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Strong Convergence of the iteration schemes for two asymptotically strict pseudo-contractions

by

Yeon Kyung Kim

Graduate School of Education
Pukyong National University

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Strong Convergence of the iteration schemes for two asymptotically strict pseudo-contractions
(두 개의 점근적 순 준-축소사상에 대한 반복구조의 강수렴)

Advisor: Prof. Tae Hwa Kim

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Yeon Kyung Kim

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Yeon Kyung Kim

Approved by :

(Chairman) Nak Eun Cho, Ph. D.

(Member) Jin Mun Jeong, Ph. D.

(Member) Tae Hwa Kim, Ph. D.

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두 개의 점근적 순 준-축소사상에 대한 반복구조의 강수렴

김 연 경

부경대학교 교육대학원 수학교육전공

요 약

집합 C가 Hilbert공간 H의 공집합이 아닌 닫힌 볼록 부분집합이라 할 때, 사상 $T: C \to C$가 점근적 순 준-축소사상(asymptotically strict pseudo-contraction)이라 함은 상수 $\kappa \in (0,1)$와 0에 수렴하는 어떤 수열 $\{\gamma_n\}$가 존재하여 모든 $x, y \in C$와 $n \geq 1$에 대하여 다음 부등식을 만족함을 뜻한다.

$$\|T^n x - T^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa\|(I-T^n)x - (I-T^n)y\|^2.$$

특히, $\kappa = 0$일 때 사상 T는 κ-순 준-축소사상(strict pseudo-contraction)이라 한다. 또한 집합 $F(T) = \{x \in C : Tx = x\}$을 T의 부동점들의 집합이다.

$T_i : C \to C\ (i=1,2)$는 두 개의 점근적 κ_i-순 준-축소사상($0 \leq \kappa_i < 1$)이라 하자. T_1과 T_2의 공통부동점들의 집합 $F = F(T_1) \cap F(T_2)$는 C의 닫힌 볼록 부분집합이다. 만약 F가 공집합이 아닌 유계집합이고, 매개변수들의 통제조건으로 다음 두 조건이 만족한다고 가정하자.

(i) 모든 $\alpha_n \in [0,1)$이고 $\beta_n \in [0,1]$, $\beta_n \to 1$이다.

(ii) 적당한 상수 $L > 0$이 존재하여 모든 $x, y \in C$에 대하여 $\|T_1 x - T_2 y\| \leq L\|x-y\|$이다.

본 논문에서는

$$
\begin{cases}
x_0 \in C, \\
z_n = \beta_n x_n + (1-\beta_n) T_1^n x_n, \\
y_n = \alpha_n x_n + (1-\alpha_n) T_2^n z_n, \\
C_n = \{ z \in C : \|v - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)\theta_n \\
\qquad + (1-\alpha_n)(1+\gamma_{n)(\kappa_1} - \beta_n)\|x_n - T_1^n x_n\|^2 \\
\qquad + (1-\alpha_n)[\kappa_2 \|z_n - T_2^n z_n\|^2 - \alpha_n \|x_n - T_2^n z_{n\|}^2] \} \\
Q_n = \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \leq 0 \}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{cases}
$$

위와 같이 반복적으로 정의된 수열 $\{x_n\}$이 T_1과 T_2의 공통부동점에 강수렴(strong convergence)함을 밝혔다. 엄밀히 말하자면, $x_n \rightarrow P_F x_0$이다. 여기서, P_K는 H에서 닫힌 볼록 부분집합 $K(\subset H)$ 위로의 거리사영(metric projection)이다.

1 Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to C$ be a self-mapping of C. We use $Fix(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$. (Throughout this paper, we always assume that $F(T) \neq \emptyset$.)

