Thesis for the Degree of Master of Science

Weak and Strong Forms of b-irresolute Functions

by

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b-irresolute 함수의 약과 강한 형태들

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b-irresolute 함수의 약과 강한 형태들

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요약

본 논문에서는 약한 b-irresolute 함수와 강한 b-irresolute 함수의 기본적인 성질과 다른 함수들과의 관계, 위상적 성질인 피복성 및 분리공리를 다음 내용을 중심으로 조사한다.

첫째로, 위상공간 X의 부분집합 A에 있어서의 b-open, b-closed, b-regular의 정의와 성질들과 그에 따른 정리들을 알아본다.

둘째로, 약하고 강한 b-irresolute 함수와 b-연속 함수들과의 관계를 예제를 이용하여 살펴보고, 약한 b-irresolute 함수의 성질들을 알아본다.

셋째로, 강한 b-irresolute 함수의 약한 b-irresolute 함수, b-irresolute 함수, graph function g와의 관계와 b-T_1, b-T_2 공간에서의 성질들을 조사한다.

넷째로, 강한 b-irresolute 함수의 강한 b-regular 함수, b-irresolute 함수, α-open 함수, α-연속 함수 와의 관계를 조사하고, 강한 b-irresolute 성과 피복성과의 관계를 조사한다.

1 Introduction

Andrijević [4] introduced the notion of b-open sets which is weaker than those of both preopen sets [10] and semiopen sets [9] and is stronger than that of β-open sets [1]. El-Atik [7] and Dontchev and Przemski [6] called b-open sets by sp-open sets and γ-open sets, respectively. By using b-open sets, Nasef [12] introduced the notions of b-locally closed sets and b-LC-continuity and discussed some of their properties. El-Atik [7] used b-open sets to define b-continuity in topological spaces. Dontchev and Przemski [6] called b-continuity by sp-continuity and used this notion to obtain a decomposition of precontinuity [10]. The notion of b-irresoluteness in topological spaces is introduced by Ha [8]. Recently, Park [14] introduced the notions of b-θ-open sets and strong θ-b-continuity and obtained some characterizations and several properties concerning strongly θ-b-continuous functions.

The purpose of this thesis is to introduce and investigate some of the fundamental properties of weakly b-irresolute and strongly b-irresolute functions. The relations with above-mentioned notions directly or indirectly connected with weak and strong b-irresoluteness are investigated. In Section 3, we obtain characterizations and basic properties of weakly b-irresolute functions. In Section 4, we investigate relationships between weak b-irresoluteness and separation axioms and between weak b-irresoluteness and b-θ-closed graphs, respectively. In Section 5, we obtain characterizations of strongly b-irresolute functions and investigate relationships between strong b-irresoluteness and separation axioms. In the last section, we investigate relationships between strong b-irresoluteness and covering properties.

2 Preliminaries

Throughout this thesis, spaces X and Y always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. We denote the closure and the interior of a set A by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$, respectively. A point x of X is called a θ-cluster [18] point of A if $\mathrm{cl}(U) \cap A \neq \emptyset$ for every open set U of X containing x. The set of all θ-cluster points of A is called the θ-closure [18] of A and is denoted by $\mathrm{cl}_\theta(A)$. A subset A is said to be θ-closed [18] if $\mathrm{cl}_\theta(A) = A$. The complement of a θ-closed set is said to be θ-open.

A subset A is said to be *α-open* [13] (resp. *preopen* [10], *semi-open* [9], *b-open* [4], *semi-preopen* [3] or *β-open* [1]) if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ (resp. $A \subset \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$). The complement of an α-open (resp. preopen, semi-open, b-open, β-open) set is said to be *α-closed* (resp. *preclosed*, *semi-closed*, *b-closed*, *β-closed*). The intersection of all b-closed sets of X containing A is called the *b-closure* [4] of A and is denoted by $\mathrm{bcl}(A)$. The semi-closure and preclosure are similarly defined and are denoted by $\mathrm{scl}(A)$ and $\mathrm{pcl}(A)$. The union of all b-open sets of X contained in A is called *b-interior* [4] and is denoted by $\mathrm{bint}(A)$. A subset A is said to be *b-regular* [14] if it is b-open and b-closed. The family of all b-open (resp. b-closed, b-regular) sets of X is denoted by $\mathrm{BO}(X)$ (resp. $\mathrm{BC}(X)$, $\mathrm{BR}(X)$) and the family of all b-open (resp. b-regular) sets of X containing a point $x \in X$ is denoted by $\mathrm{BO}(X, x)$ (resp. $\mathrm{BR}(X, x)$).

The following basic properties of b-closure are useful in the sequel:

Lemma 2.1 (Andrijevic [4]) *For a subset A of a space X, the following hold:*

(a) $\mathrm{bcl}(A) = \mathrm{scl}(A) \cap \mathrm{pcl}(A)$;
(b) $\mathrm{bint}(A) = \mathrm{sint}(A) \cup \mathrm{pint}(A)$;
(c) $\mathrm{bcl}(X \setminus A) = X \setminus \mathrm{bint}(A)$;
(d) $x \in \mathrm{bcl}(A)$ *if and only if* $A \cap U \neq \emptyset$ *for every* $U \in \mathrm{BO}(X, x)$;
(e) $A \in \mathrm{BC}(X)$ *if and only if* $A = \mathrm{bcl}(A)$.

