Argument properties of multivalent functions associated with the fractional differintegral operator
(분수 미분적분 연산자와 관련된 다엽함수들의 편각 추정)

Advisor : Prof. Nak Eun Cho

by

A Reum Park

A thesis submitted in partial fulfillment of the requirement
for the degree of

Master of Education

Graduate School of Education
Pukyong National University

February 2015

Argument properties of multivalent functions associated with the fractional differintegral operator

A dissertation

by

A Reum Park

Approved by:

(Chairman) Tae Hwa Kim

(Member) Jin Mun Jung

(Member) Nak Eun Cho

February 27. 2015

CONTENTS

분수 미분적분 연산자와 관련된 다엽함수들의 편각 추정

박 아 름

부경대학교 교육대학원 수학교육전공

요 약

기하 함수 이론에 관한 여러 성질들은 지금까지 많은 학자들에 의하여 연구되었다. 특히, Patel과 Mishra는 확장된 분수 미분적분 연산자를 이용한 다엽함수들의 족들을 소개하고, 족들 사이의 포함관계 등 여러 가지 성질들을 조사하였다.

본 논문에서는 Patel과 Mishra에 의하여 소개된 연산자를 이용하여 새로운 다엽함수들의 족들을 소개하고, Miller와 Mocanu, Nunokawa의 결과들을 응용하여 다엽함수들의 편각추정과 함께 족들 사이의 포함관계들을 연구하였다. 그리고 본 논문에서 소개된 결과들을 이용하여 기존에 알려진 여러 결과들을 발전시켰다.

1. Introduction

Let $\mathcal{A}_{p,m}$ denote the class of functions f defined by

$$f(z) = z^p + \sum_{k=m}^{\infty} a_{k+p} z^{k+p} \quad (p, m \in \mathbb{N} = \{1, 2, \cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}_{p,1} \equiv \mathcal{A}_p$. If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathbb{U}$ such that $f(z) = g(w(z))$ (see [19] and [23]). We denote by $\mathcal{S}^*_{p,m}(\eta)$ and $\mathcal{C}_{p,m}(\eta)$ the subclasses of $\mathcal{A}_{p,m}$ consisting of all analytic functions which are, respectively, p-valent starlike of order $\eta (0 \leq \eta < p)$ in $\mathbb{U}$ and p-valent convex of order $\eta (0 \leq \eta < p)$ in $\mathbb{U}$ (see, e.g., Miller and Mocanu [9]).

Let $\mathcal{M}$ be the class of analytic functions h with $h(0) = 1$, and let $\mathcal{N}$ be the subclass of $\mathcal{M}$ which is convex and univalent in $\mathbb{U}$ and $\mathrm{Re}\{h(z)\} > 0$ $(z \in \mathbb{U})$.

By using the subordination principle between analytic functions, we define each of the following subclasses of $\mathcal{A}_{p,m}$:

$$\mathcal{S}^*_{p,m}(\eta;\ h) := \left\{ f \in \mathcal{A}_{p,m} : \frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec h(z)\ (0 \leq \eta < p;\ z \in \mathbb{U}) \right\}$$

and

$$\mathcal{C}_{p,m}(\eta;\ h) := \left\{ f \in \mathcal{A}_{p,m} : \frac{1}{p-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec h(z)\ (0 \leq \eta < p;\ z \in \mathbb{U}) \right\}.$$

In particular, we set

$$\mathcal{S}^*_{p,m}\left(\eta;\ \left(\frac{1+z}{1-z}\right)^{\alpha}\right) =: \mathcal{S}^*_{p,m}(\eta;\ h_\alpha) \quad (0 \leq \eta < p;\ 0 < \alpha \leq 1;\ z \in \mathbb{U})$$

and

$$\mathcal{C}_{p,m}\left(\eta;\ \left(\frac{1+z}{1-z}\right)^{\alpha}\right) =: \mathcal{C}_{p,m}(\eta;\ h_\alpha) \quad (0 \leq \eta < p;\ 0 < \alpha \leq 1;\ z \in \mathbb{U})$$

It is noted that $f \in \mathcal{C}_{p,m}(\eta;\ h)$ if and only if $zf'/p \in \mathcal{S}^*_{p,m}(\eta;\ h)$. We also see that $\mathcal{S}^*_{p,m}(\eta;\ h_1) = \mathcal{S}^*_{p,m}(\eta)$ and $\mathcal{C}_{p,m}(\eta;\ h_1) = \mathcal{C}_{p,m}(\eta)$. The classes $\mathcal{S}^*_{1,1}(\eta;\ h)$ and $\mathcal{C}_{1,1}(\eta;\ h)$ were studied by Ma and Minda [8]. Furthermore, $\mathcal{S}^*_{1,1}(0;\ h_\alpha)$ and

$\mathcal{C}_{1,1}(0;\ h_\alpha)$, which are the classes of strongly starlike and strongly convex functions of order α in $\mathbb{U}$, respectively, have been extensively investigated by Mocanu [10] and Nunokawa [13].