Iterative methods are often used to solve the fixed point equation $Tx = x$. The most well-known method is perhaps the Picard successive iteration method when T is a contraction. Picard's method generates a sequence $\{x_n\}$ successively as $x_n = Tx_{n-1}$ for $n \geq 1$ with x_0 arbitrary, and this sequence converges in norm to the unique fixed point of T. However, if T is not a contraction (for instance, if T is nonexpansive), then Picard's successive iteration fails, in general, to converge. Instead, Mann's iteration method [14] or Ishikawa's iteration method [6] prevails. First, Mann's method, an averaged process in nature, generates a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.1}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0, 1]$. Ishikawa's averaged process [6] is also defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, \quad n \geq 0, \end{cases} \tag{1.2}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ lie in the interval $[0, 1]$.

Mann's iteration method (1.1) or Ishikawa's iteration method [6] has been proved to be a powerful method for solving nonlinear operator equations involving nonexpansive mappings, asymptotically nonexpansive mappings, and other type of nonlinear mappings; see [1, 2, 5, 8, 9, 12, 15, 17, 18, 19, 20, 23, 24, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35] and the references therein.

Recall that a mapping $T : C \to C$ is said to be a strict pseudo-contraction [1] if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \tag{1.3}$$

for all $x, y \in C$. (If (1.3) holds, we also say that T is a κ-strict pseudo-contraction.)

A 0-strict pseudo-contraction T is nonexpansive; that is, T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$.

Recall also that a mapping $T : C \to C$ is said to be an asymptotically κ-strict pseudo-contraction [22] if, there exists a constant $\kappa \in [0, 1)$ satisfying

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + \kappa\|(I - T^n)x - (I - T^n)y\|^2 \tag{1.4}$$

for all $x, y \in C$ and all integers $n \geq 1$, where $\gamma_n \geq 0$ for all n and such that $\gamma_n \to 0$ as $n \to \infty$; see also [10] or [21]. Note that if $\kappa = 0$, then T is an asymptotically nonexpansive mapping with $k_n := \sqrt{1 + \gamma_n}$, a concept introduced by Geobel and Kirk [4] in 1972. That is, T is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ and such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\| \tag{1.5}$$

for all $x, y \in C$ and all integers $n \geq 1$. Notice also that taking both $\gamma_n = 0$ and $T^n = T$ in (1.4) for all $n \geq 1$ reduces to (1.3).

Our iteration method to find a fixed point of an asymptotically κ-strict pseudo-contraction T is the modified Mann's iteration method studied in [26, 27, 31, 11] which generates a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \geq 0, \tag{1.6}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0,1]$.

It is known that Mann's iteration method (1.1) is in general not strongly convergent [3] for either nonexpansive mappings or strict pseudo-contractions. Similarly, the modified Mann's iteration method (1.6) is in general not strongly convergent for either asymptotically nonexpansive mappings or asymptotically strict pseudo-contractions. So to get strong convergence, one has to modify the iteration method (1.6). In 2003, such an attempt has firstly been proposed by Nakajo and Takahashi [18] for a single nonexpansive mapping T in Hilbert spaces, namely, the fact that if the $(n+1)$th iterate x_{n+1} is defined as the projection of the initial guess x_0 onto the intersection of two closed convex subsets C_n and Q_n which are appropriately constructed from the n-th iterate x_n, such constructed sequence $\{x_n\}$ is strongly convergent.

It is also known that if T is a nonexpansive mapping with a fixed point and if the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by Mann's algorithm (1.1) converges weakly to a fixed point of T. (This is indeed true in a uniformly convex Banach space with a Frechet differentiable norm; see [23]). This result has recently been extended to the class of κ-strict pseudo-contractions T by Marino and Xu [16] as follows.

Theorem MX (see Theorem 4.1 of [16]). *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strict pseudo-contraction for some*

$0 \le \kappa < 1$ *and assume that the fixed point set* $F(T)$ *of* T *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by the following* (CQ) *algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.7}$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ *is chosen so that* $\alpha_n < 1$ *for all* n*. Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$*, where* P_K *denotes the nearest point projection (or metric projection) from* H *onto a closed convex subset* K *of* H*.*

Very recently, Theorem MX was carried over the wider class of asymptotically strict pseudo-contractions as follows.