Theorem 2.2 (Park [14]) *Let A be a subset of a space X. Then*

(a) $A \in \mathrm{BO}(X)$ *if and only if* $\mathrm{bcl}(A) \in \mathrm{BR}(X)$.

(b) $A \in \mathrm{BC}(X)$ *if and only if* $\mathrm{bint}(A) \in \mathrm{BR}(X)$.

A point x of X is called a *b-θ-cluster point* [14] of A if $\mathrm{bcl}(U) \cap A \neq \emptyset$ for every $U \in \mathrm{BO}(X, x)$. The set of all b-θ-cluster points of A is called *b-θ-closure* [14] of A and denoted by $\mathrm{bcl}_\theta(A)$. A subset A is said to be *b-θ-closed* [14] if $A = \mathrm{bcl}_\theta(A)$. The complement of a b-θ-closed set is said to be *b-θ-open* [14].

Theorem 2.3 (Park [14]) *Let A and A_α $(\alpha \in I)$ be any subsets of a space X. Then the following properties hold:*

(a) *A is b-θ-open in X if and only if for each $x \in A$ there exists $V \in \mathrm{BR}(X, x)$ such that $x \in V \subset A$,*

(b) $\mathrm{bcl}_\theta(A)$ *is b-θ-closed,*

(c) *if A_α is b-θ-open in X for each $\alpha \in I$, then $\bigcup_{\alpha \in I} A_\alpha$ is b-θ-open in X.*

Theorem 2.4 (Park [14]) *For a subset A of a space X, the following properties hold:*

(a) *if $A \in \mathrm{BO}(X)$, then* $\mathrm{bcl}(A) = \mathrm{bcl}_\theta(A)$,

(b) $A \in \mathrm{BR}(X)$ *if and only if A is b-θ-open and b-θ-closed.*

3 Characterizations of weakly b-irresolute functions

Definition 3.1 A function $f : X \to Y$ is said to be

(a) *b-continuous* [7] if $f^{-1}(V) \in \mathrm{BO}(X)$ for each open set V of Y;

(b) *almost b-continuous* if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \mathrm{BO}(X, x)$ such that $f(U) \subset \mathrm{int}(\mathrm{cl}(V))$;

(c) *strongly θ-b-continuous* [14] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset V$.

Definition 3.2 A function $f : X \to Y$ is said to be

(a) *b-irresolute* [8] if $f^{-1}(V) \in \mathrm{BO}(X)$ for each $V \in \mathrm{BO}(Y)$;

(b) *strongly b-irresolute* if for each $x \in X$ and each $V \in \mathrm{BO}(Y, f(x))$, there exists a $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset V$;

(c) *weakly b-irresolute* if for each $x \in X$ and each $V \in \mathrm{BO}(Y, f(x))$, there exists a $U \in \mathrm{BO}(X, x)$ such that $f(U) \subset \mathrm{bcl}(V)$.

Remark 3.3 From Definitions 3.1 and 3.2, we have the following diagram for a function $f : X \to Y$:

$$\begin{array}{ccccc}
\text{strongly } b\text{-irresolute} & \Rightarrow & b\text{-irresolute} & \Rightarrow & \text{weakly } b\text{-irresolute} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{strongly } \theta\text{-}b\text{-continuous} & \Rightarrow & b\text{-continuous} & \Rightarrow & \text{almost } b\text{-continuous}
\end{array}$$

However, none of these implications is reversible as shown by the following examples. Moreover, strong θ-b-continuity and weak b-irresoluteness are independent of each other as the following examples show.

Example 3.4 Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{a\}, \{b, c\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is strongly θ-b-continuous but it is not weakly b-irresolute.

Example 3.5 Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{a, b\}\}$. Then the identity function $f : (X, \tau) \to (X, \tau)$ is b-irresolute but it is not strongly θ-b-continuous.

Example 3.6 Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{X, \emptyset, \{c\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is weakly b-irresolute but it is not b-continuous.

Theorem 3.7 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is weakly b-irresolute;*

(b) *$f^{-1}(V) \subset \mathrm{bint}(f^{-1}(\mathrm{bcl}(V)))$ for each $V \in \mathrm{BO}(Y)$;*

(c) *$\mathrm{bcl}(f^{-1}(V)) \subset f^{-1}(\mathrm{bcl}(V))$ for each $V \in \mathrm{BO}(Y)$.*

Proof (a)$\Rightarrow$(b): Let $V \in \mathrm{BO}(Y)$ and $x \in f^{-1}(V)$. Then by (a), there exists $U \in \mathrm{BO}(X, x)$ such that $f(U) \subset \mathrm{bcl}(V)$. Therefore, we have $U \subset f^{-1}(\mathrm{bcl}(V))$ and $x \in U \subset \mathrm{bint}(f^{-1}(\mathrm{bcl}(V)))$. This shows that $f^{-1}(V) \subset \mathrm{bint}(f^{-1}(\mathrm{bcl}(V)))$.