With a view to introducing an extended fractional differintegral operator, we begin by recalling the following definitions of fractional calculus (that is, fractional intgral and fractional derivative of an arbitrary order) considered by Owa [14] (see also [15] and [23]).

Definition 1.1 The fractional integral of order $\lambda(\lambda > 0)$ is defined, for a function f, analytic in a simply-connected region of the complex plane containing the origin by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta,$$

where the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2. Under the Definition 1.1, the fractional derivative of f of order $\lambda(\lambda \geq 0)$ is defined by

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta & (0 \leq \lambda < 1) \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) & (n \leq \lambda < n+1; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{cases}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed as in Definition 1.1.

We observe that, for a function f, given by (1.1), we have

$$D_z^{\lambda} f(z) = \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)} z^{p-\lambda} + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)}{\Gamma(k+p+1-\lambda)} a_{k+p} z^{k+p-\lambda}, \tag{1.2}$$

provided that $z \in \tilde{\mathbb{U}}$, where $\tilde{\mathbb{U}} = \mathbb{U}$ if $-\infty < \lambda \leq p$ and $\tilde{\mathbb{U}} = \mathbb{U} \backslash \{0\}$ if $p < \lambda < p+1$, and $D_z^{\lambda} f(z)$ is, respectively, the fractional integral of f of order $-\lambda$ when $-\infty < \lambda < 0$ and the fractional derivative of f of order λ when $0 \leq \lambda < p+1$.

In view of (1.2), Patel and Mishra [17] introduce the extended fractional differintegral operator $\Omega_z^{\lambda,p} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ for a function f of the form (1.1) and for a real number $\lambda(-\infty < \lambda < p+1)$ by

$$\Omega_z^{\lambda,p}f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)}z^{\lambda}D_z^{\lambda}f(z)$$
$$= z^p + \sum_{k=1}^{\infty}\frac{\Gamma(k+p+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+p+1-\lambda)}a_{k+p}z^{k+p}. \qquad (1.3)$$

It is easily seen from (1.3) that

$$z(\Omega_z^{\lambda,p}f(z))' = (p-\lambda)\Omega_z^{\lambda+1,p}f(z) + \lambda\Omega_z^{\lambda,p}f(z) \;\; (-\infty < \lambda < p;\; z \in \mathbb{U}). \qquad (1.4)$$

We also note that

$$\Omega_z^{0,p}f(z) = f(z), \quad \Omega_z^{1,p}f(z) = \frac{zf'(z)}{p},$$

and, in general

$$\Omega_z^{n,p}f(z) = \frac{(p-n)z^n f^{(n)}(z)}{p!} \;\; (n \in \mathbb{N};\; n < p+1).$$

The fractional differential operator $\Omega_z^{\lambda,p}$ with $0 \le \lambda < 1$ was investigated by Srivastava and Aouf [21]. More recently, Srivastava and Mishra [22] obtained several interesting properties and characteristics for certain subclasses of p-valent analytic functions involving the differintegral operator $\Omega_z^{\lambda,p}$ when $-\infty < \lambda < 1$. We further observe that $\Omega_z^{\lambda,1}$ is the operator introduced by Owa and Srivastava [15].

Now, by using the fractional differintegral operator $\Omega_z^{\lambda,p}$, we define the following subclasses of functions in $\mathcal{A}_{p,m}$.

Definition 1.3. We note that for suitably chosen parameters λ and h, the class $\mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$ reduces some favoured subclasses of multivalent functions mentioned above. For examples, we see easily that $\mathcal{S}_{p,m}^{0}(\eta;\ h) = \mathcal{S}_{p,m}^{*}(\eta;\ h)$ and $\mathcal{S}_{p,m}^{1}(\eta;\ h) = \mathcal{C}_{p,m}(\eta;\ h)$.

