Thesis for the Degree of Doctor of Philosophy

Strong Convergence of Iterative Algorithms for Nonlinear Self-Mappings in Banach Spaces

by

Hwa Jung Lee

Department of Applied Mathematics

The Graduate School

Pukyong National University

August 2008

Strong Convergence of Iterative Algorithms for Nonlinear Self-mappings in Banach Spaces
(Banach공간 내에서 비선형자기사상들에 대한 반복알고리즘들의 강수렴)

Advisor : Tae Hwa Kim

by

Hwa Jung Lee

A thesis submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in the Department of Applied Mathematics, Graduate School,
Pukyong National University

August 2008

Iterative Algorithms and Their Convergence Theorems

A dissertation

by

Kui Yeon Kim

Approved by:

(Chairman) Il Bong Jung, Ph. D.

(Member) Oh Sang Kwon, Ph. D.

(Member) Nak Eun Cho, Ph. D.

(Member) Jin Mun Jeong, Ph. D.

(Member) Tae Hwa Kim, Ph. D.

August 27, 2008

Contents

Banach공간 내에서 비선형자기사상에 대한 반복알고리즘들의 강수렴

이 화 정

부경대학교 대학원 응용수학과

요 약

집합 $C(\neq\varnothing)$을 Banach공간 X의 닫힌볼록집합이라 할 때, 사상 $T: C\to C$가 모든 $x,y\in C$에 대하여 $\|Tx-Ty\|\le\|x-y\|$을 만족할 때 비확대(nonexpansive)라 한다. 또한 $F(T)=\{x\in C: Tx=x\}$는 T의 부동점들의 집합이고 $\hat{F}(T)$는 T의 근사부동점(approximating fixed point)들의 집합이다. 이제 X가 매끄러운 공간이고, 함수 $\phi: X\times X\to\mathbb{R}$가

$$\phi(x,y)=\|x\|^2-2\langle x,Jy\rangle+\|y\|^2 \quad (x,y\in X)$$

라 할 때, 사상 $T: C\to C$가 (ⅰ) $F(T)\neq\varnothing$, (ⅱ) $\hat{F}(T)=F(T)$, 그리고 (ⅲ) $\phi(p,Tx)\le\phi(p,x)$ $(p\in F(T),\ x\in C)$을 만족할 때 상대적비확대(relatively nonexpansive)라 한다. 한편, 쌍대개념으로, 사상 $T: C\to C$가 (ⅲ) 대신에 (ⅲ)′ $\phi(Tx,p)\le\phi(x,p)$ $(p\in F(T),\ x\in C)$을 만족할 때 일반화된 비확대(generalized nonexpansive)라 한다.

비확대사상에 대한 근사부동점의 구축은 이미지 복구, 부호처리, 균형문제 등 다양하게 응용되며, 근사부동점을 구축하는 방법으로는 Picard 반복구조($x_0\in C,\ x_n=Tx_{n-1}$)가 잘 알려져 있지만, T가 비확대사상이면 그러한 반복구조는 일반적으로 수렴하지 않는다. 반면에, 평균반복방법(average iteration method)을 적용하는 세 종류의 반복알고리즘, 즉 Halpern, Mann, IshiKawa방법이 있다. $x_1:=u\in C$을 닻으로 고정하여 반복적으로 $x_{n+1}=t_nu+(1-t_n)Tx_n,\ \{t_n\}\subset[0,1]$처럼 구축되는 Halpern방법은 HIlbert공간에서 강수렴을 하지만, Mann과 IshiKawa구조는 일반적으로 그렇지 않다.

$$x_{n+1}=\alpha_nx_n+(1-\alpha_n)Tx_n \quad (\{\alpha_n\}\subset[0,1],\ x_1=u\in C) \qquad \text{(Mann)}$$

$$x_{n+1}=\alpha_nx_n+(1-\alpha_n)T(y_n=\beta_nx_n+(1-\beta_n)Tx_n)\ (\alpha_n,\beta_n\subset[0,1]) \qquad \text{(IsjiKawa)}$$

2003년 NaKajo와 TaKahashi는 혼합형의 방법(hybrid method)을 적용하여, Mann방법이 강수렴함을 처음으로 밝혔다. 그 후 많은 수학자들이 그들의 방법을 이용하여 더 일반적인 비선형사상으로 확장시켜왔다.

본 연구는 이러한 측면에서 일반적인 Banach공간 내에서 위에 소개한 상대적비확대사상과 일반화된 비확대사상에 대한 강수렴 문제를 연구하였다. 특히, 본 논문의 3장에서는 상대적 비확대사상들의 유한 족에 대한 Mann형의 반복방법에 대한 강수렴 정리를 밝혔고, 4장에서는 일반화된 비확대사상에 대한 강수렴 문제를 연구했다. 더욱, 5장에서는 4장에서 연구한 결과를 일반화된 비확대사상들의 유한 족에 대한 강수렴 정리로 확장시켰다.

Chapter 1

Introduction

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping. We say that T is a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in C$. In particular, a Lipschitzian mapping T is called *nonexpansive* if $k_n = 1$ for all n and *asymptotically nonexpansive* [14] if $\lim_{n\to\infty} k_n = 1$, respectively. A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an *asymptotic fixed point* of T [33] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Let X be a smooth Banach space and let X^* be the dual of X. The gauge function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in X$, where J is the *normalized* duality mapping from X to its dual space X^* such that

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

for each $x \in X$. We say that a mapping $T : C \to C$ is *relatively nonexpansive* [6, 7, 9] if

(i) $F(T)$ is nonempty,

(ii) $\hat{F}(T) = F(T)$, and

(iii) $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$, $p \in F(T)$;

see also [26]. As its dual concept, $T : C \to C$ is said to be *generalized nonexpansive* if (i), (ii) and the following dual property (iii)$'$ instead of (iii) are satisfied:

(iii)$'$ $\phi(Tx, p) \leq \phi(x, p)$ for all $x \in C$, $p \in F(T)$;

see [18] for definition with no condition (ii). Then it is well known in [26] that if X is strictly convex and T is relatively nonexpansive, then $F(T)$ is closed and convex.

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing (see, e.g., [8, 28, 35, 42, 43]). However, the sequence $\{T^n x\}$ of Picard iterates of the mapping T at a point $x \in C$ may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [16] and is defined as follows: Take an initial guess $x_1 = u \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n u + (1 - t_n) T x_n, \quad n \geq 1, \tag{1.1}$$

where $\{t_n\}$ is a sequence in the interval $[0, 1]$.

The second iteration process is now known as Mann's iteration process [24] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 1, \tag{1.2}$$

where the initial guess $x_1 = u$ is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0, 1]$.

The third iteration process is referred to as Ishikawa's iteration process [17] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 1, \tag{1.3}$$

where the initial guess $x_1 = u$ is taken in C arbitrarily and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0, 1]$. By taking $\beta_n = 1$ for all n in (1.3), Ishikawa's iteration process reduces to the Mann's iteration process (1.2). It is known in [10] that the process (1.2) may fail to converge while the process (1.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (1.1) has been proved to be strongly convergent in both Hilbert spaces [16, 23, 38] and uniformly smooth Banach spaces [31, 36, 41], while Mann's iteration (1.2) has only weak convergence even in a Hilbert space [13].

Attempts to modify the Mann iteration method (1.2) or the Ishikawa iteration method (1.3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [27] proposed the following modification of Mann's iteration process (1.2) for a single nonexpansive mapping T with $F(T) \neq \emptyset$ in a

Hilbert space H:

$$\begin{cases} u \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, u - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} u, \end{cases} \tag{1.4}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}u$. A recent extension of the process (1.4) to asymptotically nonexpansive mappings can be found in [21]. See also [20] for another modification of the Mann iteration process (1.2) which also has strong convergence. Very recently, Martinez-Yanez and Xu [25] generalized Nakajo and Takahashi's iteration process (1.4) to the following modification of Ishikawa's iteration process (1.3) for a nonexpansive mapping $T : C \to C$ with $F(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} u \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\ z_n = \beta_n x_n + (1-\beta_n) T x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|z_n - v\|^2\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - u \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} u, \end{cases} \tag{1.5}$$

and proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}u$ provided the sequence $\{\alpha_n\}$ is bounded above from one and $\lim_{n\to\infty} \beta_n = 1$.

On the other hand, Matsushita and Takahashi [26] extended Nakajo and Takahashi's iteration process (1.4) to the following modification of Mann's iteration process (1.2) using the hybrid method in mathematical programming for a relatively nonexpansive mapping $T : C \to C$ in a uniformly convex and uniformly smooth Banach space X:

$$\begin{cases} u \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Ju - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} u, \end{cases} \tag{1.6}$$

where J is the normalized duality mapping. Then they proved that if the sequence $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\limsup_{n\to\infty} \alpha_n < 1$, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to $\prod_{F(T)} u$, where $\prod_K$ denotes the generalized projection from X onto a closed convex subset K of X.

The paper is organized as follows. In the following chapter we give some preparations relating to four projections in Banach spaces which play crucial roles for our argument. In Chapter 3, motivated and inspired by their ideas due to Martinez-Yanez and Xu [25] and Matsushita and Takahashi [26], we shall prove some strong convergence theorems for a pair of relatively nonexpansive mappings in Banach spaces. This chapter is organized as follows. In the section 3.1 we give a new equivalent to the Kadec-Klee property in a Banach space. In Section 3.2, motivated by [25, 26], we extend Matsushita and Takahashi's iteration process (1.6) to the Mann or Ishikawa iteration type process for a pair of relatively nonexpansive mappings.

In Chapter 4, we employ ideas due to Matsushita and Takahashi [26] and Ibaraki and Takahashi [18] to prove some strong convergence theorems for generalized nonexpansive mappings in uniformly convex Banach spaces, as analogues of recent results due to Matsushita and Takahashi [26]. Finally, in section 4.2, some applications are added.

Finally, in Chapter 5, we shall discuss the strong convergence problems relating to the previous chapter for a finite family of generalized nonexpansive mappings in uniformly convex Banach spaces.

Chapter 2

Preliminaries

In this chapter, we introduce some notations and prerequisites which are used in the subsequent chapters.

2.1 Geometrical properties

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}.$$

The normalized duality mapping J from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$.

Now we summarize some well known properties of the duality mapping J for our further argument.

Proposition 2.1.1. ([12, 32, 37]) *Let X be a real Banach space. The normalized duality mapping J from X to X^* satisfies the following basic properties:*

(1) *Jx is nonempty, bounded, closed and convex (hence weakly compact) for all $x \in X$.*

(2) *$J0 = 0$.*

(3) *$J(\lambda x) = \lambda Jx$ for $x \in X$ and real λ.*

(4) *J is monotone, that is, $\langle x-y, j_x-j_y\rangle \geq 0, \quad \forall x, y \in X, \forall j_x \in Jx, \forall j_y \in Jy$.*

(5) *$\|x\|^2 - \|y\|^2 \geq 2\langle x-y, j\rangle$ for $x, y \in X$ and $j \in Jy$.*

Recall that a Banach space X is said to be *strictly convex* (SC) [4] if any non-identically zero continuous linear functional takes maximum value on the closed unit ball at most at one point. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$.

We introduce some equivalent properties of strict convexity of X; see Proposition 2.1.1 in [4] for the detailed proof.

Proposition 2.1.2. ([4]) *A linear normed space X is strictly convex if and only if one of the following equivalent properties holds:*

(a) *if $\|x+y\| = \|x\| + \|y\|$ and $x \neq 0$, then $y = tx$ for some $t \geq 0$;*

(b) *if $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$;*

(c) *if $\|x\| = \|y\| = 1$ and $x \neq y$, then $\|(x+y)/2\| < 1$;*

(d) *the function $x \to \|x\|^2$, $x \in X$, is strictly convex.*

Proof. The detailed proof of (b) $\Leftrightarrow$ (d) will be given for our reference. It suffices to show (b) $\Rightarrow$ (d). Let $x \neq y \in X$ and $\lambda \in (0,1)$. First, let $\|x\| = \|y\| = r > 0$. By (b), we obtain

$$\|\lambda x + (1-\lambda)y\|^2 < r^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \tag{2.1}$$

for all $x, y \in X$ with $\|x\| = \|y\|$. Next $\|x\| \neq \|y\|$. Then, from the equality

$$\lambda\|x\|^2 + (1-\lambda)\|y\|^2 = (\lambda\|x\| + (1-\lambda)\|y\|)^2 + \lambda(1-\lambda)(\|x\| - \|y\|)^2$$

it follows that

$$\begin{aligned} \|\lambda x + (1-\lambda)y\|^2 &\leq (\lambda\|x\| + (1-\lambda)\|y\|)^2 \\ &< \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \end{aligned} \tag{2.2}$$

for all $x, y \in X$ with $\|x\| \neq \|y\|$. By (2.1) and (2.2), (d) is satisfied. $\square$

Let $S(X) := \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each x, $y \in S(X)$. In this case, the norm of X is said to be *Gâteaux differentiable.* The space X is said to be a *uniformly Gâteaux differentiable norm* if for each $y \in S(X)$, the limit (2.3) is attained uniformly for $x \in S(X)$. the norm of X is said to be *Fréchet differentiable* if for each $x \in S(X)$, the limit (2.3) is attained uniformly for $y \in S(X)$. The norm of X said to be *uniformly Fréchet differentiable* (or X is said to be *uniformly smooth*) if the limit in (2.3) is attained uniformly for $x, y \in S(X)$.

A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [12, 37] for more details.

Again, we introduce some well known properties of the duality mapping J relating to geometrical properties of X.

Proposition 2.1.3. ([12, 32, 37])

(1) *X is smooth if and only if J is single valued. In this case, J is norm-to-weak* continuous;*

(2) *if X is strictly convex, then J is one to one (or injective), i.e.,*

$$x \neq y \quad \Rightarrow \quad Jx \cap Jy = \emptyset.$$

(3) *X is strictly convex if and only if J is a strictly monotone operator, i.e.,*

$$x \neq y,\ j_x \in Jx,\ j_y \in Jy \quad \Rightarrow \quad \langle x - y, j_x - j_y \rangle > 0.$$

(4) *if X is reflexive, then J is a mapping of X onto X^*.*

(5) *if X^* is strictly convex (resp., smooth), then X is smooth (resp., strictly convex). Further, the converse is satisfied if X is relexive.*

(6) *if X has a Fréchet differentiable norm, then J is norm-to-norm continuous.*

(7) *if X has a uniformly Gâteaux differentiable norm, then J is norm-to-weak* uniformly continuous on each bounded subset of X.*

(8) *if X is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of X.*

Finally, we shall add the well-known properties between X and its dual X^*.

(9) X is uniformly convex if and only if X^* is uniformly smooth.

(10) X is reflexive, strictly convex, and has the Kadec-Klee property if and only if X^* has a Fréchet differentiable norm.

2.2 Four projections on Banach spaces

Let C be a nonempty subset of a real Banach space X. We say that C is said to be a *Chebyshev set* with respect to the function f if to each $x \in X$ there exists a unique $u \in C$ such that

$$\|x - u\| = d(x, C) = \inf_{y \in C} \|x - y\|.$$

In this case, we may define the *nearest point projection* (or called *metric projection*) $P_C : X \to C$ by assigning u to x. Then we have the following

Proposition 2.2.1. ([15]; see Proposition 3.4; pp.13) *Let C be a convex Chebyshev set in X and $x \in X$. Then,*

$$u = P_C x \quad \Leftrightarrow \quad \exists j \in J(x - u) \ \text{s.t.} \ \langle y - u, j \rangle \le 0, \quad \forall y \in C. \tag{2.4}$$

Here, let us introduce the following well known existence theorem; see Theorem 1.3.11 in [37] or Theorem 1.2 and Remark 1.2 in [4].

