Thesis for the Degree of Doctor of Philosophy

Various Measures of Interval-valued Intuitionistic Fuzzy Sets

by

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Department of Applied Mathematics

The Graduate School

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구간값 직관적 퍼지집합의 다양한 측도에 관한 연구

Advisor : Prof. Jin Han Park

by

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구간값 직관적 퍼지집합의 다양한 측도에 관한 연구

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요약

본 논문에서는 구간값 직관적 퍼지집합에 대한 거리, entropy, 닮음측도, subsethood 측도와 같은 여러 가지 측도를 소개하고 그들의 연관성을 밝혔다.

첫째로, 구간값 퍼지집합을 묘사하는 3가지 변수를 택한 기하적인 해석을 기초로 하여 Burillo와 Bustince가 소개한 거리측도의 보완된 거리측도를 소개하고, Burillo와 Bustince 및 Grzegorzewski가 소개한 거리측도들과의 상호관계를 비교 조사하였으며, 이들을 구간값 직관적 퍼지집합으로 확장하고 구간값 퍼지집합의 경우와 유사한 결과를 구하였다.

둘째로, 구간값 직관적 퍼지집합에 대한 entrory와 닮음측도를 소개하고, 이들의 정의로부터 서로를 유도해 낼 수 있는 몇 가지 정리를 소개하였으며, 실제로 entropy와 닮음측도를 계산할 수 있는 식을 제안하였다.

셋째로, 구간값 직관적 퍼지집합에 대한 subsethood 측도와 cardinality의 개념을 소개하고, subsethood 측도로부터 entropy를 유도할 수 있는 entropy-subsethood 정리를 확장하였다. 또한, 평균가능한 cardinality에 기초하여 퍼지 entropy 정리를 확장하고, 제안된 측도와 연관된 측도사이의 관계를 제공하였다.

Chapter 1

Introduction

Most of problems in real life situation such as economics, engineering, environment, social sciences and medical sciences not always involve crisp data. So we cannot successfully use the traditional methods because of various types of uncertainties presented in those problems. Since Zadeh [61] introduced fuzzy sets in 1965, many new approaches and theories treating imprecision and uncertainty have been proposed. Some of these theories, such as intuitionistic fuzzy set theory pioneered by Atanassov [1, 2] and the generalized theory of uncertainty (GTU) introduced by Zadeh [67] and interval-valued fuzzy set theory introduced by Zadeh [62, 63, 64, 65, 66] and interval-valued intuitionistic fuzzy set theory introduced by Atanassov and Gargov [5], are extensions of fuzzy set theory and have allowed people to deal with uncertainty and information in much broader perspective, and the others try to handle imprecision and uncertainty in different ways. Some authors [5, 17] pointed out that there is strong connection between intuitionistic fuzzy sets and interval-valued fuzzy sets, i.e., intuitionistic fuzzy set theory and interval-valued fuzzy set theory are equipollent generalizations of fuzzy set theory.

The similarity measure, the distance measure, the subsethood measure and the entropy of fuzzy sets are four important topics in fuzzy set theory. The similarity measure indicates the similar degree of two fuzzy sets. Wang [53] first put forward the concept of fuzzy set' similarity measure and gave a computation formula. Since then, similarity measure of fuzzy sets has attracted some researchers' interest and has been investigated further. For example, Li and Dick [32] proposed the similarity measure for fuzzy rulebases based on linguistic gradients to reveal linguistic structure. Mitchell [37] introduced the similarity of type-II fuzzy sets and applied it to the classification problem in pattern recognition represented by natural language. The similarity measure of fuzzy sets is now being extensively applied in many research fields such as fuzzy clustering, image processing, fuzzy reasoning and fuzzy neural network [25, 53].

The subsethood measure of fuzzy sets indicate the degree to which a fuzzy set is contained in another fuzzy set. Subsethood measures are also called inclusion measure. Zadeh [61] first gave the definition of fuzzy set inclusion and pointed out that inclusion was a crisp relation. In other words, a fuzzy set is either included or included in another fuzzy set. After that, several researchers used the axiomatic approach to study the inclusion measure of fuzzy sets. They provided a list of properties ('axioms') that a 'reasonable' inclusion measure of fuzzy sets should satisfy, and proposed some methods to calculate the inclusion measure of fuzzy sets. For example, Sinha and Dougherty [43] introduced an axiomatic definition of the inclusion measure of fuzzy sets, and Young [60] proposed a different axiomatic definition from Sinha and Doughtery's and showed the significance of fuzzy subsethood by demonstrating how it is connected with fuzzy entropy, probability and fuzzy logic. Later, Cornelis et al. [12] revised Sinha and Doughtery's axiom. Kehagias and Konstantinidou [26] introduced the concept of L-fuzzy valued inclusion measure and investigated the relationship among the L-fuzzy valued inclusion measure, the L-fuzzy similarity measure and the L-fuzzy distance.

The entropy of a fuzzy set describes the fuzziness degree of a fuzzy set and was first introduced by Zadeh [61] in 1965. Several scholars have studied it from different points of view. For example, in 1972, De Luca and Termini [13] introduced some axioms which captured people's intuitive comprehension to describe the fuzziness degree of a fuzzy set. Kaufmann [25] proposed a method for measuring the fuzziness degree of a fuzzy set by a metric distance between its membership function and the membership function of its nearest crisp set. Another method proposed by Yager [58] was to view the fuzziness degree of a fuzzy set in terms of a lack of distinction between the fuzzy set and its complement. Based on these concepts and their axiomatic definitions, Zeng and Li [69] investigated the relationship among the inclusion measure, the similarity measure and the entropy of fuzzy sets.

The distance measure is term that describes the difference between fuzzy sets. Distance measure can be considered as a dual concept of similarity measure. Several researchers, such as Yager [58], Kosko [29] and Kaufmann [25] had used distance measure to define fuzzy entropy. Liu [35] extended Yager's formula to give a general relationship among distance measure, entropy and similarity measure. Fan et al. [18, 19] gave some properties of distance measure and some new formulas of fuzzy entropy induced by distance measure and extended Kaufmann's formula to some extent. Recently several researchers, mainly [11, 49, 16, 57, 52, 59], focused on computing the distance between fuzzy numbers. Diamond [16] defined a measure for fuzzy numbers in Euclidean space. Yang [59] modified Diamond's proposed measure with a function that captures more information about vagueness. In [49] Tran and Duckstein introduced distance concept based on the interval-numbers where the fuzzy number has transformed into an interval number on the basis of the α-cut. Cheng [11] has proposed a distance index based on centroid points. Voxman [52] first introduced the concept of fuzzy distance for fuzzy numbers. Chakraborty and Chakraborty [10] proposed new fuzzy distance measure and showed that the proposed method computes a fuzzy distance value with less fuzziness and ambiguity as compared that of Vaxman.

Aimed at these important numerical indexes in the fuzzy set theory, some researchers extended these concepts to the interval-valued fuzzy set theory and intuitionistic fuzzy set theory and investigated their related topics from different points of view. For example, Bulliro and Bustince [6, 9] defined distance measure between intuitionistic fuzzy sets and interval-valued fuzzy sets, such as Hamming distance and Euclidean distance, and gave an axiomatic definition of intuitionistic fuzzy entropy. The intuitionistic fuzzy entropy is magnitude which allow us to measure the degree of intuitionism of an intuitionistic fuzzy set. Szimdt and Kacprzyk [45] gave the three-dimension representation of an intuitionistic fuzzy set and proposed new definitions of distances between intuitionistic fuzzy sets by taking into account the three parameter characterization of intuitionistic fuzzy sets. In [47], they also defined a different entropy of intuitionistic fuzzy sets from Bulliro and Bustince's. This entropy was a result of a geometric interpretation of intuitionistic fuzzy sets, used a ratio of distances [45] between them, and showed that the proposed entropy can be stated as ratio of the intuitionistic fuzzy cardinalities. Li and Cheng [31] proposed similarity measures of intuitionistic fuzzy sets and applied these measures to pattern recognition. Liang and Shi [33], Mitchell [36] and Park et al. [40] pointed out that Li and Cheng's measures are not always effective in some cases and made some modifications, respectively. Zeng and Li [70] introduced the entropy of interval-valued fuzzy set by using a different method and a general definition of the similarity measure of the interval-valued fuzzy sets. In [69], they also investigated the relationship between the similarity measure and the entropy of interval-valued fuzzy sets. Wang and Li [54] studied the integral representation of the interval-valued fuzzy degree and the interval-valued similarity measure. Grzegorzewski [21] proposed a definition of the interval-valued fuzzy sets distance based on the Hausdorff metric. Vlachos and Sergiadis [51] proposed a definition of subsethood of the interval-valued fuzzy sets and discussed its relationship with entropy and cardinality. Bustince [8] studied the indicator of inclusion grade for interval-valued fuzzy sets and applied it to approximate reasoning of interval-valued fuzzy sets. Zeng and Guo [68] introduced an axiomatic definition of the inclusion measure of interval-valued fuzzy sets which is different from Bustince's [8] and investigated the relationships among the similarity measure, the inclusion measure and the entropy of interval-valued fuzzy sets.

In this thesis, we consider some measures such as distance measure, similarity measure, entropy and subsethood measure in interval-valued intuitionistic fuzzy set theory. We briefly summarize the contents of each chapter.

We, in Chapter 2, fristly review some definitions and related results. In Chapter 3, we give a geometrical interpretation of the interval-valued fuzzy set and take into account all three parameters describing the interval-valued fuzzy set. So, based on the geometrical background, we propose new distance measures between interval-valued fuzzy sets and compare these measures with above-mentioned distance measures proposed by Burillo and Bustince [6] and Grzegorzewski [21], respectively. Furthermore, we extend three methods for measuring distances between interval-valued fuzzy sets to interval-valued intuitionistic fuzzy sets.

In Chapter 4, we study the relationship between entropy and similarity measure of interval-valued intuitionistic fuzzy sets, give three theorems that entropy and similarity measure of interval-valued intuitionistic fuzzy sets can be transformed by each other based on their axiomatic definitions and propose some formulas to calculate the entropy and the similarity measure of interval-valued intuitionistic fuzzy sets.

Finally, in Chapter 5, we establish a unified framework between the concepts of subsethood, entropy and cardinality for interval-valued intuitionistic fuzzy sets. Then we review the axioms of subsethood for interval-valued intuitionistic fuzzy sets and propose an alternative axiomatic skeleton, in order for subsethood to reduce to entropy. Based on the axioms, we also prove an interval-valued intuitionistic version of the entropy-subsethood theorem and derive new measures of subsethood and entropy for interval-valued intuitionistic fuzzy sets. Furthermore, the concepts of cardinality and average possible cardinality of interval-valued intuitionistic fuzzy sets is presented. We carry out an algebraic and geometrical analysis, which demonstrates a connection between the above-mentioned cardinality and the least and biggest cardinalities. Finally, based on the average possible cardinality, we extend the fuzzy entropy theorem in the interval-valued intuitionistic fuzzy setting and provide connections between the proposed measure and corresponding measures for interval-valued fuzzy sets and fuzzy sets.

Chapter 2

Preliminaries

Throughout this thesis, X denotes the discourse set, IVIFS(X), IVFS(X) and IFS(X) stand for the set of all interval-valued intuitionistic fuzzy sets, interval-valued fuzzy sets and intuitionistic fuzzy sets on X, respectively. The operation "c" is the complement of interval-valued intuitionistic fuzzy set or interval-valued fuzzy set or intuitionistic fuzzy set on X and $\emptyset$ stands for the empty set.

Let $I = [0, 1]$ and $[I]$ be the set of all closed subintervals of the interval $[0, 1]$. Then, by Zadeh's extension principle [61], we can popularize the operations such as $\vee$, $\wedge$ and c to $[I]$ and thus $([I], \vee, \wedge, c)$ is a complete lattice with a minimal element $\bar{0} = [0, 0]$ and a maximal element $\bar{1} = [1, 1]$. Furthermore, let $\bar{a} = [a^-, a^+]$, $\bar{b} = [b^-, b^+]$, then we have $\bar{a} = \bar{b} \Longleftrightarrow a^- = b^-, a^+ = b^+$, $\bar{a} \leq \bar{b} \Longleftrightarrow a^- \leq b^-, a^+ \leq b^+$, and $\bar{a} < \bar{b} \Longleftrightarrow \bar{a} \leq \bar{b}$ and $\bar{a} \neq \bar{b}$.

We recall the notion of interval-valued fuzzy set or Φ-fuzzy set introduced by Zadeh [63, 62] and Sambuc [41].

Definition 2.0.1 An *interval-valued fuzzy set* (IVFS) A on X is defined as

$$A = \{(x, M_A(x)) : x \in X\},$$

where the function $M_A : X \to [I]$ defines the degree of membership of an element x to A.

For each $A \in \text{IVFS}(X)$, let $A(x) = [M_A^-(x), M_A^+(x)]$, where $M_A^-(x) \leq M_A^+(x)$ for any $x \in X$. Then fuzzy set $M_A^- : X \to I$ and $M_A^+ : X \to I$ are called a lower fuzzy set of A and a upper fuzzy set of A, respectively.

For $A, B \in \text{IVFS}(X)$, the basic operations such as union, intersection and complement are defined as follows: for all $x \in X$,

- $A \cup B(x) = [\max(M_A^-(x), M_B^-(x)), \max(M_A^+(x), M_B^+B(x))]$,
- $A \cap B(x) = [\min(M_A^-(x), M_B^-(x))), \min((M_A^+(x), M_B^+B(x))]$,
- $A \subset B \Longleftrightarrow M_A^-(x) \leq M_B^-(x),\ M_A^+(x) \leq M_B^+(x)$,
- $A = B \Longleftrightarrow A \subset B,\ B \subset A$,
- $A^c(x) = [1 - M_A^+(x), 1 - M_A^-(x)]$.

Note that IVFSs are called grey sets by Deng [14].

Zeng and Li [70] introduced the concepts of entropy and similarity measure of IVFSs as follows:

Definition 2.0.2 [70] A real function $E : \text{IVFS}(X) \to [0, 1]$ is called *entropy* on $\text{IVFS}(X)$ if E satisfies the following properties:

(1) $E(A) = 0 \Longleftrightarrow A$ is a crisp set;

(2) $E(A) = 1 \Longleftrightarrow M_A^-(x) = M_A^+(x)$ for every $x \in X$;

(3) $E(A) \leq E(B)$ if A is less fuzzy than B, i.e., $M_A^-(x) \leq M_B^-(x)$ and $M_A^+(x) \leq M_B^+(x)$ for $M_B^-(x) + M_B^+(x) \leq 1$ or $M_A^-(x) \geq M_B^-(x)$ and $M_A^+(x) \geq M_B^+(x)$ for $M_B^-(x) + M_B^+(x) \geq 1$;

(4) $E(A) = E(A^c)$.