(b)$\Rightarrow$(c): Let $V \in \mathrm{BO}(Y)$ and let $x \notin f^{-1}(\mathrm{bcl}(V))$. Then $f(x) \notin \mathrm{bcl}(V)$. There exists $W \in \mathrm{BO}(Y, f(x))$ such that $W \cap V = \emptyset$. Since $V \in \mathrm{BO}(Y)$, we have $\mathrm{bcl}(W) \cap V = \emptyset$ and hence $\mathrm{bint}(f^{-1}(\mathrm{bcl}(W))) \cap f^{-1}(V) = \emptyset$. By (b), we have

$$x \in f^{-1}(W) \subset \mathrm{bint}(f^{-1}(\mathrm{bcl}(W))) \in \mathrm{BO}(X).$$

Therefore, we obtain $x \notin \mathrm{bcl}(f^{-1}(V))$. This shows that $\mathrm{bcl}(f^{-1}(V)) \subset f^{-1}(\mathrm{bcl}(V))$.

(c)$\Rightarrow$(a): Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$. By Theorem 2.2, we have

$$\mathrm{bcl}(V) \in \mathrm{BR}(Y) \text{ and } x \notin f^{-1}(\mathrm{bcl}(Y \setminus \mathrm{bcl}(V))).$$

Since $Y \setminus \mathrm{bcl}(V) \in \mathrm{BO}(Y)$, by (c) we have $x \notin \mathrm{bcl}(f^{-1}(Y \setminus \mathrm{bcl}(V)))$. Hence there exists $U \in \mathrm{BO}(X, x)$ such that $U \cap f^{-1}(Y \setminus \mathrm{bcl}(V)) = \emptyset$. Therefore, we obtain $f(U) \cap (Y \setminus \mathrm{bcl}(V)) = \emptyset$ and hence $f(U) \subset \mathrm{bcl}(V)$. This shows that f is weakly b-irresolute.

Theorem 3.8 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is weakly b-irresolute;*

(b) *$\mathrm{bcl}(f^{-1}(B)) \subset f^{-1}(\mathrm{bcl}_\theta(B))$ for each subset B of Y;*

(c) *$f(\mathrm{bcl}(A)) \subset \mathrm{bcl}_\theta(f(A))$ for each subset A of X;*

(d) *$f^{-1}(F) \in \mathrm{BC}(X)$ for each b-θ-closed set F of Y;*

(e) *$f^{-1}(V) \in \mathrm{BO}(X)$ for each b-θ-open set V of Y.*

Proof (a)⇒(b): Let B be any subset of Y and $x \notin f^{-1}(\mathrm{bcl}_\theta(B))$. Then $f(x) \notin \mathrm{bcl}_\theta(B)$ and there exists $V \in \mathrm{BO}(Y, f(x))$ such that $\mathrm{bcl}(V) \cap B = \emptyset$. By (a), there exists $U \in \mathrm{BO}(X, x)$ such that $f(U) \subset \mathrm{bcl}(V)$. Therefore, we have $f(U) \cap B = \emptyset$ and $U \cap f^{-1}(B) = \emptyset$. Consequently, we have $x \notin \mathrm{bcl}(f^{-1}(B))$.

(b)⇒(c): Let A be any subset of X. Then by (b), we have

$$\mathrm{bcl}(A) \subset \mathrm{bcl}(f^{-1}(f(A))) \subset f^{-1}(\mathrm{bcl}_\theta(f(A)))$$

and hence $f(\mathrm{bcl}(A)) \subset \mathrm{bcl}_\theta(f(A))$.

(c)⇒(d): Let F be any b-θ-closed set of Y. Then by (c), we have

$$f(\mathrm{bcl}(f^{-1}(F))) \subset \mathrm{bcl}_\theta(f(f^{-1}(F))) \subset \mathrm{bcl}_\theta(F) = F.$$

Therefore, we have $\mathrm{bcl}(f^{-1}(F)) \subset f^{-1}(F)$ and hence $\mathrm{bcl}(f^{-1}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F) \in \mathrm{BC}(X)$.

(d)⇒(e): This proof is obvious and is omitted.

(e)⇒(a): Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$. By Theorems 2.2 and 2.4, $\mathrm{bcl}(V)$ is b-θ-open in Y. Put $U = f^{-1}(\mathrm{bcl}(V))$. Then by (e), we have $U \in \mathrm{BO}(X, x)$ and $f(U) \subset \mathrm{bcl}(V)$. This shows that f is weakly b-irresolute.

Theorem 3.9 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is weakly b-irresolute;*

(b) *for each $x \in X$ and each $V \in \mathrm{BO}(Y, f(x))$, there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset \mathrm{bcl}(V)$;*

(c) *$f^{-1}(F) \in \mathrm{BR}(X)$ for each $F \in \mathrm{BR}(Y)$.*

Proof (a)⇒(b): Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$. By Theorems 2.2 and 2.4, $\mathrm{bcl}(V)$ is b-θ-open and b-θ-closed in Y. Now, put $U = f^{-1}(\mathrm{bcl}(V))$. Then by Theorem 3.8, we have $U \in \mathrm{BR}(X)$. Therefore, we obtain $U \in \mathrm{BO}(X, x)$, $U = \mathrm{bcl}(U)$ and $f(\mathrm{bcl}(U)) \subset \mathrm{bcl}(V)$.