Theorem KX (see Theorem 4.1 of [10]). *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be an asymptotically* κ*-strict pseudo-contraction for some* $0 \le \kappa < 1$*. Assume that the fixed point set* $F(T)$ *of* T *is nonempty and bounded. Let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by the following* (CQ) *algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - T^n x_n\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.8}$$

where

$$\theta_n = \triangle_n^2(1-\alpha_n)\gamma_n \to 0 \text{ as } n \to \infty, \qquad \triangle_n = \sup\{\|x_n - z\|^2 : z \in F(T)\} < \infty.$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ *is chosen so that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

In this paper, we first consider the following modified Ishikawa type iteration method (1.2) for two asymptotically κ_1, κ_2-strict pseudo-contractions T_1 and T_2, respectively:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_n = \beta_n x_n + (1-\beta_n)T_1^n x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_2^n z_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)\theta_n \\ \qquad +(1-\alpha_n)(1+\gamma_n)(1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2 \\ \qquad +(1-\alpha_n)[\kappa_2\|z_n - T_2^n z_n\|^2 - \alpha_n\|x_n - T_2^n z_n\|^2]\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{1.9}$$

where

$$\theta_n = \gamma_n[1 + (1-\beta_n)(1+\gamma_n)] \cdot \sup\{\|x_n - z\|^2 : z \in F\} \to 0$$

as $n \to \infty$ and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$, and next prove the strong convergence of the sequence $\{x_n\}$ to a common fixed point of T_1 and T_2 under some suitable conditions of parameters and mappings. Also, some corrections and modifications of typing errors in [10] are done, and applications are added.

2 Preliminaries

Let H be a real Hilbert space with the duality product $\langle \cdot, \cdot \rangle$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}.$$

We now need some facts and tools in a real Hilbert space H which are listed as lemmas below (see [17] for necessary proofs of Lemmas 2.2 and 2.4).

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities (which will be used in the various places in the proofs of the results of this paper).*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \quad \forall x, y \in H.$

(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2 \quad \forall t \in [0,1], \forall x, y \in H.$

(iii) *If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z - y\|^2 \quad \forall y \in H.$$

Lemma 2.2. *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\left\{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\right\}$$

is convex (and closed).

Recall that given a closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted $P_K x$ in K from x to K; that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x - y\| \quad \text{for all } y \in K.$$

Lemma 2.3. *Let K be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x - z, y - z \rangle \leq 0 \quad \text{for all } y \in K.$$

Lemma 2.4 *Let K be a closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_K u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \quad \text{for all } n. \tag{2.1}$$

Then $x_n \to q$.

We also need the following lemma (see [30]).

Lemma 2.5 *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 + \gamma_n) a_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

We need the following useful properties of asymptotically strict pseudo-contractions which was proven in Kim and Xu [10].

Proposition 2.6 ([10]). *Assume C is a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction.*

(i) For each $n \geq 1$, T^n satisfies the Lipschitz condition:

$$\|T^n x - T^n y\| \leq L_n(T)\|x - y\| \quad \forall x, y \in C, \tag{2.2}$$

where $L_n(T) = \frac{\kappa + \sqrt{1+\gamma_n(1-\kappa)}}{1-\kappa}$.

(ii) The demiclosedness principle holds for $I - T$ in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$. In particular,

$$x_n \rightharpoonup x \quad \text{and} \quad (I - T)x_n \to 0 \quad \Rightarrow \quad (I - T)x = 0.$$

(iii) The fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well-defined.