Definition 2.0.3 [70] A real function $S : \text{IVFS}(X) \times \text{IVFS}(X) \to [0, 1]$ is called *similarity measure* of IVFSs if S satisfies the following properties:

(1) $S(A, A^c) = 0$ if A is a crisp set;

(2) $S(A, B) = 1 \Longleftrightarrow A = B$;

(3) $S(A, B) = S(B, A)$;

(4) for all $A, B, C \in \text{IVFS}(X)$, if $A \subset B \subset C$, then $S(A, C) \leq S(A, B)$, $S(A, C) \leq S(B, C)$.

Intuitionistic fuzzy sets constitute a generalization of the notion of a fuzzy set and introduced by Atanassov [1]. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy set give both a degree of membership and a degree of non-membership. In [1, 2] intuitionistic fuzzy sets are defined as follows:

Definition 2.0.4 An *intuitionistic fuzzy set* (IFS) A on X is an object of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

where $\mu_A, \nu_A : X \to I$ denote, respectively, the membership function and non-membership function of A and satisfy $\mu_A(x) + \nu_A(x) \leq 1$ for any $x \in X$.

The operations such as union, intersection and complement are defined as follows: let $A, B \in \mathrm{IFS}(X)$, then

- $A \cup B = \{\langle x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x))\rangle : x \in X\}$,
- $A \cap B = \{\langle x, \min(\mu_A(x), \mu_B(x)), \max(\mu_A(x), \mu_B(x))\rangle : x \in X\}$,
- $A \subset B \Longleftrightarrow \mu_A(x) \leq \mu_B(x),\ \nu_A(x) \geq \nu_B(x),$ for all $x \in X$,
- $A = B \Longleftrightarrow A \subset B,\ B \subset A$,
- $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$.

Bustince and Burillo [7] showed that the notion of vague sets defined by Gau and Buehrer [20] is the same that of IFSs.

Atanassov and Gargov [5] prove that IFSs and IVFSs are equivalent generalizations of fuzzy sets, using the following maps:

(a) the map f assigns to every IVFS $A(= [M_A^-, M_A^+])$ an IFS $B = f(A)$ given by

$$\mu_B(x) = M_A^-(x) \quad \text{and} \quad \nu_B(x) = 1 - M_A^+(x)$$

(b) the map g assigns to every IFS $B(= \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\})$ an IVFS $A = g(B)$ given by

$$A(x) = [\mu_B(x), 1 - \nu_B(x)].$$

De Luca and Termini [13] first axiomatized non-probabilistic entropy. Szmidt and Kacprzyk [47] extended De Luca and Termini axioms for fuzzy set to introduce entropy of IFSs as follows:

Definition 2.0.5 [47] A real function E : IFS$(X) \to [0,1]$ is called *entropy* on IFS(X) if E satisfies the following properties:

(1) $E(A) = 0 \Longleftrightarrow A$ is a crisp set;

(2) $E(A) = 1 \Longleftrightarrow \mu_A(x) = \nu_A(x)$ for every $x \in X$;

(3) $E(A) \leq E(B)$ if A is less fuzzy than B, i.e., $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for $\mu_B(x) \leq \nu_B(x)$ or $\mu_A(x) \geq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for $\mu_B(x) \geq \nu_B(x)$;

(4) $E(A) = E(A^c)$.

Definition 2.0.6 A real function S : IFS$(X) \times$ IFS$(X) \to [0,1]$ is called *similarity measure* of IFSs if S satisfies the following properties:

(1) $S(A, A^c) = 0$ if A is a crisp set;

(2) $S(A, B) = 1 \Longleftrightarrow A = B$;

(3) $S(A, B) = S(B, A)$;

(4) for all $A, B, C \in$ IFS(X), if $A \subset B \subset C$, then $S(A, C) \leq S(A, B)$, $S(A, C) \leq S(B, C)$.

As a generalization of the notion of IFSs, Atanassov and Gargov [5] introduced the notion of interval-valued intuitionistic fuzzy sets in the spirit of IVFSs.

Definition 2.0.7 An *interval-valued intuitionistic fuzzy set* (IVIFS) A on X is defined as

$$A = \{(x, M_A(x), N_A(x)) : x \in X\},$$

where $M_A : X \to [I]$ and $N_A : X \to [I]$ denote, respectively, membership function and non-membership function of A and satisfy $0 \leq M_A^+(x) + N_A^+(x) \leq 1$ for any $x \in X$. For simplicity, we often denote $A = (M_A, N_A)$.

The basic operations are defined as follows: let $A, B \in \text{IVIFS}(X)$, then

- $A \cup B = \{(x, [\max(M_A^-(x), M_B^-(x)), \max(M_A^+(x), M_B^+B(x))],$
  $[\min(N_A^-(x), N_B^-(x)), \min(N_A^+(x), N_B^+B(x))]) : x \in X\},$
- $A \cap B = \{(x, [\min(M_A^-(x), M_B^-(x)), \min(M_A^+(x), M_B^+B(x))],$
  $[\max(N_A^-(x), N_B^-(x)), \max(N_A^+(x), N_B^+B(x))]) : x \in X\},$
- $A \subset B \Longleftrightarrow M_A(x) \leq M_B(x),\ N_A(x) \geq N_B(x),$ for all $x \in X$,
- $A = B \Longleftrightarrow A \subset B,\ B \subset A,$
- $A^c = \{(x, N_A(x), M_A(x)) : x \in X\}.$

Definition 2.0.8 Let $A, B \in \text{IVIFS}(X)$. We call that A *refines* B (i.e., A is *less fuzzy than* B), denoted as $A \leq B$, if the following conditions are satisfied: for every $x \in X$,

(a) If $M_B(x) \geq N_B(x)$, then $M_A(x) \geq M_B(x)$ and $N_A(x) \leq N_B(x)$;

(b) If $M_B(x) \leq N_B(x)$, then $M_A(x) \leq M_B(x)$ and $N_A(x) \geq N_B(x)$.

Theorem 2.0.9 *If A refines B, $A, B \in \text{IVIFS}(X)$, then we have*

$$A \cap A^c \subset B \cap B^c \quad \text{and} \quad A \cup A^c \supset B \cup B^c.$$

Proof We prove only $A \cap A^c \subset B \cap B^c$.

(a) When $M_B(x) \geq N_B(x)$, then for every $x \in X$, we have $N_A(x) \leq N_B(x) \leq M_B(x) \leq M_A(x)$. Hence we get

$$\begin{aligned}
M_{A\cap A^c}(x) &= M_A(x) \cap N_A(x) = N_A(x),\\
M_{B\cap B^c}(x) &= M_B(x) \cap N_B(x) = N_B(x),\\
N_{A\cap A^c}(x) &= N_A(x) \cup M_A(x) = M_A(x),\\
N_{B\cap B^c}(x) &= N_B(x) \cup M_B(x) = M_B(x)
\end{aligned}$$

and thus $M_{A\cap A^c}(x) \leq M_{B\cap B^c}(x)$ and $N_{A\cap A^c}(x) \geq N_{B\cap B^c}(x)$.

(b) When $M_B(x) \leq N_B(x)$, then for every $x \in X$, we have $N_A(x) \geq N_B(x) \geq M_B(x) \geq M_A(x)$. Hence we get

$$\begin{aligned}
M_{A\cap A^c}(x) &= M_A(x) \cap N_A(x) = M_A(x),\\
M_{B\cap B^c}(x) &= M_B(x) \cap N_B(x) = M_B(x),\\
N_{A\cap A^c}(x) &= N_A(x) \cup M_A(x) = N_A(x),\\
N_{B\cap B^c}(x) &= N_B(x) \cup M_B(x) = N_B(x)
\end{aligned}$$

and thus $M_{A\cap A^c}(x) \leq M_{B\cap B^c}(x)$ and $N_{A\cap A^c}(x) \geq N_{B\cap B^c}(x)$.

Therefore, in both cases, we obtain $A \cap A^c \subset B \cap B^c$. □

Corollary 2.0.10 *If A refines B, $A, B \in \mathrm{IVIFS}(X)$, then we have*

$$A \cap A^c \subset B \cap B^c \subset B \cup B^c \subset A \cup A^c.$$

Chapter 3

Distances between Interval-valued Fuzzy Sets and Interval-valued Intuitionistic Fuzzy Sets

In this chapter, we give a geometrical interpretation of the IVFSs. So, based on the geometrical background, we propose new distance measures between IVFSs and compare these measures with distance measures proposed by Burillo and Bustince [6] and Grzegorzewski [21], respectively. Furthermore, we extend three methods for measuring distances between IVFSs to IVIFSs.

3.1 A geometrical interpretation of IVFSs

For each $A \in \mathrm{IVFS}(X)$, we will call the *amplitude* of membership of the element x in the set A the following expression

$$W_A(x) = M_A^+(x) - M_A^-(x) \tag{3.1}$$

evidently $0 \leq W_A(x) \leq 1$ for all $x \in X$.

If A is a fuzzy set, for each $x \in X$, $M_A^-(x) = M_A^+(x)$, i.e. $W_A(x) = 0$. So, we should present the amplitude of membership for handling an IVFS but not a fuzzy set.

To present the geometrical interpretation of the IVFS we consider a universe X and subset Y in the Euclidean plane with Cartesian coordinates.

For a fixed IVFS A, a function f_A from X to Y can be constructed, such that if $x \in X$, then

$$y = f_A(x) \in Y,$$

and the point $y \in Y$ has the coordinates $(M_A^-(x), M_A^+(x))$ for which

$$0 \leq M_A^-(x) \leq M_A^+(x) \leq 1.$$

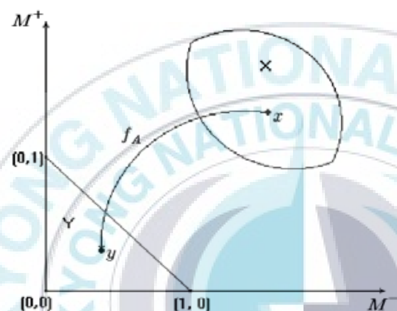

Figure 3.1: A geometrical interpretation of an IVFS

The above geometrical interpretation can be used as an example when considering a situation at beginning of negotiations - cf. Fig. 3.2 (applications of interval-valued fuzzy sets for group decision making, negotiations and other real situations are presented in real life). Each expert i is represented as a point having coordinates $(M^-(i), M^+(i), W(i))$. Expert A: $(1, 1, 0)$ - fully accepts a discussed idea. Expert B: $(0, 0, 0)$ - fully rejects it. The experts placed on the segment AB fixed their points of view (their amplitude margins equal zero for segment AB, so each expert is convinced that the extent $M^-(i)$ is equal to the extent $M^+(i)$; segment AB represents a fuzzy set). Expert C: $(0, 1, 1)$ is absolutely hesitant, i.e. undecided - he or she is the most open to the influence of the arguments presented. A line parallel to a segment AB describes a set of experts with the same level of amplitude. For example, in Fig. 3.2, two sets presented with amplitudes equal to $W(m)$ and $W(n)$, where $0 < W(m) < W(n) < 1$.

In other words, Fig. 3.2 (the triangle ABD) is an orthogonal projection of the real situation (the triangle ABC) presented in Fig. 3.3.

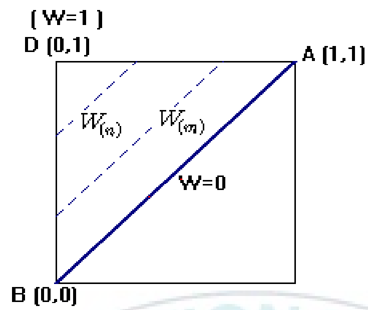

Figure 3.2: An orthogonal projection of the real (three-dimension) representation (triangle ABD in Fig. 3.3) of an IVFS

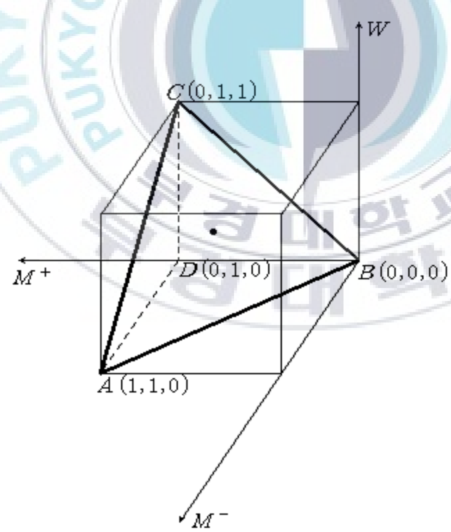

Figure 3.3: A three-dimension representation of an IVFS

An element of an IVFS has three coordinates $(M^-(i), M^+(i), W(i))$ (cf. (3.1)), hence the most natural representation of an IVFS is to draw a cuboid (with edge length equal to 1), and because of (3.1), the triangle ABC (Fig. 3.3) represents an IVFS. As before (Fig. 3.2), the triangle ABD is the orthogonal projection of the triangle ABC.

This representation of an IVFS will be another point of departure for considering the distances and entropy for IVFSs.

3.2 Distances between IVFSs

Burillo and Bustince [6] suggested some methods for measuring distances between IVFSs that are generalizations of the well known Hamming distance, Euclidean distance and their normalized forms as follows: For any two IVFSs $A = \{(x_i, M_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$,

- the Hamming distance $d'(A, B)$:

$$d'(A,B) = \frac{1}{2}\sum_{i=1}^{n}[|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)|], \tag{3.2}$$

- the normalized Hamming distance $l'(A, B)$:

$$l'(A,B) = \frac{1}{2n}\sum_{i=1}^{n}[|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)|], \tag{3.3}$$

- the Euclidean distance $e'(A, B)$:

$$e'(A,B) = \Big\{\frac{1}{2}\sum_{i=1}^{n}[(M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2]\Big\}^{\frac{1}{2}}, \tag{3.4}$$

- the normalized Euclidean distance $q'(A, B)$:

$$q'(A,B) = \Big\{\frac{1}{2n}\sum_{i=1}^{n}[(M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2]\Big\}^{\frac{1}{2}}. \tag{3.5}$$

Now we modify these distances. So, we propose to take into account the three parameter characterization of IVFSs: the lower degree of membership $M_A^-(x)$, the upper degree of membership $M_A^+(x)$ and the amplitude margin $W_A(x)$.