Definition 1.4. We say that a function $f \in \mathcal{A}_{p,m}$ is in the class $\mathcal{K}_{p,m}^{\lambda}(\eta,\gamma;\ \delta;\ h)$ if it is satisfies the following argument condition

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta.$$
$$(0 \le \eta,\gamma < p;\; 0 < \delta \le 1;\; g \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ h);\; z \in \mathbb{U}).$$

We also denote by $\mathcal{K}_{p,m}^{\lambda}(\eta,\gamma;\ \delta;\ A,B)$ the subclass $\mathcal{K}_{p,m}^{\lambda}(\eta,\gamma;\ \delta;\ h)$ by taking

$$h(z)=\frac{1+Az}{1+Bz}\ (-1\leq B<A\leq 1;\ z\in\mathbb{U}).$$

We note that $\mathcal{K}_{1,1}^{0}(\eta,\gamma;\ 1;\ 1,-1)$ and $\mathcal{K}_{1,1}^{1}(\eta,\gamma;\ 1;\ 1,-1)$ are the classes of close-to-convex functions of order γ and type η and quasi convex functions of order γ and type η, respectively, studied by Silverman [19] and Noor and Alkhorasani [12]. Furthermore, $\mathcal{K}_{1,1}^{0}(0,0;\ 1;\ 1,-1)$ is the class of strongly close-to-convex functions of order δ (see [16]).

The purpose of the present paper is to investigate some arguments properties of multivalent functions belonging to $\mathcal{A}_{p,m}$ which contain the basic inclusion relationships related to the classes $\mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$ and $\mathcal{K}_{p,m}^{\lambda}(\eta,\gamma;\ \delta;\ A,B)$. The integral preserving properties in connection with the operator $\Omega_z^{\lambda,p}$ defined by (1.3) are also considered. In particular, we obtain the previous results by Bernardi [1], Libera [7], Noor [11], Noor and Alkhorasani [12] and Sakaguchi [18] as special cases of the results presented in this paper. Furthermore, we remark in passing that the readers may refer the literature [2-4] of Cho *et al.* and the references cited therein for more detailed information in connection with the results of the thesis.

2. A set of lemmas

Lemma 2.1 [5]. *Let h be convex univalent in $\mathbb{U}$ with $h(0)=1$ and* Re $\{\kappa h(z)+\nu\}>0(\kappa,\nu\in\mathbb{C})$. *If q is analytic in $\mathbb{U}$ with $q(0)=1$, then*

$$q(z)+\frac{zq'(z)}{\kappa q(z)+\nu}\prec h(z)\quad (z\in\mathbb{U})$$

implies

$$q(z)\prec h(z)\quad (z\in\mathbb{U}).$$

Lemma 2.2 [9]. *Let h be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with* $\mathrm{Re}\{\omega(z)\}\geq 0$. *If q is analytic in $\mathbb{U}$ and $q(0)=h(0)$, then*

$$q(z)+\omega(z)zq'(z)\prec h(z)\quad (z\in\mathbb{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.3 [13]. *Let* $p(z)$ *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$, $p^{(s)}(0) = 0$ $(0 \leq s \leq n-1;\ n \in \mathbb{N})$ *and let* $p(z) \neq 0$ $(z \in \mathbb{U})$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\left|\arg\ p(z)\right| \ < \ \frac{\pi}{2}\alpha\ (|z| < |z_0|) \tag{2.1}$$

and

$$\left|\arg\ p(z_0)\right| \ = \ \frac{\pi}{2}\alpha \tag{2.2}$$

for some $\alpha > 0$, *then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} \ = \ is\alpha, \tag{2.3}$$

where

$$s \geq \frac{n}{2}\left(a + \frac{1}{a}\right) \geq n \ \text{when}\ \ \arg\ p(z_0) = \frac{\pi}{2}\alpha \tag{2.4}$$

$$s \leq -\frac{n}{2}\left(a + \frac{1}{a}\right) \leq -n \ \text{when}\ \ \arg\ p(z_0) = -\frac{\pi}{2}\alpha \tag{2.5}$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia \ \ \text{and}\ \ a > 0. \tag{2.6}$$

Proof. We use the same manner which was used by Nunokawa [13] for the proof of the lemma. Let us put

$$q(z) = p^{1/\alpha}. \tag{2.7}$$

Then we see that

$$Re\{q(z)\} > 0\ (|z| < |z_0|),$$

$$Re\{q(z_0)\} = 0,\ q(0) = 1 \text{ and } q^{(s)}(0) = 0\ (0 \leq s \leq n-1,\ n \in \mathbb{N}).$$

