Thesis for the Degree of Master of Science

Control problems for semilinear second order equations with cosine families

by

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(코사인 족을 포함한 준선형 이계방정식에 대한 제어문제)

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코사인 족을 포함한 준선형 이계방정식에 대한 제어문제

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요　약

본 논문에서는 코사인 족과 그에 관계된 사인 족을 포함한 이계 비선형 제어 시스템에 대한 근사적인 제어성을 얻기 위한 논문이다.

우선적으로 다음의 Banach 공간 X 상에서 주어진 추상적 준 선형 이계 초기치 문제:

$$\begin{cases} \frac{d^2w(t)}{dt} = w(t) + F(t,w) + F(t), \qquad 0 < t < T, \\ w(0) = x_0, \quad \frac{d}{dt}w(0) = y_0 \end{cases}$$

에서 주작용소 A에의해 구성되는 코사인족과 사인족에 대한 기본적인 성질들을 이용하여 해의 존재성과 정칙성에 대한 결과를 유도하였다.

두 번째로 제어항을 포함한 준성형 시스템:

$$\begin{cases} \frac{d^2w(t)}{dt^2} = Aw(t) + F(t,w) + B(t), \qquad 0 < t < T, \\ w(0) = x_0, \quad \frac{d}{dt}w(0) = y_0 \end{cases}$$

에서 제어기 B에 대한 근사적인 제어성에 대해 생각한다. 주어진 제어변수공간으로부터 도달 가능한 집합을 조사하여 주어진 공간을 근사적으로 제어 가능하도록 새로운 기법을 사용하여 응용가능한 제어기의 충분조건을 유도하였다.

1 Introduction

The first part of this paper gives some basic results on the regularity of solutions of abstract semilinear second order initial value problem

$$\begin{cases} \frac{d^2 w(t)}{dt^2} = Aw(t) + F(t,w) + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad \frac{d}{dt}w(0) = y_0 \end{cases} \tag{1.1}$$

in a Banach space X. Here, the nonlinear part is given by

$$F(t,w) = \int_0^t k(t-s)g(s,w(s))ds$$

where k belongs to $L^2(0,T)$ and $g : [0,T] \times X \longrightarrow X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuous. In (1.1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$.

Let E be a subspace of all $x \in X$ which $C(t)x$ is a once continuously differentiable function of t.

In [1], when $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A), y_0 \in E$, and $k \in W^{1,2}(0,T)$, the existence of a solution $w \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$ of (1.1) for each $T > 0$ is given. Moreover, they have been established a variation of constant formula for solutions of the second order nonlinear system (1.1).

The work presented in this paper, based on the regularity for solution of (1.1), investigates necessary and sufficient conditions for the approximate controllability for (1.1) with the strict range condition on B even though the system (1.1) contains unbounded principal operators and the convolution nonlinear term, which is more flexible necessary assumption than one in [2].

We will make use of some of the basic ideas from cosine family referred to [3, 4] and the regular properties for solutions in [1, 5] for a discussion of the control results. In [6, 7] a one-dimensional nonlinear hyperbolic equation of convolution type which is nonlinear in the partial differential equation part and linear in the hereditary part is treated.

As a second part in this paper, we consider the approximate controllability for the nonlinear second order control system

$$\begin{cases} \frac{d^2 w(t)}{dt^2} = Aw(t) + F(t,w) + Bu(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad \frac{d}{dt}w(0) = y_0 \end{cases} \tag{1.2}$$

in a Banach space X where the controller B is bounded linear operator from some Banach space U to X. In [2, 8, 9] the approximate controllability for (1.2) was studied under the particular range conditions of the controller B depending on the time T.

In Section 4 we establish to the approximate controllability for the second order nonlinear system (1.2) under a condition for the range of the controller B without the inequality condition independent to the time T, and see that the necessary assumption is more flexible than one in [2, 9]. Finally, we give a simple example to which our main result can be applied.

2 Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas. Let X be a Banach space with norm denoted by $||\cdot||$.

Definition 2.1. [1] *A one parameter family $C(t), t \in \mathbb{R}$, of bounded linear operators in X is called a strongly continuous cosine family if*

c(1) $C(s+t)+C(s-t)=2C(s)C(t)$, *for all* $s,\ t \in \mathbb{R}$,

c(2) $C(0)=I$,

c(3) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$.

If $C(t),\ t \in \mathbb{R}$ is a strongly continuous cosine family in X , then $S(t),\ t \in \mathbb{R}$ is the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)xds,\ \ x \in X,\ \ t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in \mathbb{R}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2}C(0)x.$$

We endow with the domain $D(A) = \{x \in X : C(t)x$ is a twice continuously differentiable function of $t\}$ with norm

$$||x||_{D(A)} = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\} + ||Ax||.$$

We shall also make use of the set

$$E = \{x \in X : C(t)x \ \text{ is a once continuously differentiable function of } t\}$$

with norm

$$||x||_E = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\}.$$

It is not difficult to show that $D(A)$ and E with given norms are Banach spaces.