• the Hamming distance $d''(A,B)$:

$$d''(A,B) = \frac{1}{2}\sum_{i=1}^{n}(|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| \\ +|W_A(x_i) - W_B(x_i)|), \tag{3.6}$$

• the normalized Hamming distance $l''(A,B)$:

$$l''(A,B) = \frac{1}{2n}\sum_{i=1}^{n}(|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| \\ +|W_A(x_i) - W_B(x_i)|), \tag{3.7}$$

• the Euclidean distance $e''(A,B)$:

$$e''(A,B) = \Big\{\frac{1}{2}\sum_{i=1}^{n}((M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 \\ +(W_A(x_i) - W_B(x_i))^2)\Big\}^{\frac{1}{2}}, \tag{3.8}$$

• the normalized Euclidean distance $q''(A,B)$:

$$q''(A,B) = \Big\{\frac{1}{2n}\sum_{i=1}^{n}((M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 \\ +(W_A(x_i) - W_B(x_i))^2)\Big\}^{\frac{1}{2}}. \tag{3.9}$$

We claim that our approach ensures that the distances for fuzzy sets and IVFSs can be easily compared since it reflects distances in three dimensional space, while distances due to Burillo and Bustince [6] are orthogonal projections of the real distances. Obviously, these distances satisfy the conditions of the metric.

Example 3.2.1 Let us consider following IVF sets A, B, C, G and E of $X = \{x\}$:

$$A = \{(x,[1,1])\},\ B = \{(x,[0,0])\},\ C = \{(x,[0,1])\}, \\ G = \Big\{(x,[\frac{1}{2},\frac{1}{2}])\Big\},\ E = \Big\{(x,[\frac{1}{4},\frac{3}{4}])\Big\}$$

and their geometrical interpretation is presented in Fig. 3.4. We calculate the Euclidean distances between the above IVFSs using the formula (3.4) (i.e. omitting the third parameter):

$$e'(A,C) = e'(B,C) = \sqrt{\frac{1}{2}},$$
$$e'(A,B) = 1,$$
$$e'(A,G) = e'(B,G) = e'(C,G) = \frac{1}{2},$$
$$e'(E,G) = \frac{1}{4}.$$

These results are not of the sort that one can agree with. As Fig. 3.3, the triangle ABC (Fig. 3.4) has all edges equal to $\sqrt{2}$. So we should obtain $e'(A,C) = e'(B,C) = e'(A,B)$. But Burillo and Bustince's results show only that $e'(A,C) = e'(B,C)$, but $e'(A,C) \neq e'(A,B)$ and $e'(B,C) \neq e'(A,B)$. Also $e'(E,G)$, i.e., it is half of the height of triangle ABC multiplied by $\frac{1}{\sqrt{2}}$, is not the value we want.

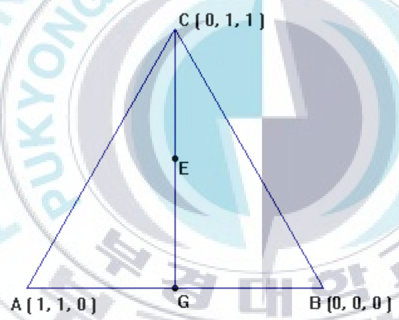

Figure 3.4: An geometrical interpretation of the IVFSs in Example 1

Let us calculate the same Euclidean distances using (3.8). Then we obtain

$$e''(A,C) = e''(B,C) = e''(A,B) = 1,$$
$$e''(A,G) = e''(B,G) = \frac{1}{2},$$
$$e''(E,G) = \frac{\sqrt{3}}{4}$$

$$e''(C,G)=\frac{\sqrt{3}}{2}.$$

Formula (3.8) gives the results we expect, i.e.

$$e''(A,C)=e''(B,C)=e''(A,B)=2e''(A,G)=2e''(B,G)$$

and $e''(E,G)$ is equal to half of the height of the triangle with all edges equal to $\sqrt{2}$ multiplied by $\frac{1}{\sqrt{2}}$, i.e. $\frac{\sqrt{3}}{4}$.

Besides Hamming distance and Euclidean distance, some distances based on the Hausdorff metric are also used in the fuzzy sets theory. For any two subsets U and V of a Banach space X the Hausdorff metric is defined by

$$d_H(U,V)=\max\left\{\sup_{u\in U}\inf_{v\in V}|u-v|,\sup_{v\in V}\inf_{u\in U}|u-v|\right\}. \tag{3.10}$$

If $X=\mathbf{R}$ and $U=[u_1,u_2]$ and $V=[v_1,v_2]$ are intervals, then (3.10) reduces to

$$d_H(U,V)=\max\{|u_1,-v_1|,|u_2,-v_2|\}. \tag{3.11}$$

Grzegorzewski [21] suggested how to measure the distance between IVFSs on arbitrary finite universe of discourse utilizing the Hausdorff metric. For any two IVFss $A=\{(x_i,M_A(x_i)):x_i\in X\}$ and $B=\{(x_i,M_B(x_i)):x_i\in X\}$ of the universe of discourse $X=\{x_1,x_2,\cdots,x_n\}$,

- the Hamming distance $d_h(A,B)$:

$$d_h(A,B)=\sum_{i=1}^{n}\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}, \tag{3.12}$$

- the normalized Hamming distance $l_h(A,B)$:

$$l_h(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}, \tag{3.13}$$

- the Euclidean distance $e_h(A,B)$:

$$e_h(A,B)=\Big\{\sum_{i=1}^{n}\max\{(M_A^-(x_i)-M_B^-(x_i))^2,(M_A^+(x_i)-M_B^+(x_i))^2\}\Big\}^{\frac{1}{2}}, \tag{3.14}$$

• the normalized Euclidean distance $q_h(A,B)$:

$$q_h(A,B)=\Big\{\frac{1}{n}\sum_{i=1}^{n}\max\{(M_A^-(x_i)-M_B^-(x_i))^2,(M_A^+(x_i)-M_B^+(x_i))^2\}\Big\}^{\frac{1}{2}}. \quad (3.15)$$

Now, we give some results on elementary properties of these concepts.

Proposition 3.2.2 *Let $X=\{x_1,x_2,\cdots,x_n\}$ be a finite universe of discourse. Then function $d_h,l_h,e_h,q_h:\mathrm{IVF}(X)\to\mathbf{R}^+\cup\{0\}$ given by (3.12)–(3.15), respectively, are metrics.*

Proof We give only the proof for d_h.

Let $A=\{(x_i,[M_A^-(x_i),M_A^+(x_i)]):x_i\in X\}$, $B=\{(x_i,[M_B^-(x_i),M_B^+(x_i)]):x_i\in X\}$ and $C=\{(x_i,[M_C^-(x_i),M_C^+(x_i)]):x_i\in X\}$ be IVFSs of X.

(a) $d_h(A,B)$ given by (3.12) is positive definite, i.e. $d_h(A,B)\geq 0$ because of the absolute value properties.

(b) If $A=B$ then $M_A^-(x_i)=M_B^-(x_i)$ and $M_A^+(x_i)=M_B^+(x_i)$ for each $x_i\in X$ and hence $d_h(A,B)=0$. Conversely, if $d_h(A,B)=0$ then for each $x_i\in X$ we have $\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}=0$. Then both $M_A^-(x_i)-M_B^-(x_i)=0$ and $M_A^+(x_i)-M_B^+(x_i)=0$ and hence $A=B$.

(c) The symmetry property $d_h(A,B)=d_h(B,A)$ holds because $|M_A^-(x_i)-M_B^-(x_i)|=|M_B^-(x_i)-M_A^-(x_i)|$ and $|M_A^+(x_i)-M_B^+(x_i)|=|M_B^+(x_i)-M_A^+(x_i)|$ for each $x_i\in X$.

(d) Since for any nonnegative numbers a_1,a_2,a_3,b_1,b_2,b_3 such that $a_1+a_2\geq a_3$ and $b_1+b_2\geq b_3$ we have $\max\{a_1,b_1\}+\max\{a_2,b_2\}\geq\max\{a_3,b_3\}$, we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}\\&\qquad+\sum_{i=1}^{n}\max\{|M_B^-(x_i)-M_C^-(x_i)|,|M_B^+(x_i)-M_C^+(x_i)|\}\\&\quad\geq\sum_{i=1}^{n}\max\{|M_A^-(x_i)-M_C^-(x_i)|,|M_A^+(x_i)-M_C^+(x_i)|\}.\end{aligned}$$

Thus $d_h(A,B)+d_h(B,C)\geq d_h(A,C)$, i.e. the triangle inequality holds. □

Proposition 3.2.3 *For any two* IVFSs $A = \{(x_i, M_A(x_i)) : x_i \in X\}$ *and* $B = \{(x_i, M_B(x_i)) : x_i \in X\}$ *of the universe of discourse* $X = \{x_1, x_2, \cdots, x_n\}$*, the following inequalities hold:*

$$d_h(A, B) \le n, \tag{3.16}$$

$$l_h(A, B) \le 1, \tag{3.17}$$

$$e_h(A, B) \le \sqrt{n}, \tag{3.18}$$

$$q_h(A, B) \le 1. \tag{3.19}$$

Proof Since $|M_A^-(x_i) - M_B^-(x_i)| \le 1$ and $|M_A^+(x_i) - M_B^+(x_i)| \le 1$ for each $x_i \in X$, we have $d_h(A, B) \le \sum_{i=1}^n 1 = n$, $l_h(A, B) \le \frac{1}{n}\sum_{i=1}^n 1 = 1$, $e_h(A, B) \le \sqrt{\sum_{i=1}^n 1} = \sqrt{n}$ and $q_h(A, B) \le \sqrt{\frac{1}{n}\sum_{i=1}^n 1} = 1$. □

Proposition 3.2.4 *For any two* IVFSs $A = \{(x_i, M_A(x_i)) : x_i \in X\}$ *and* $B = \{(x_i, M_B(x_i)) : x_i \in X\}$ *of the universe of discourse* $X = \{x_1, x_2, \cdots, x_n\}$*, the following inequalities hold:*

$$d'(A, B) \le d_h(A, B) \le d''(A, B), \tag{3.20}$$

$$l'(A, B) \le l_h(A, B) \le l''(A, B), \tag{3.21}$$

$$e'(A, B) \le e_h(A, B) \le e''(A, B), \tag{3.22}$$

$$q'(A, B) \le q_h(A, B) \le q''(A, B). \tag{3.23}$$

Proof We give only the proof for e_h and the other cases are left.

Since $\frac{1}{2}(a+b) \le \max\{a, b\}$ for any two numbers a and b, we have $e'(A, B) \le e_h(A, B)$. Moreover, since $W_A(x_i) = M_A^+(x_i) - M_A^-(x_i)$ and $W_B(x_i) = M_B^+(x_i) - M_B^-(x_i)$ for each $x_i \in X$, then

$$\begin{aligned}
&\frac{1}{2}\left((M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 + (W_A(x_i) - W_B(x_i))^2\right) \\
&\quad = \frac{1}{2}\left((M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2\right. \\
&\qquad\qquad \left.+ (M_B^-(x_i) - M_A^-(x_i) + M_A^+(x_i) - M_B^+(x_i))^2\right) \\
&\quad = \frac{1}{2}\left((M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 + (M_B^-(x_i) - M_A^-(x_i))^2\right.
\end{aligned}$$

$$+2(M_B^-(x_i) - M_A^-(x_i))(M_A^+(x_i) - M_B^+(x_i)) + (M_A^+(x_i) - M_B^+(x_i))^2\Big)$$
$$= (M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2$$
$$+(M_B^-(x_i) - M_A^-(x_i))(M_A^+(x_i) - M_B^+(x_i))$$
$$\geq \max\{(M_A^-(x_i) - M_B^-(x_i))^2, (M_A^+(x_i) - M_B^+(x_i))^2\}$$

for each $x_i \in X$. Hence we have $e''(A, B) \geq e_h(A, B)$. □

Example 3.2.5 Let us consider the IVF sets A, B, C, E and G on $X = \{x\}$ given in Example 1. Let us calculate the same Euclidean distances using (3.14). Then we get

$$e_h(A, C) = e_h(B, C) = e_h(A, B) = 1,$$
$$e_h(A, G) = e_h(B, G) = e_h(C, G) = \frac{1}{2},$$
$$e_h(E, G) = e_h(C, E) = \frac{1}{4},$$
$$e_h(A, E) = e_h(B, E) = \frac{3}{4}.$$

Formula (3.14) also gives the results we expect as formula (3.8), i.e.

$$e''(A, C) = e''(B, C) = e''(A, B) = 2e''(A, G) = 2e''(B, G)$$

and $e''(E, G)$ is equal to half of the height of the triangle with all edges equal to $\sqrt{2}$ multiplied by $\frac{1}{\sqrt{6}}$, i.e. $\frac{1}{4}$.

3.3 Distances between IVIFSs

Even though we can represent a fuzzy set in an intuitionistic-type representation, we can not always represent any IVFS in interval-valued intuitionistic-type representation. For example, let A be an IVFS on $X = \{x\}$ such that $M_A = [\frac{1}{4}, \frac{1}{2}]$. Then $(M_A, \bar{M}_A) = ([\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}])$ is not IVIFS because $M_A^+ + \bar{M}_{AU} = \frac{1}{2} + \frac{3}{4} \not\leq 1$. However, if an IVFS A satisfy the condition $M_A^+ + \bar{M}_A^+ \leq 1$, i.e. $M_A^+ + 1 - M_A^- \leq 1$, then the IVFS A can represent interval-valued intuitionistic-type representation $(M_A, \bar{M}_A)$.