Theorem 2.2.2. ([4, 37]) *Let X be a reflexive Banach space and let C be a closed convex subset of X. Let f be a proper convex lower semicontinuous function of C into $(-\infty, \infty]$ and suppose $f(x_n) \to \infty$ as $\|x_n\| \to \infty$. Then, there exists $u \in C$ such that*

$$f(u) = \inf_{y \in C} f(y). \tag{2.5}$$

Let X be a reflexive and strictly convex Banach space and let C be a nonempty closed convex subset of X. For an arbitrary (fixed) point $x \in X$, consider $f_x(y) = \|x - y\|^2$ for $y \in C$. Then $f_x : C \to [0, \infty)$ is a proper strictly convex and continuous function and $f_x(y) \to \infty$ as $\|y\| \to \infty$. By Theorem 2.2.2, there exists $u \in C$ such that

$$f_x(u) = \inf_{y \in C} f_x(y). \tag{2.6}$$

Since X is strictly convex, $f_x(\cdot)$ is a strictly convex function; see (d) of Proposition 2.1.1. Therefore, such a $u \in C$ is uniquely determined. Note that (2.6) is equivalent to

$$\|x - u\| = \inf_{y \in C} \|x - y\| = d(x, C).$$

So, the closed convex subset C of a reflexive and strictly convex Banach space X is a Chebyshev set and hence $P_C : X \to C$ is a nearest point projection (or metric projection). Combined with Proposition 2.2.1, we have the following

Proposition 2.2.3. *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space X. Then*

$$u = P_C x \quad \Leftrightarrow \quad \langle y - u, J(x - u) \rangle \le 0, \quad \forall y \in C. \tag{2.7}$$

On the other hand, let X be a smooth Banach space. Recall [2] that the gauge function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in X$; see also [19]. It is obvious from the definition of ϕ that, for $x, y, z \in X$,

(a) $(\|y\| - \|x\|)^2 \le \phi(y, x) \le (\|y\| + \|x\|)^2$,

(b) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,

(c) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \le \|x\| \, \|Jx - Jy\| + \|y - x\| \, \|y\|$,

(d) if X is strictly convex, then $\phi(x, y) = 0 \Leftrightarrow x = y$,

(e) $\phi(\cdot, x)$ is a strictly convex function if and only if X is strictly convex,

(f) both $\phi(\cdot, x)$ and $\phi(x, \cdot)$ are continuous functions on X; further, $\phi(\cdot, x)$ is convex, while $\phi(x, \cdot)$ is not convex.

Then applying for Theorem 2.2.2 again, we have the following result due to Kamimura and Takahashi [19].

Proposition 2.2.4. ([19]; see Proposition 3) *Let X be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of X, and let $x \in X$. Then there exists a unique element $u \in C$ such that*

$$\phi(u, x) = \inf_{z \in C} \phi(z, x).$$

Let C and X be as Proposition 2.2.4. For $x \in X$, define

$$\prod\nolimits_C x = u.$$

Then a mapping $\prod_C : X \to C$ is well-defined (called the *generalized projection* from X onto C); see [2, 3, 19]. In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [2, 3, 19]).

Proposition 2.2.5. ([2, 3, 19]) *Let C be a nonempty closed convex subset of a smooth Banach space X, $x \in X$ and $u \in C$. Then*

$$u = \prod\nolimits_C x \quad \Leftrightarrow \quad \langle y - u, Jx - Ju \rangle \leq 0, \quad \forall y \in C. \tag{2.8}$$

Let C be a nonempty closed convex subset of a normed linear space X and let F be a nonempty subset of C. We say that $R : C \to F$ is *retraction* if $R(x) = x$ for all $x \in F$. A retraction $R : C \to F$ is said to be *sunny* [29] if whenever $z \in C$ is on the ray from Rx to $x\,(\in C)$, we have $Rz = Rx$, that is, $u = Rx$ implies $R(u + \lambda(x - u)) = u$ for all $x \in C$ and $\lambda \in [0, 1]$.

The following result is well known (see [29]). For our convenience, the proof will be included.

Proposition 2.2.6. ([29]) *Let C be a nonempty closed convex subset of a normed linear space X whose norm is Gâteaux differentiable, and let F be a nonempty subset of C. Let $R : C \to F$ be a retraction. Then the followings are equivalent:*

(a) $\langle x - Rx, J(y - Rx)\rangle \le 0$ *for* $x \in C$ *and* $y \in F$*;*

(b) $\|Rx - Ry\|^2 \le \langle x - y, J(Rx - Ry)\rangle$ *for* $x, y \in C$*;*

(c) R *is both sunny and nonexpansive.*

Proof. First, we show (a) $\Leftrightarrow$ (b). Suppose (a) holds. Then, for $x, y \in C$, we have

$$\begin{aligned} \langle y - Ry, J(Rx - Ry)\rangle &\le 0, \\ \langle x - Rx, J(Ry - Rx)\rangle &\le 0. \end{aligned}$$

Summing both sides yields

$$\langle y - x + Rx - Ry, J(Rx - Ry)\rangle \le 0,$$

which is equivalent to

$$\|Rx - Ry\|^2 \le \langle x - y, J(Rx - Ry)\rangle,$$

and so (b) is fulfilled. Now suppose (b) holds and let $x \in C$, $y \in F$. Since $y = Ry$, it follows from (b) that

$$\|Rx - y\|^2 \leq \langle x - y, J(Rx - y)\rangle,$$

which immediately reduces to (a). Next we show (c) $\Leftrightarrow$ (a). Suppose R is both sunny and nonexpansive. Then, for $x \in C$ and $y \in F$, put $v = Rx$ and $K = \{v + \lambda(x - v) : 0 \leq \lambda \leq 1\} \subset C$. If $w \in K$, then sunny nonexpansive retraction of R gives

$$\|v - y\| = \|Rw - Ry\| \leq \|w - y\|, \quad \forall w \in C,$$

and so K is a Chebyshev set in X, by Proposition 2.2.3,

$$\langle w - v, J(y - v)\rangle \leq 0, \quad \forall w \in K.$$

In particular, taking $w = x$ (when $\lambda = 1$)$\in K$ gives

$$\langle x - v, J(y - v)\rangle \leq 0, \quad \forall y \in F.$$

Therefore (a) is obtained. Now suppose (a) holds, then it immediately follows from (b) that R is nonexpansive. Finally to prove that R is sunny, let $v = Px$, and $w = v + \lambda(x - v)$, $\lambda \in [0, 1]$. Then,

$$\langle x - v, J(Rw - v)\rangle \leq 0.$$

Firstly multiplying by λ and next inserting $\lambda(x - v) = w - v$, we have

$$\langle w - v, J(Rw - v)\rangle \leq 0. \tag{2.9}$$

On the other hand, since $v = Rx \in C$, it follows from (a) again that

$$\langle w - Rw, J(v - Rw)\rangle \leq 0. \tag{2.10}$$

Summing (2.9) and (2.10) gives $\|v - Rw\|^2 = \langle v - Rw, J(v - Rw)\rangle \leq 0$, and so $v = Rw$. Then the proof is completed. $\square$

Remark 2.2.7. Note that (b) of Proposition 2.2.6 holds if and if only R is *firmly nonexpansive.*

Let X be a smooth Banach space and let C be a nonempty closed convex subset of X. Recall that a mapping $T : C \to C$ is said to be *generalized nonexpansive* [18] if $F(T) \neq \emptyset$ and

$$\phi(Tx, q) \leq \phi(x, q), \quad \forall x \in C,\ q \in F(T), \tag{2.11}$$

where $F(T)$ is the set of fixed points of T.

The following result is well known (see [18]). For our convenience, we shall give the detail proof.

Proposition 2.2.8. ([18]; see Proposition 4.2) *Let C be a nonempty closed convex subset of a smooth and strictly convex Banach space X and let F be a nonempty subset of C. Let R_F be a retraction of C onto F. Then R_F is sunny and generalized nonexpansive if and only if*

$$\langle x - R_F x, Jy - JR_F x\rangle \leq 0, \quad \forall x \in C,\ y \in F. \tag{2.12}$$

Proof. $(\Rightarrow)$ Let $x \in C$ and let $y \in F \in F(R)$. Putting

$$K = \{R_F x + t(x - R_F x) : t \in [0, 1]\},$$

we get

$$\phi(R_F x, y) = \phi(R_F w, y) \leq \phi(w, y), \quad \forall w \in K.$$

Since $\phi(\cdot, y)$ is continuous for each $y \in F$, we have

$$\phi(R_F x, y) = \min_{w \in K} \phi(w, y).$$

Since $R_F x = \prod_K y$ from Proposition 2.2.5, we have $\langle w - R_F x, Jy - JR_F x\rangle \leq 0$ for all $w \in K$. In particular, taking $w = x$ (with $t = 1$) yields

$$\langle x - R_F x, Jy - JR_F x\rangle \leq 0$$

for all $x \in C$ and $y \in F$.

($\Leftarrow$) Let $x \in C$ and $p \in F = F(R_F)$. Then it follows from the properties (a)-(b) of ϕ and (2.12) that

$$\begin{aligned}\phi(x,p) &= \phi(x,R_Fx)+\phi(R_Fx,p)+\langle x-R_Fx, JR_Cx-Jp\rangle \\ &\geq \phi(x,R_Fx)+\phi(R_Fx,p)\geq \phi(R_Fx,p)\end{aligned}$$

and so R_F is generalized nonexpansive. To prove that R_F is sunny, let $x \in C$ and $x_t = R_Fx + t(x - R_Fx)$, $t \in [0,1]$. From (2.12), we have

$$\langle x_t - R_Fx_t, JR_Fx - JR_Fx_t\rangle \leq 0 \tag{2.13}$$

and

$$\langle x - R_Fx, JR_Fx_t - JR_Fx\rangle \leq 0. \tag{2.14}$$

Since $x_t - R_Fx = t(x - R_Fx)$, it follows from (2.14) that

$$\langle R_Fx - x_t, JR_Fx - JR_Fx_t\rangle \leq 0. \tag{2.15}$$

Combining (2.13) and (2.15) gives $\langle R_Fx - R_Fx_t, JR_Fx - JR_Fx_t\rangle \leq 0$. Since X is strictly convex, this shows $R_Fx_t = R_Fx$ and so R_F is sunny. □

Remark 2.2.9. Let C, F and X be as in Proposition 2.2.8. Notice that such a sunny generalized nonexpansive retraction of C onto F is unique; see Ibaraki and Takahashi [18].

2.3 Properties relating to the gauge function ϕ

We begin with the following well known result; see, for example, [2, 3, 19]).

Proposition 2.3.1. ([2, 3, 19]) *Let K be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

(a) *If X is smooth, then, $\tilde{x} = \prod_K x$ if and only if $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$ for $y \in K$; see also Proposition 2.2.5.*

(b) *If X is reflexive, strictly convex and smooth, then, for all $y \in K$ the following inequality holds:*

$$\phi(y, \textstyle\prod_K x) + \phi(\prod_K x, x) \leq \phi(y, x).$$

Lemma 2.3.2. *Let X be a smooth Banach space. Then, for any fixed $x \in X$, $\phi(\cdot, x)$ is weakly lower semicontinuous on X; moreover, it is continuous and convex on X.*

Proof. Fix $x \in X$ and let $x_n \rightharpoonup p \in X$. Clearly, $\langle x_n, Jx \rangle \to \langle p, Jx \rangle$, and using the weakly lower semicontinuity of the norm, we have

$$\begin{aligned}\phi(p, x) &= \|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 \\ &\leq \liminf_{n\to\infty} \left(\|x_n\|^2 - 2\langle x_n, Jx \rangle + \|x\|^2\right) \\ &= \liminf_{n\to\infty} \phi(x_n, x).\end{aligned}$$

Hence $\phi(\cdot, x)$ is weakly lower semicontinuous on X. Obviously, the continuity and convexity of the function $\phi(\cdot, x)$ follow from the continuity and convexity of $\|\cdot\|^2$ and the linearity of Jx. □

Motivated by Lemmas 1.3 and 1.5 of Martinez-Yanes and Xu [25] in Hilbert spaces, we present the following two lemmas.

Lemma 2.3.3. *Let C be a nonempty closed convex subset of a smooth Banach space X, $x, y, z \in X$ and $\lambda \in [0, 1]$. Given also a real number $a \in \mathbb{R}$, the set*

$$D := \{v \in C : \phi(v, z) \leq \lambda\phi(v, x) + (1 - \lambda)\phi(v, y) + a\}$$

is closed and convex.

Proof. The closedness of D is obvious from the continuity of $\phi(\cdot, x)$ for $x \in X$. Now we show that D is convex. As a matter of fact, the defining inequality in D is equivalent to the inequality

$$\langle v, \lambda Jx + (1-\lambda)Jy - Jz\rangle \leq \frac{1}{2}(\lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \|z\|^2 + a).$$

This inequality is affine in v and hence the set D is convex. $\square$

Lemma 2.3.4. *Let X be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let K be a nonempty closed convex subset of X. Let $u \in X$ and $q := \prod_K u$, where $\prod_K$ denotes the generalized projection from X onto K. If $\{x_n\}$ is a sequence in X such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\phi(x_n, u) \leq \phi(q, u) \tag{2.16}$$

for all n. Then $x_n \to q = \prod_K u$.

Proof. By (2.16), $\{\phi(x_n, u)\}$ is bounded and, by the property (a) of ϕ in Chapter 2, $\{x_n\}$ is bounded; so $\omega_w(x_n) \neq \emptyset$ by reflexivity of X. Since $\phi(\cdot, u)$ is weakly lower semicontinuous on X by Lemma 2.2.2, and, by using (2.16) again, we get $\phi(v, u) \leq \phi(q, u)$ for all $v \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset K$ and $q = Q_K u$, we must have $v = q$ for all $v \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{q\}$ and $x_n \rightharpoonup q$. On the other hand, using the weakly lower semicontinuity of $\phi(\cdot, u)$ again, we have

$$\begin{aligned} \phi(q, u) &\leq \liminf_{n\to\infty} \phi(x_n, u) \\ &\leq \limsup_{n\to\infty} \phi(x_n, u) \\ &\leq \phi(q, u) \qquad \text{by (2.16)} \end{aligned}$$

and so $\lim_{n\to\infty} \phi(x_n, u) = \phi(q, u)$. This implies $\lim_{n\to\infty} \|x_n\| = \|q\|$. By the Kadec-Klee property of X, we have $x_n \to q$. $\square$

Lemma 2.3.5. ([40]) *Let X be a uniformly convex Banach space and let $B_r = \{x \in X : \|x\| \leq r\}$ be a closed ball with radius $r > 0$ in X. Then* *there is a continuous, strictly increasing and convex function* $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, *such that*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|) \tag{2.17}$$

for all x, $y \in B_r$ *and* $\alpha \in [0, 1]$.

Recently, Kamimura and Takahashi [19] proved the following result, which plays a crucial role in our discussion.

Proposition 2.3.6. ([19]) *Let* X *be a uniformly convex and smooth Banach space and let* $\{x_n\}, \{z_n\}$ *be two sequences of* X. *If* $\phi(x_n, z_n) \to 0$ *and either* $\{x_n\}$ *or* $\{z_n\}$ *is bounded, then* $x_n - z_n \to 0$.

Here we give the following converse of Proposition 2.3.6.

