f \in \Sigma_k(\eta;\phi)\},$$

and

$$\Sigma_c(\lambda,\mu;\ \eta,\beta;\ \phi,\psi) := \{f \in \Sigma : I_{\lambda,\mu}f \in \Sigma_c(\eta,\beta;\phi,\psi)\}.$$

We also note that

$$f(z) \in \Sigma_k(\lambda,\mu;\ \eta;\ \phi) \iff -zf'(z) \in \Sigma^*(\lambda,\mu;\ \eta;\ \phi). \tag{1.5}$$

In particular, we set

$$\Sigma^*\left(\lambda,\mu;\ \eta;\ \frac{1+Az}{1+Bz}\right) = \Sigma^*(\lambda,\mu;\ \eta;\ A,B)\ \ (-1 \le B < A \le 1)$$

and

$$\Sigma_k\left(\lambda,\mu;\ \eta;\ \frac{1+Az}{1+Bz}\right) = \Sigma_k(\lambda,\mu;\ \eta;\ A,B)\ \ (-1 \le B < A \le 1).$$

In this paper, we investigate several inclusion properties of the classes $\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$, $\Sigma_k(\lambda,\mu;\ \eta;\ \phi)$ and $\Sigma_c(\lambda,\mu;\ \eta,\beta;\ \phi,\psi)$ associated with the operator $I_{\lambda,\mu}$. Some applications involving integral operators are also considered.

2. Inclusion Properties Involving the Operator $I_{\lambda,\mu}$

The following results will be required in our investigation.

Lemma 2.1 [3]. *Let ϕ be convex univalent in $\mathcal{U}$ with $\phi(0) = 1$ and $\mathrm{Re}\{\kappa\phi(z)+\nu\} > 0$ $(\kappa,\nu \in \mathbb{C})$. If p is analytic in $\mathcal{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\kappa p(z)+\nu} \prec \phi(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathcal{U}).$$

Lemma 2.2 [10]. *Let ϕ be convex univalent in $\mathcal{U}$ and ω be analytic in $\mathcal{U}$ with* $\mathrm{Re}\{\omega(z)\} \geq 0$. *If p is analytic in $\mathcal{U}$ and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathcal{U}).$$

At first, with the help of Lemma 2.1, we obtain the following

Theorem 2.1. *Let $\phi \in \mathcal{N}$ with* $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\} < \min\{(\mu+1-\eta)/(1-\eta), (\lambda+2-\eta)/(1-\eta)\}$ $(\lambda > -1;\ \mu > 0;\ 0 \leq \eta < 1)$. *Then*

$$\Sigma^*(\lambda, \mu+1;\ \eta;\ \phi) \subset \Sigma^*(\lambda, \mu;\ \eta;\ \phi) \subset \Sigma^*(\lambda+1, \mu;\ \eta;\ \phi).$$

Proof. First of all, we will show that

$$\Sigma^*(\lambda, \mu+1;\ \eta;\ \phi) \subset \Sigma^*(\lambda, \mu;\ \eta;\ \phi).$$

Let $f \in \Sigma^*(\lambda, \mu+1;\ \eta;\ \phi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(-\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}f(z)} - \eta\right), \tag{2.1}$$

where p is analytic in $\mathcal{U}$ with $p(0) = 1$. Applying (1.4) and (2.1), we obtain

$$-\mu\frac{I_{\lambda,\mu+1}f(z)}{I_{\lambda,\mu}f(z)} = (1-\eta)p(z) - (\mu+1-\eta). \tag{2.2}$$

Taking the logarithmic differentiation on both sides of (2.2) and multiplying by z, we have

$$\frac{1}{1-\eta}\left(-\frac{z(I_{\lambda,\mu+1}f(z))'}{I_{\lambda,\mu+1}f(z)}-\eta\right)=p(z)+\frac{zp'(z)}{-(1-\eta)p(z)+\mu+1-\eta}\ \ (z\in\mathcal{U}). \tag{2.3}$$

Since $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\}<(\mu+1-\eta)/(1-\eta)$, we see that

$$\mathrm{Re}\{-(1-\eta)\phi(z)+\mu+1-\eta\}>0\ \ (z\in\mathcal{U}).$$

Applying Lemma 2.1 to (2.3), it follows that $p\prec\phi$, that is, $f\in\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$.