The following Lemma is from Proposition 2.1 and Proposition 2.2 of [1].

Lemma 2.1. *Let $C(t)(t \in \mathbb{R})$ be a strongly continuous cosine family in X. The following are true :*

c(4) $C(t) = C(-t)$ *for all* $t \in \mathbb{R}$,

c(5) $C(s), S(s), C(t)$ *and* $S(t)$ *commute for all* $s, t \in \mathbb{R}$,

c(6) $S(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$,

c(7) *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that*

$$||C(t)|| \leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R},$$

$$||S(t_1) - S(t_2)|| \leq K\left|\int_{t_2}^{t_1} e^{\omega|s|}ds\right| \text{ for all } t_1, t_2 \in \mathbb{R},$$

c(8) *if* $x \in E$, *then* $S(t)x \in D(A)$ *and*

$$\frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

c(9) *if* $x \in D(A)$, *then* $C(t)x \in D(A)$ *and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

c(10) *if* $x \in X$ *and* $r, s \in \mathbb{R}$, *then*

$$\int_r^s S(\tau)xd\tau \in D(A) \quad \text{and} \quad A(\int_r^s S(\tau)xd\tau) = C(s)x - C(r)x,$$

c(11) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$,

c(12) $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in \mathbb{R}$,

c(13) $C(s+t) = C(t)C(s) - S(t)S(s)$ *for all* $s, t \in \mathbb{R}$,

c(14) $C(s+t) - C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$.

The following results are crucial in discussing regular problem for the linear case(for proof one can see [1])

Proposition 2.1. *Let $f : R \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$. Then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}$$

is a solution of the following equation

$$\frac{d^2w(t)}{dt^2} = Aw(t) + f(t), \ \ t \in R, \ \ w(0) = x_0, \ \ \dot{w}(0) = y_0. \tag{2.2}$$

belonging to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$. Moreover, we have that there exists a positive constant C_1 such that for any $T > 0$,

$$||w||_{L^2(0,T;D(A))} \leq C_1(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{2.3}$$

If f is continuously differentiable and $(x_0, y_0) \in D(A) \times E$, it is easily shown that w is continuously differentiable and satisfies

$$\dot{w}(t) = AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)f(s)ds, \ \ t \in \mathbb{R}.$$

Let us remark that if w is a solution of (2.2) in an interval $[0, t_1 + t_2]$ with $t_1, t_2 > 0$. Then when $t \in [0, t_1 + t_2]$, from c(11-15), we have

$$\begin{aligned}
w(t) &= C(t-t_1)w(t_1) + S(t-t_1)\dot{w}(t_1) + \int_{t_1}^t S(t-s)f(s)ds \\
&= C(t-t_1)\{C(t_1)x_0 + S(t_1)y_0 + \int_0^{t_1} S(t_1-\tau)f(\tau)d\tau\} \\
&\quad + S(t-t_1)\{AS(t_1)x_0 + C(t_1)y_0 + \int_0^{t_1} C(t_1-\tau)f(\tau)d\tau\} \\
&\quad + \int_{t_1}^t S(t-s)f(s)ds \\
&= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds,
\end{aligned}$$

here, we used the relation

$$S(t)AS(s) = AS(t)S(s) = \frac{1}{2}C(t+s) - \frac{1}{2}C(t-s) = C(t+s) - C(t)C(s)$$

for all $s, t \in \mathbb{R}$. This mean the mapping $t \mapsto w(t_1 + t)$ is a solution of (2.2) in $[0, t_1 + t_2]$ with initial data $(w(t_1), \dot{w}(t_1)) \in D(A) \times E$.

From now on, we introduce the regularity of solutions of abstract semilinear second order initial value problem (1.1) in a Banach space X. Let $g : [0,T] \times D(A) \to X$ be a nonlinear mapping such that $t \mapsto g(t, w)$ is measurable and

$$\begin{cases} ||g(t, w_1) - g(t, w_2)||_{D(A)} \le L||w_1 - w_2|| \\ g(t, 0) = 0 \end{cases}$$

for a positive constant L.

For $w \in L^2(0, T; D(A))$, we set

$$F(t, w) = \int_0^t k(t-s) g(s, w(s)) ds$$

where k belongs to $L^2(0, T)$. We will seek a mild solution of (1.1), that is, a solution of the integral equation

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s, w) + f(s)\} ds. \tag{2.4}$$

Remark 2.1. *If $g : [0, T] \times X \to X$ is a nonlinear mapping satisfying*

$$||g(t, w_1) - g(t, w_2)|| \le L||w_1 - w_2||$$

for a positive constant L, then our results can be obtained immediately.