(b)⇒(c): Let $F \in \mathrm{BR}(Y)$ and $x \in f^{-1}(F)$. Then $f(x) \in F$. By (b), there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset F$. Therefore, we have $x \in U \subset \mathrm{bcl}(U) \subset f^{-1}(F)$ and hence $f^{-1}(F) \in \mathrm{BO}(X)$. Since $Y \setminus F \in \mathrm{BR}(Y)$, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \in \mathrm{BO}(X)$. Thus $f^{-1}(F) \in \mathrm{BC}(X)$ and hence $f^{-1}(F) \in \mathrm{BR}(X)$.

(c)⇒(a): Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$. By Theorem 2.2, $\mathrm{bcl}(V) \in \mathrm{BR}(Y, f(x))$ and $f^{-1}(\mathrm{bcl}(V)) \in \mathrm{BR}(X, x)$. Put $U = f^{-1}(\mathrm{bcl}(V))$. Then $U \in \mathrm{BO}(X, x)$ and $f(U) \subset \mathrm{bcl}(V)$. This shows that f is weakly b-irresolute.

Similarly to Theorems 3.7 and 3.8, we can obtain the characterizations of weakly b-irresolute functions as follows.

Theorem 3.10 *For a function $f : X \to Y$, the following are equivalent:*
(a) *f is weakly b-irresolute;*
(b) $f^{-1}(V) \subset \mathrm{bint}_\theta(f^{-1}(\mathrm{bcl}_\theta(V)))$ *for each* $V \in \mathrm{BO}(Y)$*;*
(c) $\mathrm{bcl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{bcl}_\theta(V))$ *for each* $V \in \mathrm{BO}(Y)$.

Theorem 3.11 *For a function $f : X \to Y$, the following are equivalent:*
(a) *f is weakly b-irresolute;*
(b) $\mathrm{bcl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{bcl}_\theta(B))$ *for each subset B of Y;*
(c) $f(\mathrm{bcl}_\theta(A)) \subset \mathrm{bcl}_\theta(f(A))$ *for each subset A of X;*
(d) $f^{-1}(F)$ *is b-θ-closed in X for each b-θ-closed set F of Y;*
(e) $f^{-1}(V)$ *is b-θ-open in X for each b-θ-open set V of Y.*

4 Properties of weakly b-irresolute functions

Definition 4.1 A space X is said to be *strongly b-regular* if for each $F \in \mathrm{BC}(X)$ and each $x \in X \setminus F$, there exist disjoint b-open sets U and V such that $x \in U$ and $F \subset V$.

Lemma 4.2 *For a space X the following are equivalent:*

(a) *X is strongly b-regular;*

(b) *for each $U \in \mathrm{BO}(X)$ and each $x \in U$, there exists $V \in \mathrm{BO}(X)$ such that $x \in V \subset \mathrm{bcl}(V) \subset U$;*

(c) *for each $U \in \mathrm{BO}(X)$ and each $x \in U$, there exists $V \in \mathrm{BR}(X)$ such that $x \in V \subset U$;*

(d) *for each subset A of X and each $F \in \mathrm{BC}(X)$ such that $A \cap F = \emptyset$, there exist disjoint $U, V \in \mathrm{BO}(X)$ such that $A \cap U \neq \emptyset$ and $F \subset V$;*

(e) *for each $F \in \mathrm{BC}(X)$, $F = \bigcap\{\mathrm{bcl}(V) : F \subset V \text{and } V \in \mathrm{BO}(X)\}$.*

Proof It follows from Theorem 2.2.

Theorem 4.3 *Let Y be a strongly b-regular space. Then the function $f : X \to Y$ is weakly b-irresolute if and only if it is b-irresolute.*

Proof Suppose that $f : X \to Y$ is weakly b-irresolute. Let $V \in \mathrm{BO}(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$ and since Y is b-regular, by Lemma 4.2, there exists $W \in \mathrm{BO}(Y)$ such that $f(x) \in W \subset \mathrm{bcl}(W) \subset V$. Since f is weakly b-irresolute, there exists $U \in \mathrm{BO}(X, x)$ such that $f(U) \subset \mathrm{bcl}(W)$. Therefore, we have $x \in U \subset f^{-1}(V)$ and hence $f^{-1}(V) \in \mathrm{BO}(X)$. This shows that f is b-irresolute. The converse is obvious.

Theorem 4.4 *A function $f : X \to Y$ is weakly b-irresolute if the graph function $g : X \to X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, is weakly b-irresolute.*

Proof Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$, Then $X \times V \in \mathrm{BO}(X \times Y)$ and $g(x) \in X \times V$. Since g is weakly b-irresolute, there exists $U \in \mathrm{BO}(X, x)$ such that $g(U) \subset \mathrm{bcl}(X \times V) \subset X \times \mathrm{bcl}(V)$. Therefore, we have $f(U) \subset \mathrm{bcl}(V)$.