3 Strong convergence

In an infinite-dimensional Hilbert space, both Mann's iteration method (1.1) and Ishikawa's iteration method (1.2) has only weak convergence, in general, even for nonexpansive mappings (see the example in [3]). Hence attempts have recently been made to modify (1.1) and (1.2) in order to get strong convergence; see such modifications in [18, 8, 9, 17, 33]) for nonexpansive mappings, in [9] for asymptotically nonexpansive mappings, and in [16, 13] for strict pseudo-contractions. In this section we prove strong convergence of a modification of the modified Ishikawa's iteration method (1.2) for two asymptotically strict pseudo-contractions, thus extending the corresponding result in [9] for asymptotically nonexpansive mappings. (Some related modifications for maximal operators can be found in [29, 7, 15].)

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H and, for each $i \in \{1,2\}$, let $T_i : C \to C$ be an asymptotically κ_i-strict pseudo-contraction for some $0 \le \kappa_i < 1$. Assume that the common fixed point set $F := F(T_1) \cap F(T_2)$ of T_1 and T_2 is nonempty and bounded, and also that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0,1]$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the following (CQ) algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ z_n = \beta_n x_n + (1-\beta_n) T_1^n x_n, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_2^n z_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)\theta_n \\ \qquad +(1-\alpha_n)(1+\gamma_n)(1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2 \\ \qquad +(1-\alpha_n)[\kappa_2 \|z_n - T_2^n z_n\|^2 - \alpha_n \|x_n - T_2^n z_n\|^2]\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3.1}$$

where

$$\theta_n = \gamma_n [1 + (1-\beta_n)(1+\gamma_n)] \cdot \sup\{\|x_n - z\|^2 : z \in F\} \to 0$$

as $n \to \infty$. Assume that the following conditions are satisfied:

(i) *$\alpha_n < 1$ for all $n \ge 1$ and $\lim_{n\to\infty} \beta_n = 1$;*

(ii) *there exists a positive real number L such that, for all distinct $x, y \in C$*

$$\|T_1 x - T_2 y\| \le L \|x - y\|. \tag{3.2}$$

Then $\{x_n\}$ converges strongly to $P_F x_0$.

Proof. First observe that C_n is convex by Lemma 2.2. Next we show that $F \subset C_n$ for all n. Indeed, we have, for all $p \in F$ and n,

$$
\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(T_2^n z_n - p)\|^2 \\
&= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|T_2^n z_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - T_2^n z_n\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)[(1+\gamma_n)\|z_n - p\|^2 + \kappa_2\|z_n - T_2^n z_n\|^2] \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_2^n z_n\|^2 \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|z_n - p\|^2 = \|\beta_n(x_n - p) + (1-\beta_n)(T_1^n x_n - p)\|^2 \\
=\ & \beta_n\|x_n - p\|^2 + (1-\beta_n)\|T_1^n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T_1^n x_n\|^2 \\
\leq\ & \beta_n\|x_n - p\|^2 + (1-\beta_n)[(1+\gamma_n)\|x_n - p\|^2 + \kappa_1\|x_n - T_1^n x_n\|^2] \\
& -\beta_n(1-\beta_n)\|x_n - T_1^n x_n\|^2 \\
=\ & [1 + (1-\beta_n)\gamma_n]\|x_n - p\|^2 + (1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2. \qquad (3.4)
\end{aligned}
$$

Now substituting (3.4) into (3.3) yields

$$
\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big[(1+\gamma_n) \\
&\quad ([1 + (1-\beta_n)\gamma_n]\|x_n - p\|^2 + (1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2) \\
&\quad +\kappa_2\|z_n - T_2^n z_n\|^2\big] - \alpha_n(1-\alpha_n)\|x_n - T_2^n z_n\|^2 \\
&= \|x_n - p\|^2 + (1-\alpha_n)\gamma_n\big[1 + (1-\beta_n)(1+\gamma_n)\big]\|x_n - p\|^2 \\
&\quad +(1-\alpha_n)(1+\gamma_n)(1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2 \\
&\quad +(1-\alpha_n)[\kappa_2\|z_n - T_2^n z_n\|^2 - \alpha_n\|x_n - T_2^n z_n\|^2] \\
&\leq \|x_n - p\|^2 + (1-\alpha_n)\theta_n \\
&\quad +(1-\alpha_n)(1+\gamma_n)(1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2 \\
&\quad +(1-\alpha_n)[\kappa_2\|z_n - T_2^n z_n\|^2 - \alpha_n\|x_n - T_2^n z_n\|^2]
\end{aligned}
$$

and hence $p \in C_n$, which shows $F \subset C_n$ for each $n \geq 0$.