Proposition 2.3.7. *Let* X *be a smooth Banach space and let* $\{x_n\}, \{z_n\}$ *be two sequences in* X. *If* $x_n - z_n \to 0$ *and either* $\{x_n\}$ *or* $\{z_n\}$ *is bounded, then* $\phi(x_n, z_n) \to 0$.

Proof. Since $x_n - z_n \to 0$, it is not hard to see that if either $\{x_n\}$ or $\{z_n\}$ is bounded, then the other is also bounded. Now let $x \in X$ be fixed. Then noticing that

$$\begin{aligned}
|\phi(x_n, x) - \phi(z_n, x)| &= \left|\, \|x_n\|^2 - \|z_n\|^2 + 2\langle z_n - x_n, Jx\rangle \,\right| \\
&\leq |\|x_n\| - \|z_n\||(\|x_n\| + \|z_n\|) + 2\|z_n - x_n\|\,\|x\| \\
&\leq \|x_n - z_n\|(\|x_n\| + \|z_n\| + 2\|x\|) \to 0
\end{aligned}$$

and using the identity equation the property (b) of ϕ in Chapter 2, we have

$$\begin{aligned}
\phi(x_n, z_n) &= \phi(x_n, x) - \phi(z_n, x) + 2\langle x_n - z_n, Jx - Jz_n\rangle \\
&\leq |\,\phi(x_n, x) - \phi(z_n, x)\,| + 2\|x_n - z_n\|(\|x\| + \|z_n\|) \to 0
\end{aligned}$$

and the proof is complete. □

Now combing Proposition 2.3.6 and 2.3.7 gives the following equivalent form in uniformly convex and smooth Banach spaces. This property will be also used for proving our main result.

Proposition 2.3.8. *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_n) \to 0$ if and only if $x_n - z_n \to 0$.*

As a easy observation of Proposition 2.3.8, we first prove the following results.

Proposition 2.3.9. *Let C be a closed convex subset of a uniformly convex and smooth Banach space X and $T : C \to C$ be a relatively nonexpansive mapping. Then T is continuous on $F(T)$.*

Proof. Let $p \in F(T)$ and let $x_n \to p$. To claim that $Tx_n \to p$, by Proposition 2.3.8, it suffices to show that $\phi(p, Tx_n) \to 0$. Indeed, since J is norm-to-weak* continuous, $Jx_n \xrightarrow{*} Jp$; in particular, $\langle p, Jx_n\rangle \to \langle p, Jp\rangle$. Hence

$$\phi(p, x_n) = \|p\|^2 - 2\langle p, Jx_n\rangle + \|x_n\|^2 \to \|p\|^2 - 2\langle p, Jp\rangle + \|p\|^2 = 0.$$

Now using the relative nonexpansivity of T, we get

$$\phi(p, Tx_n) \le \phi(p, x_n) \to 0.$$

and so $Tx_n \to Tp = p$. $\square$

Chapter 3

Relatively nonexpansive mappings and strong convergence

In this paper, motivated by an idea due to Matsushida and Takahashi [26], we prove some strong convergence theorems of modified Ishikawa type iteration processes for a pair of relatively nonexpansive mappings in Banach spaces, which extend the recent result due to Matsushida and Takahashi in Banach spaces. Also some applications for nonexpansive mappings in Hilbert spaces are added.

3.1 Kadec-Klee property and its equivalence

Recall that a Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [12, 37] for more details.

In this section we consider the relationship between the Kadec-Klee property and the following weak property which is motivated by Proposition 2.3.8:

(KT) *Given a sequence* $\{x_n\}$ *in a smooth Banach space* X *and* $x(\neq 0) \in X$*,* $\phi(x_n, x) \to 0$ if and only if $x_n \to x$.

Here, we prove that the property (KT) is equivalent to the Kadec-Klee property in a reflexive, strictly convex and smooth Banach space.

Theorem 3.1.1. *Let* X *be a smooth Banach space. Then,*

(a) $(KT) \ \Rightarrow \ (Kadec-Klee)$.

(b) *if* X *is reflexive and strictly convex,* $(Kadec-Klee) \ \Rightarrow \ (KT)$.

Proof. (a) Let $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. Assume without loss of generality that $x \neq 0$. Then, we have

$$\phi(x_n, x) = \|x_n\|^2 - 2\langle x_n, Jx\rangle + \|x_n\|^2 \to \|x\|^2 - 2\langle x, Jx\rangle + \|x\|^2 = 0.$$

From (KT), it follows that $x_n \to x$. Hence X satisfies the Kadec-Klee property.

(b) Let $x\,(\neq 0) \in X$. Then it suffices to show that if $\phi(x_n, x) \to 0$, then $x_n \to x$. Now let $\phi(x_n, x) \to 0$. Clearly, $\{\phi(x_n, x)\}$ is bounded; by the property (a) of ϕ in Chapter 2, $\{x_n\}$ is bounded and so $\omega_w(x_n) \neq \emptyset$. Now if $x_{n_k} \rightharpoonup v \in \omega_w(x_n)$, then, since $\phi(\cdot, x)$ is weakly lower semicontinuous by Lemma 2.2.2,

$$\phi(v, x) \le \liminf_{k\to\infty} \phi(x_{n_k}, x) = \lim_{k\to\infty} \phi(x_{n_k}, x) = 0,$$

which says that $\phi(v, x) = 0$. By strict convexity of X, we have $v = x$ for all $v \in \omega_w(x_n)$. Therefore, $\omega_w(x_n) = \{x\}$; so $x_n \rightharpoonup x$. On the other hand, since

$$(\|x_n\| - \|x\|)^2 \le \phi(x_n, x) \to 0,$$

we have $\|x_n\| \to \|x\|$. By the Kadec-Klee property, we conclude that $x_n \to x$. $\square$

3.2 Strong convergence theorems

In this section we first propose a modification of Ishikawa's iteration process (1.3), motivated by the idea due to [25, 26], to prove strong convergence for a pair of relatively nonexpansive mappings in a Banach space.

Theorem 3.2.1. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X. Let $\Im = \{T_1, T_2 : C \to C\}$ be a pair of relatively nonexpansive mappings with $F \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)e_n, \\ y_n = J^{-1}(\alpha_n JT_2 z_n + (1-\alpha_n)JT_1 x_n), \\ H_n = \{v \in C : \phi(v, y_n) \leq \alpha_n \phi(v, z_n) + (1-\alpha_n)\phi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Ju\rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} u, \qquad n \geq 1, \end{cases}$$

where J is the normalized duality mapping and $\{e_n\}$ is a bounded sequence in C. If T_2 is uniformly continuous on C, then $x_n \to \prod_F u$, where $\prod_F$ is the generalized projection from X onto F.

Proof. We employ the methods of the proofs in [26] and [25]. First, observe that H_n is closed and convex by Lemma 2.3.3, and that W_n is obviously closed and convex for each n. Next we show that $F \subset H_n$ for all n. Indeed, for all $p \in F$, we have, using convexity of $\|\cdot\|^2$ and relative nonexpansivity of T_i, $i = 1, 2$

(noticing that $z_n \in C$),

$$\begin{aligned}
&\phi(p, y_n) = \phi(p, J^{-1}(\alpha_n JT_2 z_n + (1-\alpha_n)JT_1 x_n)) \\
= \;& \|p\|^2 - 2\langle p, \alpha_n JT_2 z_n + (1-\alpha_n)JT_1 x_n\rangle + \|\alpha_n JT_2 z_n + (1-\alpha_n)JT_1 x_n\|^2 \\
\leq \;& \|p\|^2 - 2\alpha_n\langle p, JT_2 z_n\rangle - 2(1-\alpha_n)\langle p, JT_1 x_n\rangle + \alpha_n\|T_2 z_n\|^2 + (1-\alpha_n)\|T_1 x_n\|^2 \\
= \;& \alpha_n\phi(p, T_2 z_n) + (1-\alpha_n)\phi(p, T_1 x_n) \\
\leq \;& \alpha_n\phi(p, z_n) + (1-\alpha_n)\phi(p, x_n). \qquad (3.1)
\end{aligned}$$

So $p \in H_n$ for all n. Moreover, we show that

$$F \subset H_n \cap W_n \qquad (3.2)$$

for all n. It suffices to show that $F \subset W_n$ for all n. We prove this by induction. For $n = 1$, we have $F \subset C = W_1$. Assume that $F \subset W_k$ for some $k \geq 2$. Since x_{k+1} is the generalized projection of u onto $H_k \cap W_k$, by Proposition 2.3.1 (a) we have

$$\langle x_{k+1} - z, Ju - Jx_{k+1}\rangle \geq 0$$

for all $z \in H_k \cap W_k$. As $F \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F$. This together with the definition of W_{k+1} implies that $F \subset W_{k+1}$. Hence (3.2) holds for all n. So, $\{x_n\}$ is well defined. Obviously, since $x_n = \prod_{W_n} u$ by the definition of W_n and Proposition 2.3.1 (a), and since $F \subset W_n$, it follows from the definition of $\prod_{W_n}$ that $\phi(x_n, u) \leq \phi(p, u)$ for all $p \in F$. In particular, we obtain that for all n,

$$\phi(x_n, u) \leq \phi(q, u), \qquad \text{where } q := \textstyle\prod_F u. \qquad (3.3)$$

Therefore, $\{\phi(x_n, u)\}$ is bounded; so is $\{x_n\}$ by the property (a) of ϕ in Chapter 2. Since $\{e_n\}$ is bounded, $\{z_n\}$ is also bounded. Noticing that $\phi(p, T_i x_n) \leq \phi(p, x_n)$ for all $p \in F(T_i)$, $\{T_i x_n\}$ is also bounded for $i = 1, 2$.

Now we show that

$$\|x_{n+1} - x_n\| \to 0. \qquad (3.4)$$

Indeed, by the definition of W_n and Proposition 2.3.1 (a), we have $x_n = \prod_{W_n} u$ which together with the fact that $x_{n+1} \in H_n \cap W_n \subset W_n$ implies that

$$\phi(x_n, u) = \min_{z \in W_n} \phi(z, u) \leq \phi(x_{n+1}, u),$$

which shows that the sequence $\{\phi(x_n, u)\}$ is nondecreasing and so the $\lim_{n\to\infty} \phi(x_n, u)$ exists. Simultaneously, from Proposition 2.3.1 (b), we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi\left(x_{n+1}, \textstyle\prod_{W_n} u\right) \leq \phi(x_{n+1}, u) - \phi(\textstyle\prod_{W_n} u, u) \\ &= \phi(x_{n+1}, u) - \phi(x_n, u) \to 0. \end{aligned} \tag{3.5}$$

Hence, (3.4) is satisfied from Proposition 2.3.8.

Since $\beta_n \to 1$, and $\{x_n\}, \{e_n\}$ are bounded, we have

$$\|x_n - z_n\| = (1 - \beta_n)\|x_n - e_n\| \to 0. \tag{3.6}$$

Combining with (3.4) gives $\|x_{n+1} - z_n\| \to 0$, which is equivalent to $\phi(x_{n+1}, z_n) \to 0$ by Proposition 2.3.8. Now since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \leq \alpha_n \phi(x_{n+1}, z_n) + (1 - \alpha_n)\phi(x_{n+1}, x_n) \to 0,$$

hence $\phi(x_{n+1}, y_n) \to 0$. Using Proposition 2.3.8 again, we obtain $\|x_{n+1} - y_n\| \to 0$. This, together with (3.4), implies that $\|x_n - y_n\| \to 0$ and also $\|z_n - y_n\| \to 0$.

Next, we show that $\|x_n - T_i x_n\| \to 0$ for all $i = 1, 2$. Since $\{T_1 x_n\}$ and $\{T_2 z_n\}$ are bounded, there exists $r > 0$ such that $\{T_1 x_n\} \cup \{T_2 z_n\} \subset B_r$. Applying for Lemma 2.3.5 yields

$$\begin{aligned} \|\alpha_n J T_2 z_n + (1 - \alpha_n) J T_1 x_n\|^2 &\leq \alpha_n \|T_2 z_n\|^2 + (1 - \alpha_n)\|T_1 x_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n) g(\|J T_2 z_n - J T_1 x_n\|), \end{aligned} \tag{3.7}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$. Using (3.7) instead of convexity of $\|\cdot\|^2$ in (3.1), we have

$$\phi(p, y_n) \leq \alpha_n \phi(p, z_n) + (1 - \alpha_n)\phi(p, x_n) - \alpha_n(1 - \alpha_n) g(\|J T_2 z_n - J T_1 x_n\|)$$

and so

$$\begin{aligned}
& \alpha_n(1-\alpha_n)g(\|JT_2z_n - JT_1x_n\|) \\
\leq \;& \alpha_n\phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) - \phi(p, y_n).
\end{aligned} \tag{3.8}$$

Notice that, for $p \in F$, using the property (b) of ϕ in Chapter 2 repeatedly,

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, z_n) + \phi(z_n, y_n) + 2\langle p - z_n, Jz_n - Jy_n\rangle, \\
&= \phi(p, z_n) + c_n
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, x_n) + \phi(x_n, y_n) + 2\langle p - x_n, Jx_n - Jy_n\rangle \\
&= \phi(p, x_n) + d_n,
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
c_n &:= \phi(z_n, y_n) + 2\langle p - z_n, Jz_n - Jy_n\rangle \to 0, \\
d_n &:= \phi(x_n, y_n) + 2\langle p - x_n, Jx_n - Jy_n\rangle \to 0,
\end{aligned}$$

respectively, from Proposition 2.3.8. After multiplying α_n and $1-\alpha_n$ in (3.9) and (3.10), respectively, summing both sides yields

$$\phi(p, y_n) = \alpha_n\phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) + \alpha_n c_n + (1-\alpha_n)d_n.$$

Since $c_n,\ d_n \to 0$, we obtain

$$\alpha_n\phi(p, z_n) + (1-\alpha_n)\phi(p, x_n) - \phi(p, y_n) \to 0.$$

Then it follows from (3.8), together with $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, that

$$\lim_{n\to\infty} g(\|JT_2z_n - JT_1x_n\|) = 0.$$

Since g is continuous, strictly increasing and $g(0) = 0$, we have

$$\lim_{n\to\infty} \|JT_2z_n - JT_1x_n\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|T_2 z_n - T_1 x_n\| \to 0.$$

Immediately, using convexity of $\|\cdot\|^2$ and Proposition 2.3.8 again, we have

$$\begin{aligned}
\phi(T_1 x_n, y_n) &= \|T_1 x_n\|^2 - 2\langle T_1 x_n, \alpha_n J T_2 z_n + (1-\alpha_n) J T_1 x_n \rangle \\
&\quad + \|\alpha_n J T_2 z_n + (1-\alpha_n) J T_1 x_n\|^2 \\
&\leq \alpha_n \phi(T_1 x_n, T_2 z_n) \to 0.
\end{aligned}$$

Using Proposition 2.3.8 once more gives $\|T_1 x_n - y_n\| \to 0$, this combined with $\|y_n - x_n\| \to 0$ implies

$$\|T_1 x_n - x_n\| \to 0. \tag{3.11}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\|Jx_n - Jy_n\| \to 0, \quad \|JT_1 x_n - Jx_n\| \to 0. \tag{3.12}$$

On the other hand, notice that

$$\begin{aligned}
Jx_n - Jy_n &= Jx_n - (\alpha_n J T_2 z_n + (1-\alpha_n) J T_1 x_n) \\
&= \alpha_n (Jx_n - JT_2 z_n) + (1-\alpha_n)(Jx_n - JT_1 x_n)
\end{aligned} \tag{3.13}$$

from the definition of y_n. Then using (3.12) and $\liminf_{n\to\infty} \alpha_n > 0$ yields

$$\begin{aligned}
\|Jx_n - JT_2 z_n\| &= \frac{1}{\alpha_n} \|(Jx_n - Jy_n) + (1-\alpha_n)(JT_1 x_n - Jx_n)\| \\
&\leq \frac{1}{\alpha_n} (\|Jx_n - Jy_n\| + (1-\alpha_n)\|JT_1 x_n - Jx_n\|) \to 0.
\end{aligned}$$

Again, since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\|x_n - T_2 z_n\| \to 0.$$

Since $\|z_n - x_n\| \to 0$ and T_2 is uniformly continuous, this yields

$$\|x_n - T_2 x_n\| \leq \|x_n - T_2 z_n\| + \|T_2 z_n - T_2 x_n\| \to 0. \tag{3.14}$$

With the help of (3.11) and (3.14), we have

$$\omega_w(x_n) \subset \hat{F}(T_1) \cap \hat{F}(T_2) = F(T_1) \cap F(T_2) = F.$$

Joining with (3.3) and Lemma 2.3.4 (with $K := F$), we conclude that $x_n \to q = \prod_F u$.