Remark 4.5 The converse of Theorem 4.4 is not necessarily true as the following example shows.

Example 4.6 Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Define a function $f : (X, \tau) \to (X, \tau)$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is b-irresolute and hence weakly b-irresolute but the graph function g is not weakly b-irresolute.

Definition 4.7 A space X is said to be

(a) b-T_1 [15] if for each pair of distinct points x and y in X there exist $U \in \mathrm{BO}(X)$ containing x but not y and $V \in \mathrm{BO}(X)$ containing y but not x;

(b) b-T_2 [14] if for each pair of distinct points x and y in X, there exist $U \in \mathrm{BO}(X, x)$ and $V \in \mathrm{BO}(X, y)$ such that $U \cap V = \emptyset$.

In [14], Park obtained the following interesting result which is useful in the sequel:

Lemma 4.8 *A space X is b-T_2 if and only if for each pair of distinct points x and y in X, there exist $U \in \mathrm{BO}(X, x)$ and $V \in \mathrm{BO}(X, y)$ such that* $\mathrm{bcl}(U) \cap \mathrm{bcl}(V) = \emptyset$.

Theorem 4.9 *If Y is a b-T_2 space and $f : X \to Y$ is weakly b-irresolute injection, then X is b-T_2.*

Proof Let x, y be any distinct points of X. Then $f(x) \neq f(y)$. Since Y is b-T_2, by Lemma 4.8 there exist $V \in \mathrm{BO}(Y, f(x))$ and $W \in \mathrm{BO}(Y, f(y))$ such that $\mathrm{bcl}(V) \cap \mathrm{bcl}(W) = \emptyset$. Since f is weakly b-irresolute, there exist $G \in \mathrm{BO}(X, x)$ and $H \in \mathrm{BO}(X, y)$ such that $f(G) \subset \mathrm{bcl}(V)$ and $f(H) \subset \mathrm{bcl}(W)$. Hence we obtain $G \cap H = \emptyset$. This shows that X is b-T_2.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$ is called the graph of f is denoted by $G(f)$.

Definition 4.10 A function $f : X \to Y$ is said to have a *b-θ-closed graph* if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in \mathrm{BO}(X, x)$ and $V \in \mathrm{BO}(Y, y)$ such that $[\mathrm{bcl}(U) \times \mathrm{bcl}(V)] \cap G(f) = \emptyset$.

Theorem 4.11 *If Y is a b-T_2 space and $f : X \to Y$ is weakly b-irresolute, then $G(f)$ is b-θ-closed.*

Proof Let $(x, y) \in (X \times Y) \setminus G(f)$. Then $y \neq f(x)$ and by Lemma 4.8 there exist $V \in \mathrm{BO}(Y, f(x))$ and $W \in \mathrm{BO}(Y, y)$ such that $\mathrm{bcl}(V) \cap \mathrm{bcl}(W) = \emptyset$. Since f is weakly b-irresolute, by Theorem 3.9 there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset \mathrm{bcl}(V)$. Therefore, we obtain $f(\mathrm{bcl}(U)) \cap \mathrm{bcl}(W) = \emptyset$ and hence $[\mathrm{bcl}(U) \times \mathrm{bcl}(W)] \cap G(f) = \emptyset$. This shows that $G(f)$ is b-θ-closed in $X \times Y$.

Definition 4.12 A space X is said to be *b-connected* if it cannot be written as the union of two nonempty disjoint b-open sets.

Theorem 4.13 *If a function $f : X \to Y$ is a weakly b-irresolute surjection and X is b-connected, then Y is b-connected.*

Proof Suppose that Y is not b-connected. There exist nonempty b-open sets V and W of Y such that $V \cup W = Y$ and $V \cap W = \emptyset$. Then we have V, $W \in \mathrm{BR}(Y)$. Since f is weakly b-irresolute, by Theorem 3.9, we have $f^{-1}(V)$, $f^{-1}(W) \in \mathrm{BR}(X)$. Moreover, we have $f^{-1}(V) \cup f^{-1}(W) = X$, $f^{-1}(V) \cap f^{-1}(W) = \emptyset$, and $f^{-1}(V)$ and $f^{-1}(W)$ are nonempty. Therefore, X is not b-connected.

5 Strongly b-irresolute functions

Theorem 5.1 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is strongly b-irresolute;*

(b) *for each $x \in X$ and each $V \in \mathrm{BO}(Y, f(x))$, there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}_\theta(U)) \subset V$;*

(c) *for each $x \in X$ and each $V \in \mathrm{BO}(Y, f(x))$, there exists $U \in \mathrm{BR}(X, x)$ such that $f(U) \subset V$;*

(d) *for each $x \in X$ and each $V \in \mathrm{BO}(Y, f(x))$, there exists a b-θ-open set U of X such that $f(U) \subset V$;*

(e) *$f^{-1}(V)$ is b-θ-open in X for each $V \in \mathrm{BO}(Y)$;*

(f) *$f^{-1}(V)$ is b-θ-closed in X for each $V \in \mathrm{BC}(Y)$;*

(g) *$f(\mathrm{bcl}_\theta(A)) \subset \mathrm{bcl}(f(A))$ for each subset A of X;*

(h) *$\mathrm{bcl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{bcl}(B))$ for each subset B of Y.*

Proof (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d): It follows from Theorems 2.1 and 2.4.