Next we show that

$$F \subset Q_n \quad \text{for all } n \geq 0. \tag{3.5}$$

We prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that $F \subset Q_n$. Since x_{n+1} is the projection of x_0 onto $C_n \cap Q_n$, by Lemma 2.3 we have

$$\langle x_{n+1} - z, x_0 - x_{n+1} \rangle \geq 0 \quad \forall z \in C_n \cap Q_n.$$

As $F \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence (3.5) holds for all $n \geq 0$.

Notice that the definition of Q_n actually implies $x_n = P_{Q_n} x_0$. This together with that fact $F \subset Q_n$ further implies

$$\|x_n - x_0\| \leq \|p - x_0\| \quad \text{for all } p \in F.$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad \text{where } q = P_F x_0. \tag{3.6}$$

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. This together with Lemma 2.1 (i) implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.7}$$

This implies that the sequence $\{\|x_n - x_0\|\}$ is increasing. Since it is also bounded, we get that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Note that since $\{x_n\}$ is bounded, so are

$\{T_i^n x_n\}$, $\{z_n\}$, and $\{T_i^n z_n\}$, $i = 1, 2$. Now it turns out from (3.7) that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.8}$$

Since $z_n = \beta_n x_n + (1 - \beta_n) T_1^n x_n$ and $\beta_n \to 1$, we see that

$$\|x_n - z_n\| = (1 - \beta_n)\|T_1^n x_n - x_n\| \to 0. \tag{3.9}$$

Since T_2 is uniformly Lipschitzian, it easily follows from (3.8) and (3.9) that

$$\|T_2^{n+1} x_n - T_2^{n+1} x_{n+1}\| \to 0 \quad \text{and} \quad \|T_2^n x_n - T_2^n z_n\| \to 0. \tag{3.10}$$

By the fact $x_{n+1} \in C_n$ we get

$$\begin{aligned} \|y_n - x_{n+1}\|^2 \;\leq\; & \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\theta_n \\ & + (1 - \alpha_n)(1 + \gamma_n)(1 - \beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2 \\ & + (1 - \alpha_n)[\kappa_2 \|z_n - T_2^n z_n\|^2 - \alpha_n \|x_n - T_2^n z_n\|^2]. \end{aligned} \tag{3.11}$$

On the other hand, since $y_n = \alpha_n x_n + (1 - \alpha_n) T_2^n z_n$, we have, using (ii) of Lemma 2.1

$$\begin{aligned} \|y_n - x_{n+1}\|^2 &= \|\alpha_n (x_n - x_{n+1}) + (1 - \alpha_n)(T_2^n z_n - x_{n+1})\|^2 \\ &= \alpha_n \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\|T_2^n z_n - x_{n+1}\|^2 \\ &\quad - \alpha_n (1 - \alpha_n)\|x_n - T_2^n z_n\|^2. \end{aligned}$$

Substituting this equality into (3.11) and dividing by $(1 - \alpha_n)$ (note that $\alpha_n < 1$ for all $n \geq 1$), we get

$$\begin{aligned} \|x_{n+1} - T_2^n z_n\|^2 \;\leq\; & \|x_{n+1} - x_n\|^2 + \theta_n + \kappa_2 \|z_n - T_2^n z_n\|^2 \\ & + (1 + \gamma_n)(1 - \beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2. \end{aligned} \tag{3.12}$$