Defining the function $\phi(z)$ by

$$\phi(z) = \frac{1 - q(z)}{1 + q(z)}, \tag{2.8}$$

we have that $\phi(0) = 0$, $\phi^{(s)}(0) = 0\ (0 \leq s \leq n-1,\ n \in \mathbb{N})$,

$$|\phi(z)| < 1\ (|z| < |z_0|) \text{ and } |\phi(z_0)| = 1.$$

In view of Fukui and Sakaguchi [6], we know that

$$\begin{aligned}\frac{z_0\phi'(z_0)}{\phi(z_0)} &= \frac{-2z_0q'(z_0)}{1-\{q(z_0)\}^2}\\ &= \frac{-2z_0q'(z_0)}{1+|q(z_0)|^2} \geq n. \end{aligned} \tag{2.9}$$

It follows from (2.9) that

$$-z_0q'(z_0) \geq \frac{n}{2}\ \left(1+|q(z_0)|^2\right) \tag{2.10}$$

and $z_0q'(z_0)$ is a negative real number. Since $q(z_0)$ is a non-vanishing pure imaginary number, we can put $q(z_0) = ia$, where a is a non-vanishing real number. We have, for $a > 0$,

$$\begin{aligned}\operatorname{Im}\left(\frac{z_0q'(z_0)}{q(z_0)}\right) &= \operatorname{Im}\left(-\frac{iz_0q'(z_0)}{|q(z_0)|}\right)\\ &\geq \frac{n}{2}\left(\frac{1+a^2}{a}\right) \geq n. \end{aligned} \tag{2.11}$$

and, for $a < 0$,

$$\begin{aligned}\operatorname{Im}\left(\frac{z_0q'(z_0)}{q(z_0)}\right) &= \operatorname{Im}\left(\frac{iz_0q'(z_0)}{|q(z_0)|}\right)\\ &\leq -\frac{n}{2}\left(\frac{1+a^2}{a}\right) \leq -n. \end{aligned} \tag{2.12}$$

On the other hand, it follows that

$$\frac{z_0q'(z_0)}{q(z_0)} = \frac{1}{\alpha}\frac{z_0p'(z_0)}{p(z_0)}. \tag{2.13}$$

This completes the proof of Lemma 2.3.

3. Main Results

Theorem 3.1. *Let* $h \in \mathcal{M}$ *with*

$$\operatorname{Re}\{h(z)\} > (\lambda-\eta)/(p-\eta)\ \ (-\infty < \lambda < p;\ 0 \leq \eta < p).$$

Then

$$\mathcal{S}_{p,m}^{\lambda+1}(\eta;\ h) \subset \mathcal{S}_{p,m}^{\lambda}(\eta;\ h).$$

Proof. Let $f \in \mathcal{S}_{p,m}^{\lambda+1}(\eta;\ h)$ and set

$$q(z) = \frac{1}{p-\eta}\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}f(z)} - \eta\right), \tag{3.1}$$

where q is analytic in $\mathbb{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. Applying (1.4) and (3.1), we obtain

$$(p-\lambda)\frac{\Omega_z^{\lambda+1,p}f(z)}{\Omega_z^{\lambda,p}f(z)} = (p-\eta)q(z) - \lambda + \eta. \tag{3.2}$$

Taking the logarithmic differentiation on both sides of (3.2) and multiplying by z, we have

$$\frac{1}{p-\eta}\left(\frac{z(\Omega_z^{\lambda+1,p}f(z))'}{\Omega_z^{\lambda,p}f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z)-\lambda+\eta} \quad (z \in \mathbb{U}). \tag{3.3}$$

Applying Lemma 2.1 to (3.3), it follows that $q \prec h$, that is, $f \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$. The proof of Theorem 3.1 is thus completed.

Taking

$$h(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1;\ z \in \mathbb{U})$$

in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let* $(p-\eta)(1-A) > (\lambda-\eta)(1-B)$ $(-1 \leq B < A \leq 1;\ -\infty < \lambda < p;\ 0 \leq \eta < p)$. *Then*

$$\mathcal{S}_{p,m}^{\lambda+1}(\eta;\ A, B) \subset \mathcal{S}_{p,m}^{\lambda}(\eta;\ A, B).$$

Theorem 3.2. *If* $f \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$ *with*

$$\mathrm{Re}\{h(z)\} > -(\mu+\eta)/(p-\eta) \quad (h \in \mathcal{M};\ \mu > -p;\ 0 \leq \eta < p),$$

then $F_\mu(f) \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$, *where* F_μ *is the integral operator defined by*

$$F_\mu(f) := F_\mu(f)(z) = \frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1}f(t)dt \quad (\mu > -p). \tag{3.4}$$