(d)$\Rightarrow$(e): Let $V \in \mathrm{BO}(Y)$ and $x \in f^{-1}(V)$. Then $f(x) \in V$ and by (d), there exists a b-θ-open set U of X containing x such that $f(U) \subset V$. Therefore, we have $x \in U \subset f^{-1}(V)$. Since the union of b-θ-open sets is b-θ-open [Theorem 2.3], $f^{-1}(V)$ is b-θ-open in X.

(e)$\Rightarrow$(f): Obvious.

(f)$\Rightarrow$(g): Let A be any subset of X. Since $\mathrm{bcl}(f(A))$ is b-closed in Y, by (f) we have $f^{-1}(\mathrm{bcl}(f(A)))$ is b-θ-closed in X and

$$\mathrm{bcl}_\theta(A) \subset \mathrm{bcl}_\theta(f^{-1}(f(A))) \subset \mathrm{bcl}_\theta(f^{-1}(\mathrm{bcl}(f(A)))) = f^{-1}(\mathrm{bcl}(f(A))).$$

Therefore, we obtain $f(\mathrm{bcl}_\theta(A)) \subset \mathrm{bcl}(f(A))$.

(g)$\Rightarrow$(h): Let B be any subset of Y. By (g), we obtain

$$f(\mathrm{bcl}_\theta(f^{-1}(B))) \subset \mathrm{bcl}(f(f^{-1}(B))) \subset \mathrm{bcl}(B)$$

and hence $\mathrm{bcl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{bcl}(B))$.

(h)$\Rightarrow$(a): Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$. Since $Y \setminus V \in \mathrm{BC}(Y)$, we have $\mathrm{bcl}_\theta(f^{-1}(Y \setminus V)) \subset f^{-1}(\mathrm{bcl}(Y \setminus V)) = f^{-1}(Y \setminus V)$. Therefore, $f^{-1}(Y \setminus V)$ is

b-θ-closed in X and hence $f^{-1}(V)$ is b-θ-open in X and $x \in f^{-1}(V)$. Then there exists $U \in \mathrm{BO}(X, x)$ such that $\mathrm{bcl}(U) \subset f^{-1}(V)$ and thus $f(\mathrm{bcl}(U)) \subset V$. This shows that f is strongly b-irresolute.

Theorem 5.2 *A b-irresolute function $f : X \to Y$ is strongly b-irresolute if and only if X is strongly b-regular.*

Proof *Necessity.* Let $f : X \to Y$ be identity function. Then f is b-irresolute and strongly b-irresolute by our hypothesis. For any $U \in \mathrm{BO}(X)$ and any point $x \in U$, we have $f(x) = x \in U$ and there exists $G \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(G)) \subset U$. Therefore, we have $x \in G \subset \mathrm{bcl}(G) \subset U$. It follows from Lemma 4.2 that X is strongly b-regular.

Sufficiency. Suppose that $f : X \to Y$ is b-irresolute and X is strongly b-regular. For any $x \in X$ and any $V \in \mathrm{BO}(Y, f(x))$, $f^{-1}(V)$ is b-open set containing x. Since X is strongly b-regular, there exists $U \in \mathrm{BO}(X)$ such that $x \in U \subset \mathrm{bcl}(U) \subset f^{-1}(V)$. Therefore, we have $f(\mathrm{bcl}(U)) \subset V$. This shows that f is strongly b-irresolute.

Theorem 5.3 *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function of f. If g is strongly b-irresolute, then f is strongly b-irresolute and X is strongly b-regular.*

Proof Suppose that g is strongly b-irresolute. First, we show that f is strongly b-irresolute. Let $x \in X$ and $V \in \mathrm{BO}(Y, f(x))$. Then $X \times V$ is a b-open set of $X \times Y$ containing $g(x)$. Since g is strongly b-irresolute, there exists $U \in \mathrm{BO}(X, x)$ such that $g(\mathrm{bcl}(U)) \subset X \times V$. Therefore, we obtain $f(\mathrm{bcl}(U)) \subset V$. This shows that f is strongly b-irresolute. Next, we show that X is strongly b-regular. Let $U \in \mathrm{BO}(X)$ and $x \in U$. Since $g(x) \in U \times Y$ and $U \times Y$ is b-open in $X \times Y$, there exists $G \in \mathrm{BO}(X, x)$ such that $g(\mathrm{bcl}(G)) \subset U \times Y$. Therefore, we obtain $x \in G \subset \mathrm{bcl}(G) \subset U$ and hence by Lemma 4.2, X is strongly b-regular.

Remark 5.4 The converse of Theorem 5.3 is not true because, in Example 4.6, f is strongly b-irresolute and X is strongly b-regular but g is strongly b-irresolute.

Lemma 5.5 (Nasef [12]) *If X_0 is α-open in X, then* $\mathrm{BO}(X_0) = \mathrm{BO}(X) \cap X_0$.