We extend the Burillo and Bustince's distances to IVIFSs as (3.2)-(3.5). For any two IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ and $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ of the universe of discourse $X = \{x_1, x_2, \cdots, x_n\}$,

- the Hamming distance $d_1'(A, B)$:

$$d_1'(A,B) = \frac{1}{4}\sum_{i=1}^{n}[|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| + |N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|], \quad (3.24)$$

- the normalized Hamming distance $l_1'(A, B)$:

$$l_1'(A,B) = \frac{1}{4n}\sum_{i=1}^{n}[|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| + |N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|], \quad (3.25)$$

- the Euclidean distance $e_1'(A, B)$:

$$e_1'(A,B) = \Big\{\frac{1}{4}\sum_{i=1}^{n}[(M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 + (N_A^-(x_i) - N_B^-(x_i))^2 + (N_A^+(x_i) - N_B^+(x_i))^2]\Big\}^{\frac{1}{2}}, \quad (3.26)$$

- the normalized Euclidean distance $q_1'(A, B)$:

$$q_1'(A,B) = \Big\{\frac{1}{4n}\sum_{i=1}^{n}[(M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 + (N_A^-(x_i) - N_B^-(x_i))^2 + (N_A^+(x_i) - N_B^+(x_i))^2]\Big\}^{\frac{1}{2}}. \quad (3.27)$$

Now, we consider the amplitude margin to modify these distances as (3.6)-(3.9).

- the Hamming distance $d_1''(A, B)$:

$$d_1''(A,B) = \frac{1}{4}\sum_{i=1}^{n}[|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| + |N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)| + |W_{M_A}(x_i) - W_{M_B}(x_i)| + |W_{N_A}(x_i) - W_{N_B}(x_i)|], \quad (3.28)$$

• the normalized Hamming distance $l_1''(A,B)$:

$$l_1''(A,B)=\frac{1}{4n}\sum_{i=1}^{n}[|M_A^-(x_i)-M_B^-(x_i)|+|M_A^+(x_i)-M_B^+(x_i)|$$
$$+|N_A^-(x_i)-N_B^-(x_i)|+|N_A^+(x_i)-N_B^+(x_i)|$$
$$+|W_{M_A}(x_i)-W_{M_B}(x_i)|+|W_{N_A}(x_i)-W_{N_B}(x_i)|], \quad (3.29)$$

• the Euclidean distance $e_1''(A,B)$:

$$e_1''(A,B)=\Big\{\frac{1}{4}\sum_{i=1}^{n}((M_A^-(x_i)-M_B^-(x_i))^2+(M_A^+(x_i)-M_B^+(x_i))^2$$
$$+(N_A^-(x_i)-N_B^-(x_i))^2+(N_A^+(x_i)-N_B^+(x_i))^2$$
$$+(W_{M_A}(x_i)-W_{M_B}(x_i))^2+(W_{N_A}(x_i)-W_{N_B}(x_i))^2\Big\}^{\frac{1}{2}}, \quad (3.30)$$

• the normalized Euclidean distance $q_1''(A,B)$:

$$q_1''(A,B)=\Big\{\frac{1}{4n}\sum_{i=1}^{n}((M_A^-(x_i)-M_B^-(x_i))^2+(M_A^+(x_i)-M_B^+(x_i))^2$$
$$+(N_A^-(x_i)-N_B^-(x_i))^2+(N_A^+(x_i)-N_B^+(x_i))^2$$
$$+(W_{M_A}(x_i)-W_{M_B}(x_i))^2+(W_{N_A}(x_i)-W_{N_B}(x_i))^2\Big\}^{\frac{1}{2}}. \quad (3.31)$$

Clearly these distances satisfy the conditions of the metric (cf. [24]). Finally, we extend the Grzegorzewski's distances to IVIFSs as (3.12)-(3.15). For any two IVIFSs $A=\{(x_i,M_A(x_i),N_A(x_i)):x_i\in X\}$ and $B=\{(x_i,M_B(x_i),\ N_B(x_i)):x_i\in X\}$ of the universe of discourse $X=\{x_1,x_2,\cdots,x_n\}$,

• the Hamming distance $d_H(A,B)$:

$$d_H(A,B)=\frac{1}{2}\sum_{i=1}^{n}[\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}$$
$$+\max\{|N_A^-(x_i)-N_B^-(x_i)|,|N_A^+(x_i)-N_B^+(x_i)|\}], \quad (3.32)$$

• the normalized Hamming distance $l_H(A,B)$:

$$l_H(A,B)=\frac{1}{2n}\sum_{i=1}^{n}[\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}$$
$$+\max\{|N_A^-(x_i)-N_B^-(x_i)|,|N_A^+(x_i)-N_B^+(x_i)|\}], \quad (3.33)$$

• the Euclidean distance $e_H(A,B)$:

$$e_H(A,B)=\Big\{\frac{1}{2}\sum_{i=1}^{n}[(\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\})^2 \\ +(\max\{|N_A^-(x_i)-N_B^-(x_i)|,|N_A^+(x_i)-N_B^+(x_i)|\})^2]\Big\}^{\frac{1}{2}}, \quad (3.34)$$

• the normalized Euclidean distance $q_H(A,B)$:

$$q_H(A,B)=\Big\{\frac{1}{2n}\sum_{i=1}^{n}[(\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\})^2 \\ +(\max\{|N_A^-(x_i)-N_B^-(x_i)|,|N_A^+(x_i)-N_B^+(x_i)|\})^2]\Big\}^{\frac{1}{2}}. \quad (3.35)$$

Proposition 3.3.1 *Let* $X=\{x_1,x_2,\cdots,x_n\}$ *be a finite universe of discourse. Then function* d_H,l_H,e_H,q_H : IVIFS(X) $\to \mathbf{R}^+\cup\{0\}$ *given by* (3.32)–(3.35), *respectively, are metrics.*

Proof We give only the proof for e_H.

Let $A=\{(x_i,M_A(x_i),N_A(x_i)):x_i\in X\}$, $B=\{(x_i,M_B(x_i),N_B(x_i)):x_i\in X\}$ and $C=\{(x_i,M_C(x_i),N_C(x_i)):x_i\in X\}$ be IVIFSs of X.

(a) $e_H(A,B)$ given by (3.34) is positive definite, i.e. $e_H(A,B)\geq 0$ because of the absolute value properties.

(b) If $A=B$ then $M_A(x_i)=M_B(x_i)$ and $N_A(x_i)=N_B(x_i)$ for each $x_i\in X$ and hence $e_H(A,B)=0$. Conversely, if $e_H(A,B)=0$ then for each $x_i\in X$ we have $\max\{|M_A^-(x_i)-M_B^-(x_i)|,|M_A^+(x_i)-M_B^+(x_i)|\}=0$ and $\max\{|N_A^-(x_i)-N_B^-(x_i)|,|N_A^+(x_i)-N_B^+(x_i)|\}=0$. Then both $M_A^-(x_i)-M_B^-(x_i)=0$, $M_A^+(x_i)-M_B^+(x_i)=0$, $N_A^-(x_i)-N_B^-(x_i)=0$ and $N_A^+(x_i)-N_B^+(x_i)=0$ and hence $A=B$.

(c) The symmetry property $e_H(A,B)=e_H(B,A)$ holds because $|M_A^-(x_i)-M_B^-(x_i)|=|M_B^-(x_i)-M_A^-(x_i)|$, $|M_A^+(x_i)-M_B^+(x_i)|=|M_B^+(x_i)-M_A^+(x_i)|$, and $|N_A^-(x_i)-N_B^-(x_i)|=|N_B^-(x_i)-N_A^-(x_i)|$ and $|N_A^+(x_i)-N_B^+(x_i)|=|N_B^+(x_i)-N_A^+(x_i)|$ for each $x_i\in X$.

(d) Since for any nonnegative numbers a_i,b_i,c_i,d_i $(i=1,2,3)$ such that $a_1+a_2\geq a_3$, $b_1+b_2\geq b_3$, $c_1+c_2\geq c_3$ and $d_1+d_2\geq d_3$ we have $\max\{a_1,b_1\}+$

$\max\{a_2, b_2\} + \max\{c_1, c_2\} + \max\{d_1, d_2\} \geq \max\{a_3, b_3\} + \max\{c_3, d_3\}$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} [(\max\{|M_A^-(x_i) - M_B^-(x_i)|, |M_A^+(x_i) - M_B^+(x_i)|\})^2 \\
&\qquad +(\max\{|N_A^-(x_i) - N_B^-(x_i)|, |N_A^+(x_i) - N_B^+(x_i)|\})^2] \\
&\qquad + \sum_{i=1}^{n} [(\max\{|M_B^-(x_i) - M_C^-(x_i)|, |M_B^+(x_i) - M_C^+(x_i)|\})^2 \\
&\qquad +(\max\{|N_B^-(x_i) - N_C^-(x_i)|, |N_B^+(x_i) - N_C^+(x_i)|\})^2] \\
&\quad \geq \sum_{i=1}^{n} [(\max\{|M_A^-(x_i) - M_C^-(x_i)|, |M_A^+(x_i) - M_C^+(x_i)|\})^2 \\
&\qquad +(\max\{|N_A^-(x_i) - N_C^-(x_i)|, |N_A^+(x_i) - N_C^+(x_i)|\})^2].
\end{aligned}$$

Thus $e_H(A, B) + e_H(B, C) \geq e_H(A, C)$, i.e. the triangle inequality holds. □

Proposition 3.3.2 *For any two* IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ *and* $B = \{(x_i, M_B(x_i), N_A(x_i)) : x_i \in X\}$ *of the universe of discourse* $X = \{x_1, x_2, \cdots, x_n\}$, *the following inequalities hold:*

$$d_H(A, B) \leq n, \tag{3.36}$$
$$l_H(A, B) \leq 1, \tag{3.37}$$
$$e_H(A, B) \leq \sqrt{n}, \tag{3.38}$$
$$q_H(A, B) \leq 1. \tag{3.39}$$

Proof The proof is similar to that of Proposition 3.3.3. □

Proposition 3.3.3 *For any two* IVIFSs $A = \{(x_i, M_A(x_i), N_A(x_i)) : x_i \in X\}$ *and* $B = \{(x_i, M_B(x_i), N_B(x_i)) : x_i \in X\}$ *of the universe of discourse* $X = \{x_1, x_2, \cdots, x_n\}$, *the following inequalities hold:*

$$d_1'(A, B) \leq d_H(A, B) \leq d_1''(A, B), \tag{3.40}$$
$$l_1'(A, B) \leq l_H(A, B) \leq l_1''(A, B), \tag{3.41}$$
$$e_1'(A, B) \leq e_H(A, B) \leq e_1''(A, B), \tag{3.42}$$
$$q_1'(A, B) \leq q_H(A, B) \leq q_1''(A, B). \tag{3.43}$$

Proof We give only the proof for e_H and the other cases are left.

Since $\frac{1}{4}(a+b+c+d) \leq \frac{1}{2}(\max\{a,b\}+\max\{c,d\})$ for any four numbers a, b, c and d, we have $e_1'(A,B) \leq e_H(A,B)$. Moreover, since $W_{M_A}(x_i) = M_A^+(x_i) - M_A^-(x_i)$, $W_{N_A}(x_i) = N_A^+(x_i) - N_A^-(x_i)$, $W_{M_B}(x_i) = M_B^+(x_i) - M_B^-(x_i)$ and $W_{N_B}(x_i) = N_B^+(x_i) - N_B^-(x_i)$ for each $x_i \in X$, then

$$\begin{aligned}
&\frac{1}{4}[(M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 + (N_A^-(x_i) - N_B^-(x_i))^2 \\
&\quad +(N_A^+(x_i) - N_B^+(x_i))^2 + (W_{M_A}(x_i) - W_{M_B}(x_i))^2 + (W_{N_A}(x_i) - W_{N_B}(x_i))^2] \\
&\quad = \frac{1}{2}[(M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2 \\
&\quad\quad +(M_B^-(x_i) - M_A^-(x_i))(M_A^+(x_i) - M_B^+(x_i)) \\
&\quad\quad +(N_A^-(x_i) - N_B^-(x_i))^2 + (N_A^+(x_i) - N_B^+(x_i))^2 \\
&\quad\quad +(N_B^-(x_i) - N_A^-(x_i))(N_A^+(x_i) - N_B^+(x_i))] \\
&\quad \geq \frac{1}{2}[\max\{(M_A^-(x_i) - M_B^-(x_i))^2, (M_A^+(x_i) - M_B^+(x_i))^2\} \\
&\quad\quad + \max\{(N_A^-(x_i) - N_B^-(x_i))^2, (N_A^+(x_i) - N_B^+(x_i))^2\}]
\end{aligned}$$

for each $x_i \in X$. Hence we have $e_1''(A,B) \geq e_H(A,B)$. □

When generalizing any notion it is desirable that the new object should be consistent with the primary one and it should reduce to that primary one in some particular cases. As it was mentioned above each IVFS can be IVIFS under some conditions. Thus it would be desirable that our definitions (3.24)-(3.35) should reduce to the Burillo and Bustince's distances (3.2)-(3.5), our distances (3.6)-(3.9) and Grzegorzewski's distances (3.12)-(3.15), respectively, for ordinary IVFSs. One can check easily that

Proposition 3.3.4 *For any two* IVIFS*s* $A, B \in X = \{x_1, \cdots, x_n\}$ *such that* $A = \{(x_i, M_A(x_i), \bar{M}_A(x_i)) : x_i \in X\}$ *and* $B = \{(x_i, M_B(x_i), \bar{M}_B(x_i)) : x_i \in X\}$, *the following equalities hold:*

$$d'(A,B) = d_1'(A,B),\ l'(A,B) = l_1'(A,B),$$
$$e'(A,B) = e_1'(A,B),\ q'(A,B) = q_1'(A,B),$$

$$d''(A,B) = d''_1(A,B),\ \ l''(A,B) = l''_1(A,B),$$
$$e''(A,B) = e''_1(A,B),\ \ q''(A,B) = q''_1(A,B),$$
$$d_h(A,B) = d_H(A,B),\ \ l_h(A,B) = l_H(A,B),$$
$$e_h(A,B) = e_H(A,B),\ \ q_h(A,B) = q_H(A,B).$$

Remark 3.3.5 Since IFSs and IVFSs are equipollent generalizations of fuzzy sets, our definitions (3.24)-(3.35) should also reduce to the Szmidt and Kacprzyk's distances [45] and Grzegorzewski's distances [21], respectively, for ordinary IFSs.

Chapter 4

Entropy and Similarity Measure of Interval-valued Intuitionistic Fuzzy Sets

In this chapter, we introduce concepts of entropy and similarity measure of IVIFSs, discuss their relationship between similarity measure and entropy of IVIFSs, show that similarity measure and entropy of IVIFSs can be transformed by each other based on their axiomatic definitions and give some formulas to calculate entropy and similarity measure of IVIFSs.