Proof. Let $f \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$ and set

$$q(z) = \frac{1}{p-\eta}\left(\frac{z(\Omega_z^{\lambda,p}F_\mu(f)(z))'}{\Omega_z^{\lambda,p}F_\mu(f)(z)} - \eta\right), \tag{3.5}$$

where q is analytic in $\mathbb{U}$ with $p(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathbb{U}$. From (2.17), we have

$$z(\Omega_z^{\lambda,p}F_\mu(f)(z))' = (\mu + p)\Omega_z^{\lambda,p}f(z) - \mu\Omega_z^{\lambda,p}F_\mu(f)(z). \tag{3.6}$$

Then, by applying (3.6) to (3.5), we get

$$(\mu + p)\frac{\Omega_z^{\lambda,p}f(z)}{\Omega_z^{\lambda,p}F_\mu(f)(z)} = (p-\eta)q(z) + \mu + \eta. \tag{3.7}$$

Making use of the logarithmic differentiation on both sides of (3.7) and multiplying by z, we have

$$\frac{1}{p-\eta}\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z)+\mu+\eta}\ (z \in \mathbb{U}).$$

Hence, by virtue of Lemma 2.1, we conclude that $q \prec h$ in $\mathbb{U}$, which implies that $F_\mu(f) \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$.

Letting

$$h(z) = \frac{1+Az}{1+Bz}\ (-1 \leq B < A \leq 1;\ z \in \mathbb{U})$$

in Theorem 3.2, we immediately get the following result.

Corollary 3.2. *If* $f \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ A, B)$ *with* $(p-\eta)(1-A) > -(\mu+\eta)(1-B)$ $(-1 \leq B < A \leq 1;\ \mu > -p;\ 0 \leq \eta < p)$, *then* $F_\mu(f) \in \mathcal{S}_{p,m}^{\lambda}(\eta;\ A, B)$, *where* F_μ *is the integral operator defined by* (3.4).

Theorem 3.3. *Let* $f \in \mathcal{A}_{p,m},\ h \in \mathcal{N},\ -\infty < \lambda \leq 0,\ 0 \leq \gamma,\ \eta < p$ *and* $0 < \delta \leq 1$. *If*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda+1,p})f(z))'}{\Omega_z^{\lambda+1,p}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in \mathcal{S}_{p,m}^{\lambda+1}(\eta;\ h)$, then

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where $\alpha(0<\alpha\leq 1)$ is the solution of the equation :

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha n\cos\frac{\pi}{2}t_1}{(p-\eta)\sup_{z\in\mathbb{U}}|h(z)|+\eta-\lambda+\alpha n\sin\frac{\pi}{2}t_1}\right) \tag{3.8}$$

when

$$t_1=\sup_{z\in\mathbb{U}}\left|\arg\{(p-\eta)h(z)+\eta-\lambda\}\right|. \tag{3.9}$$

Proof. Let

$$q(z)=\frac{1}{p-\gamma}\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)}-\gamma\right).$$

Then q is analytic in $\mathbb{U}$ with $q(0)=1$. By using (1.4), we obtain

$$((p-\gamma)q(z)+\gamma)\Omega_z^{\lambda,p}g(z)=(p-\lambda)\Omega_z^{\lambda+1,p}(a,c)f(z)+\lambda\Omega_z^{\lambda,p}f(z). \tag{3.10}$$

Differentiating both sides of (3.10) and multiplying the resulting equation by z, we find that

$$\begin{aligned}(p-\gamma)zq'(z)\Omega_z^{\lambda,p}g(z)+((p-\gamma)q(z)+\gamma)z(\Omega_z^{\lambda,p}(a,c)g(z))'\\=(p-\lambda)z(\Omega_z^{\lambda+1,p}f(z))'+\lambda z(\Omega_z^{\lambda,p}f(z))'.\end{aligned} \tag{3.11}$$

Since $g\in\mathcal{S}_{p,m}^{\lambda+1}(\eta;\ h)$, by Theorem 3.1, we know that $g\in\mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$.
Let

$$r(z)=\frac{1}{p-\eta}\left(\frac{z(\Omega_z^{\lambda,p}g(z))'}{\Omega_z^{\lambda,p}g(z)}-\eta\right). \tag{3.12}$$

Then, using (1.4) once again, we have

$$(p-\lambda)\frac{\Omega_z^{\lambda+1,p}g(z)}{\Omega_z^{\lambda,p}g(z)}=(p-\eta)r(z)+\eta-\lambda. \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$\frac{1}{p-\gamma}\left(\frac{z(\Omega_z^{\lambda+1,p}f(z))'}{\Omega_z^{\lambda+1,p}g(z)}-\gamma\right)=q(z)+\frac{zq'(z)}{(p-\eta)r(z)+\eta-\lambda}.$$