□

Remark 3.2.2. Note that if $T_2 = I$, the processes of (3.7)-(3.11) are abundant. Also, the parameter assumption $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ in Theorem 3.1 can be weaken with $\limsup_{n\to\infty} \alpha_n < 1$ as readily seen in (3.13) to get $\|x_n - T_1 x_n\| \to 0$.

Taking $\beta_n = 1$ for $n \geq 1$ in Theorem 3.2.1, we have the following modification of Mann's iteration process (1.2) to prove strong convergence for a pair of relatively nonexpansive mappings in a Banach space.

Theorem 3.2.3. *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of X. Let $\Im = \{T_1, T_2 : C \to C\}$ be a pair of relatively nonexpansive mappings with $F \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n J T_2 x_n + (1-\alpha_n) J T_1 x_n), \\ H_n = \{v \in C : \phi(v, y_n) \leq \phi(v, x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Ju \rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} u, \qquad n \geq 1, \end{cases}$$

where J is the normalized duality mapping. If either T_1 or T_2 is uniformly continuous on C, then $x_n \to \prod_F u$, where $\prod_F$ is the generalized projection from X onto F.

Now taking $T_2 = I$, the identity operator of X and $T_1 = T$ in Theorem 3.2.3, by Remark 3.2.2, we have the following result due to Matsushita and Takahashi [26].

Corollary 3.2.4. ([26]) *Let X be a uniformly convex and uniformly smooth Banach space, let C be a nonempty bounded closed convex subset of X and let $T : C \to C$ be a relatively nonexpansive mapping. Assume that $\{\alpha_n\}$ is a sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ generated by the algorithm (1.6) converges in norm to $\prod_{F(T)} u$, where $\prod_{F(T)}$ is the generalized projection from C onto $F(T)$.*

In Hilbert spaces, noticing that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, we see that $\|Tx - Ty\| \le \|x-y\|$ is equivalent to $\phi(Tx,Ty) \le \phi(x,y)$. Also, the demiclosedness principle of a nonexpansive mapping T yields that $\hat{F}(T) = F(T)$. Therefore, every nonexpansive mapping is relatively nonexpansive (for more details, see the proof of Theorem 4.1 in [26]). Now we have the following two variants of Theorem 3.2.1 and 3.2.3 for a pair of nonexpansive mappings in Hilbert spaces.

Theorem 3.2.5. *Let C be a closed convex subset of a Hilbert space H and let $\Im = \{T_1, T_2 : C \to C\}$ be a pair of nonexpansive mappings such that $F \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n)e_n, \\ y_n = \alpha_n T_2 z_n + (1-\alpha_n) T_1 x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|z_n - v\|^2\} \\ Q_n = \{v \in C : \langle x_n - v, x_n - u \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} u, \qquad n \ge 1, \end{cases}$$

where $\{e_n\}$ *is a bounded sequence in* C. *Then the sequence* $\{x_n\}$ *converges in norm to* $P_F u$.

Theorem 3.2.6. *Let* C *be a closed convex subset of a Hilbert space* H *and let* $\Im = \{T_1, T_2 : C \to C\}$ *be a pair of nonexpansive mappings such that* $F \neq \emptyset$. *Assume that* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *such that* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n T_2 x_n + (1-\alpha_n) T_1 x_n, \\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\} \\ Q_n = \{v \in C : \langle x_n - v, x_n - u \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} u, \qquad n \geq 1, \end{cases}$$

Then the sequence $\{x_n\}$ *converges in norm to* $P_F u$.

As recalling Remark 3.2.2 again, taking $T_2 = I$, $T_1 = T$ and the term $e_n = Tx_n$ for $n \geq 1$ in Theorem 3.2.5, and taking $T_2 = I$ and $T_1 = T$ in Theorem 3.2.6, respectively, we obtain the following subsequent results due to Martinez-Yanez and Xu [25] and Nakajo and Takahashi [27], respectively.

Corollary 3.2.7. ([25]) *Let* C *be a nonempty closed convex subset of a Hilbert space* H, *and let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0,1]$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\beta_n \to 1$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (1.5) converges in norm to* $P_{F(T)} u$.

Corollary 3.2.8. ([27]) *Let* C *be a nonempty closed convex subset of a Hilbert space* H, *and let* $T : C \to C$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Assume that* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *such that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (1.4) converges in norm to* $P_{F(T)} u$.

Now we propose another modification of Ishikawa's iteration process (1.3) to have strong convergence for a pair of relatively nonexpansive mappings defined on a Banach space.

Theorem 3.2.9. *Let X be a uniformly convex and uniformly smooth Banach space, and let $\Im = \{T_1, T_2 : X \to X\}$ be a pair of relatively nonexpansive mappings. Assume that T_2 is uniformly continuous and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$ and $\beta_n \to 1$. Define a sequence $\{x_n\}$ by the algorithm:*

$$\begin{cases} x_1 = u \in X \text{ chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n JT_2 z_n + (1-\alpha_n)JT_1 x_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)Je_n), \\ H_n = \{v \in X : \phi(v, y_n) \le \alpha_n \phi(v, x_n) + (1-\alpha_n)\phi(v, z_n)\}, \\ W_n = \{v \in X : \langle x_n - v, Jx_n - Ju\rangle \le 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n} u, \qquad n \ge 1, \end{cases}$$

where J is the normalized duality mapping and $\{e_n\}$ is a bounded sequence in X. Then $\{x_n\}$ converges in norm to $\prod_F u$, where $\prod_F$ is the generalized projection from X onto F.

Proof. Use the following (3.15)-(3.17) to prove $\|x_n - z_n\| \to 0$ of (3.6) in the proof of Theorem 3.2.1. Since $x_{n+1} \in H_n$, we have

$$\phi(x_{n+1}, y_n) \le \alpha_n \phi(x_{n+1}, x_n) + (1-\alpha_n)\phi(x_{n+1}, z_n). \tag{3.15}$$

However, using the convexity of $\|\cdot\|^2$ for the first inequality, and $\beta_n \to 1$,

$\phi(x_{n+1}, x_n) \to 0$ and the boundedness of $\{x_n\}$ and $\{e_n\}$, we get

$$\begin{aligned}\phi(x_{n+1}, z_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, \beta_n Jx_n + (1-\beta_n)Je_n\rangle \\ &\quad + \|\beta_n Jx_n + (1-\beta_n)Je_n\|^2 \\ &\le \|x_{n+1}\|^2 - 2\beta_n\langle x_{n+1}, Jx_n\rangle - 2(1-\beta_n)\langle x_{n+1}, Je_n\rangle \\ &\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|e_n\|^2 \\ &= \beta_n\phi(x_{n+1}, x_n) + (1-\beta_n)\phi(x_{n+1}, e_n) \to 0. \end{aligned} \tag{3.16}$$

Therefore, the right hand of (3.15) converges to 0; hence $\phi(x_{n+1}, y_n) \to 0$. Also, from Proposition 2.3.8, $\phi(x_{n+1}, z_n) \to 0$ implies that $\|x_{n+1} - z_n\| \to 0$, and this, together with (3.4), gives that

$$\|x_n - z_n\| \to 0. \tag{3.17}$$

Now repeating the remaining part of the proof of Theorem 3.2.1, we can prove that $x_n \to \prod_F u$. $\square$

Finally, we shall give examples of relatively nonexpansive self-mappings which are not nonexpansive. This is motivated by the example in the Hilbert space ℓ^2 of Goebel and Kirk [14].

Example 3.2.10. Let B denote the unit ball in the space $X = \ell^p$, where $1 < p < \infty$. Obviously, X is uniformly convex and uniformly smooth. Let $T : B \to B$ be defined by

$$Tx = (0, x_1^2, \lambda_2 x_2, \lambda_3 x_3, \ldots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in B$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \ge 2$ (hence $\prod_{n=2}^{\infty} \lambda_n = \frac{1}{2}$). Then T is Lipschitzian, i.e., $\|Tx - Ty\| \le 2\|x - y\|$ for all $x, y \in B$. Noticing that, for $x = (x_1, x_2, \ldots) \in B$,

$$T^n x = \Big(\overbrace{0, \ldots, 0}^{n}, \prod_{i=2}^{n} \lambda_i\, x_1^2, \prod_{i=2}^{n+1} \lambda_i\, x_2, \prod_{i=3}^{n+2} \lambda_i\, x_3, \ldots\Big)$$

and also for each $n \geq 2$, since $\prod_{i=2}^{n} \lambda_i = \frac{1}{2}\left(1 + \frac{1}{n}\right)$ and

$$\prod_{i=k}^{n+k-1} \lambda_i = \left(1 - \frac{1}{k}\right)\left(\frac{n+k}{n+k-1}\right) \uparrow 1$$

as $k \to \infty$, we have

$$2\prod_{i=2}^{n} \lambda_i = 1 + \frac{1}{n} \geq \prod_{i=k}^{n+k-1} \lambda_i$$

for all $k \geq 2$. Thus we have $\|T^n x - T^n y\| \leq 2\prod_{i=2}^{n} \lambda_i \|x - y\|$ for all $n \geq 2$. Obviously, since $2\prod_{i=2}^{n} \lambda_i \downarrow 1$, T is asymptotically nonexpansive. On the other hand, since $\|Tx - Ty\| = \frac{3}{4} > \frac{1}{2} = \|x - y\|$ for $x = (1, 0, 0, \ldots)$ and $y = (1/2, 0, 0, \ldots)$, T is not nonexpansive. But T is relatively nonexpansive. Indeed, since $\|Tx\| \leq \|x\|$ for $x \in B$ and $F(T) = \{0\}$, where $0 = (0, 0, \ldots) \in B$, we can see that

$$\phi(0, Tx) = \|Tx\|^2 \leq \|x\|^2 = \phi(0, x)$$

for all $x \in B$. Also, from the demiclosedness principle of the asymptotically nonexpansive mapping T (see Theorem 2 of [39]) it follows immediately that $\hat{F}(T) \subset F(T)$. Since the converse inclusion always holds true, it must be $\hat{F}(T) = F(T)$. Therefore, T is relatively nonexpansive.

Next, consider an example in case that $F(T)$ is not singleton set.

Example 3.2.11. Let $X = \ell^p$, where $2 < p < \infty$, and

$$C = \{x = (x_1, x_2, \ldots) \in X : \ 0 \leq x_n \leq 1\}.$$

Then C is a closed convex subset of X. Note that C is not bounded. Let $T : C \to C$ be defined by

$$Tx = (x_1, 0, x_2^2, \lambda_2 x_3, \lambda_3 x_4, \ldots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in C$, where $\lambda_n = 1 - \frac{1}{n^2}$ for $n \geq 2$ as in Example 3.2.10. In a similar way to Example 3.2.10, we see that T is Lipschitzian, asymptotically nonexpansive, but not nonexpansive. Obviously,

$$F(T) = \{p = (p_1, 0, 0, \dots) : 0 \le p_1 \le 1\}$$

and

$$Jx = \frac{1}{\|x\|^{p-2}}(|x_1|^{p-1}\mathrm{sign}\, x_1, |x_2|^{p-1}\mathrm{sign}\, x_2, \dots)$$

for $x = (x_1, x_2, \dots) \in X$. Now we claim that T is relatively nonexpansive. Indeed, since $\|Tx\| \le \|x\|$ for $x \in C$, for $p = (p_1, 0, \dots) \in F(T)$ and $x = (x_1, x_2, \dots) \in C$, we have

$$\begin{aligned} \langle p, JTx\rangle &= p_1 x_1^{p-1}/\|Tx\|^{p-2} \\ &\ge p_1 x_1^{p-1}/\|x\|^{p-2} = \langle p, Jx\rangle, \end{aligned}$$

and so

$$\phi(p, Tx) = \|p\|^2 - 2\langle p, JTx\rangle + \|Tx\|^2 \le \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 = \phi(p, x).$$

Similarly to the argument of Example 3.2.10, we have $\hat{F}(T) = F(T)$. Thus, T is relatively nonexpansive.

Chapter 4

Generalized nonexpansive mappings and strong convergence

In this chapter, motivated by ideas due to Matsushida and Takahashi [26] and Ibaraki and Takahashi [18], we prove some strong convergence theorems of modified Mann type iteration processes for generalized nonexpansive mappings in uniformly convex Banach spaces. Some applications are also added.

4.1 Strong convergence theorems

We begin with the following lemma, which is very important for our argument.

Lemma 4.1.1. *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space X with the Kadec-Klee property, and let K be a nonempty subset of C. Assume that the normalized duality mapping J is weakly sequentially continuous. Let $u \in C$ and $q := R_K u$, where R_K denotes the sunny generalized nonexpansive retraction of C onto K. If $\{x_n\}$ is a bounded sequence in C such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\langle u - x_n, Jy - Jx_n \rangle \leq 0, \quad \forall y \in K. \tag{4.1}$$

Then $x_n \to q\,(= R_K u)$.

Proof. Since $\{x_n\}$ is bounded, $\omega_w(x_n) \neq \emptyset$ by reflexivity of X. Let $v \in \omega_w(x_n)$, that is, $x_{n_k} \rightharpoonup v$. As an equivalent form of (4.1), notice that

$$\|x_n\|^2 \leq \langle u, Jx_n\rangle - \langle u - x_n, Jy\rangle, \quad \forall y \in K. \tag{4.2}$$

After substituting $\{x_n\}$ in (4.2) for $\{x_{n_k}\}$, by using weakly lower semicontinuity of $\|\cdot\|$ and weakly sequential continuity of J, we have

$$\begin{aligned} \|v\|^2 &\leq \liminf_{k\to\infty} \|x_{n_k}\|^2 \leq \limsup_{k\to\infty} \|x_{n_k}\|^2 \\ &\leq \limsup_{k\to\infty} [\langle u, Jx_{n_k}\rangle - \langle u - x_{n_k}, Jy\rangle] \\ &= \langle u, Jv\rangle - \langle u - v, Jy\rangle \end{aligned}$$

for all $y \in K$. Equivalently,

$$\langle u - v, Jy - Jv\rangle \leq 0, \quad \forall y \in K,$$

in particular, since $q \in K$,

$$\langle u - v, Jq - Jv\rangle \leq 0. \tag{4.3}$$

On the other hand, since $q = R_K u$, by Proposition 2.2.8,

$$\langle u - q, Jy - Jq\rangle \leq 0, \quad \forall y \in K;$$

especially, since $v \in K$,

$$\langle u - q, Jv - Jq\rangle \leq 0. \tag{4.4}$$

Now summing both sides of (4.3) and (4.4) yields

$$\langle q - v, Jq - Jv\rangle \leq 0.$$

Since X is strictly convex (hence J is strictly monotone; see [37]), we obtain $v = q$ for all $v \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{q\}$ and $x_n \rightharpoonup q$. After taking $y = q$

in (4.2), applying for weakly lower semicontinuity of $\|\cdot\|$ and weakly sequential continuity of J again gives

$$\begin{aligned}\|q\|^2 &\leq \liminf_{n\to\infty} \|x_n\|^2 \leq \limsup_{n\to\infty} \|x_n\|^2 \\ &\leq \limsup_{n\to\infty} [\langle u, Jx_n\rangle - \langle u - x_n, Jq\rangle] \\ &= \langle u, Jq\rangle - \langle u - q, Jq\rangle = \|q\|^2,\end{aligned}$$

and so $\lim_{n\to\infty} \|x_n\| = \|q\|$. By the Kadec-Klee property of X, we have $x_n \to q$. The proof is complete. $\square$

Now we prove strong convergence for generalized nonexpansive mappings in uniformly convex Banach spaces.