To prove the second part, let $f\in\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$ and put

$$s(z)=\frac{1}{1-\eta}\left(-\frac{z(I_{\lambda+1,\mu}f(z))'}{I_{\lambda+1,\mu}f(z)}-\eta\right),$$

where s is analytic function with $s(0)=1$. Then, by using the arguments similar to those detailed above with (1.3), it follows that $s\prec\phi$ in $\mathcal{U}$, which implies that $f\in\Sigma^*(\lambda+1,\mu;\ \eta;\ \phi)$. Therefore we complete the proof of Theorem 2.1.

Theorem 2.2. *Let* $\phi\in\mathcal{N}$ *with* $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\}<\min\{(\mu+1-\eta)/(1-\eta),(\lambda+2-\eta)/(1-\eta)\}$ $(\lambda>-1;\ \mu>0;\ 0\le\eta<1)$. *Then*

$$\Sigma_k(\lambda,\mu+1;\ \eta;\ \phi)\subset\Sigma_k(\lambda,\mu;\ \eta;\ \phi)\subset\Sigma_k(\lambda+1,\mu;\ \eta;\ \phi).$$

Proof. Applying (1.5) and Theorem 2.1, we observe that

$$\begin{aligned}
f(z) \in \Sigma_k(\lambda, \mu+1;\ \eta;\ \phi) &\Longleftrightarrow I_{\lambda,\mu+1}f(z) \in \Sigma_k(\eta;\ \phi) \\
&\Longleftrightarrow -z(I_{\lambda,\mu+1}f(z))' \in \Sigma^*(\eta;\ \phi) \\
&\Longleftrightarrow I_{\lambda,\mu+1}(-zf'(z)) \in \Sigma^*(\eta;\ \phi) \\
&\Longleftrightarrow -zf'(z) \in \Sigma^*(\lambda, \mu+1;\ \eta;\ \phi) \\
&\Longrightarrow -zf'(z) \in \Sigma^*(\lambda, \mu;\ \eta;\ \phi) \\
&\Longleftrightarrow I_{\lambda,\mu}(-zf'(z)) \in \Sigma^*(\eta;\ \phi) \\
&\Longleftrightarrow -z(I_{\lambda,\mu}f(z))' \in \Sigma^*(\eta;\ \phi) \\
&\Longleftrightarrow I_{\lambda,\mu}f(z) \in \Sigma_k(\eta;\ \phi) \\
&\Longleftrightarrow f(z) \in \Sigma_k(\lambda, \mu;\ \eta;\ \phi),
\end{aligned}$$

and

$$\begin{aligned}
f(z) \in \Sigma_k(\lambda, \mu;\ \eta;\ \phi) &\Longleftrightarrow -zf'(z) \in \Sigma^*(\lambda, \mu;\ \eta;\ \phi) \\
&\Longrightarrow -zf'(z) \in \Sigma^*(\lambda+1, \mu;\ \eta;\ \phi) \\
&\Longleftrightarrow -z(I_{\lambda+1,\mu}f(z))' \in \Sigma^*(\eta;\ \phi) \\
&\Longleftrightarrow I_{\lambda+1,\mu}f(z) \in \Sigma_k(\eta;\ \phi) \\
&\Longleftrightarrow f(z) \in \Sigma_k(\lambda+1, \mu;\ \eta;\ \phi),
\end{aligned}$$

which evidently proves Theorem 2.2.

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1;\ z \in \mathcal{U})$$

in Theorem 2.1 and Theorem 2.2, we have

Corollary 2.1. *Let* $(1+A)/(1+B) < \min\{(\mu+1-\eta)/(1-\eta), (\lambda+2-\eta)/(1-\eta)\}(\lambda > -1;\ \mu > 0;\ 0 \leq \eta < 1;\ -1 < B < A \leq 1)$. *Then*

$$\Sigma^*(\lambda, \mu+1;\ \eta;\ A, B) \subset \Sigma^*(\lambda, \mu;\ \eta;\ A, B) \subset \Sigma^*(\lambda+1, \mu;\ \eta;\ A, B)$$

and

$$\Sigma_k(\lambda, \mu+1;\ \eta;\ A, B) \subset \Sigma_k(\lambda, \mu;\ \eta;\ A, B) \subset \Sigma_k(\lambda+1, \mu;\ \eta;\ A, B).$$

Next, by using Lemma 2.2, we obtain the following inclusion relation for the class $\Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi)$.