Then we obtain

$$(p-\eta)r(z)+\eta-\lambda=\rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} (p-\eta)\inf_{z\in\mathbb{U}}|h(z)|+\eta-\lambda \;<\; \rho \;<\; (p-\eta)\sup_{z\in\mathbb{U}}|h(z)|+\eta-\lambda \\ -t_1 \;<\; \phi \;<\; t_1, \end{cases}$$

when

$$t_1=\sup_{z\in\mathbb{U}}|\arg\{(p-\eta)h(z)+\eta-\lambda\}|.$$

We also note that q is analytic in $\mathbb{U}$ with $q(0)=1$, and so $\mathrm{Re}\{q(z)\}>0$ in $\mathbb{U}$ by applying the assumption and Lemma 2.2 with $\omega(z)=1/((p-\eta)r(z)+\eta-\lambda)$. Hence $q(z)\neq 0$ in $\mathbb{U}$.

If there exists a point $z_0\in\mathbb{U}$ such that the conditions (2.1) and (2.2) are satisfied, then(by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}}=ia$ $(a>0)$. Then we obtain

$$\begin{aligned}
&\arg\left(q(z_0)+\frac{z_0q'(z_0)}{(p-\eta)r(z_0)+\eta-\lambda}\right)\\
&=\frac{\pi}{2}\alpha+\arg\left(1+i\alpha s(\rho e^{i\frac{\pi\phi}{2}})^{-1}\right)\\
&\geq\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha s\sin\frac{\pi}{2}(1-\phi)}{\rho+\alpha s\cos\frac{\pi}{2}(1-\phi)}\right)\\
&\geq\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha n\cos\frac{\pi}{2}t_1}{(p-\eta)\sup_{z\in\mathbb{U}}|h(z)|+\eta-\lambda+\alpha n\sin\frac{\pi}{2}t_1}\right)\\
&=\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (3.8) and (3.9), respectively. These evidently contradict the assumption of Theorem 3.3.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}}=-ia$ $(a>0)$. Applying the same method as the above, we have

$$\begin{aligned}
&\arg\left(q(z_0)+\frac{z_0q'(z_0)}{(p-\eta)r(z_0)+\eta-\lambda}\right)\\
&\leq -\frac{\pi}{2}\alpha-\tan^{-1}\left(\frac{\alpha n\cos\frac{\pi}{2}t_1}{(p-\eta)\sup_{z\in\mathbb{U}}|h(z)|+\eta-\lambda+\alpha n\sin\frac{\pi}{2}t_1}\right)\\
&= -\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (3.8) and (3.9), respectively. These also are contradiction to the assumption of Theorem 3.1. Therefore we complete the proof of Theorem 3.3.

If we take

$$h(z)=\frac{1+Az}{1+Bz}\ (-1\leq B<A\leq 1;\ z\in\mathbb{U})$$

in Theorem 3.3, then we also note that from the known result given earlier by Silverman and Silvia [20], we have

$$\left|r(z)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2}\qquad (z\in\mathbb{U}\ ;\ B\neq -1) \tag{3.14}$$

and

$$\mathrm{Re}\,\{r(z)\}>\frac{1-A}{2}\qquad (z\in\mathbb{U}\ ;\ B=-1), \tag{3.15}$$

where $r(z)$ is given in (3.12). Therefore we have the following Corollary 3.3.

Corollary 3.3. *Let* $f\in\mathcal{A}_{p,m}$, $h\in\mathcal{N}$, $-\infty<\lambda\leq 0$, $0\leq\gamma$, $\eta<p$ *and* $0<\delta\leq 1$. *If*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda+1,p})f(z))'}{\Omega_z^{\lambda+1,p}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

for some $g\in\mathcal{S}_{p,m}^{\lambda+1}(\eta;\ A,B)$, then

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where $\alpha(0<\alpha\leq 1)$ is the solution of the equation :

$$\delta=\begin{cases}\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_2}{\left(\frac{(p-\eta)(1+A)}{1+B}+\eta-\lambda\right)+\alpha\sin\frac{\pi}{2}t_2}\right) & \text{for } B\neq -1,\\ \alpha & \text{for } B=-1,\end{cases} \tag{3.16}$$

when b is given by (2.3) *and*

$$t_2 = \frac{2}{\pi}\sin^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta-\lambda)(1-B^2)}\right). \tag{3.17}$$

From corollary 3.3, we see easily the following result.