4.1 Entropy of IVIFSs

De Luca and Termini [13] first axiomatized non-probabilistic entropy. Szmidt and Kacprzyk [47] extended De Luca and Termini axioms for fuzzy set to introduce entropy of IFSs. Based on this view point of Szmidt and Kacprzyk, Zeng and Li [70] introduced the concept of entropy of IVFSs which is different from Bustince and Burillo [6]. We introduce the concept of entropy of IVIFSs.

Definition 4.1.1 A real function E : IVIFS$(X) \to [0, 1]$ is called *entropy* on IVIFS(X) if E satisfies the following properties:

(E1) $E(A) = 0$ if A is a crisp set;
(E2) $E(A) = 1 \Longleftrightarrow M_A(x) = N_A(x)$ for every $x \in X$;
(E3) $E(A) \leq E(B)$ if A refines B;
(E4) $E(A) = E(A^c)$.

Then we can give the following formulas to calculate entropy of IVIFS A on X:

$$E_1(A) = 1 - \frac{1}{2n}\sum_{i=1}^{n}(|M_A^-(x_i) - N_A^-(x_i)| + |M_A^+(x_i) - N_A^+(x_i)|), \tag{4.1}$$

$$E_2(A) = 1 - \sqrt{\frac{1}{2n}\sum_{i=1}^{n}((M_A^-(x_i) - N_A^-(x_i))^2 + (M_A^+(x_i) - N_A^+(x_i))^2)}, \tag{4.2}$$

$$E_3(A) = 1 - \frac{1}{2(b-a)}\int_a^b(|M_A^-(x) - N_A^-(x)| + |M_A^+(x) - N_A^+(x)|)dx, \tag{4.3}$$

$$E_4(A) = \frac{\int_a^b(\min(M_A^-(x), N_A^-(x)) + \min(M_A^+(x), N_A^+(x)))dx}{\int_a^b(\max(M_A^-(x), N_A^-(x)) + \max(M_A^+(x), N_A^+(x)))dx}, \tag{4.4}$$

where in Equations (4.1) and (4.2), $X = \{x_1, x_2, \ldots, x_n\}$ is finite and in Equations (4.3) and (4.4), $M_A^-(x)$, $M_A^+(x)$, $N_A^-(x)$ and $N_A^+(x)$ are continuous on the closed interval $X = [a, b]$ and the integral is Lebesgue integral.

Definition 4.1.2 A real function S : IVIFS$(X) \times$ IVIFS$(X) \to [0, 1]$ is called *similarity measure* of IVIFSs if S satisfies the following properties:
(S1) $S(A, A^c) = 0$ if A is a crisp set;
(S2) $S(A, B) = 1 \Longleftrightarrow A = B$;
(S3) $S(A, B) = S(B, A)$;
(S4) for all $A, B, C \in$ IVIFS(X), if $A \subset B \subset C$, then $S(A, C) \leq S(A, B)$, $S((A, C) \leq S(B, C)$.

Then we can give the following formulas to calculate similarity measure of IVIFSs A and B in finite set $X = \{x_1, x_2, \ldots, x_n\}$:

$$S_1(A, B) = 1 - \frac{1}{4n}\sum_{i=1}^{n}(|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)|$$

$$+|N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|\ ). \tag{4.5}$$

$$S_2(A,B) = 1 - \Big\{\frac{1}{4n}\sum_{i=1}^{n}((M_A^-(x_i) - M_B^-(x_i))^2 + (M_A^+(x_i) - M_B^+(x_i))^2$$
$$+(N_A^-(x_i) - N_B^-(x_i))^2 + (N_A^+(x_i) - N_B^+(x_i))^2\Big\}^{\frac{1}{2}}. \tag{4.6}$$

Obviously, the axiomatic definition of similarity measure and entropy of IVIFSs can be extended from IFS theory or IVFS theory. Specially, if IVIFSs A and B become IFSs (resp. IVFSs), then $S(A, B)$ is similarity measure of IFSs (resp. IVFSs). Based on this point of view, we have the following results.

Theorem 4.1.3 *If* $S : \mathrm{IFS}(X) \times \mathrm{IFS}(X) \to [0, 1]$ *is a similarity measure, then* $E : \mathrm{IVIFS}(X) \to [0, 1]$ *given by*

$$E(A) = \frac{S(A_-, (A_-)^c) + S(A_+, (A_+)^c)}{2}$$

is an entropy of IVIFS A, *where* A_- *and* A_+ *are* IFS*s given by* $A_- = \langle M_A^-, N_A^-\rangle$ *and* $A_+ = \langle M_A^+, N_A^+\rangle$, *respectively.*

Proof (E1) If A is a crisp set, then for every $x \in X$, we have either $M_A(x) = \bar{1}$ and $N_A(x) = \bar{0}$, or $M_A(x) = \bar{0}$ and $N_A(x) = \bar{1}$. It means that A_- and A_+ are crisp sets, and then $S(A_-, (A_-)^c) = S(A_+, (A_+)^c) = 0$. Hence $E(A) = 0$.

(E2) By Definition 2.0.6 of similarity measure of IFSs, we have

$$\begin{aligned} E(A) = 1 &\iff S(A_-, (A_-)^c) = S(A_+, (A_+)^c) = 1 \\ &\iff A_- = (A_-)^c, A_+ = (A_+)^c \\ &\iff M_A^-(x) = N_A^-(x), M_A^+(x) = N_A^+(x) \text{ for every } x \in X \\ &\iff M_A = N_A. \end{aligned}$$

(E3) Since $M_B(x) \le M_A(x)$ and $N_A(x) \le N_B(x)$ for $M_B(x) \ge N_B(x)$ implies $N_A(x) \le N_B(x) \le M_B(x) \le M_A(x)$. It means $(A_-)^c \subset (B_-)^c \subset B_- \subset A_-$ and $(A_+)^c \subset (B_+)^c \subset B_+ \subset A_+$. By Definition 2.0.6, we have $S(A_-, (A_-)^c) \le S(A_-, (B_-)^c) \le S(B_-, (B_-)^c)$ and $S(A_+, (A_+)^c) \le S(A_+, (B_+)^c) \le S(B_+, (B_+)^c)$. Hence $E(A) \le E(B)$.

When $M_A(x) \leq M_B(x)$ and $N_A(x) \geq N_B(x)$ for $M_B(x) \leq N_B(x)$, with the same reason, we can obtain $E(A) \leq E(B)$.

(E4) Since $A^c = (N_A, M_A)$, by Definition 2.0.6, we have

$$\begin{aligned} E(A) &= \frac{S(A_-, (A_-)^c) + S(A_+, (A_+)^c)}{2} \\ &= \frac{S((A_-)^c, A_-) + S((A_+)^c, A_+)}{2} \\ &= \frac{S((A_-)^c, ((A_-)^c)^c) + S((A_+)^c, ((A_+)^c)^c)}{2} \\ &= E(A^c). \end{aligned}$$

□

Theorem 4.1.4 *If* $S : \mathrm{IVFS}(X) \times \mathrm{IVFS}(X) \to [0,1]$ *is a similarity measure and* $A \in \mathrm{IVIFS}(X)$ *satisfies that* IVFS N_A *is the complement of* IVFS M_A*, then* $E : \mathrm{IVIFS}(X) \to [0,1]$ *given by* $E(A) = S(M_A, N_A)$ *is an entropy.*

Proof (E1) If A is a crisp set, then for every $x \in X$, we have either $M_A(x) = \bar{1}$ and $N_A(x) = \bar{0}$, or $M_A(x) = \bar{0}$ and $N_A(x) = \bar{1}$. It means that M_A and N_A are crisp sets, and hence $E(A) = S(M_A, N_A) = S(M_A, (M_A)^c) = 0$.

(E2) By Definition 2.0.3 of similarity measure of IVFSs, we have

$$E(A) = S(M_A, N_A) = 1 \Longleftrightarrow M_A = N_A.$$

(E3) Since $M_B(x) \leq M_A(x)$ and $N_A(x) \leq N_B(x)$ for $M_B(x) \geq N_B(x)$ implies $N_A(x) \leq N_B(x) \leq M_B(x) \leq M_A(x)$. That is, $N_A \subset N_B \subset M_B \subset M_A$. By Definition 2.0.3, we have $S(M_A, N_A) \leq S(M_A, N_B) \leq S(M_B, N_B)$ and hence $E(A) \leq E(B)$.

When $M_A(x) \leq M_B(x)$ and $N_A(x) \geq N_B(x)$ for $M_B(x) \leq N_B(x)$, with the same reason, we can obtain $E(A) = S(M_A, N_A) \leq S(M_B, N_B) = E(B)$.

(E4) Since $A^c = (N_A, M_A)$, by Definition 2.0.3, we have $E(A) = S(M_A, N_A) = S(N_A, M_A) = E(A^c)$. □

Remark 4.1.5 The hypothesis in Theorem 4.1.4 leads to intuitionistic fuzzy case: $N_A = M_A^c$ implies $N_A^- = 1 - M_A^+$ and $N_A^+ = 1 - M_A^-$; the condition $M_A^+ + N_A^+ \leq 1$ implies $M_A^+ \leq M_A^-$, that is $M_A^- = M_A^+$ and $N_A^- = N_A^+$.

4.2 Relationship between similarity measure and entropy of IVIFSs

Having in mind that the real functions of similarity measure and entropy of IVIFSs are not unique as in shown in Section 4.1, we will discuss the relationship between similarity measure and entropy of IVIFSs based on their axiomatic definitions.

First, we propose a transform method of setting up similarity measure of IVIFSs based on entropy of IVIFSs. For $A, B \in \text{IVIFS}(X)$, we define an IVIFS $I(A, B)$ on X as follows: for every $x \in X$,

$$\begin{aligned}
M^-_{I(A,B)}(x) &= \frac{1}{2}\Big[1 + \min\{\min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \\
&\qquad \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}\Big], \\
M^+_{I(A,B)}(x) &= \frac{1}{2}\Big[1 + \max\{\min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \\
&\qquad \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}\Big], \\
N^-_{I(A,B)}(x) &= \frac{1}{2}\Big[1 - \max\{\max(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \\
&\qquad \max(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}\Big], \\
N^+_{I(A,B)}(x) &= \frac{1}{2}\Big[1 - \max\{\min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \\
&\qquad \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)\}\Big].
\end{aligned}$$

Then we have the following result.

Theorem 4.2.1 *If* $E : \text{IVIFS}(X) \to [0, 1]$ *is an entropy, then* $S : \text{IVIFS}(X) \times \text{IVIFS}(X) \to [0, 1]$ *given by* $S(A, B) = E(I(A, B))$ *is a similarity measure.*

Proof (S1) If A is a crisp set, then, for every $x \in X$, we have either $M_A(x) = \bar{1}$ and $N_A(x) = \bar{0}$, or $M_A(x) = \bar{0}$ and $N_A(x) = \bar{1}$, it means that, for every $x \in X$, we have $|M^-_A(x) - M^-_{A^c}(x)| = 1$, $|M^+_A(x) - M^+_{A^c}(x)| = 1$, $|N^-_A(x) - N^-_{A^c}(x)| = 1$ and $|N^+_A(x) - N^+_{A^c}(x)| = 1$. Therefore, $M^-_{I(A,A^c)}(x) = M^+_{I(A,A^c)}(x) = 1$ and

$N^{-}_{I(A,A^c)}(x) = N^{+}_{I(A,A^c)}(x) = 0$, it shows $I(A, A^c) = X$ is a crisp set and hence $S(A, B) = E(I(A, A^c)) = 0$.

(S2) By Definition 4.1.2 of entropy of IVIFSs, we have

$$\begin{aligned}
S(A,B) = 1 \iff & E(I(A,B)) = 1 \\
\iff & M_{I(A,B)}(x) = N_{I(A,B)}(x) \text{ for every } x \in X \\
\iff & |M^-_A(x) - M^-_B(x)| = 0, |M^+_A(x) - M^+_B(x)| = 0, \\
& |N^-_A(x) - N^-_B(x)| = 0, |N^+_A(x) - N^+_B(x)| = 0 \text{ for every } x \in X \\
\iff & M^-_A = M^-_B, M^+_A = M^+_B, N^-_A = N^-_B, N^+_A = N^+_B \\
\iff & M_A = M_B, N_A = N_B \\
\iff & A = B.
\end{aligned}$$

(S3) By the definition of $I(A, B)$, since $I(A, B) = I(B, A)$ is obvious, we have $S(A, B) = E(I(A, B)) = E(I(B, A)) = S(B, A)$.

(S4) Since $A \subset B \subset C$, for every $x \in X$, we have $M_A(x) \leq M_B(x) \leq M_C(x)$ and $N_A(x) \geq N_B(x) \geq N_C(x)$. Thus, for every $x \in X$, we get $|M^-_A(x) - M^-_C(x)| \geq |M^-_A(x) - M^-_B(x)|$, $|M^+_A(x) - M^+_C(x)| \geq |M^+_A(x) - M^+_B(x)|$, $|N^-_A(x) - N^-_C(x)| \geq |N^-_A(x) - N^-_B(x)|$ and $|N^+_A(x) - N^+_C(x)| \geq |N^+_A(x) - N^+_B(x)|$.

Further, for every $x \in X$, we have

$$\begin{aligned}
&\min(|M^-_A(x) - M^-_C(x)|, |M^+_A(x) - M^+_C(x)|) \\
&\quad \geq \min(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \\
&\max(|M^-_A(x) - M^-_C(x)|, |M^+_A(x) - M^+_C(x)|) \\
&\quad \geq \max(|M^-_A(x) - M^-_B(x)|, |M^+_A(x) - M^+_B(x)|), \\
&\min(|N^-_A(x) - N^-_C(x)|, |N^+_A(x) - N^+_C(x)|) \\
&\quad \geq \min(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|)
\end{aligned}$$

and

$$\begin{aligned}
&\max(|N^-_A(x) - N^-_C(x)|, |N^+_A(x) - N^+_C(x)|) \\
&\quad \geq \max(|N^-_A(x) - N^-_B(x)|, |N^+_A(x) - N^+_B(x)|).
\end{aligned}$$

Therefore, for every $x \in X$, we get $M^-_{I(A,C)}(x) \geq M^-_{I(A,B)}(x)$, $M^+_{I(A,C)}(x) \geq M^+_{I(A,B)}(x)$, $N^-_{I(A,C)}(x) \leq N^-_{I(A,B)}(x)$ and $N^+_{I(A,C)}(x) \leq N^+_{I(A,B)}(x)$ and by the definition of $I(A,B)$, we have $N^-_{I(A,B)}(x) \leq M^-_{I(A,B)}(x)$ and $N^+_{I(A,B)}(x) \leq M^+_{I(A,B)}(x)$ for every $x \in X$, i.e., $N_{I(A,B)}(x) \leq M_{I(A,B)}(x)$ for every $x \in X$. Hence, by Definition 4.1.2 of entropy of IVIFSs, $S(A,C) = E(I(A,C)) \leq E(I(A,B)) = S(A,B)$.