Theorem 4.1.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let a mapping $T : C \to C$ be generalized nonexpansive with $F(T) \neq \emptyset$. Assume that the normalized duality mapping J is weakly sequentially continuous and also that $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ H_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n\rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} u, \qquad n \geq 1. \end{cases}$$

Then $x_n \to R_{F(T)}u$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of C onto $F(T)$.

Proof. First, we show that $F(T) \subset H_n$ for all n. Indeed, for all $p \in F(T)$, using convexity of $\phi(\cdot, p)$ and generalized nonexpansivity of T, we get

$$\begin{aligned} \phi(y_n, p) &= \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, p)) \\ &\leq \alpha_n \phi(x_n, p) + (1-\alpha_n)\phi(Tx_n, p) \\ &\leq \alpha_n \phi(x_n, p) + (1-\alpha_n)\phi(x_n, p) = \phi(x_n, p). \end{aligned}$$

So $p \in H_n$ for all n. Next, we show that

$$F(T) \subset W_n \tag{4.5}$$

for all n. We prove this by induction. For $n = 1$, we have $F(T) \subset C = W_1$. Assume that $F(T) \subset W_k$ for some $k \geq 1$. Since x_{k+1} is the sunny generalized nonexpansive retraction of u onto $H_k \cap W_k$, it follows from Proposition 2.2.8 that

$$\langle u - x_{k+1}, Jz - Jx_{k+1} \rangle \leq 0$$

for all $z \in H_k \cap W_k$. As $F(T) \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F(T)$. This together with the definition of W_{k+1} implies that $F(T) \subset W_{k+1}$. Hence (4.5) holds for all n. So, $\{x_n\}$ is well defined. Obviously, from the construction of W_n, we see that

$$\langle u - x_n, Jz - Jx_n \rangle \leq 0, \quad \forall z \in W_n, \tag{4.6}$$

and, in particular,

$$\langle u - x_n, Jp - Jx_n \rangle \leq 0, \quad \forall p \in F(T) \tag{4.7}$$

because $F(T) \subset W_n$. Putting $q := R_{F(T)}u \in F(T) \subset W_n$, this immediately implies that

$$\begin{aligned} \phi(x_n, q) + \phi(u, x_n) &= \|q\|^2 + \|u\|^2 + 2(\|x_n\|^2 - \langle x_n, Jq \rangle - \langle u, Jx_n \rangle) \\ &= \|q\|^2 + \|u\|^2 + 2\langle u - x_n, Jq - Jx_n \rangle - 2\langle u, Jq \rangle \\ &\leq \|q\|^2 + \|u\|^2 - 2\langle u, Jq \rangle \qquad \text{by (4.7)}. \end{aligned}$$

Since both $\{\phi(x_n, q)\}$ and $\{\phi(u, x_n)\}$ are nonnegative by the property (a) of ϕ in Chapter 2, they are bounded; so is $\{x_n\}$. Since $\phi(Tx_n, q) \leq \phi(x_n, q)$, the sequence $\{Tx_n\}$ is bounded and so is $\{y_n\}$. Now we show that

$$\|x_n - x_{n+1}\| \to 0. \tag{4.8}$$

Indeed, since $x_{n+1} \in W_n$, it follows from (4.6) with $z = x_{n+1}$ that

$$\langle u - x_n, Jx_{n+1} - Jx_n \rangle \leq 0 \tag{4.9}$$

and so $\langle u, Jx_{n+1} - Jx_n \rangle \leq \langle x_n, Jx_{n+1} - Jx_n \rangle$. Then, we have

$$\begin{aligned} \phi(u, x_n) - \phi(u, x_{n+1}) &= 2\langle u, Jx_{n+1} - Jx_n \rangle + \|x_n\|^2 - \|x_{n+1}\|^2 \\ &\leq 2\langle x_n, Jx_{n+1} - Jx_n \rangle + \|x_n\|^2 - \|x_{n+1}\|^2 \\ &= 2\langle x_n, Jx_{n+1} \rangle - \|x_n\|^2 - \|x_{n+1}\|^2 \leq 0, \end{aligned}$$

which shows that $\{\phi(u, x_n)\}$ is nondecreasing and so the $\lim_{n\to\infty} \phi(u, x_n)$ exists. Simultaneously, using the property (b) of ϕ in Chapter 2 and (4.9), we obtain

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, x_n) + \phi(x_n, x_{n+1}) + 2\langle u - x_n, Jx_n - Jx_{n+1} \rangle \\ &\geq \phi(u, x_n) + \phi(x_n, x_{n+1}) \end{aligned} \tag{4.10}$$

and thus

$$0 \leq \phi(x_n, x_{n+1}) \leq \phi(u, x_{n+1}) - \phi(u, x_n) \to 0.$$

Hence, $\phi(x_n, x_{n+1}) \to 0$ and (4.8) is obtained by Proposition 2.3.8.

Now since $x_{n+1} \in H_n$, we have

$$\phi(y_n, x_{n+1}) \leq \phi(x_n, x_{n+1}) \to 0,$$

hence $\phi(y_n, x_{n+1}) \to 0$. Using Proposition 2.3.8 again, we obtain $\|y_n - x_{n+1}\| \to 0$. This, together with (4.8), implies that $\|x_n - y_n\| \to 0$.

Next, we show that $\|x_n - Tx_n\| \to 0$. Noticing that

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - y_n\| + \|y_n - Tx_n\| \\ &= \|x_n - y_n\| + \alpha_n \|x_n - Tx_n\|, \end{aligned}$$

which is equivalent to

$$\|x_n - Tx_n\| \le \frac{1}{1-\alpha_n}\|x_n - y_n\| \to 0 \tag{4.11}$$

because $\limsup_{n\to\infty} \alpha_n < 1$. So, we have $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. Joining with (4.7) and Lemma 4.1.1 (with $K := F(T)$), we conclude that $x_n \to q = R_{F(T)}u$. This completes the proof. □

Recently, Martinez-Yanez and Xu [25] modified the Halpern's iteration method (1.1) to enhance the convergence rate of the algorithm (1.1) in Hilbert spaces. More precisely, they defined a sequence $\{x_n\}$ recursively in a Hilbert space H by

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = t_n u + (1-t_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + t_n(\|u\|^2 + 2\langle x_n - u, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - u\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} u, \qquad n \ge 1. \end{cases} \tag{4.12}$$

Then they proved that if C is a nonempty closed convex subset of H, $T : C \to C$ is a nonexpansive mapping such that $F(T) \neq \emptyset$, and if $\{t_n\} \subset (0,1)$ is such that $t_n \to 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}u$.

Here we propose some modification for the process (4.12), and discuss the problem of strong convergence concerning generalized nonexpansive mappings in uniformly convex Banach spaces.

Theorem 4.1.3. *Let C, X, T and J as be in Theorem 4.1.2. Assume that $\{t_n\}$ is a sequence in $(0,1]$ such that $t_n \to 0$. Define a sequence $\{x_n\}$ in C by the* *algorithm:*

$$\begin{cases} u \in C \text{ chosen arbitrarily}, \\ y_n = t_n u + (1-t_n)Tx_n, \\ H_n = \{z \in C : \phi(y_n, z) \leq t_n \phi(u, z) + (1-t_n)\phi(x_n, z) + t_n \|x_n\|^2\} \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} u. \end{cases}$$

Then $x_n \to R_{F(T)}u$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of C onto $F(T)$.

Proof. The proof is similar to one of Theorem 4.1.2. We sketch the differences briefly. Since

$$\phi(y_n, p) \leq t_n \phi(u, p) + (1-t_n)\phi(x_n, p)$$

for $p \in F(T)$, we have $p \in H_n$ for all n. All the processes of (4.5)-(4.10) are similarly satisfied. Now since $x_{n+1} \in H_n$, $\phi(x_n, x_{n+1}) \to 0$, $t_n \to 0$, and $\{x_n\}$ is bounded, we have

$$\phi(y_n, x_{n+1}) \leq t_n \phi(u, x_{n+1}) + (1-t_n)\phi(x_n, x_{n+1}) + t_n \|x_n\|^2 \to 0.$$

Using Proposition 2.3.8 again, it follows that

$$\|y_n - x_{n+1}\| \to 0. \tag{4.13}$$

On the other hand, by the definition of y_n we have

$$\|y_n - Tx_n\| = t_n \|u - Tx_n\| \to 0.$$

Since $\|x_n - y_n\| \to 0$ in the process of the proof of Theorem 4.1.2, this implies

$$\|x_n - Tx_n\| \leq \|x_n - y_n\| + \|y_n - Tx_n\| \to 0. \tag{4.14}$$

By (4.14), $\omega_w(x_n) \subset \hat{F}(T) = F(T)$. Joining with (4.7) and Lemma 4.1.1 (with $K := F(T)$), we conclude that $x_n \to q = R_{F(T)}u$. $\square$

Remark 4.1.4. Note that all our results remain true with no changes of the proof for non-self mappings $T_i : C \to X$, $1 \le i \le N$.

Finally we give an example of generalized nonexpansive mappings which is not nonexpansive.

Example 4.1.5. *Let X, C and T as in Example 3.2.11. Then recall that $T : C \to C$ is relatively nonexpansive but not nonexpansive. Also, we observe*

$$F(T) = \{p = (p_1, 0, 0, \dots) \,:\, 0 \le p_1 \le 1\}$$

and

$$Jx = \frac{1}{\|x\|^{p-2}}(|x_1|^{p-1} sign\, x_1, |x_2|^{p-1} sign\, x_2, \dots)$$

for all $x = (x_1, x_2, \dots) \in X$. Now we claim that T is generalized nonexpansie. Indeed, for $p = (p_1, 0, \dots) \in F(T)$ and $x = (x_1, x_2, \dots) \in C$, observing that

$$\langle Tx, Jp\rangle = x_1 p_1^{p-1}/\|p\|^{p-2} = x_1 p_1^{p-2} = \langle x, Jp\rangle,$$

and $\|Tx\| \le \|x\|$ for all $x \in C$, we have

$$\begin{aligned} \phi(Tx, p) &= \|Tx\|^2 - 2\langle Tx, Jp\rangle + \|p\|^2 \\ &\le \|x\| - 2\langle x, Jp\rangle + \|p\|^2 = \phi(x, p). \end{aligned}$$

Similarly to the argument of Example 3.2.10, we have $\hat{F}(T) = F(T)$. Hence, T is generalized nonexpansive.

4.2 Some applications

Let X be a reflexive, strictly convex and smooth Banach space and let $A \subset X^* \times X$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in X$, since the set

$$J_\lambda x := \{z \in X : x \in z + \lambda AJz\}$$

consists of exactly one point, the mapping J_λ is well-defined with the domain $D(J_\lambda) = R(I + \lambda AJ)$ and the range $R(J_\lambda) = D(AJ)$ of J_λ, where I is the identity. Such a J_λ is called the *generalized resolvent* of A and is denoted by

$$J_\lambda = (I + \lambda AJ)^{-1}.$$

For more details, see [18]. For some applications of our theorem 4.1.2, we need the following modification of Proposition 4.1 of Ibaraki and Takahashi [18].

Proposition 4.2.1. *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space X and let $A \subset X^* \times X$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. Then $F(J_\lambda|_C) = (AJ)^{-1}0 \cap C$ for each $\lambda > 0$, where $J_\lambda|_C$ means the restriction of J_λ to C. Moreover, if the normalized duality mapping J is weakly sequentially continuous, then $J_\lambda|_C : C \to X$ is generalized nonexpansive.*

Proof. Let $\lambda > 0$. We claim that $J_\lambda|_C$ is generalized nonexpansive. Then it suffices to show that $\hat{F}(J_\lambda|_C) \subset F(J_\lambda|_C)$. Indeed, let $p \in \hat{F}(J_\lambda|_C)$. Then there exists a sequence $\{u_n\}$ in C such that $u_n \rightharpoonup p$ and $u_n - J_\lambda|_C u_n \to 0$. Since $\frac{1}{\lambda}(u_n - J_\lambda|_C u_n) \in AJ J_\lambda|_C u_n$, monotonicity of A gives

$$\langle \frac{1}{\lambda}(u_n - J_\lambda|_C u_n) - \tilde{w},\ JJ_\lambda|_C u_n - Jw \rangle \geq 0 \tag{4.15}$$

for all $w \in X$ and $\tilde{w} \in AJw$. Note that $J : X \to X^*$ is a bijection mapping under our assumptions. Since $u_n \rightharpoonup p$ and $u_n - J_\lambda|_C u_n \to 0$, we get $J_\lambda|_C u_n \rightharpoonup p$ and the weakly sequential continuity of J implies $JJ_\lambda|_C u_n \overset{*}{\rightharpoonup} Jp$. Now letting $n \to \infty$ in (4.15), we have

$$\langle 0 - \tilde{w},\ Jp - Jw \rangle \geq 0$$

for all $w \in X$ and $\tilde{w} \in AJw$. Then it follows from the maximality of A that $Jp \in A^{-1}0$, which is equivalent to $p \in (AJ)^{-1}0$ and so $p \in (AJ)^{-1}0 \cap C = F(J_\lambda|_C)$. $\square$

Now we have the following result from Proposition 4.2.1 and Remark 4.1.4 following Theorem 4.1.2.

Theorem 4.2.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $A \subset X^* \times X$ be a maximal monotone operator with $(AJ)^{-1}0 \cap C \neq \emptyset$. Let J_λ be the generalized resolvent of A for $\lambda > 0$. Assume that the normalized duality mapping J is weakly sequentially continuous and that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) J_\lambda|_C x_n, \\ H_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} u, \qquad n \geq 1. \end{cases}$$

Then $x_n \to R_{(AJ)^{-1}0\cap C}\, u$, where $R_{(AJ)^{-1}0\cap C}$ is the sunny generalized nonexpansive retraction of C onto $(AJ)^{-1}0 \cap C$.