Corollary 3.4. *Let* $f \in \mathcal{A}_{p,m}$, $-\infty < \lambda \leq 0$, $0 \leq \gamma$, $\eta < p$ *and* $0 < \delta \leq 1$. *Then*

$$\mathcal{K}^{\lambda+1}_{p,m}(\eta,\gamma;\ \delta;\ A,B) \subset \mathcal{K}^{\lambda}_{p,m}(\eta,\gamma;\ \delta;\ A,B).$$

Remark 3.1. If we put $\lambda = 0$, $m = 1$, $p = 1$, $A = 1$, $B = -1$ and $\delta = 1$ in Theorem 3.3, then we see that every quasi-convex function of order γ and type η in $\mathbb{U}$ is close-to-convex function of order γ and type η in $\mathbb{U}$ as proven earlier by Noor [11] and Sakaguchi [18].

Letting $\gamma = 0$, $B \to A(A < 1)$ and $g(z) = z^p$ in Theorem 3.3, we obtain

Corollary 3.5. *Let* $f \in \mathcal{A}_{p,m}$ *and* $-\infty < \lambda \leq 0$, $0 < \delta \leq 1$. *If*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda+1,p}f(z))'}{z^p}\right)\right| < \frac{\pi}{2}\delta,$$

then

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{z^p}\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation :*

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha}{p-\lambda}.$$

Finally, we prove an argument property asserted by Theorem 3.4 below.

Theorem 3.4. *Let* $f \in \mathcal{A}_{p,m}$, $h \in \mathcal{N}$, $\mu \geq 0$, $0 \leq \gamma$, $\eta < p$ *and* $0 < \delta \leq 1$. *If*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in \mathcal{S}^{\lambda}_{p,m}(\eta;\ h)$, *then*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}F_\mu(f)(z))'}{\Omega_z^{\lambda,p}F_\mu(g)(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F_μ is the integral operator defined by (3.4), *and* $\alpha(0<\alpha\le 1)$ *is the solution of the equation* (3.8) *with* $\lambda=-\mu$.

Proof. Let

$$q(z)=\frac{1}{p-\gamma}\left(\frac{z(\Omega_z^{\lambda,p}F_\mu(f)(z))'}{\Omega_z^{\lambda,p}F_\mu(g)(z)}-\gamma\right).$$

Since $g\in\mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$, we see from Corollary 3.2 that $F_\mu(g)\in\mathcal{S}_{p,m}^{\lambda}(\eta;\ h)$. Using (3.6), we have

$$((p-\gamma)q(z)+\gamma)\Omega_z^{\lambda,p}F_\mu(g)(z)=(\mu+p)\Omega_z^{\lambda,p}f(z)-\mu\Omega_z^{\lambda,p}F_\mu(f)(z).$$

Then, by a simple calculation, we get

$$(\mu+p)\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}F_\mu(g)(z)}=(p-\gamma)zq'(z)+((p-\gamma)q(z)+\gamma)((p-\eta)r(z)+\eta+\mu),$$

where

$$r(z)=\frac{1}{p-\eta}\left(\frac{z(\Omega_z^{\lambda,p}F_\mu(g)(z))'}{\Omega_z^{\lambda,p}F_\mu(g)(z)}-\gamma\right).$$

Hence we have

$$\frac{1}{p-\gamma}\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)}-\gamma\right)=q(z)+\frac{zq'(z)}{(p-\eta)r(z)+\eta+\mu}.$$

The remaining part of the proof in Theorem 3.4 is similar to that of Theorem 3.1 and so we omit it.

If we take

$$h(z)=\frac{1+Az}{1+Bz}\ (-1\le B<A\le 1;\ z\in\mathbb{U}),$$

we have the following result.

Corollary 3.6. *Let* $f\in\mathcal{A}_{p,m}$, $h\in\mathcal{N}$, $\mu\ge 0$, $0\le\gamma,\ \eta<p$ *and* $0<\delta\le 1$. *If*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}f(z))'}{\Omega_z^{\lambda,p}g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

for some $g\in\mathcal{S}_{p,m}^{\lambda}(\eta;\ A,B)$, *then*

$$\left|\arg\left(\frac{z(\Omega_z^{\lambda,p}F_\mu(f)(z))'}{\Omega_z^{\lambda,p}F_\mu(g)(z)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F_μ *is the integral operator defined by* (3.4), *and* $\alpha(0<\alpha\le 1)$ *is the solution of the equation :*

$$\delta=\begin{cases}\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_2}{\left(\frac{(p-\eta)(1+A)}{1+B}+\eta+\mu\right)+\alpha\sin\frac{\pi}{2}t_2}\right) & \text{for } B\neq -1,\\ \alpha & \text{for } B=-1,\end{cases}$$

when t_2 *is* t_1 *with* $\lambda=-\mu$ *given by* (3.17).