With the same reason, we obtain $S(A,C) \leq S(B,C)$. □

Corollary 4.2.2 *If* $E : \mathrm{IVIFS}(X) \to [0,1]$ *is an entropy, then* $S : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0,1]$ *given by* $S(A,B) = E((I(A,B))^c)$ *is a similarity measure.*

For $A, B \in \mathrm{IVIFS}(X)$, we define an IVIFS $J(A,B)$ on X as follows: for every $x \in X$ and $p > 0$,

$$\begin{aligned}
M^-_{J(A,B)}(x) &= \frac{1}{2}\Big[1 + \min\{\min(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \\
&\qquad \min(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}\Big], \\
M^+_{J(A,B)}(x) &= \frac{1}{2}\Big[1 + \max\{\min(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \\
&\qquad \min(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}\Big], \\
N^-_{J(A,B)}(x) &= \frac{1}{2}\Big[1 - \max\{\max(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \\
&\qquad \max(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}\Big], \\
N^+_{J(A,B)}(x) &= \frac{1}{2}\Big[1 - \max\{\min(|M^-_A(x) - M^-_B(x)|^p, |M^+_A(x) - M^+_B(x)|^p), \\
&\qquad \min(|N^-_A(x) - N^-_B(x)|^p, |N^+_A(x) - N^+_B(x)|^p)\}\Big].
\end{aligned}$$

Then we have

Corollary 4.2.3 *If* $E : \mathrm{IVIFS}(X) \to [0,1]$ *is an entropy, then* $S : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0,1]$ *given by* $S(A,B) = E(J(A,B))$ *or* $S(A,B) = E((J(A,B))^c)$ *is a similarity measure.*

Example 4.2.4 Let $X = \{x_1, x_2, \cdots, x_n\}$, $A, B \in \mathrm{IVIFS}(X)$ and

$$E(A) = 1 - \frac{1}{2n}\sum_{i=1}^{n}(|M^-_A(x_i) - N^-_A(x_i)| + |M^+_A(x_i) - N^+_A(x_i)|).$$

Then

$$\begin{aligned}
S(A,B) &= E(I(A,B)) \\
&= 1 - \frac{1}{4n}\sum_{i=1}^{n}\Big(\min(|M_A^-(x_i) - M_B^-(x_i)|, |M_A^+(x_i) - M_B^+(x_i)|, \\
&\qquad |N_A^-(x_i) - N_B^-(x_i)|, |N_A^+(x_i) - N_B^+(x_i)|) \\
&\quad + \max(|M_A^-(x_i) - M_B^-(x_i)|, |M_A^+(x_i) - M_B^+(x_i)|, \\
&\qquad |N_A^-(x_i) - N_B^-(x_i)|, |N_A^+(x_i) - N_B^+(x_i)|) \\
&\quad +2\max\{\min(|M_A^-(x_i) - M_B^-(x_i)|, |M_A^+(x_i) - M_B^+(x_i)|, \\
&\qquad \min(|N_A^-(x_i) - N_B^-(x_i)|, |N_A^+(x_i) - N_B^+(x_i)|)\}\Big)
\end{aligned}$$

is a similarity measure of IVIFSs A and B.

Example 4.2.5 Let $X = [a,b]$, $A, B \in \text{IVIFS}(X)$ and

$$E(A) = 1 - \frac{1}{2(b-a)}\int_a^b (|M_A^-(x) - N_A^-(x)| + |M_A^+(x) - N_A^+(x)|)\, dx,$$

where $M_A^-(x)$, $M_A^+(x)$, $N_A^-(x)$ and $N_A^+(x)$ are continuous on $X = [a,b]$ and the integral is Lebesgue integral. Then

$$\begin{aligned}
S(A,B) &= E(I(A,B)) \\
&= 1 - \frac{1}{4(b-a)}\int_a^b \Big(\min(|M_A^-(x) - M_B^-(x)|, |M_A^+(x) - M_B^+(x)|, \\
&\qquad |N_A^-(x) - N_B^-(x)|, |N_A^+(x) - N_B^+(x)|) \\
&\quad + \max(|M_A^-(x) - M_B^-(x)|, |M_A^+(x) - M_B^+(x)|, \\
&\qquad |N_A^-(x) - N_B^-(x)|, |N_A^+(x) - N_B^+(x)|) \\
&\quad +2\max\{\min(|M_A^-(x) - M_B^-(x)|, |M_A^+(x) - M_B^+(x)|, \\
&\qquad \min(|N_A^-(x) - N_B^-(x)|, |N_A^+(x) - N_B^+(x)|)\}\Big)\, dx
\end{aligned}$$

is a similarity measure of IVIFSs A and B.

Next, we propose another method of setting up an entropy of IVIFSs based on similarity measure of IVIFSs. For $A \in \text{IVIFS}(X)$, we define IVIFS $m(A)$ and

$n(A)$ on X as follows: for every $x \in X$,

$$
\begin{aligned}
M^-_{m(A)}(x) &= \frac{1+(M^-_A(x)-N^-_A(x))^4}{2}, \quad M^+_{m(A)}(x) = \frac{1+(M^-_A(x)-N^-_A(x))^2}{2},\\
N^-_{m(A)}(x) &= \frac{1-|M^-_A(x)-N^-_A(x)|}{2}, \quad N^+_{m(A)}(x) = \frac{1-(M^-_A(x)-N^-_A(x))^2}{2},\\
M^-_{n(A)}(x) &= \frac{1-|M^+_A(x)-N^+_A(x)|}{2}, \quad M^+_{n(A)}(x) = \frac{1-(M^+_A(x)-N^+_A(x))^2}{2},\\
N^-_{n(A)}(x) &= \frac{1+(M^+_A(x)-N^+_A(x))^4}{2}, \quad N^+_{n(A)}(x) = \frac{1+(M^+_A(x)-N^+_A(x))^2}{2}.
\end{aligned}
$$

Then we have the following result.

Theorem 4.2.6 *If* $S : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0,1]$ *is a similarity measure, then* $E : \mathrm{IVIFS}(X) \to [0,1]$ *given by* $E(A) = S(m(A), n(A))$ *is an entropy.*

Proof (E1) If A is a crisp set, then for every $x \in X$, we have either $M_A(x) = \bar{1}$, $N_A(x) = \bar{0}$ or $M_A(x) = \bar{0}$, $N_A(x) = \bar{1}$. Hence for every $x \in X$, we get $|M^-_A(x) - N^-_A(x)| = 1$ and $|M^+_A(x) - N^+_A(x)| = 1$, it means that $M_{m(A)}(x) = \bar{1}$, $N_{m(A)}(x) = \bar{0}$, $M_{n(A)}(x) = \bar{0}$ and $N_{n(A)}(x) = \bar{1}$. It shows that $m(A) = X$ and $n(A) = \emptyset$ are crisp sets. Therefore, $E(A) = S(m(A), n(A)) = 0$.

(E2) By the definitions of $m(A)$ and $n(A)$, $m(A)$ and $n(A)$ are IVIFSs and thus

$$
\begin{aligned}
E(A) = S((m(A), n(A)) = 1&\\
\Longleftrightarrow\ & m(A) = n(A)\\
\Longleftrightarrow\ & M^-_A(x) = N^-_A(x), M^+_A(x) = N^+_A(x) \text{ for every } x \in X\\
\Longleftrightarrow\ & M_A = N_A.
\end{aligned}
$$

(E3) Since $M_B(x) \leq M_A(x)$ and $N_A(x) \leq N_B(x)$ for $M_B(x) \geq N_B(x)$ implies $N_A(x) \leq N_B(x) \leq M_B(x) \leq M_A(x)$. Hence, we get $|N^-_A(x) - M^-_A(x)| \geq |N^-_B(x) - M^-_B(x)|$ and $|N^+_A(x) - M^+_A(x)| \geq |N^+_B(x) - M^+_B(x)|$. It means that for every $x \in X$, $M_{m(A)}(x) \geq M_{m(B)}(x)$, $N_{m(A)}(x) \leq N_{m(B)}(x)$, $M_{n(A)}(x) \leq M_{n(B)}(x)$ and $N_{n(A)}(x) \geq N_{n(B)}(x)$. Then, we get $n(A) \subset n(B) \subset m(B) \subset m(A)$ and thus we obtain $E(A) = S(m(A), n(A)) \leq S(m(B), n(A)) \leq S(m(B), n(B)) = E(B)$.

When $M_A(x) \le M_B(x)$ and $N_A(x) \ge N_B(x)$ for $M_B(x) \le N_B(x)$, with the same reason, we obtain $E(A) = S(m(A), n(A)) \le S(m(B), n(B)) = E(B)$.

(E4) By the definitions of $m(A)$ and $n(A)$, we have $m(A) = m(A^c)$ and $n(A) = n(A^c)$. Therefore, $E(A) = S(m(A), n(A)) = S(m(A^c), n(A^c)) = E(A^c)$. □

Corollary 4.2.7 *If* $S : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0,1]$ *is a similarity measure, then* $E : \mathrm{IVIFS}(X) \to [0,1]$ *given by* $E(A) = S((m(A))^c, (n(A))^c)$ *is an entropy.*

Example 4.2.8 Let $X = \{x_1, x_2, \cdots, x_n\}$, $A, B \in \mathrm{IVIFS}(X)$ and

$$\begin{aligned} S(A,B) = 1 - \frac{1}{4n}\sum_{i=1}^{n} (&|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| \\ &+|N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|). \end{aligned}$$

Then

$$\begin{aligned} E(A) &= S(m(A), n(A)) \\ &= 1 - \frac{1}{8n}\sum_{i=1}^{n} ((M_A^-(x_i) - N_A^-(x_i))^4 + 2(M_A^-(x_i) - N_A^-(x_i))^2 \\ &\quad +|M_A^-(x_i) - N_A^-(x_i)| + (M_A^+(x_i) - N_A^+(x_i))^4 \\ &\quad +2(M_A^+(x_i) - N_A^+(x_i))^2 + |M_A^+(x_i) - N_A^+(x_i)|) \end{aligned}$$

is an entropy of IVIFS A.

Example 4.2.9 Let $X = [a, b]$, $A, B \in \mathrm{IVIFS}(X)$ and

$$\begin{aligned} S(A,B) = 1 - \frac{1}{4(b-a)}\int_a^b (&|M_A^-(x) - M_B^-(x)| + |M_A^+(x) - M_B^+(x)| \\ &+|N_A^-(x) - N_B^-(x)| + |N_A^+(x) - N_B^+(x)|)dx, \end{aligned}$$

where $M_A^-(x)$, $M_A^+(x)$, $N_A^-(x)$, $N_A^+(x)$, $M_B^-(x)$, $M_B^+(x)$, $N_B^-(x)$ and $N_B^+(x)$ are continuous on $X = [a, b]$ and the integral is Lebesgue integral. Then

$$E(A) = S(m(A), n(A))$$

$$= 1 - \frac{1}{8(b-a)} \int_a^b \Big((M_A^-(x) - N_A^-(x))^4 + 2(M_A^-(x) - N_A^-(x))^2 + |M_A^-(x) - N_A^-(x)| + (M_A^+(x) - N_A^+(x))^4 + 2(M_A^+(x) - N_A^+(x))^2 + |M_A^+(x) - N_A^+(x)| \Big) dx$$

is an entropy of IVIFS A.

Theorem 4.2.10 *If* $S : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0,1]$ *is a similarity measure, then* $E : \mathrm{IVIFS}(X) \to [0,1]$ *given by* $E(A) = S(A, A^c)$ *is an entropy.*

Proof (E1) If A is a crisp set, then by Definition 4.1.2 of similarity of IVIFSs, we have $E(A) = S(A, A^c) = 0$.

(E2) By Definition 4.1.2, we have

$$E(A) = S(A, A^c) = 1 \Longleftrightarrow M_A = M_{A^c}, N_A = N_{A^c} \Longleftrightarrow M_A = N_A.$$

(E3) Since $M_B(x) \le M_A(x)$ and $N_A(x) \le N_B(x)$ for $M_B(x) \ge N_B(x)$ implies $N_A(x) \le N_B(x) \le M_B(x) \le M_A(x)$. It means that we have $A^c \subset B^c \subset B \subset A$. Therefore, by Definition 4.1.2, we have $E(A) = S(A, A^c) \le S(B, A^c) \le S(B, B^c) = E(B)$.

When $M_A(x) \le M_B(x)$ and $N_A(x) \ge N_B(x)$ for $M_B(x) \le N_B(x)$, with the same reason, we obtain $E(A) \le E(B)$.

(E4) $E(A) = S(A, A^c) = S(A^c, A) = E(A^c)$ is obvious. □

Example 4.2.11 Let $X = \{x_1, x_2, \cdots, x_n\}$, $A, B \in \mathrm{IVIFS}(X)$ and

$$S(A,B) = 1 - \frac{1}{4n} \sum_{i=1}^{n} (|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| + |N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|).$$

Then

$$\begin{aligned} E(A) &= S(A, A^c) \\ &= 1 - \frac{1}{2n} \sum_{i=1}^{n} (|M_A^-(x_i) - N_A^-(x_i)| + |M_A^+(x_i) - N_A^+(x_i)|) \\ &= E_1(A) \end{aligned}$$

is an entropy of IVIFS A.

Example 4.2.12 Let $X=[a,b]$, $A,B\in$ IVIFS(X) and

$$S(A,B)=1-\frac{1}{4(b-a)}\int_a^b(|M_A^-(x)-M_B^-(x)|+|M_A^+(x)-M_B^+(x)|$$
$$+|N_A^-(x)-N_B^-(x)|+|N_A^+(x)-N_B^+(x)|)dx,$$

where $M_A^-(x)$, $M_A^+(x)$, $N_A^-(x)$, $N_A^+(x)$, $M_B^-(x)$, $M_B^+(x)$, $N_B^-(x)$ and $N_B^+(x)$ are continuous on $X=[a,b]$ and the integral is Lebesgue integral. Then

$$\begin{aligned}E(A)&=S(A,A^c)\\&=1-\frac{1}{2(b-a)}\int_a^b(|M_A^-(x)-N_A^-(x)|+|M_A^+(x)-N_A^+(x)|)dx\\&=E_3(A)\end{aligned}$$

is an entropy of IVIFS A.