Theorem 2.3. *Let* $\phi, \psi \in \mathcal{N}$ *with* $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\} < \min\{(\mu+1-\eta)/(1-\eta), (\lambda+2-\eta)/(1-\eta)\}$ $(\lambda > -1;\ \mu > 0;\ 0 \le \eta < 1)$. *Then*

$$\Sigma_c(\lambda, \mu+1;\ \eta, \beta;\ \phi, \psi) \subset \Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi) \subset \Sigma_c(\lambda+1, \mu;\ \eta, \beta;\ \phi, \psi).$$

Proof. We begin by proving that

$$\Sigma_c(\lambda, \mu+1;\ \eta, \beta;\ \phi, \psi) \subset \Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi).$$

Let $f \in \Sigma_c(\lambda, \mu+1;\ \eta, \beta;\ \phi, \psi)$. Then, in view of the definition of the class $\Sigma_c(\lambda, \mu+1;\ \eta, \beta;\ \phi, \psi)$, there exists a function $r \in \Sigma^*(\eta;\ \phi)$ such that

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu+1}f(z))'}{r(z)} - \beta\right) \prec \psi(z)\ \ (z \in \mathcal{U}).$$

Choose the function g such that $I_{\lambda,\mu+1}g(z) = r(z)$. Then $g \in \Sigma^*(\lambda, \mu+1;\ \eta;\ \phi)$ and

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu+1}f(z))'}{I_{\lambda,\mu+1}g(z)} - \beta\right) \prec \psi(z)\ \ (z \in \mathcal{U}). \tag{2.4}$$

Now let

$$p(z) = \frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}g(z)} - \beta\right), \tag{2.5}$$

where p is analytic in $\mathcal{U}$ with $p(0) = 1$. Using (1.4), we obtain

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu+1}f(z))'}{I_{\lambda,\mu+1}g(z)}-\beta\right)=\frac{1}{1-\beta}\left(\frac{I_{\lambda,\mu+1}(-zf'(z))}{I_{\lambda,\mu+1}g(z)}-\beta\right)$$
$$=\frac{1}{1-\beta}\left(\frac{z(I_{\lambda,\mu}(-zf'(z)))'+(\mu+1)I_{\lambda,\mu}(-zf'(z))}{z(I_{\lambda,\mu}g(z))'+(\mu+1)I_{\lambda,\mu}g(z)}-\beta\right) \tag{2.6}$$
$$=\frac{1}{1-\beta}\left(\frac{\frac{z(I_{\lambda,\mu}(-zf'(z)))'}{I_{\lambda,\mu}g(z)}+(\mu+1)\frac{I_{\lambda,\mu}(-zf'(z))}{I_{\lambda,\mu}g(z)}}{\frac{z(I_{\lambda,\mu}g(z))'}{I_{\lambda,\mu}g(z)}+\mu+1}-\beta\right).$$

Since $g\in\Sigma^*(\lambda,\mu+1;\ \eta;\ \phi)\subset\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$, by Theorem 2.1, we set

$$q(z)=\frac{1}{1-\eta}\left(-\frac{z(I_{\lambda,\mu}g(z))'}{I_{\lambda,\mu}g(z)}-\eta\right),$$

where $q\prec\phi$ in $\mathcal{U}$ with the assupption for $\phi\in\mathcal{N}$. Then, by virtue of (2.5) and (2.6), we observe that

$$I_{\lambda,\mu}(-zf'(z))=(1-\beta)p(z)I_{\lambda,\mu}g(z)+\beta I_{\lambda,\mu}g(z) \tag{2.7}$$

and

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu+1}f(z))'}{I_{\lambda,\mu+1}g(z)}-\beta\right)=\frac{1}{1-\beta}\left(\frac{\frac{z(I_{\lambda,\mu}(-zf'(z)))'}{I_{\lambda,\mu}g(z)}+(\mu+1)(1-\beta)p(z)+\beta)}{-(1-\eta)q(z)+\mu+1-\eta}-\beta\right). \tag{2.8}$$

Upon differentiating both sides of (2.7), we have

$$\frac{z(I_{\lambda,\mu}(-zf'(z)))'}{I_{\lambda,\mu}g(z)}=(1-\beta)zp'(z)-((1-\beta)p(z)+\beta)((1-\eta)q(z)+\eta). \tag{2.9}$$

Making use of (2.4), (2.8) and (2.9), we get

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu+1}f(z))'}{I_{\lambda,\mu+1}g(z)}-\beta\right)=p(z)+\frac{zp'(z)}{-(1-\eta)q(z)+\mu+1-\eta}\prec\psi(z)\ \ (z\in\mathcal{U}). \tag{2.10}$$

Since $\mu>0$ and $q\prec\phi$ in $\mathcal{U}$ with $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\}<(\mu+1-\eta)/(1-\eta)$,

$$\mathrm{Re}\{-(1-\eta)q(z)+\mu+1-\eta\}>0 \ \ (z\in\mathcal{U}).$$

Hence, by taking

$$\omega(z)=\frac{1}{-(1-\eta)q(z)+\mu+1-\eta},$$

in (2.10), and applying Lemma 2.2, we can show that $p \prec \psi$ in $\mathcal{U}$, so that $f\in\Sigma_c(\lambda,\mu;\ \eta,\beta;\ \phi,\psi)$.