Proof. Notice that

$$x \in (AJ)^{-1}0 \quad \Leftrightarrow \quad Jx \in A^{-1}0.$$

From Proposition 4.2.1, $J_\lambda|_C : C \to X$ is generalized nonexpansive, and $F(J_\lambda|_C) = (AJ)^{-1}0 \cap C$ for each $\lambda > 0$. Therefore our conclusion immediately follows from Theorem 4.1.2 and Remark 4.1.4. □

In Hilbert spaces, recalling that $\phi(x,y) = \|x-y\|^2$ for all $x, y \in H$, we see that $\|Tx - Ty\| \leq \|x - y\|$ is equivalent to $\phi(Tx, Ty) \leq \phi(x, y)$. Also, the demiclosedness principle of a nonexpansive mapping T yields that $\hat{F}(T) = F(T)$. Therefore, every nonexpansive mapping is both relatively nonexpansive and generalized nonexpansive. Also, recall that both generalized projection $\prod_{F(T)}$ and sunny nonexpansive retraction $R_{F(T)}$ coincide with the metric projection $P_{F(T)}$ of X onto $F(T)$ in Hilbert space settings.

First, as an application of Theorem 4.1.2, we have the following result due to Nakajo and Takahashi [27].

Corollary 4.2.3. ([27]) *Let C be a closed convex subset of a Hilbert space H and let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\alpha_n \leq 1-\delta$ for some $\delta \in (0,1]$. Then the sequence $\{x_n\}$ generated by the algorithm (1.4) converges in norm to $P_{F(T)}u$.*

Next, as a consequence of Theorem 4.1.3, we obtain the following corollary due to Martinez-Yanez and Xu [25] in Hilbert spaces.

Corollary 4.2.4. ([25]) *Let H be a real Hilbert space, C a closed convex subset of H and $T: C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{t_n\} \subset (0,1]$ is such that $t_n \to 0$. Then the sequence $\{x_n\}$ generated by (1.5) or (1.5) converges strongly to $P_{F(T)}u$.*

Chapter 5

Strong convergence for a finite family of generalized nonexpansive mappings

In this chapter, motivated by ideas due to Matsushida and Takahashi [26], Ibaraki and Takahashi [18], and Acedo and Xu [1], we prove some strong convergence theorems of modified Mann type iteration processes for a finite family of generalized nonexpansive self mappings in uniformly convex Banach spaces as analogues of the recent results due to Acedo and Xu [1] for strict pseudo-contractions in Hilbert spaces. Some applications are also added.

5.1 Strong convergence theorems

Let C be a nonempty closed convex subset of a real Banach space X. Recall that $T : C \to C$ is called *generalized nonexpansive* if the following conditions (i)-(iii) are fulfilled.

(a) $F(T)$ is nonempty,

(b) $\hat{F}(T) = F(T)$, and

(c) $\phi(Tx, p) \le \phi(x, p)$ for all $x \in C$, $p \in F(T)$.

We begin with following useful lemma for our argument.

Lemma 5.1.1. *Let C be a nonempty closed convex subset of a smooth and strictly convex Banach space X. Given an integer $N \ge 1$, let $\{T_i\}_{i=1}^N$ be a finite family of generalized nonexpansive mappings from C into itself with $F := \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $A : C \to C$ be defined by*

$$Ax = \sum_{i=1}^N \lambda_i T_i x \tag{5.1}$$

for all $x \in C$, where $\{\lambda_i\}$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. Then $A : C \to C$ is generalized nonexpansive, that is,

(i) $F(A) = F$,

(ii) $\widehat{F}(A) = F(A)$, *and*

(iii) $\phi(Ax, p) \le \phi(x, p)$ *for all $x \in C$ and $p \in F$.*

Proof. To prove (i), it suffices to show $\subset$. For this end, let $Ax = x$. Then for $p \in F$, use the convexity of $\phi(\cdot, p)$ and the property (c) for T_i, $1 \le i \le N$, to derive

$$\phi(x, p) = \phi(Ax, p) \le \sum_{i=1}^N \lambda_i \phi(T_i x, p) \le \sum_{i=1}^N \lambda_i \phi(x, p) = \phi(x, p),$$

which shows

$$\sum_{i=1}^N \lambda_i \phi(T_i x, p) = \phi(x, p).$$

Using the definition of the gauge function ϕ and $Ax = x$, this gives

$$\begin{aligned} 0 &= \sum_{i=1}^{N} \lambda_i(\phi(T_i x, p) - \phi(x, p)) \\ &= \sum_{i=1}^{N} \lambda_i(\|T_i x\|^2 - 2\langle T_i x - x, p\rangle - \|x\|^2) \\ &= \sum_{i=1}^{N} \lambda_i \|T_i x\|^2 - 2\langle Ax - x, p\rangle - \|x\|^2 \\ &= \sum_{i=1}^{N} \lambda_i \|T_i x\|^2 - \|x\|^2. \end{aligned}$$

This jointed with $Ax = x$ again yields

$$\begin{aligned} \sum_{i=1}^{N} \lambda_i \phi(T_i x, x) &= \sum_{i=1}^{N} \lambda_i(\|T_i x\|^2 - 2\langle T_i x, Jx\rangle + \|x\|^2) \\ &= \sum_{i=1}^{N} \lambda_i \|T_i x\|^2 - 2\langle Ax, Jx\rangle + \|x\|^2 \\ &= \|x\|^2 - 2\langle x, Jx\rangle + \|x\|^2 = 0. \end{aligned}$$

By the property (a) of the gauge function ϕ and hypothesis, $\phi(T_i x, x) \geq 0$ and $\lambda_i > 0$ for $1 \leq i \leq N$. Therefore we obtain $\phi(T_i x, x) = 0$ for all $1 \leq i \leq N$. Since X is strictly convex, $T_i x = x$ for all $1 \leq i \leq N$. Hence $x \in F$ and so (i) is proven.

Now to prove (ii), we first claim: if $\{x_n\}$ is a bounded sequence such that $\|x_n - Ax_n\| \to 0$, then $\|x_n - T_i x_n\| \to 0$ for all $1 \leq i \leq N$. Indeed, for $p \in F$, we observe

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i \|T_i x_n\|^2 - \|x_n\|^2 - 2\langle Ax_n - x_n, Jp\rangle \\ =\; &\sum_{i=1}^{N} \lambda_i(\|T_i x_n\|^2 - \|x_n\|^2 - 2\langle T_i x_n - x_n, Jp\rangle) \\ =\; &\sum_{i=1}^{N} \lambda_i[\,\phi(T_i x_n, p) - \phi(x_n, p)\,] \leq 0, \end{aligned}$$

which implies

$$\sum_{i=1}^{N} \lambda_i^{(n)} \|T_i x_n\|^2 - \|x_n\|^2 \quad \le \quad 2\langle Ax_n - x_n, Jp\rangle.$$

Since $\|x_n - Ax_n\| \to 0$ and $\{x_n\}$ is bounded by assumption, this gives

$$\begin{aligned}
\sum_{i=1}^{N} \lambda_i \phi(T_i x_n, x_n) &= \sum_{i=1}^{N} \lambda_i \|T_i x_n\|^2 - 2\langle Ax_n, Jx_n\rangle + \|x_n\|^2 \\
&= \sum_{i=1}^{N} \lambda_i \|T_i x_n\|^2 - \|x_n\|^2 - 2\langle Ax_n - x_n, Jx_n\rangle \\
&\le 2(\langle Ax_n - x_n, Jp\rangle - \langle Ax_n - x_n, Jx_n\rangle) \\
&\le 2\|Ax_n - x_n\|(\|p\| + \|x_n\|) \to 0.
\end{aligned}$$

As in the last proof of (i), since all $\lambda_i > 0$ and $\phi(T_i x_n, x_n) \ge 0$, we have $\phi(T_i x_n, x_n) \to 0$ for all $1 \le i \le N$. Since X is uniformly convex, by Proposition 2.3.8 we arrive at the conclusion:

$$\|x_n - T_i x_n\| \to 0, \qquad 1 \le i \le N.$$

To complete the proof of (ii), it suffices to show: $\widehat{F}(A) \subset F$. For this end, let $x \in \widehat{F}(A)$, that is, there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup x$ and $\|x_n - Ax_n\| \to 0$. Then $\|x_n - T_i x_n\| \to 0$ for $1 \le i \le N$. Therefore $x \in \widehat{F}(T_i) = F(T_i)$ for $1 \le i \le N$. Hence, $x \in \cap_{i=1}^{N} F(T_i) = F$.

(iii) is easily obtained from convexity of $\phi(\cdot, p)$ for each fixed $p \in F$. □

As a direct consequence of Theorem 4.1.2 and Lemma 5.1.1, we have strong convergence of the following modified parallel algorithm.

Theorem 5.1.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\{T_i\}_{i=1}^{N}$ be a finite family of generalized nonexpansive self-mappings of C with $F := \cap_{i=1}^{N} F(T_i) \ne \emptyset$. Assume that the normalized duality mapping J is weakly sequentially continuous and also that $\{\alpha_n\}$ is a* *sequence in* $[0,1]$ *such that* $\limsup_{n\to\infty}\alpha_n < 1$*. Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_1 := u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N}\lambda_i T x_n, \\ H_n = \{z \in C : \phi(y_n, z) \le \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n\rangle \le 0\}, \\ x_{n+1} = R_{H_n\cap W_n} u, \end{cases} \tag{5.2}$$

where $\{\lambda_i\}$ *is a finite sequence of positive numbers such that* $\sum_{i=1}^{N}\lambda_i = 1$*. Then* $x_n \to R_F u$*, where* R_F *is the sunny generalized nonexpansive retraction of* C *onto* F*.*

In the algorithm (5.14), the weight $\{\lambda_n\}_{i=1}^{N}$ are constant in the sense that they are independent of n, the number of steps of the iterative process. Below we consider a more general case by allowing the weights $\{\lambda_i\}$ to be step dependent.

Theorem 5.1.3. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* X*. Let* $\{T_i\}_{i=1}^{N}$ *be a finite family of generalized nonexpansive self-mappings of* C *with* $F := \cap_{i=1}^{N}F(T_i) \neq \emptyset$*. Assume that the normalized duality mapping* J *is weakly sequentially continuous and also that* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *such that* $\limsup_{n\to\infty}\alpha_n < 1$*. Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_1 := u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N}\lambda_i^{(n)} T x_n, \\ H_n = \{z \in C : \phi(y_n, z) \le \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n\rangle \le 0\}, \\ x_{n+1} = R_{H_n\cap W_n} u, \end{cases} \tag{5.3}$$

where $\{\lambda_i^{(n)}\}$ *is a finite sequence of positive numbers such that* $\sum_{i=1}^{N}\lambda_i^{(n)} = 1$ *for each* n *and* $\inf_{n\ge 1}\lambda_i^{(n)} > 0$ *for* $1 \le i \le N$*. Then* $x_n \to R_F u$*, where* R_F *is the* *sunny generalized nonexpansive retraction of C onto F.*

Proof. On setting $A_nx = \sum_{i=1}^{N} \lambda_i^{(n)}Tx$ for all $x \in C$, we see that

$$\phi(A_nx, p) \leq \phi(x, p) \tag{5.4}$$

for $x \in C$ and $p \in F$. First we show that $F \subset H_n$ for all n. Indeed, for all $p \in F$, using convexity of $\phi(\cdot, p)$ and (5.4), we have

$$\begin{aligned}
\phi(y_n, p) &= \phi(\alpha_n x_n + (1-\alpha_n)A_nx_n, p)) \\
&\leq \alpha_n\phi(x_n, p) + (1-\alpha_n)\phi(A_nx_n, p) \\
&\leq \alpha_n\phi(x_n, p) + (1-\alpha_n)\phi(x_n, p) = \phi(x_n, p).
\end{aligned}$$

So $p \in H_n$ for all n. Next, we claim that

$$F \subset W_n \tag{5.5}$$

for all $n \geq 0$. We prove this by induction. For $n = 0$, we have $F \subset C = W_0$. Assume that $F \subset W_k$ for some $k \geq 1$. Since x_{k+1} is the sunny generalized nonexpansive retraction of u onto $H_k \cap W_k$, it follows from Proposition 2.2.8 that

$$\langle u - x_{k+1}, Jz - Jx_{k+1}\rangle \leq 0$$

for all $z \in H_k \cap W_k$. As $F \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F$. This together with the definition of W_{k+1} implies that $F \subset W_{k+1}$. Hence (5.5) holds for all $n \geq 0$. So, $\{x_n\}$ is well defined. Obviously, from the construction of W_n, we see that

$$\langle u - x_n, Jz - Jx_n\rangle \leq 0 \tag{5.6}$$

for all $z \in W_n$ and, in particular,

$$\langle u - x_n, Jp - Jx_n\rangle \leq 0 \tag{5.7}$$

for all $p \in F$ because $F \subset W_n$. Putting $q := R_F u \in F \subset W_n$, this immediately implies that

$$\begin{aligned}
\phi(x_n, q) + \phi(u, x_n) &= \|q\|^2 + \|u\|^2 + 2(\|x_n\|^2 - \langle x_n, Jq\rangle - \langle u, Jx_n\rangle) \\
&= \|q\|^2 + \|u\|^2 + 2\langle u - x_n, Jq - Jx_n\rangle - 2\langle u, Jq\rangle \\
&\leq \|q\|^2 + \|u\|^2 - 2\langle u, Jq\rangle \qquad \text{by (5.7).}
\end{aligned}$$

Since both $\{\phi(x_n, q)\}$ and $\{\phi(u, x_n)\}$ are nonnegative by the property (a) of the gauge function ϕ, they are bounded; so is $\{x_n\}$. Since $\phi(A_n x_n, q) \leq \phi(x_n, q)$, the sequence $\{A_n x_n\}$ is bounded and so is $\{y_n\}$. Now we show that

$$\|x_n - x_{n+1}\| \to 0. \tag{5.8}$$

Indeed, since $x_{n+1} \in W_n$, it follows from (5.6) with $z = x_{n+1}$ that

$$\langle u - x_n, Jx_{n+1} - Jx_n\rangle \leq 0 \tag{5.9}$$

and so $\langle u, Jx_{n+1} - Jx_n\rangle \leq \langle x_n, Jx_{n+1} - Jx_n\rangle$. Then, we have

$$\begin{aligned}
\phi(u, x_n) - \phi(u, x_{n+1}) &= 2\langle u, Jx_{n+1} - Jx_n\rangle + \|x_n\|^2 - \|x_{n+1}\|^2 \\
&\leq 2\langle x_n, Jx_{n+1} - Jx_n\rangle + \|x_n\|^2 - \|x_{n+1}\|^2 \\
&= 2\langle x_n, Jx_{n+1}\rangle - \|x_n\|^2 - \|x_{n+1}\|^2 \leq 0.
\end{aligned}$$

Therefore, the sequence $\{\phi(u, x_n)\}$ is nondecreasing and so the $\lim_{n\to\infty} \phi(u, x_n)$ exists. Simultaneously, using the property (b) of the gauge function ϕ and (5.9), we obtain

$$\begin{aligned}
\phi(u, x_{n+1}) &= \phi(u, x_n) + \phi(x_n, x_{n+1}) + 2\langle u - x_n, Jx_n - Jx_{n+1}\rangle \\
&\geq \phi(u, x_n) + \phi(x_n, x_{n+1}) \qquad (5.10)
\end{aligned}$$

and thus

$$0 \leq \phi(x_n, x_{n+1}) \leq \phi(u, x_{n+1}) - \phi(u, x_n) \to 0.$$

Hence, $\phi(x_n, x_{n+1}) \to 0$ and (5.8) is satisfied from Proposition 2.3.8.

Now since $x_{n+1} \in H_n$, we have

$$\phi(y_n, x_{n+1}) \leq \phi(x_n, x_{n+1}) \to 0,$$

hence $\phi(y_n, x_{n+1}) \to 0$. Using Proposition 2.3.8 again, we obtain $\|y_n - x_{n+1}\| \to 0$. This, together with (5.8), implies that $\|x_n - y_n\| \to 0$.