Also, since

$$\begin{aligned}\|x_{n+1} - T_2^n z_n\|^2 &= \|x_{n+1} - x_n\|^2 + \|x_n - T_2^n z_n\|^2 - 2\langle x_n - x_{n+1}, x_n - T_2^n z_n\rangle \\ &= \|x_{n+1} - x_n\|^2 + \|x_n - T_2^n x_n\|^2 + \|T_2^n x_n - T_2^n z_n\|^2 \\ &\quad -2(\langle T_2^n x_n - x_n, T_2^n x_n - T_2^n z_n\rangle + \langle x_n - x_{n+1}, x_n - T_2^n z_n\rangle)\end{aligned}$$

and

$$\begin{aligned}\|z_n - T_2^n z_n\|^2 &= \|z_n - x_n\|^2 + \|x_n - T_2^n x_n\|^2 + \|T_2^n x_n - T_2^n z_n\|^2 \\ &\quad +2(\langle z_n - x_n, x_n - T_2^n z_n\rangle + \langle x_n - T_2^n x_n, T_2^n x_n - T_2^n z_n\rangle)\end{aligned}$$

by the parallelogram law, substituting these two equalities into (3.12) again and doing the simple calculation yield that

$$\begin{aligned}&(1-\kappa_2)\|x_n - T_2^n x_n\|^2 \le (1-\kappa_2)(\|x_n - T_2^n x_n\|^2 + \|T_2^n x_n - T_2^n z_n\|^2) \\ \le\ &\kappa_2\|x_n - z_n\|^2 + 2\kappa_2\big(\|z_n - x_n\|\,\|x_n - T_2^n z_n\| + \|x_n - T_2^n x_n\|\,\|T_2^n x_n - T_2^n z_n\|\big) \\ &+\theta_n + (1+\gamma_n)(1-\beta_n)(\kappa_1 - \beta_n)\|x_n - T_1^n x_n\|^2 \\ &+2\big(\|T_2^n x_n - x_n\|\,\|T_2^n x_n - T_2^n z_n\| + \|x_n - x_{n+1}\|\,\|x_n - T_2^n z_n\|\big).\end{aligned}$$

Using (3.8)-(3.10), $\beta_n \to 1$ and $\theta_n \to 0$, we get

$$\lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0. \tag{3.13}$$

Since

$$\begin{aligned}\|x_n - T_2 x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_2^{n+1} x_{n+1}\| \\ &\quad +\|T_2^{n+1} x_{n+1} - T_2^{n+1} x_n\| + \|T_2^{n+1} x_n - T_2 x_n\| \\ &\le (1 + L_{n+1}(T_2))\|x_n - x_{n+1}\| + \|x_{n+1} - T_2^{n+1} x_{n+1}\| \\ &\quad +L_1(T_2)\|T_2^n x_n - x_n\|,\end{aligned}$$

Using (3.8) and (3.10), this gives

$$\|x_n - T_2 x_n\| \to 0. \tag{3.14}$$

By the condition (ii) and (3.9), we have

$$\begin{aligned} \|x_n - T_1 x_n\| &\le \|x_n - T_2 x_n\| + \|T_2 x_n - T_2 z_n\| + \|T_2 z_n - T_1 x_n\| \\ &\le \|x_n - T_2 z_n\| + [L_1(T) + L]\|z_n - x_n\| \to 0. \end{aligned} \tag{3.15}$$

Proposition 2.6(ii), (3.14) and (3.15) then guarantee that every weak limit point of $\{x_n\}$ is a common fixed point of T_1 and T_2. That is,

$$\omega_w(x_n) \subset F = F(T_1) \cap F(T_2)).$$

This fact, the inequality (3.6) and Lemma 2.4 ensure the strong convergence of $\{x_n\}$ to $q = P_F x_0$. □

4 Applications

Taking $T_1 = T_2 := T$ in Theorem 3.1, we immediately obtain the strong convergence of the following modified Ishikawa's iteration process for asymptotically κ-strict pseudo-contraction.