From corollary 3.6, we see easily the following result.

Corollary 3.7. *If* $f\in\mathcal{K}_{p,m}^{\lambda}(\eta,\gamma;\ \delta;\ A,B)$, *then* $F_\mu(f)\in\mathcal{K}_{p,m}^{\lambda}(\eta,\gamma;\ \delta;\ A,B)$, *where* F_μ *is the integral operator defined by* (3.4).

Remark 3.2. If we take $\lambda=0$ and $\lambda=1$ with $m=1,\ p=1,\ A=1,\ B=-1$ and $\delta=1$ in Corollary 3.7, respectively, then we have the corresponding results obtained by Noor and Alkhorasani [12]. Furthermore, taking $\lambda=0,\ p=1,\gamma=0$, $A=1,\ \ B=-1$ and $\delta=1$ in Corollary 3.7, we obtain the classical results given earlier by Bernardi [1] and Libera [7].

References

1. S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **35**(1969), 429-446.

2. N. E. Cho, The noor integral operator and strongly close-to-convex functions, *J. Math. Anal. Appl.*, **283**(2003), 202-212.

3. N. E. Cho, T. H. Kim, Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.*, **40**(2003), 399-410.

4. N. E. Cho, O. S. Kwon, H. M. Srivastava, Inclusion relationship and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, *J. Math. Anal. Appl.*, **292**(2004), 470-483.

5. P. Enigenberg, S. S. Miller, P. T. Mocanu and M. O. Reade, On a Briot-Bouquet differential subordination, *General Inequalities*, **3**(1983)(Birkhauser Verlag-Basel), 339-348.

6. S. Fukui and K. Sakaguchi, An extension of a theorem of S. Ruscheweyh, *Bull. Fac. Ed. Wakayama Univ. Natur. Sci.* , **29**(1980), 1-3.

7. R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, **16**(1965), 755-758.

8. W. C. Ma and D. Minda, An internal geometric characterization of strongly starlike functions, *Ann. Univ. Mariae Curie-Sklodowska Sect. A*, **45**(1991), 89-97.

9. S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, *Michigan Math. J.*, **28**(1981), 157-171.

10. P. T. Mocanu, Alpha-convex integral operators and strongly starlike functions, *Studia Univ. Babes-Bolyai Mathematica*, **34**(1989), 18-24.

11. K. I. Noor, On quasi-convex functions and related topics, *Internat. J. Math. Math. Sci.*, **10**(1987), 241-258.

12. K. I. Noor and H. A. Alkhorasani, Properties of close-to-convexity preserved by some integral operators, *J. Math. Anal. Appl.*, 112(1985), 509-516.

13. M. Nunokawa, On the order of strongly starlikeness of strongly convex functions, *Proc. Japan Acad. Ser. A Math. Sci.*, **69**(1993), 234-237.

14. S. Owa, On the distortion theorems I, *Kyungpook Math. J.*, **18**(1978), 53-59.

15. S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39**(1987), 1057-1077.

16. Ch. Pommerenke, On close-to-convex analytic functions, *Trans. Amer. Math. Soc.*, **114**(1965), 176-186.

17. J. Patel and A. K. Mishra, On certain subclasses of multivalent functions associated with an extended fractional differential operator, *J. Math. Appl.*, **332**(2007), 109-122.

18. K. Sakaguchi, On a certain univalent mapping, *J. Math. Soc. Japan*, **2**(1959), 72-75.

19. H. Silverman, On a class of close-to-convex schlicht functions, *Proc. Amer. Math. Soc.*, **36**(1972), 477-484.

20. H. Silverman and E. M. Silvia, Subclasses of starlike functions subordinate to convex functions, *Can. J. Math.*, **37**(1985), 48-61.

21. H. M. Srivastava and M. K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II , *J. Math. anal. Appl.*, **171**(1992), 1-13; *ibid.* **192**(1995), 673-688.

22. H. M. Srivastava and A. K. Mishra, A fractional differintegral operator and its applications to a nested class of multivalent functions with negative coefficients, *Adv. Stud. Contemp. Math.*, **7**(2003), 203-214.

23. H. M. Srivastava and S. Owa (Editors), Univalent functions, Fractional Calculus, and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, 1989.