Next, we show that $\|x_n - A_n x_n\| \to 0$. Noticing that

$$\begin{aligned} \|x_n - A_n x_n\| &\leq \|x_n - y_n\| + \|y_n - A_n x_n\| \\ &= \|x_n - y_n\| + \alpha_n \|x_n - A_n x_n\|, \end{aligned}$$

which is equivalent to

$$\|x_n - A_n x_n\| \leq \frac{1}{1-\alpha_n}\|x_n - y_n\| \to 0 \tag{5.11}$$

because $\limsup_{n\to\infty} \alpha_n < 1$.

Next we prove that $\omega_w(x_n) \subset F$. To see this, let $u \in \omega_w(x_n)$, say $x_{n_k} \rightharpoonup u$. Without no loss of generality, we may assume that

$$\lim_{k\to\infty} \lambda_i^{(n_k)} = \lambda_i, \qquad 1 \leq i \leq N. \tag{5.12}$$

It is obvious to see that each $\lambda_i > 0$ and $\sum_{i=1}^N \lambda_i = 1$. We also have

$$Ax = \lim_{k\to\infty} A_{n_k} x$$

for all $x \in C$, where $A = \sum_{i=1}^N \lambda_i T_i$. Note that by Lemma 5.1.1, A is generalized nonexpansive and $F(A) = F$. Using (5.11) and (5.12) gives

$$\begin{aligned} \|x_{n_k} - A x_{n_k}\| &\leq \|x_{n_k} - A_{n_k} x_{n_k}\| + \|A_{n_k} x_{n_k} - A x_{n_k}\| \\ &\leq \|x_{n_k} - A_{n_k} x_{n_k}\| + \sum_{i=1}^N |\lambda_i^{(n_k)} - \lambda_i| \|T_i x_n\| \to 0 \end{aligned}$$

as $k \to \infty$, noticing that $\{T_i x_n\}$ is also bounded for $1 \leq i \leq N$ because $\phi(T_i x_n, p) \leq \phi(x_n, p)$ for $p \in F$. This with (ii) of Lemma 5.1.1 implies

$u \in \widehat{F}(A) = F$. Hence $\omega_w(x_n) \subset F$. Joining with (5.7) and Lemma 4.1.1 (with $K := F$), we conclude that $x_n \to q = R_F u$. $\square$

Lopez Acedo and Xu [1] recently studied the convergence problems for the following cyclic algorithm:

$$\begin{aligned}
&x_1 := u \in C \text{ chosen arbitrarily},\\
&x_2 = \alpha_1 x_1 + (1-\alpha_1)T_1 x_1,\\
&x_3 = \alpha_2 x_2 + (1-\alpha_2)T_2 x_2,\\
&\quad\vdots\\
&x_{N+1} = \alpha_N x_N + (1-\alpha_N)T_N x_N,\\
&x_{N+2} = \alpha_{N+1} x_{N+1} + (1-\alpha_{N+1})T_1 x_{N+1},\\
&\quad\vdots
\end{aligned}$$

where $\{\alpha_n\}$ be a sequence in $[0,1]$. The above cyclic algorithm can be written in a more compact form as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n, \quad n \geq 1, \tag{5.13}$$

where $T_{[k]} = T_{k \bmod N}$ for integer $k \geq 1$. The mod function takes values in the set $\{1, 2, \cdots, N\}$ as

$$T_{[k]} = \begin{cases} T_N, & \text{if } q = 0;\\ T_q, & \text{if } 0 < q < N \end{cases}$$

for $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$.

Then, we similarly have the following analogue for the cyclic algorithm (5.13).

Theorem 5.1.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly continuous and generalized nonexpansive mappings from C into itself with $F := \cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that the normalized duality mapping J is weakly sequentially continuous and also that $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Define* *a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_1 := u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_{[n]} x_n, \\ H_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} u, \end{cases} \tag{5.14}$$

where $\{\lambda_i^{(n)}\}$ *is a finite sequence of positive numbers such that* $\sum_{i=1}^N \lambda_i^{(n)} = 1$ *for each* n *and* $\inf_{n\geq 1} \lambda_i^{(n)} > 0$ *for* $1 \leq i \leq N$*. Then* $x_n \to R_F u$*, where* R_F *is the sunny generalized nonexpansive retraction of* C *onto* F*.*

Proof. First, to claim the following observations (i)-(vi), simply replace A_n with $T_{[n]}$ in the proof of Theorem 5.1.3.

(i) x_n is well defined for all $n \geq 1$.

(ii) $\langle u - x_n, Jp - Jx_n \rangle \leq 0$ for all $p \in F$.

(iii) $\|x_{n+1} - x_n\| \to 0$, furthermore, $\|x_{n+i} - x_n\| \to 0$ for $1 \leq i \leq N$.

(iv) $\|x_n - T_{[n]} x_n\| \to 0$, in particular, $\|x_{n+i} - T_{[n+i]} x_{n+i}\| \to 0$ for $1 \leq i \leq N$.

Since all T_k, $1 \leq k \leq N$, are uniformly continuous, (iii) implies that

$$\|T_k x_{n+i} - T_k x_n\| \to 0, \quad 1 \leq i,\, k \leq N,$$

equivalently,

$$\|T_{[n+i]} x_{n+i} - T_{[n+i]} x_n\| \to 0, \quad 1 \leq i \leq N \tag{5.15}$$

as $n \to \infty$. Use (iii), (iv) and (5.15) to derive the convergence to 0 as

$$\begin{aligned} \|x_n - T_{[n+i]} x_n\| \;\; &\leq \;\; \|x_n - x_{n+i}\| + \|x_{n+i} - T_{[n+i]} x_{n+i}\| \\ &\quad + \|T_{[n+i]} x_{n+i} - T_{[n+i]} x_n\| \to 0. \end{aligned}$$

For simplicity, put $c_i^n := \|x_n - T_i x_n\|$ for $1 \le i \le N$ and $n \ge 1$. After first taking $i = N,\, N-1, \cdots, 1$ in the set $\{\|x_n - T_{[n+i]}x_n\|\}$ and next enumerating for $n \ge 1$ in turn, we have the following enumeration with N-rows.

$$\begin{array}{ccccccccccccc}
c_1^1 & c_2^2 & c_3^3 & \cdots & c_N^N & c_1^{N+1} & c_2^{N+2} & \cdots & c_N^{2n} & c_1^{2N+1} & \cdots & \to & 0 \\
c_N^1 & c_1^2 & c_2^3 & \cdots & c_{N-1}^N & c_N^{N+1} & c_1^{N+2} & \cdots & c_{N-1}^{2n} & c_N^{2N+1} & \cdots & \to & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \to & 0 \\
c_2^1 & c_3^2 & c_4^3 & \cdots & c_1^N & c_2^{N+1} & c_3^{N+2} & \cdots & c_1^{2n} & c_2^{2N+1} & \cdots & \to & 0
\end{array} \tag{5.16}$$

It is not hard to find a sequence $\{c_1^n\}$ positioned at each N-diagonal repeatedly such that $c_1^n = \|x_n - T_1 x_n\| \to 0$. Moving each row downwards once and the last row to the first cyclically, we found the sequence $\{c_2^n\}$ at the same position with $\{c_1^n\}$ such that $c_2^n = \|x_n - T_2 x_n\| \to 0$. Repeating these processes, we have

$$\|x_n - T_i x_n\| \to 0, \quad 1 \le i \le N. \tag{5.17}$$

Now we claim: $\omega_w(x_n) \subset F$. Indeed, assume $u \in \omega_w(x_n)$, say $x_{n_k} \rightharpoonup u$. By (5.17), it follows that $u \in \cap_{j=1}^N \widehat{F}(T_j) = \cap_{j=1}^N F(T_j) = F$. Joining with (ii) and Lemma 4.1.1 (with $K := F$), we conclude that $x_n \to q := R_F u$. $\square$

Remark 5.1.5. (a) Taking $T_i = T$ for $1 \le i \le N$ in Theorem 5.1.3 and 5.1.4, our results then reduce to Theorem 4.1.2.

(b) Note that all our results remain true with no changes of the proof for non-self mappings $T_i : C \to X$, $1 \le i \le N$.

5.2 Further development

Using the well known inequality due to Xu [40], we can easily observe the following.

Lemma 5.2.4. *Let X be a uniformly convex Banach space and let $B_r(0)$ be a closed ball of X with center zero and radius $r > 0$. Then there exists a continuous* *strictly increasing convex function* $g : [0,\infty) \to [0,\infty)$ *with* $g(0) = 0$ *such that*

$$\|\sum_{i=1}^{n} \lambda_i x_i\|^2 \le \sum_{i=1}^{n} \lambda_i \|x_i\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|) \tag{5.18}$$

for all $n \ge 1$ *and some fixed* i, j *with* $i \ne j$, *where all* $x_i \in B_r(0)$ *and* $\lambda_i \in [0,1]$ *with* $\sum_{i=1}^{n} \lambda_i = 1$.

Proof. It suffices to show that (5.18) holds true for $i = 1$ and $j = 2$. For $n = 2$, (5.18) is obviously satisfied; see Theorem 2 in [40] with $p = 2$. By mathematical induction, assume that (5.18) holds true for some $k \ge 2$, that is,

$$\|\sum_{i=1}^{k} \beta_i x_i\|^2 \le \sum_{i=1}^{k} \beta_i \|x_i\|^2 - \beta_1 \beta_2 g(\|x_1 - x_2\|) \tag{5.19}$$

for all $x_i \in B_r(0)$ and $\beta_i \in [0,1]$ with $\sum_{i=1}^{k} \beta_i = 1$. Then we claim that (5.18) holds true for $k+1$. Indeed, for all $x_i \in B_r(0)$ and $\lambda_i \in [0,1]$ with $\sum_{i=1}^{k+1} \lambda_i = 1$,

$$\begin{aligned}
\|\sum_{i=1}^{k+1} \lambda_i x_i\|^2 &= \|(1-\lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i + \lambda_{k+1} x_{k+1}\|^2 \\
&\le (1-\lambda_{k+1}) \|\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} x_i\|^2 + \lambda_{k+1} \|x_{k+1}\|^2.
\end{aligned}$$

On taking $\beta_i = \frac{\lambda_i}{1-\lambda_{k+1}}$, we see that all $\beta_i \in [0,1]$ and $\sum_{i=1}^{k} \beta_i = 1$. Hence this combined with (5.19) implies that

$$\begin{aligned}
\|\sum_{i=1}^{k+1} \lambda_i x_i\|^2 &\le \|\sum_{i=1}^{k+1} \lambda_i x_i\|^2 - (1-\lambda_{k+1}) \beta_1 \beta_2 g(\|x_1 - x_2\|) \\
&\le \|\sum_{i=1}^{k+1} \lambda_i x_i\|^2 - \frac{\lambda_1 \lambda_2}{1-\lambda_{k+1}} g(\|x_1 - x_2\|) \\
&\le \|\sum_{i=1}^{k+1} \lambda_i x_i\|^2 - \lambda_1 \lambda_2 g(\|x_1 - x_2\|),
\end{aligned}$$

which completes the proof. $\square$

Remark 5.2.5. Note that if $n = 3$ the above lemma reduces to Lemma 1.4 due to Cho et al. [11].

Now we similarly have the following variant of Theorem 5.1.4.

Theorem 5.2.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $N \geq 2$ and let $\{T_i\}_{i=1}^N$ be a finite family of generalized nonexpansive self-mappings of C with $F := \cap_{i=1}^N F(T_i) \neq \emptyset$. Assume that the normalized duality mapping J is weakly sequentially continuous. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} x_1 = u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n) \sum_{i=0}^N \beta_n^{(i)} T_i x_n, \\ H_n = \{z \in C : \phi(y_n, z) \leq \phi(x_n, z)\}, \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} u, \qquad n \geq 1, \end{cases}$$

where $T_0 = I$ and J denote the identity mapping of X and the normalized duality mapping, respectively. Assume that all control sequences $\{\alpha_n\}$ and $\{\beta_n^{(i)}\}$ in $[0,1]$ satisfy the followings:

(C1) $\limsup_{n\to\infty} \alpha_n < 1$;

(C2) $\lim_{n\to\infty} \beta_n^{(0)} = 0$ *and* $\sum_{i=0}^N \beta_n^{(i)} = 1$;

(C3) $\liminf_{n\to\infty} \beta_n^{(i)} \beta_n^{(j)} > 0$ *for any* $i, j = 1, 2, \ldots, N$ *with* $i \neq j$.

Then $x_n \to R_F u$, where R_F is the sunny generalized nonexpansive retraction of C onto F.

Proof. Put $z_n := \sum_{i=0}^N \beta_n^{(i)} T_i x_n$ and we first show that $F \subset H_n$ for all n. Indeed, for all $p \in F$, using convexity of $\phi(\cdot, p)$ and generalized nonexpansivity of T_i $(i = 1, 2, \ldots, N)$, we get

$$\begin{aligned}\phi(z_n, p) &= \phi\left(\sum_{i=0}^{N} \beta_n^{(i)} T_i x_n, p\right) \leq \sum_{i=0}^{N} \beta_n^{(i)} \phi\left(T_i x_n, p\right) \\ &\leq \sum_{i=0}^{N} \beta_n^{(i)} \phi(x_n, p) = \phi(x_n, p) \end{aligned} \tag{5.20}$$

and so

$$\phi(y_n, p) \leq \alpha_n \phi(x_n, p) + (1 - \alpha_n)\phi(z_n, p) \leq \phi(x_n, p).$$

Hence $p \in H_n$ for all n. Next we claim that

$$F \subset W_n \tag{5.21}$$

for all n. We prove this by induction. For $n = 1$, we have $F \subset C = W_1$. Assume that $F \subset W_k$ for some $k \geq 2$. Since x_{k+1} is the sunny generalized nonexpansive retraction of u onto $H_k \cap W_k$, it follows from Proposition 2.2.8 that

$$\langle u - x_{k+1}, Jz - Jx_{k+1}\rangle \leq 0$$

for all $z \in H_k \cap W_k$. As $F \subset H_k \cap W_k$, the last inequality holds, in particular, for all $z \in F$. This together with the definition of W_{k+1} implies that $F \subset W_{k+1}$. Hence (5.21) holds for all n. So, $\{x_n\}$ is well defined. Obviously, from the construction of W_n, we see that

$$\langle u - x_n, Jz - Jx_n\rangle \leq 0, \quad \forall z \in W_n \tag{5.22}$$

and, in particular,

$$\langle u - x_n, Jp - Jx_n\rangle \leq 0, \quad \forall p \in F \tag{5.23}$$

because $F \subset W_n$. Putting $q := R_F u \in F \subset W_n$, this immediately implies that

$$\begin{aligned}\phi(x_n, q) + \phi(u, x_n) &= \|q\|^2 + \|u\|^2 + 2(\|x_n\|^2 - \langle x_n, Jq\rangle - \langle u, Jx_n\rangle) \\ &= \|q\|^2 + \|u\|^2 + 2\langle u - x_n, Jq - Jx_n\rangle - 2\langle u, Jq\rangle \\ &\leq \|q\|^2 + \|u\|^2 - 2\langle u, Jq\rangle \qquad \text{by (5.23)}.\end{aligned}$$

Since both $\{\phi(x_n, q)\}$ and $\{\phi(u, x_n)\}$ are nonnegative by the property (a) of ϕ in Chaper 2, they are bounded; so is $\{x_n\}$. Since $\phi(T_k x_n, q) \leq \phi(x_n, q)$ for all $k \geq 1$, the sequence $\{T_k x_n\}$ is bounded for all $k \geq 1$, and so are $\{z_n\}$ and $\{y_n\}$. Now we show that

$$\|x_n - x_{n+1}\| \to 0. \tag{5.24}$$

Indeed, since $x_{n+1} \in W_n$, it follows from (5.22) with $z = x_{n+1}$ that

$$\langle u - x_n, Jx_{n+1} - Jx_n \rangle \leq 0 \tag{5.25}$$

and so $\langle u, Jx_{n+1} - Jx_n \rangle \leq \langle x_n, Jx_{n+1} - Jx_n \rangle$. Then, we have

$$\begin{aligned} \phi(u, x_n) - \phi(u, x_{n+1}) &= 2\langle u, Jx_{n+1} - Jx_n \rangle + \|x_n\|^2 - \|x_{n+1}\|^2 \\ &\leq 2\langle x_n, Jx_{n+1} - Jx_n \rangle + \|x_n\|^2 - \|x_{n+1}\|^2 \\ &= 2\langle x_n, Jx_{n+1} \rangle - \|x_n\|^2 - \|x_{n+1}\|^2 \leq 0, \end{aligned}$$

which shows that $\{\phi(u, x_n)\}$ is nondecreasing and so the $\lim_{n\to\infty} \phi(u, x_n)$ exists. Simultaneously, using the property (b) of ϕ in Chapter 2 and (5.25), we obtain

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, x_n) + \phi(x_n, x_{n+1}) + 2\langle u - x_n, Jx_n - Jx_{n+1} \rangle \\ &\geq \phi(u, x_n) + \phi(x_n, x_{n+1}) \end{aligned}$$

and thus

$$0 \leq \phi(x_n, x_{n+1}) \leq \phi(u, x_{n+1}) - \phi(u, x_n) \to 0.$$

Hence, $\phi(x_n, x_{n+1}) \to 0$ and (5.24) is satisfied from Proposition 2.3.8.