Theorem 4.1. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction for some $0 \le \kappa < 1$. Assume that the fixed point set $F(T)$ of T is nonempty and bounded, and also that $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences in $[0, 1]$ such that $\alpha_n < 1$ for all $n \ge 1$ and $\lim_{n\to\infty} \beta_n = 1$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the following*

(CQ) *algorithm:*

$$
\begin{cases}
x_0 \in C \text{ chosen arbitrarily,} \\
z_n = \beta_n x_n + (1-\beta_n) T^n x_n, \\
y_n = \alpha_n x_n + (1-\alpha_n) T^n z_n, \\
C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)\theta_n \\
\qquad +(1-\alpha_n)(1+\gamma_n)(1-\beta_n)(\kappa-\beta_n)\|x_n - T^n x_n\|^2 \\
\qquad +(1-\alpha_n)[\kappa\|z_n - T^n z_n\|^2 - \alpha_n \|x_n - T^n z_n\|^2]\}, \\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{cases}
\tag{4.1}
$$

where

$$\theta_n = \gamma_n[1 + (1-\beta_n)(1+\gamma_n)] \cdot \sup\{\|x_n - z\|^2 : z \in F(T)\} \to 0$$

as $n \to \infty$. *Then* $\{x_n\}$ *converges strongly to* $P_F x_0$.

Especially, taking $\beta_n = 1$ in the modified Ishikawa's iteration algorithm (4.1) reduces to the following modified Mann's iteration algorithm (4.3), which was originally due to Kim and Xu [10].

Corollary 4.2 ([10]). *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be an asymptotically* κ*-strict pseudo-contraction for some* $0 \leq \kappa < 1$. *Assume that the fixed point set* $F(T)$ *of* T *is nonempty and bounded.*

Let $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by the following* (CQ) *algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)\theta_n \\ \qquad\qquad +(\kappa - \alpha_n)(1-\alpha_n)\|x_n - T^n x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0,1)$ *and*

$$\theta_n = \gamma_n \cdot \sup\{\|x_n - z\|^2 : z \in F(T)\} \to 0$$

as $n \to \infty$*. Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$*.*

Remark 4.1. Note that there are some typing errors in the statement of Theorem 4.1 in [10], which must be modified as the above Corollary 4.2.

Also, taking $\gamma_n = 1$ and $T^n = T$ in the modified Ishikawa's iteration algorithm (4.1) the result reduces to the corresponding one due to Marino and Xu [16] for strict pseudo-contractions; see Theorem MX.

Corollary 4.3 ([16]). *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be a* κ*-strict pseudo-contraction for some* $0 \le \kappa < 1$*. Assume that the fixed point set* $F(T)$ *of* T *is nonempty. Let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence* *generated by the following* (CQ) *algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ *is chosen such that* $0 \le \alpha_n < 1$*. Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$*.*

Since asymptotically nonexpansive mappings are asymptotically 0-strict pseudo-contractions, we have the following consequence which was originally studied in Kim and Xu [9].

Corollary 4.4 ([9]). *Let* C *be a closed convex subset of a Hilbert space* H *and let* $T : C \to C$ *be an asymptotically nonexpansive mapping. Assume that the fixed point set* $F(T)$ *of* T *is nonempty and bounded, and that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0, 1)$*. Let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence generated by the following* (CQ) *algorithm*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 + (1-\alpha_n)\theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where

$$\theta_n = (k_n^2 - 1) \cdot \sup\{\|x_n - z\|^2 : z \in F(T)\} \to 0$$

as $n \to \infty$*. Then* $\{x_n\}_{n=0}^{\infty}$ *strongly converges to* $P_{Fix(T)}x_0$*.*

Remark 4.2. Note that Theorem 2.2 in [9] can be modified as the above Corollary 4.4.

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