For the second part, by using the arguments similar to those detailed above with (1.3), we obtain

$$\Sigma_c(\lambda,\mu;\ \eta,\beta;\ \phi,\psi)\subset\Sigma_c(\lambda+1,\mu;\ \eta,\beta;\ \phi,\psi).$$

Therefore we complete the proof of Theorem 2.3.

3. Inclusion Properties Involving the Integral Operator F_c

In this section, we consider the integral operator F_c [1,4,5] defined by

$$F_c(f):=F_c(f)(z)=\frac{c}{z^{c+1}}\int_0^z t^c f(t)dt \quad (f\in\Sigma;\ c>0). \tag{3.1}$$

We first prove

Theorem 3.1. *Let* $\lambda>-1,\ \mu>0$ *and let* $\phi\in\mathcal{N}$ *with* $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\}<(c+1-\eta)/(1-\eta)$ $(c>0;\ 0\leq\eta<1)$*. If* $f\in\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$*, then* $F_c(f)\in\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$.

Proof. Let $f\in\Sigma^*(\lambda,\mu;\ \eta;\ \phi)$ and set

$$p(z)=\frac{1}{1-\eta}\left(-\frac{z(I_{\lambda,\mu}F_c(f)(z))'}{I_{\lambda,\mu}F_c(f)(z)}-\eta\right), \tag{3.2}$$

where p is analytic in $\mathcal{U}$ with $p(0)=1$. From (3.1), we have

$$z(I_{\lambda,\mu}F_c(f)(z))'=cI_{\lambda,\mu}f(z)-(c+1)I_{\lambda,\mu}F_c(f)(z). \tag{3.3}$$

Then, by using (3.2) and (3.3), we obtain

$$-c\frac{I_{\lambda,\mu}f(z)}{I_{\lambda,\mu}F_c(f)(z)} = (1-\eta)p(z) - (c+1-\eta). \tag{3.4}$$

Making use of the logarithemic differentiation on both sides of (3.4) and multiplying by z, we get

$$\frac{1}{1-\eta}\left(-\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{-(1-\eta)p(z)+c+1-\eta} \quad (z \in \mathcal{U}).$$

Hence, by virtue of Lemma 2.1, we conclude that $p \prec \phi$ in $\mathcal{U}$ for $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\} < (c+1-\eta)/(1-\eta)$, which implies that $F_c(f) \in \Sigma^*(\lambda,\mu;\ \eta;\ \phi)$.

Next, we derive an inclusion property involving F_c, which is given by

Theorem 3.2. *Let* $\lambda > -1,\ \mu > 0$ *and let* $\phi \in \mathcal{N}$ *with* $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\} < (c+1-\eta)/(1-\eta)$ $(c > 0;\ 0 \le \eta < 1)$*. If* $f \in \Sigma_k(\lambda,\mu;\ \eta;\ \phi)$*, then* $F_c(f) \in \Sigma_k(\lambda,\mu;\ \eta;\ \phi)$.

Proof. By applying Theorem 3.1, it follows that

$$\begin{aligned} f(z) \in \Sigma_k(\lambda,\mu;\ \eta;\ \phi) &\Longleftrightarrow -zf'(z) \in \Sigma^*(\lambda,\mu;\ \eta;\ \phi) \\ &\Longrightarrow F_c(-zf'(z))(z) \in \Sigma^*(\lambda,\mu;\ \eta;\ \phi) \\ &\Longleftrightarrow -z(F_c(f)(z))' \in \Sigma^*(\lambda,\mu;\ \eta;\ \phi) \\ &\Longleftrightarrow F_c(f)(z) \in \Sigma_k(\lambda,\mu;\ \eta;\ \phi), \end{aligned} \tag{2.6}$$

which proves Theorem 3.2.