Now since $x_{n+1} \in H_n$, we have

$$\phi(y_n, x_{n+1}) \leq \phi(x_n, x_{n+1}) \to 0,$$

hence $\phi(y_n, x_{n+1}) \to 0$. Using Proposition 2.3.8 again, we obtain $\|y_n - x_{n+1}\| \to 0$. This, together with (5.24), implies that $\|x_n - y_n\| \to 0$. Then, we have

$$\|y_n - x_n\| = (1 - \alpha_n)\|z_n - x_n\|,$$

equivalently,

$$\|z_n - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0 \tag{5.26}$$

as $n \to \infty$ by (C1).

In the processes of the proof of Proposition 3.1.7, notice that

$$\phi(x_n, p) - \phi(z_n, p) \to 0 \tag{5.27}$$

for all $p \in F$. We claim that $\|x_n - T_k x_n\| \to 0$ for all $k = 1, 2, \ldots, N$. Applying for Lemma 5.1.1 and generalized nonexpansivity of T_i $(i = 1, 2, \ldots N)$,

$$\begin{aligned}
\phi(z_n, p) &= \phi\left(\sum_{i=0}^{N} \beta_n^{(i)} T_i x_n, p\right) \\
&= \|\sum_{i=0}^{N} \beta_n^{(i)} T_i x_n\|^2 - 2\langle \sum_{i=0}^{N} \beta_n^{(i)} T_i x_n, Jp\rangle + \|p\|^2 \\
&= \sum_{i=0}^{N} \beta_n^{(i)} \|T_i x_n\|^2 - \beta_n^{(i)}\beta_n^{(j)} g(\|T_i x_n - T_j x_n\|) \\
&\quad -2\sum_{i=0}^{N} \beta_n^{(i)} \langle T_i x_n, Jp\rangle + \|p\|^2 \\
&= \sum_{i=0}^{N} \beta_n^{(i)} \phi(T_i x_n, p) - \beta_n^{(i)}\beta_n^{(j)} g(\|T_i x_n - T_j x_n\|) \\
&\le \phi(x_n, p) - \beta_n^{(i)}\beta_n^{(j)} g(\|T_i x_n - T_j x_n\|)
\end{aligned}$$

for any i, $j = 1, 2, \ldots, N$ with $i \neq j$. This joined with (5.27) gives

$$\beta_n^{(i)}\beta_n^{(j)} g(\|T_i x_n - T_j x_n\|) \le \phi(x_n, p) - \phi(z_n, p) \to 0$$

and in turn (C3) yields

$$\|T_i x_n - T_j x_n\| \to 0 \tag{5.28}$$

for any i, $j = 1, 2, \ldots, N$ with $i \neq j$. Let k be a fixed number with $1 \le k \le N$ and let $p \in F$. Applying for Proposition 3.1.7 again yields

$$\phi(T_i x_n, T_k x_n) \to 0 \tag{5.29}$$

for every $i \neq k$. Since $\phi(\cdot, x)$ is convex on X for any fixed $x \in X$, (5.30) combined with (C2) yields

$$\phi(z_n, T_k x_n) = \phi\left(\sum_{i=0}^{N} \beta_n^{(i)} T_i x_n, T_k x_n\right) \leq \sum_{i=0}^{N} \beta_n^{(i)} \phi(T_i x_n, T_k x_n)$$
$$= \beta_n^{(0)} \phi(x_n, T_k x_n) + \sum_{1 \leq i \neq k \leq N} \beta_n^{(i)} \phi(T_i x_n, T_k x_n) \to 0$$

and Proposition 2.3.8 gives $\|z_n - T_k x_n\| \to 0$. Joined with (5.26), it implies

$$\|x_n - T_k x_n\| \leq \|x_n - z_n\| + \|z_n - T_k x_n\| \to 0$$

for arbitrary fixed k with $1 \leq k \leq N$. So, we have

$$\omega_w(x_n) \subset \cap_{k=1}^{N} \hat{F}(T_k) = \cap_{k=1}^{N} F(T_k) = F.$$

Joining with (5.23) and Lemma 4.1.1 (with $K := F$), we conclude that $x_n \to q$ $= R_F u$. $\square$

Remark 5.2.7. Note that taking $\beta_n^{(0)} = 0$ for all $n \geq 1$ and $T_1 = T_2 = \cdots = T_N$ $(:= T)$, we have $\sum_{i=0}^{N} \beta_n^{(i)} T_i x_n = T x_n$ in Theorem 5.2.6 and therefore Theorem 5.2.6 directly reduces to Theorem 4.1.2.

Here we propose some modification for the process (1.5), and discuss the problem of strong convergence concerning a family of finite generalized nonexpansive mappings in uniformly convex Banach spaces.

Theorem 5.2.8. *Let C, X, $\Im$ and J as be in Theorem 5.2.6. Define a sequence $\{x_n\}$ in C by the algorithm:*

$$\begin{cases} u \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n u + (1-\alpha_n) \sum_{i=0}^{N} \beta_n^{(i)} T_i x_n, \\ H_n = \{z \in C : \phi(y_n, z) \leq \alpha_n \phi(u, z) + (1-\alpha_n)\phi(x_n, z) + \alpha_n \|x_n\|^2\} \\ W_n = \{z \in C : \langle u - x_n, Jz - Jx_n \rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} u, \end{cases}$$

where $T_0 = I$ denotes the identity mapping of X. Assume that control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$ satisfy the followings:

(C1)$'$ $\alpha_n \in (0,1]$ *and* $\lim_{n\to\infty} \alpha_n = 0$*;*

(C2) $\lim_{n\to\infty} \beta_n^{(0)} = 0$ *and* $\sum_{i=0}^{N} \beta_n^{(i)} = 1$*;*

(C3) $\liminf_{n\to\infty} \beta_n^{(i)} \beta_n^{(j)} > 0$ *for any* $i,\, j = 1, 2, \ldots, N$ *with* $i \neq j$*.*

Then $x_n \to R_F u$, where R_F is the sunny generalized nonexpansive retraction of C onto F.

Proof. The proof is similar to one of Theorem 5.2.6. We sketch the differences briefly. Since

$$\phi(y_n, p) \leq \alpha_n \phi(u, p) + (1 - \alpha_n)\phi(x_n, p)$$

for $p \in F$, we have $p \in H_n$ for all n. All the processes of (5.20)-(5.26) are similarly satisfied. Now since $x_{n+1} \in H_n$, $\phi(x_n, x_{n+1}) \to 0$, $\alpha_n \to 0$, and $\{x_n\}$ is bounded, we have

$$\phi(y_n, x_{n+1}) \leq \alpha_n \phi(u, x_{n+1}) + (1 - \alpha_n)\phi(x_n, x_{n+1}) + \alpha_n \|x_n\|^2 \to 0.$$

Using Proposition 2.3.8 again, it follows that

$$\|y_n - x_{n+1}\| \to 0. \tag{5.30}$$

On the other hand, by the definition of y_n and (C1) we have

$$\|y_n - z_n\| = \alpha_n \|u - z_n\| \to 0.$$

Since $\|x_n - y_n\| \to 0$ in the process of the proof of Theorem 5.2.6, we also have $\|x_n - z_n\| \to 0$. Now for completing the proof, mimic from (5.27) to the remaining proof of Theorem 5.2.6. $\square$

References

[1] G. L. Acedo and H. K. Xu, Iterative methods for strict pseudo-contractions in Hilbert spaces, *Nonlinear Anal.,* **67** (2007), 2258–2271.

[2] Ya. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 15–50.

[3] Ya. I. Alber and S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, *PanAmer. Math. J.,* **4** (1994), 39–54.

[4] V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach spaces*, Editura Academiei, Romania, 1986.

[5] R. E. Bruck, Nonexpansive retracts of Banach spaces, *Bull. Amer. Math. Soc.,* **76** (1970), 384–385.

[6] D. Butnariu, S. Reich and A. J. Zaslavski, Asymptotic behavior of relatively nonexpansive operators in Banach spaces, *J. Appl. Anal.,* **7** (2001), 151–174.

[7] D. Butnariu, S. Reich and A. J. Zaslavski, Weak convergence of orbits of nonlinear operators in reflexive Banach spaces, *Numer. Funct. Anal. Optim.,* **24** (2003), 489–508.

[8] C. Byrne, A unified treatment of some iterative algorithms in signal processing and image reconstruction, *Inverse Problems,* **20** (2004), 103–120.

[9] Y. Censor and S. Reich, Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, *Optimization* **37** (1996), 323–339.

[10] C. E. Chidume and S. A. Mutangadura, An example on the Mann iteration method for Lipschitz pseudocontractions, *Proc. Amer. Math. Soc.,* **129** (2001), 2359–2363.

[11] Y. J. Cho, H. Zhou and G. Guo, Weak and strong convergence theorems for three step iterations with errors for asymptotically nonexpansive mappings, *Comput. Math. Appl.,* **47** (2004), 707–717.

[12] I. Cioranescu, Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1990.

[13] A. Genel and J. Lindenstrass, An example concerning fixed points, *Israel J. Math.,* **22** (1975), 81–86.

[14] K. Goebel and W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.,* **35** (1972), 171–174.

[15] K. Goebel and S. Reich, *Uniform Continuity, Hyperbolic Geometry, and Nonexpansie Mappings*, Marcel Dekker, Inc.m New York and Basel, 1984.

[16] B. Halpern, Fixed points of nonexpanding maps, *Bull. Amer. Math. Soc.,* **73** (1967), 957–961.

[17] S. Ishikawa, Fixed points by a new iteration method, *Proc. Amer. Math. Soc.,* **44** (1974), 147–150.

[18] T. Ibaraki and W. Takahashi, A new projection and convergence theorems for the projections in Banach spaces, *J. Approx. Theory.,* (2007), doi: 10.1016/j.jat.2007.04.003.

[19] S. Kamimura and W. Takahashi, Strong convergence of a proximal-type algorithm in a Banach space, *SIAM J. Optim.,* **13** (2003), 938–945.

[20] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations, *Nonlinear Anal.,* **61** (2005), 51–60.

[21] T. H. Kim and H. K. Xu, Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups *Nonlinear Anal.,* **64** (2006), 1140–1152.

[22] T. H. Kim and H. J. Lee, Strong convergence of modified iteration processes for relatively nonexpansive mappings in Banach spaces, *Kyungpook Math. J.,* (to appear).

[23] P. L. Lions, Approximation de points fixes de contractions, *C.R. Acad. Sci. Sèr. A-B Paris,* **284** (1977), 1357–1359.

[24] W. R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.,* **4** (1953), 506–510.

[25] C. Martinez-Yanes and H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, *Nonlinear Anal.*, **64** (2006), 2400–2411.

[26] S. Matsushita and W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, *J. Approx. Theory,* **134** (2005), 257–266.

[27] K. Nakajo and W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.,* **279** (2003), 372–379.

[28] C. I. Podilchuk and R. J. Mammone, Image recovery by convex projections using a least-squares constraint, *J. Opt. Soc. Am. A,* **7** (1990), 517–521.

[29] S. Reich, Asymtotic behavior of contractions in Banach spaces, *J. Math. Anal. Appl.,* **44** (1973) 57–70.

[30] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, *J. Math. Anal. Appl.,* **75** (1980), 287–292.

[31] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.,* **67** (1979), 274–276.

[32] S. Reich, Review of Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems, Kluwer Academic Publishers, Dordrecht, 1990, *Bull. Amer. Math. Soc.,* **26** (1992), 367–370.

[33] S. Reich, A weak convergence theorem for the alternating method with Bregman distance, in:A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313–318.

[34] R. T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.,* **149** (1970), 75–88.

[35] M. I. Sezan and H. Stark, Applications of convex projection theory to image recovery in tomography and related areas, in “Image Recovery Theory and Applications,” (Edited by H. Stark), Academic Press, Orlando, 1987, pp. 415–462.

[36] N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, *Proc. Amer. Math. Soc.,* **125** (1997), 3641–3645.

[37] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[38] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.,* **58** (1992), 486–491.

[39] H. K. Xu, Existence and convergence for fixed points of mappings of asymptotically nonexpansive type, *Nonlinear Anal.,* **16** (1991), 1139–1146.

[40] H. K. Xu, Inequalities in Banach spaces with applications, *Nonlinear Anal.,* **16** (1991), 1127–1138.

[41] H. K. Xu, Iterative algorithms for nonlinear operators, *J. London Math. Soc.,* **66** (2002), 240–256.

[42] D. Youla, Mathematical theory of image restoration by the method of convex projections, in "Image Recovery Theory and Applications," (Edited by H. Stark), Academic Press, Orlando, 1987, pp. 29–77.

[43] D. Youla, On deterministic convergence of iterations of relaxed projection operators, *J. Visual Comm. Image Representation,* **1** (1990), 12–20.

Acknowledgements

I would like to express my sincere appreciation to my supervisor, Professor Tae Hwa Kim for directing this research, for his patience, academical assistance and especially valuable advice, warmhearted encouragement, and concerns during the work of dissertation and graduate course works.

I am very grateful to Professor Jeong Sheok Ume's leadership as a chairman of my doctoral committee and also thankful to my other committee members, Professors, Il Bong Jung, Gue Myung Lee and Nak Eun Cho for their careful reading of the manuscript of this dissertation and for their helpful comments and suggestion.

I also would like to thank faculty members in the Department of Applied Mathematics for their aids and comments through my study.

Especially, I am very grateful to my parent, father Ri Dong Lee and mother Yun Rye Park for sharing all joys and difficulties with me, whose love and support made this work successfully done. Also, I thank to the other members of my family, my sister Si Jung Lee and her husband Su Do Kim, my brother Chang Jin Lee and his wife Sang Hye Bae for their encouragement and support, their sacrifice and love. Without their help and love, I would not have finished this study.