Chapter 5

Subsethood Measure of Interval-valued Intuitionistic Fuzzy Sets

The main purpose of this paper is to establish a unified formulation of subsethood, entropy and cardinality for IVIFSs. We propose an axiomatic skeleton for subsethood measures in the interval-valued intuitionistic setting, in order for subsethood to reduce an entropy measure. The notion of average possible cardinality is presented and its connection to least and biggest cardinalities is established. Moreover, the entropy-subsethood and interval-valued intuitionistic fuzzy entropy theorems are algebraically proved, which generalize the Kosko's result [27, 28, 30] for fuzzy sets and the Vlachos and Sergiadis's result [51] for IVFSs. Finally, connections of the proposed entropy measure for IVIFSs with corresponding measures for fuzzy sets and IVFSs are provided.

5.1 Cardinality for IVIFSs

Szmidt and Kacprzyk [47] defined the concept of cardinality for IFSs. Vlachos and Sergiadis [51] provided an interpretation of cardinality under a geometrical framework and present the concept of average possible cardinality for IFSs. We extend these concepts in the interval-valued intuitionistic fuzzy setting.

Definition 5.1.1 For a set $A \in \text{IVIFS}(X)$ the following two cardinalities are defined:

- the *least cardinality* or *min-sigma-count*, which is given by

$$\min \sum Count(A) = \sum_{x_i \in X} \frac{M_A^+(x_i) + M_A^-(x_i)}{2} \tag{5.1}$$

- the *biggest cardinality* or *max-sigma-count* defined as

$$\max \sum Count(A) = \sum_{x_i \in X} \frac{2 - (N_A^+(x_i) + N_A^-(x_i))}{2}. \tag{5.2}$$

The *cardinality* of the IVIFS A is defined as the interval

$$card(A) = \left[\min \sum Count(A), \max \sum Count(A)\right]. \tag{5.3}$$

For the smallest IVIFS $\mathcal{O}$, for which $M_{\mathcal{O}}(x) = [0, 0]$ and $N_{\mathcal{O}}(x) = [1, 1]$ for all $x \in X$, equivalently to Vlachos and Sergiadis's definition of cardinality for IFSs, we call the magnitude $\mathcal{M}(A)$ of the vector $\overrightarrow{\mathcal{O}A}$, using Hamming distance (3.24), the *average possible cardinality* of the IVIFS A. The characterization of *average possible cardinality* will be justified by the following analysis. The Hamming distance between two IVIFSs A and B is given by

$$\begin{aligned} d_1'(A, B) = \frac{1}{4} \sum_{i=1}^{n} (&|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)| \\ &+ |N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|). \end{aligned} \tag{5.4}$$

Definition 5.1.2 For a set $A \in \text{IVIFS}(X)$ the *average possible cardinality* $\mathcal{M}(A)$ is defined as

$$\begin{aligned} \mathcal{M}(A) &= d_1'(\mathcal{O}, A) \\ &= \frac{1}{4} \sum_{x_i \in X} (M_A^-(x_i) + 1 - N_A^-(x_i) + M_A^-(x_i) + 1 - N_A^-(x_i)). \end{aligned} \tag{5.5}$$

From (5.5), taking into account (5.3), it follows that $\mathcal{M}(A)$ is the midpoint of the interval $[\min\sum Count(A), \max\sum Count(A)]$. So, (5.5) encompasses the notions of least, biggest and average possible cardinalities. Connections between the aforementioned cardinalities will be discussed and applied in interval-valued intuitionistic fuzzy decision making problems.

The axiomatic definition of cardinality and average possible cardinality of IVIFSs can be extended from IFS theory or IVFS theory. Specially, if an IVIFS A becomes an IFS, then $\mathcal{M}(A)$ is average possible cardinality of IFS. To derive connection between the cardinality of IVIFSs and that of IFSs, for $A \in \text{IVIFS}(X)$, we consider the lower IFS $A_L = \langle M_A^-, N_A^- \rangle$ of A and upper IFS $A_U = \langle M_A^+, N_A^+ \rangle$ of A. From (5.5) and Definition 18 of [51], we obtain

$$\begin{aligned}
\mathcal{M}(A) &= \frac{1}{2}\sum_{x_i \in X}\left(\frac{M_A^-(x_i) + M_A^+(x_i)}{2} + \frac{2 - N_A^-(x_i) - N_A^+(x_i)}{2}\right) \\
&= \frac{1}{2}\sum_{x_i \in X}\left(M_A^-(x_i) + \frac{\pi_{A_L}(x_i)}{2} + M_A^+(x_i) + \frac{\pi_{A_U}(x_i)}{2}\right) \\
&= \frac{\mathcal{M}(A_L) + \mathcal{M}(A_U)}{2}. \qquad (5.6)
\end{aligned}$$

Thus, the average possible cardinality of IVIFS A is half of the sum of the average possible cardinalities of A_L and A_U. Moreover, from Proposition 20 of [51], we obtain the following result

$$\mathcal{M}(A) = M(D_{0.5}(A_L)) + M(D_{0.5}(A_U)), \qquad (5.7)$$

where $M(D_{0.5}(A_L))$ and $M(D_{0.5}(A_U))$ are cardinalities of fuzzy sets $D_{0.5}(A_L)$ and $D_{0.5}(A_U)$, respectively.

Motivated by Hamming version of the Pythagorean theorem proposed by Kosko, Vlachos and Sergiadis [51] provided a geometrical interpretation of average possible cardinality of IFSs. In order to be consistent with Vlachos and Sergiadis's approach, we employ a modified Hamming distance for IVIFSs given by

$$d_1''(A, B) = \frac{1}{2}\sum_{i=1}^{n}\left(|M_A^-(x_i) - M_B^-(x_i)| + |M_A^+(x_i) - M_B^+(x_i)|\right.$$

$$+|N_A^-(x_i) - N_B^-(x_i)| + |N_A^+(x_i) - N_B^+(x_i)|). \tag{5.8}$$

Using (5.8), a modified definition of average possible cardinality is also obtained as

$$\begin{aligned}\mathcal{M}'(A) &= d_1''(\mathcal{O}, A)\\ &= \frac{1}{2}\sum_{x_i \in X}(M_A^-(x_i) + 1 - N_A^-(x_i) + M_A^-(x_i) + 1 - N_A^-(x_i)),\end{aligned} \tag{5.9}$$

which will be called *pseudo-average possible cardinality*, since $\mathcal{M}'(A)$ does not coincide with the midpoint of the interval $[\min\sum Count(A), \max\sum Count(A)]$. From (5.9), taking into account (5.6), we obtain the following

$$\mathcal{M}'(A) = \frac{\mathcal{M}'(A_L) + \mathcal{M}'(A_U)}{2}. \tag{5.10}$$

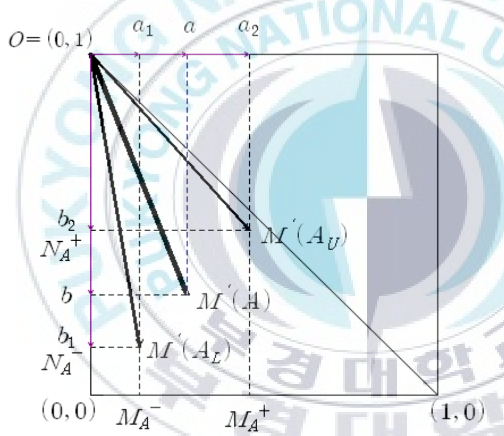

Figure 5.1: Representation of pseudo-average possible cardinality $\mathcal{M}'(A)$ in the unit square and its geometrical connection with least and biggest cardinalities.

Fig. 5.1 depicts the notion of pseudo-average possible cardinality $\mathcal{M}'(A)$ in case of an IVIFS A in $X = \{x\}$. One can observe that the vectors $\overrightarrow{\mathcal{O}A_L}$ and $\overrightarrow{\mathcal{O}A_U}$ can be projected onto the membership and non-membership axes, deriving the vectors a_1, a_2, b_1 and b_2, respectively, as illustrated in Fig. 5.1. The magnitudes of projections are

$$|a_1| = \mu_{A_L}(x) = M_A^-(x), \quad |a_2| = \mu_{A_U}(x) = M_A^+(x), \tag{5.11}$$

and

$$\begin{aligned} |b_1| &= 1 - \nu_{A_L}(x) = \mu_{A_L}(x) + \pi_{A_L}(x) = M_A^-(x) + \pi_{A_L}(x), \\ |b_2| &= 1 - \nu_{A_L}(x) = \mu_{A_L}(x) + \pi_{A_L}(x) = M_A^+(x) + \pi_{A_U}(x). \end{aligned} \tag{5.12}$$

Moreover, from (5.11) and (5.12) we deduce that the magnitudes of the projections on the membership axis equal the min-sigma-counts of A_L and A_U, respectively, in the case of an one element universe $X = \{x\}$, while the magnitudes of the projections on the non-membership axis coincide with the max-sigma-counts, respectively. Thus, from (5.10), the magnitude $|a|$ of the projection on the membership axis equal the min-sigma-count of A, while the magnitude $|b|$ of the projection on the non-membership axis coincides with the max-sigma-count.

5.2 Subsethood for IVIFSs

We axiomatize the properties of subsethood for IVIFSs, as well as extend the works of Liu and Xiong [34] and Vlachos and Sergiadis [51], in order to establish a connection between subsethood, entropy and cardinality in interval-valued intuitionistic fuzzy setting.

Definition 5.2.1 A real function S_h : IVIFS(X) $\times$ IVIFS(X) $\rightarrow [0, 1]$ is called *subsethood measure* of IVIFSs if S_h satisfies the following properties:

(SH1) $S_h(A, B) = 1$ iff $A \subset B$;

(SH2) If $A^c \subset A$, then $S_h(A, A^c) = 0$ iff $A = \mathcal{I}$;

(SH3) If $B \subset A_1 \subset A_2$, then $S_h(A_1, B) \geq S_h(A_2, B)$, and if $B_1 \subset B_2$, then $S_h(A, B_1) \leq S_h(A, B_2)$.

Theorem 5.2.2 *For two* IVIFS*s A and B on finite universe X,*

$$\begin{aligned} &S_h(A,B) \\ &\quad = 1 - \frac{\sum_{x_i \in X}(\max\{0, M_A^-(x_i) - M_B^-(x_i)\} + \max\{0, M_A^+(x_i) - M_B^+(x_i)\}}{\sum_{x_i \in X}(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)))} \\ &\qquad\quad \frac{+\max\{0, N_B^-(x_i) - N_A^-(x_i)\} + \max\{0, N_B^+(x_i) - N_A^+(x_i)\})}{\sum_{x_i \in X}(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)))} \end{aligned} \tag{5.13}$$

is a subsethood measure of IVIFS*s.*

Proof (SH1) Let $A \subset B$. Then $M_A^-(x_i) \le M_B^-(x_i)$, $M_A^+(x_i) \le M_B^+(x_i)$, $N_A^-(x_i) \ge N_B^-(x_i)$ and $N_A^+(x_i) \ge N_B^+(x_i)$ for all $x_i \in X$. Thus, we have $\max\{0, M_A^-(x_i) - M_B^-(x_i)\} = 0$, $\max\{0, M_A^+(x_i) - M_B^+(x_i)\} = 0$, $\max\{0, N_B^-(x_i) - N_A^-(x_i)\} = 0$ and $\max\{0, N_B^+(x_i) - N_A^+(x_i)\} = 0$ for all $x_i \in X$. Therefore, we obtain $S_h(A,B) = 1$. Suppose now that $S_h(A,B) = 1$. Then,

$$\begin{aligned} \sum_{x_i \in X} &(\max\{0, M_A^-(x_i) - M_B^-(x_i)\} + \max\{0, M_A^+(x_i) - M_B^+(x_i)\} \\ &+ \max\{0, N_B^-(x_i) - N_A^-(x_i)\} + \max\{0, N_B^+(x_i) - N_A^+(x_i)\}) = 0. \end{aligned}$$

Since every term of the sum is non-negative, we deduce $\max\{0, M_A^-(x_i) - M_B^-(x_i)\} = 0$, $\max\{0, M_A^+(x_i) - M_B^+(x_i)\} = 0$, $\max\{0, N_B^-(x_i) - N_A^-(x_i)\} = 0$ and $\max\{0, N_B^+(x_i) - N_A^+(x_i)\} = 0$ for all $x_i \in X$, which implies that $M_A^-(x_i) \le M_B^-(x_i)$, $M_A^+(x_i) \le M_B^+(x_i)$, $N_A^-(x_i) \ge N_B^-(x_i)$ and $N_A^+(x_i) \ge N_B^+(x_i)$. Hence $A \subset B$.

(SH2) From (5.13) we obtain that

$$\begin{aligned} &S_h(A, A^c) \\ &\quad = 1 - \frac{2\sum_{x_i \in X}(\max\{0, M_A^-(x_i) - N_A^-(x_i)\} + \max\{0, M_A^+(x_i) - N_A^+(x_i)\}}{\sum_{x_i \in X}(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))}. \end{aligned} \tag{5.14}$$

Assume that $A = \mathcal{I} \supset A^c$. Evaluating (5.14) for $A = \mathcal{I}$, we deduce that $S_h(A, A^c) = 0$. Let us now consider that $S_h(A, A^c) = 0$ and $A^c \subset A$. Then, (5.13) yields

$$\begin{aligned} \sum_{x_i \in X} &[(2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)) \\ &-2(\max\{0, M_A^-(x_i) - N_A^-(x_i)\} + \max\{0, M_A^+(x_i) - N_A^+(x_i)\})] = 0. \end{aligned} \tag{5.15}$$

Since $A^c \subset A$, we obtain that $N_A^-(x_i) \leq M_A^-(x_i)$ and $N_A^+(x_i) \leq M_A^+(x_i)$ for all x_i. Hence $\max\{0, M_A^-(x_i) - N_A^-(x_i)\} + \max\{0, M_A^+(x_i) - N_A^+(x_i)\} = (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))$ and from (5.15) we derive that $\sum_{x_i \in X}(2 - (M_A^-(x_i) - N_A^-(x_i)) - (M_A^+(x_i) - N_A^+(x_i))) = 0$. However, $2 - (M_A^-(x_i) - N_A^-(x_i)) - (M_A^+(x_i) - N_A^+(x_i)) \geq 0$ for all $x_i \in X$. Thus, every summand should equal zero, that is $2 - (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i)) = 0$ for all $x_i \in X$. Therefore, $A = \mathcal{I}$.