Lemma 5.6 (Park [14]) *If $A \subset X_0 \subset X$ and X_0 is α-open in X, then* $\mathrm{bcl}(A) \cap X_0 = \mathrm{bcl}_{X_0}(A)$, *where* $\mathrm{bcl}_{X_0}(A)$ *denote the b-closure of A in the subspace X_0.*

Theorem 5.7 *If $f : X \to Y$ is strongly b-irresolute and X_0 is an α-open subset of X, then the restriction $f|_{X_0} : X_0 \to Y$ is strongly b-irresolute.*

Proof For any $x \in X_0$ and any $V \in \mathrm{BO}(Y, f(x))$, there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset V$ since f is strongly b-irresolute. Put $U_0 = U \cap X_0$, then by Lemmas 5.5 and 5.6, $U_0 \in \mathrm{BO}(X_0, x)$ and $\mathrm{bcl}_{X_0}(U_0) \subset \mathrm{bcl}(U_0)$. Therefore, we obtain

$$(f|_{X_0})(\mathrm{bcl}_{X_0}(U_0)) = f(\mathrm{bcl}_{X_0}(U_0)) \subset f(\mathrm{bcl}(U_0)) \subset f(\mathrm{bcl}(U)) \subset V.$$

This shows that $f|_{X_0}$ is strongly b-irresolute.

In oder to obtain some properties of the compositions of strongly b-irresolute functions, we need following definitions.

Definition 5.8 A function $f : X \to Y$ is said to be

(a) *α-continuous* [11] if $f^{-1}(V)$ is α-open in X for every open set V of Y;

(b) α-open [11] if $f(U)$ is α-open in Y for every open set U of X;

(c) *pre-b-open* if $f(U) \in \mathrm{BO}(Y)$ for each $U \in \mathrm{BO}(X)$.

Lemma 5.9 (Park [14]) *If $f : X \to Y$ is an α-continuous α-open function and V is a b-θ-open set of Y, then $f^{-1}(V)$ is b-θ-open in X.*

Theorem 5.10 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then, the following properties hold:*

(a) *If f is strongly b-irresolute and g is b-irresolute, then the composition $g \circ f : X \to Z$ is strongly b-irresolute.*

(b) *If f is weakly b-irresolute and g is strongly b-irresolute, then $g \circ f : X \to Z$ is strongly b-irresolute.*

(c) *If* f *is* α*-continuous* α*-open and* g *is strongly* b*-irresolute, then* $g \circ f$ *is strongly* b*-irresolute.*

(d) *If* f *is a pre-*b*-open bijection* $g \circ f : X \to Z$ *is strongly* b*-irresolute, then* g *is strongly* b*-irresolute.*

Proof The proofs of (a), (b) and (c) follow from Theorems 3.11 and 5.1 and Lemma 5.9.

(d): Let $W \in \mathrm{BO}(Z)$. Since $g \circ f$ is strongly b-irresolute, $(g \circ f)^{-1}(W)$ is b-θ-open in X. Since f is pre-b-open and bijective, f^{-1} is b-irresolute and hence it is weakly b-irresolute. By Theorem 3.11, we have $g^{-1}(W) = f((g \circ f)^{-1}(W))$ is b-θ-open in Y. Hence, by Theorem 5.1, g is strongly b-irresolute.

Theorem 5.11 *If* $f : X \to Y$ *is a strongly* b*-irresolute injection and* Y *is* b-T_1*, then* X *is* b-T_2*.*

Proof Let x and y be any distinct points of X. Since f is injective, $f(x) \neq f(y)$ and there exist $V \in \mathrm{BO}(Y, f(x))$ and $W \in \mathrm{BO}(Y, f(y))$ such that $f(y) \notin V$ and $f(x) \notin W$. Since f is strongly b-irresolute, there exists $U \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U)) \subset V$. Therefore, we obtain $f(y) \notin f(\mathrm{bcl}(U))$. Put $G = X \setminus \mathrm{bcl}(U)$. Then $G \in \mathrm{BO}(X, y)$ and $G \cap U = \emptyset$. This shows that X is b-T_2.

Lemma 5.12 *Let* A *be a subset of* X *and* B *be a subset of* Y*. Then*

(a) (Nasef [12]) *If* $A \in \mathrm{BO}(X)$ *and* $B \in \mathrm{BO}(Y)$*, then* $A \times B \in \mathrm{BO}(X \times Y)$*.*

(b) (Park [14]) $\mathrm{bcl}(A \times B) \subset \mathrm{bcl}(A) \times \mathrm{bcl}(B)$*.*

Theorem 5.13 *If* $f : X \to Y$ *is a strongly* b*-irresolute function and* Y *is* b-T_2*, then the subset* $E = \{(x, y) : f(x) = f(y)\}$ *is* b-θ*-closed in* $X \times X$*.*

Proof Suppose that $(x, y) \notin E$. Then $f(x) \neq f(y)$. Since Y is Hausdorff, there exist b-open sets V and W of Y containing $f(x)$ and $f(y)$, respectively, such that $V \cap W = \emptyset$. Since f is strongly b-irresolute, there exist $U \in \mathrm{BO}(X, x)$ and $G \in \mathrm{BO}(X, y)$ such that $f(\mathrm{bcl}(U)) \subset V$ and $f(\mathrm{bcl}(G)) \subset W$. By Lemma 5.12, we have $(x, y) \in U \times G \in \mathrm{BO}(X \times X)$ and $\mathrm{bcl}(U \times G) \cap E \subset [\mathrm{bcl}(U) \times \mathrm{bcl}(G)] \cap E = \emptyset$. Therefore, E is b-θ-closed in $X \times X$.