From Theorem 3.1 and Theorem 3.2, we have

Corollary 3.1. *Let* $\lambda > -1,\ \mu > 0$ *and* $(1-\eta)(1+A)/(1+B) < (c+1-\eta)$ $(c > 0;\ -1 < B < A \le 1;\ 0 \le \eta < 1)$*. Then If* $f \in \Sigma^*(\lambda,\mu;\ \eta;\ A,B)$ *(or* $\Sigma_k(\lambda,\mu;\ \eta;\ A,B)$ *), then* $F_c(f) \in \Sigma^*(\lambda,\mu;\ \eta;\ A,B)$ *(or* $\Sigma^*(\lambda,\mu;\ \eta;\ A,B)$*).*

Finally, we prove

Theorem 3.3. *Let* $\lambda > -1,\ \mu > 0$ *and let* $\phi, \psi \in \mathcal{N}$ *with* $\max_{z\in\mathcal{U}}\mathrm{Re}\{\phi(z)\} < (c+1-\eta)/(1-\eta)$ $(c > 0;\ 0 \le \eta < 1)$. *If* $f \in \Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi)$, *then* $F_c(f) \in \Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi)$.

Proof. Let $f \in \Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi)$. Then, in view of the definition of the class $\Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi)$, there exists a function $g \in \Sigma^*(\lambda, \mu;\ \eta;\ \phi)$ such that

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}g(z)} - \beta\right) \prec \psi(z)\ \ (z \in \mathcal{U}). \tag{3.5}$$

Thus we set

$$p(z) = \frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu}F_c(f)(z))'}{I_{\lambda,\mu}F_c(g)(z)} - \beta\right).$$

where p is analytic in $\mathcal{U}$ with $p(0) = 1$. Applying (3.3), we get

$$\begin{aligned}
\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}g(z)} - \beta\right) &= \frac{1}{1-\beta}\left(\frac{I_{\lambda,\mu}(-zf'(z))}{I_{\lambda,\mu}g(z)} - \beta\right)\\
= \frac{1}{1-\beta}&\left(\frac{z(I_{\lambda,\mu}F_c(-zf'(z))(z))' + (c+1)I_{\lambda,\mu}F_c(-zf'(z))(z)}{z(I_{\lambda,\mu}F_c(g)(z))' + (c+1)I_{\lambda,\mu}F_c(g)(z)} - \beta\right)\\
= \frac{1}{1-\beta}&\left(\frac{\frac{z(I_{\lambda,\mu}F_c(-zf'(z))(z))'}{I_{\lambda,\mu}F_c(g)(z)} + (c+1)\frac{I_{\lambda,\mu}F_c(-zf'(z))(z)}{I_{\lambda,\mu}F_c(g)(z)}}{\frac{z(I_{\lambda,\mu}F_c(g)(z))'}{I_{\lambda,\mu}F_c(g)(z)} + c + 1} - \beta\right).
\end{aligned} \tag{3.6}$$

Since $g \in \Sigma^*(\lambda, \mu;\ \eta;\ \phi)$, we see from Theorem 3.1 that $F_c(g) \in \Sigma^*(\lambda, \mu;\ \eta;\ \phi)$. Let us now put

$$q(z) = \frac{1}{1-\eta}\left(-\frac{z(I_{\lambda,\mu}F_c(g)(z))'}{I_{\lambda,\mu}F_c(g)(z)} - \eta\right),$$

where $q \prec \phi$ in $\mathcal{U}$ with the assupption for $\phi \in \mathcal{N}$. Then, by using the same techniques as in the proof of Theorem 2.3, we conclude that from (3.5) and (3.6) that

$$\frac{1}{1-\beta}\left(-\frac{z(I_{\lambda,\mu}f(z))'}{I_{\lambda,\mu}g(z)} - \beta\right) = p(z) + \frac{zp'(z)}{-(1-\eta)q(z) + c + 1 - \eta} \prec \psi(z)\ \ (z \in \mathcal{U}). \tag{3.7}$$

Hence, upon setting

$$\omega(z) = \frac{1}{-(1-\eta)q(z) + c + 1 - \eta},$$

in (3.7), if we apply Lemma 2.2, we obtain that $p \prec \psi$ in $\mathcal{U}$, which yields that $F_c(f) \in \Sigma_c(\lambda, \mu;\ \eta, \beta;\ \phi, \psi)$. Therefore the proof of Theorem 3.3 is evidently completed.

Remark. If we take $\lambda = 1$ and $\mu = 2$ in all theorems of this section, then we extend the results by Goel and Sohi [4], which reduce the results earlier obtained by Bajpai [1].

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