(SH3) Let $A_1, A_2, B \in \mathrm{IVIFS}(X)$ such that $B \subset A_1 \subset A_2$. Since $B \subset A_1$, we obtain

$$S_h(A_1, B) = \frac{\sum_{x_i \in X}(2 + (M_B^-(x_i) - N_B^-(x_i)) + (M_B^+(x_i) - N_B^+(x_i)))}{\sum_{x_i \in X}(2 + (M_{A_1}^-(x_i) - N_{A_1}^-(x_i)) + (M_{A_1}^+(x_i) - N_{A_1}^+(x_i)))}. \tag{5.16}$$

Similarly, we get

$$S_h(A_2, B) = \frac{\sum_{x_i \in X}(2 + (M_B^-(x_i) - N_B^-(x_i)) + (M_B^+(x_i) - N_B^+(x_i)))}{\sum_{x_i \in X}(2 + (M_{A_2}^-(x_i) - N_{A_2}^-(x_i)) + (M_{A_2}^+(x_i) - N_{A_2}^+(x_i)))}. \tag{5.17}$$

Moreover since $A_1 \subset A_2$, we obtain that $M_{A_2}^-(x_i) - N_{A_2}^-(x_i) \geq M_{A_1}^-(x_i) - N_{A_1}^-(x_i)$ and $M_{A_2}^+(x_i) - N_{A_2}^+(x_i) \geq M_{A_1}^+(x_i) - N_{A_1}^+(x_i)$ for all $x_i \in X$. Hence, $\sum_{x_i \in X}(2 + (M_{A_1}^-(x_i) - N_{A_1}^-(x_i)) + (M_{A_1}^+(x_i) - N_{A_1}^+(x_i))) \leq \sum_{x_i \in X}(2 + (M_{A_2}^-(x_i) - N_{A_2}^-(x_i)) + (M_{A_2}^+(x_i) - N_{A_2}^+(x_i)))$. Thus, from (5.16) and (5.17), we obtain $S_h(A_1, B) \geq S_h(A_2, B)$.

Now assume that $B_1 \subset B_2$. Then $M_A^-(x_i) - M_{B_1}^-(x_i) \geq M_A^-(x_i) - M_{B_2}^-(x_i)$, $M_A^+(x_i) - M_{B_1}^+(x_i) \geq M_A^+(x_i) - M_{B_2}^+(x_i)$, $N_{B_1}^-(x_i) - N_A^-(x_i) \geq N_{B_2}^-(x_i) - N_A^-(x_i)$ and $N_{B_1}^+(x_i) - N_A^+(x_i) \geq N_{B_2}^+(x_i) - N_A^+(x_i)$ for all $x_i \in X$. Due to the monotonicity of the max operator, it follows that

$$\begin{aligned}
\sum_{x_i \in X} (\, & \max\{0, M_A^-(x_i) - M_{B_1}^-(x_i)\} + \max\{0, M_A^+(x_i) - M_{B_1}^+(x_i)\} \\
& + \max\{0, N_{B_1}^-(x_i) - N_A^-(x_i)\} + \max\{0, N_{B_1}^+(x_i) - N_A^+(x_i)\}) \\
\geq \sum_{x_i \in X} & (\max\{0, M_A^-(x_i) - M_{B_2}^-(x_i)\} + \max\{0, M_A^+(x_i) - M_{B_2}^+(x_i)\} \\
& + \max\{0, N_{B_2}^-(x_i) - N_A^-(x_i)\} + \max\{0, N_{B_2}^+(x_i) - N_A^+(x_i)\}).
\end{aligned}$$

Therefore, $S_h(A, B_1) \leq S_h(A, B_2)$. $\square$

Remark 5.2.3 Note that if $A = \mathcal{O}$, (5.13) is undefined, due to the fact that $\mathcal{M}(\mathcal{O}) = 0$. So, by definition, for any IVIFS B, we have that $S_h(\mathcal{O}, B) = 1$, since $\mathcal{O}$ is a proper subset of any IVIFS.

5.3 Relationship between subsethood measure and entropy of IVIFSs

Generalizing the works of Kosko [27, 28, 30] and Vlachos and Sergiadis [51], we state the entropy-subsethood theorem for IVIFSs, based on the axiomatic skeleton (SH1)-(SH3).

Theorem 5.3.1 *If* $S_h : \mathrm{IVIFS}(X) \times \mathrm{IVIFS}(X) \to [0, 1]$ *is a subsethood measure, then* $E : \mathrm{IVIFS}(X) \to [0, 1]$ *given by*

$$E(A) = S_h(A \cup A^c, A \cap A^c) \tag{5.18}$$

is an entropy of IVIFS A.

Proof (E1) Let A be a crisp set. Then $A \cup A^c = \mathcal{I}$ and $A \cap A^c = \mathcal{O}$. Since $A \cap A^c = (A \cup A^c)^c$, we have $A \cup A^c = \mathcal{I} \supset (A \cup A^c)^c$ and thus from (SH2) we obtain $E(A) = 0$. Suppose that $E(A) = 0$; that is $S(A \cup A^c, A \cap A^c) = 0$, which can be written as $(A \cup A^c, (A \cup A^c)^c) = 0$. Then, since $A \cup A^c \supset A \cap A^c = (A \cup A^c)^c$, by (SH2) we obtain $A \cup A^c = \mathcal{I}$. Hence A is a crisp set.

(E2) Let us consider that $M_A(x_i) = N_A(x_i)$ for all $x_i \in X$. Then, $A \cup A^c = A \cap A^c = A = A^c$ and thus from (SH1), we obtain $E(A) = 1$. Let us assume that $E(A) = 1$; that is $S(A \cup A^c, A \cap A^c) = 1$. Then, from (SH1), we deduce that $A \cup A^c \subset A \cap A^c$. However, for any IVIF set A it hold that $A \cup A^c \supset A \cap A^c$. Hence, $A \cup A^c = A \cap A^c$, which implies $M_A(x_i) = N_A(x_i)$ for all $x_i \in X$.

(E3) Suppose that A refines B. Then, from Corollary 2.0.10 and (SH3), we derive that $E(A) = S(A \cup A^c, A \cap A^c) \leq S(B \cup B^c, B \cap B^c) = E(B)$. Hence $E(A) \leq E(B)$.

(E4) It is evident that $E(A^c) = S(A^c \cup A, A^c \cap A) = E(A)$. $\square$

Remark 5.3.2 Theorem 5.3.1 describes an interesting relationship between the entropy and subsethood measure for IVIFSs. It states that for IVIFSs the entropy of (5.18) expresses the degree to which the superset $A \cap A^c$ is a subset of its own subset $A \cap A^c$. Evaluating (5.18) for the proposed subsethood measure S_h of (5.13), yields a new entropy for IVIFSs given by

$$E(A) = \frac{\sum_{x_i \in X}\Big(2 - \max\{M_A^-(x_i), N_A^-(x_i)\} + \min\{M_A^-(x_i), N_A^-(x_i)\} - \max\{M_A^+(x_i), N_A^+(x_i)\} + \min\{M_A^+(x_i), N_A^+(x_i)\}\Big)}{\sum_{x_i \in X}\Big(2 + \max\{M_A^-(x_i), N_A^-(x_i)\} - \min\{M_A^-(x_i), N_A^-(x_i)\} + \max\{M_A^+(x_i), N_A^+(x_i)\} - \min\{M_A^+(x_i), N_A^+(x_i)\}\Big)}. \quad (5.19)$$

Since $S_h(A,B)$ satisfies the axiomatic requirements (SH1)-(SH3), by Theorem 5.3.1, $E(A)$ is an entropy measure.

Now, we state a relationship between the entropy and average possible cardinality of IVIFSs, which generalize the works of Kosko [27, 28, 30] and Vlachos and Sergiadis [51].

Theorem 5.3.3 *If $\mathcal{M}$ is an average possible cardinality of* IVIFS*s on X and $A \in$* IVIFS*(X), then $E :$* IVIFS*$(X) \to [0,1]$ given by*

$$E(A) = \frac{\mathcal{M}(A \cap A^c)}{\mathcal{M}(A \cup A^c)} \quad (5.20)$$

is an entropy of IVIFS *A.*

Proof For $A \in \text{IVIFS}(X)$ and its complement A^c, it hold that

$$A \cup A^c = \{(x_i, [\max\{M_A^-(x_i), N_A^-(x_i)\}, \max\{M_A^+(x_i), N_A^+(x_i)\}], [\min\{N_A^-(x_i), M_A^-(x_i)\}, \min\{N_A^+(x_i), M_A^+(x_i)\}] : x_i \in X\} \quad (5.21)$$

and

$$A \cap A^c = \{(x_i, [\min\{M_A^-(x_i), N_A^-(x_i)\}, \min\{M_A^+(x_i), N_A^+(x_i)\}], [\max\{N_A^-(x_i), M_A^-(x_i)\}, \max\{N_A^+(x_i), M_A^+(x_i)\}] : x_i \in X\}. \quad (5.22)$$

From the definition of average possible cardinality we obtain that

$$\mathcal{M}(A \cup A^c) = \frac{1}{4} \sum_{x_i \in X} \Big(2 + \max\{M_A^-(x_i), N_A^-(x_i)\} - \min\{M_A^-(x_i), N_A^-(x_i)\} + \max\{M_A^+(x_i), N_A^+(x_i)\} - \min\{M_A^+(x_i), N_A^+(x_i)\}\Big) \quad (5.23)$$

and

$$\mathcal{M}(A \cap A^c) = \frac{1}{4} \sum_{x_i \in X} \Big(2 + \min\{M_A^-(x_i), N_A^-(x_i)\} - \max\{M_A^-(x_i), N_A^-(x_i)\} + \min\{M_A^+(x_i), N_A^+(x_i)\} - \max\{M_A^+(x_i), N_A^+(x_i)\}\Big). \quad (5.24)$$

Substituting (5.23) and (5.24) into (5.19) yields (5.20). This completes the proof. □

5.4 Modified entropy-subsethood theorem

In [47], the intuitionistic fuzzy entropy, as a generalized form of the fuzzy entropy, was defined as

$$E_{\mathrm{SK}}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\max Count(A_i \cap A_i^c)}{\max Count(A_i \cup A_i^c)}, \quad (5.25)$$

where n is the cardinality of the finite universe X and A_i denotes the single-element IFS corresponding to the ith element of the universe X and is described as $A_i = \{\langle x_i, \mu_{A_i}(x_i), \nu_{A_i}(x_i)\rangle\}$. Szmidt and Kacprzyk [47] used the notation $\max Count(A_i)$ instead of $\max \sum Count(A_i)$ to denote the biggest cardinality of A_i, since A_i contains only one element.

A desirable property an entropy measure should possess, is described as

$$E(A) = \sum_{i=1}^{n} E(A_i), \quad (5.26)$$

which states that the sum of the entropy of separate of a set is equal to the entropy of the set. Comparing E_{SK} with the proposed entropy E for IVIFSs, it is evident that E does not satisfy (5.26), while E_{SK} does; that is

$$E(A) \neq \sum_{i=1}^{n} E(A_i) \quad \text{and} \quad E_{\mathrm{SK}}(A) = \sum_{i=1}^{n} E_{\mathrm{SK}}(A_i). \tag{5.27}$$

In order to overcome the above-mentioned drawback, we can consider the analysis of Sections 5.2 and 5.3 to be carried out element-wisely, instead of considering the entire set. Thus, the following modified subsethood measure for IVIFSs is obtained

$$\begin{aligned} &S'_h(A,B) \\ &\quad = 1 - \frac{1}{n} \sum_{x_i \in X} \frac{(\max\{0, M_A^-(x_i) - M_B^-(x_i)\} + \max\{0, M_A^+(x_i) - M_B^+(x_i)\}}{2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))} \\ &\qquad\qquad \frac{+\max\{0, N_B^-(x_i) - N_A^-(x_i)\} + \max\{0, N_B^+(x_i) - N_A^+(x_i)\})}{2 + (M_A^-(x_i) - N_A^-(x_i)) + (M_A^+(x_i) - N_A^+(x_i))}. \end{aligned} \tag{5.28}$$

It is easy to verify that S'_h also satisfies the axiomatic properties (SH1)-(SH3). Thus, from the entropy-subsethood theorem for IVIFSs of (5.18), the following modified entropy measure E' is derived

$$\begin{aligned} E'(A) = \frac{1}{n} \sum_{x_i \in X} &\frac{(2 - \max\{M_A^-(x_i), N_A^-(x_i)\} + \min\{M_A^-(x_i), N_A^-(x_i)\}}{(2 + \max\{M_A^-(x_i), N_A^-(x_i)\} - \min\{M_A^-(x_i), N_A^-(x_i)\}} \\ &\frac{-\max\{M_A^+(x_i), N_A^+(x_i)\} + \min\{M_A^+(x_i), N_A^+(x_i)\})}{+\max\{M_A^+(x_i), N_A^+(x_i)\} - \min\{M_A^+(x_i), N_A^+(x_i)\})}, \end{aligned} \tag{5.29}$$

which can be re-written as

$$E'_{\mathrm{IVIFS}}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\mathcal{M}(A_i \cap A_i^c)}{\mathcal{M}(A_i \cup A_i^c)}. \tag{5.30}$$

From (5.29) and (5.30), we obtain $E'(A) = \sum_{i=1}^{n} E'(A_i)$. Note that in (5.30) the average possible cardinality $\mathcal{M}$ is calculated over the single-element universe $X = \{x_i\}$, where the set A_i is defined. One may easily observe the connections of (5.28) and (5.29) with (5.13) and (5.19), respectively. Finally, if A is either an IVFS such that $M_A^+ = (M_A^-)^c$ or a fuzzy set, then (5.29) and (5.30) reduce, respectively, the following modified entropy measures E'_{IVFS} proposed by Vlachos and Sergiadis [51] for IVFSs and E_{SJ} proposed by Shang and Jiang [42] for fuzzy sets

$$E'_{\mathrm{IVFS}}(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mathcal{M}(A_i \cap A_i^c)}{\mathcal{M}(A_i \cup A_i^c)} \tag{5.31}$$

and

$$E_{\mathrm{SJ}}(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\mu_{A\cap A^c}(x_i)}{\mu_{A\cup A^c}(x_i)}. \tag{5.32}$$

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