6 Covering properties

Definition 6.1 A space X is said to be

(a) *b-closed* [14] if every cover of X by b-open sets has a finite subcover whose b-closures cover X;

(b) *countably b-closed* [14] if every countable cover of X by b-open sets has a finite subcover whose b-closures cover X;

(c) *b-compact* if every cover of X by b-open sets has a finite subcover.

A subset K of a space X is said to be *b-closed relative to X* [14] (resp. *b-compact relative to X*) if for every cover $\{V_\alpha : \alpha \in I\}$ of K by b-open sets of X, there exists a finite subset I_0 of Λ such that $K \subset \cup\{\mathrm{bcl}(V_\alpha) : \alpha \in I_0\}$ (resp. $K \subset \cup\{V_\alpha : \alpha \in I_0\}$).

Theorem 6.2 *If $f : X \to Y$ is a strongly b-irresolute (resp. weakly b-irresolute) function and K is b-closed (resp. b-compact) relative to X, then $f(K)$ is b-compact (resp. b-closed) relative to Y.*

Proof Let $\{V_\alpha : \alpha \in I\}$ be a cover of $f(K)$ by b-open sets of Y. For each point $x \in K$, there exists $\alpha(x) \in I$ such that $f(x) \in V_{\alpha(x)}$. Since f is strongly b-irresolute (resp. weakly b-irresolute), there exists $U_x \in \mathrm{BO}(X, x)$ such that $f(\mathrm{bcl}(U_x)) \subset V_{\alpha(x)}$ (resp. $f(U_x) \subset \mathrm{bcl}(V_{\alpha(x)})$). The family $\{U_x : x \in K\}$ is a cover of K by b-open sets of X and hence there exists a finite subset K_0 of K such that $K \subset \cup_{x\in K_0}\mathrm{bcl}(U_x)$ (resp. $K \subset \cup_{x\in K_0}U_x$). Therefore, we obtain $f(K) \subset \cup_{x\in K_0}V_{\alpha(x)}$ (resp. $f(K) \subset \cup_{x\in K_0}\mathrm{bcl}(V_{\alpha(x)})$). This shows that $f(K)$ is b-compact (resp. b-closed) relative to Y.

Corollary 6.3 *Let $f : X \to Y$ be a surjection. Then, the following properties hold:*

(a) *If f is strongly b-irresolute and X is b-closed (resp. countably b-closed), then Y is b-compact (resp. countably b-compact).*

(b) *If f is weakly b-irresolute and X is b-compact (resp. countably b-compact), then Y is b-closed (resp. countably b-closed).*

Recall that a space X is said to be *submaximal* [17] if each dense subset of X is open in X. It is shown in [17] that a space X is submaximal if and only if every preopen set of X is open. A space X is said to be *extremally disconnected* [5] if the closure of each open set of X is open. Note that extremally disconnected space is exactly the space where every semiopen set is α-open.

Theorem 6.4 *Let X is a submaximal extremally disconnected space. If a function $f : X \to Y$ has a b-θ-closed graph, then $f^{-1}(K)$ is θ-closed in X for each subset K which is b-closed relative to Y.*

Proof Let K be a subset which is b-closed relative to Y and $x \notin f^{-1}(K)$. Then for each $y \in K$ we have $(x, y) \notin G(f)$ and there exist $U_y \in \mathrm{BO}(X, x)$ and $V_y \in \mathrm{BO}(Y, y)$ such that $f(\mathrm{bcl}(U_y)) \cap \mathrm{bcl}(V_y) = \emptyset$. The family $\{V_y : y \in K\}$ is a cover of K by b-open sets of Y and there exists a finite subset K_0 of K such that $K \subset \cup_{y \in K_0} \mathrm{bcl}(V_y)$. Since X is submaximal extremally disconnected, each U_y is open in X and $\mathrm{bcl}(U_y) = \mathrm{cl}(U_y)$. Set $U = \cap_{y \in K_0} U_y$, then U is an open set containing x and

$$f(\mathrm{cl}(U)) \cap K \subset \bigcup_{y \in K_0} [f(\mathrm{cl}(U)) \cap \mathrm{bcl}(V_y)] \subset \bigcup_{x \in K_0} [f(\mathrm{bcl}(U_y)) \cap \mathrm{bcl}(V_y)] = \emptyset.$$

Therefore, we have $\mathrm{cl}(U) \cap f^{-1}(K) = \emptyset$ and hence $x \notin \mathrm{cl}_\theta(f^{-1}(K))$. This shows that $f^{-1}(K)$ is θ-closed in X.

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