Lemma 2.2. *Let $w \in L^2(0, T; D(A))$, $T > 0$. Then $F(\cdot, w) \in L^2(0, T; X)$ and*

$$||F(\cdot, w)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)} \sqrt{T} ||w||_{L^2(0,T;D(A))}.$$

Moreover if $w_1,\ w_2 \in L^2(0, T; D(A))$, then

$$||F(\cdot, w_1) - F(\cdot, w_2)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)} \sqrt{T} ||w_1 - w_2||_{L^2(0,T;D(A))}.$$

Proof. From (g1), (g2) and using the Hölder inequality, it is easily seen that

$$\begin{aligned} ||F(\cdot, w)||^2_{L^2(0,T;X)} &\le \int_0^T || \int_0^t k(t-s) g(s, w(s)) ds||^2 dt \\ &\le ||k||^2_{L^2(0,T)} \int_0^T \int_0^t L^2 ||w(s)||^2 ds dt \\ &\le L^2 ||k||^2_{L^2(0,T)} T ||w||^2_{L^2(0,T;D(A))}. \end{aligned}$$

The proof of the second paragraph is similar. ∎

Now, as in Theorem 3.1 of [1], we give a norm estimation of the solution of (1.1) and establish the global existence of solutions with the aid of norm estimations.

Proposition 2.2. *Suppose that the assumptions (g1), and (g2) are satisfied. If $f : \mathbb{R} \longrightarrow X$ is continuously differentiable, $x_0 \in D(A), y_0 \in E$, and $k \in W^{1,2}(0,T)$, $T > 0$, then the solution w of (1.1) exists and is unique in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$, and there exists a constant C_3 depending on T such that*

$$||w||_{L^2(0,T;D(A))} \le C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{2.5}$$

3 Approximate controllability

In this section, we consider the approximate controllability for the nonlinear second order control system

$$\begin{cases} \frac{d^2w(t)}{dt^2} = Aw(t) + F(t,w) + Bu(t), & 0 < t \le T, \\ w(0) = x_0, \quad \frac{d}{dt}w(0) = y_0 \end{cases} \tag{3.1}$$

in a Banach space X where the controller B is a bounded linear operator from some Banach space U to X. Assume that

Assumption (G) The nonlinear mapping $g : [0,T] \times X \longrightarrow X$ is such that $t \mapsto g(t,w)$ is measurable and

$$||g(t,w_1) - g(t,w_2)|| \le L||w_1 - w_2||, \quad |k(t)| \le M. \tag{3.2}$$

for a positive constant L.

For (3.1), a integral equation can be written as

$$\begin{cases} w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s,w) + Bu(s)\}ds, \\ w(0) = x_0, \ \ \dot{w}(0) = y_0. \end{cases} \tag{3.3}$$

For every $u \in L^2(0,T;U)$, it is natural that the solution w of (3.3) is continuous on $[0,T]$.

Given a strongly continuous cosine family $C(t)$ $(t \in R)$, we define linear bounded operators $\hat{C}$ and $\hat{S}$ mapping $L^2(0,T:X)$ into X by

$$\hat{C}p = \int_0^T C(T-t)p(t)dt, \ \ \hat{S}p = \int_0^T S(T-t)p(t)dt,$$

for $p(\cdot) \in L^2(0,T:X)$ and $S(t)$ is the associated sine family of $C(t)$.

We define the reachable sets for the system (3.1) as follows:

Definition 3.1. *Let $w(t;F,u)$ is a solution of the (3.1) associated with nonlinear term F and control u at the time t.*

$$R_T(F) = \{(w(T;F,u),\ \dot{w}(T;F,u)) : u \in L^2(0,T;U)\} \subset X^2 = X \times X,$$
$$R_T(0) = \{(w(T;0,u),\ \dot{w}(T;0,u)) : u \in L^2(0,T;U)\} \subset X^2.$$

The nonempty subset $R_T(F)$ in X^2 consisting of all terminal states of (3.1) *is called the reachable sets at the time T of the system* (3.1). *The set $R_T(0)$ is one of the linear case where $F \equiv 0$.*

Definition 3.2. *The system* (3.1) *is said to be approximate controllable on the interval* $[0,T]$ *if*

$$\overline{R_T(F)} = X^2$$

where $\overline{R_T(F)}$ is the closure of $R_T(F)$ in X^2,
that is, for any $\epsilon > 0, \bar{x} \in D(A)$ and $\bar{y} \in E$ there exists a control $u \in L^2(0,T;U)$ such that

$$||\bar{x} - C(T)x_0 - S(T)y_0 - \hat{S}F(\cdot,w) - \hat{S}Bu|| < \epsilon,$$
$$||\bar{y} - AS(T)x_0 - C(T)y_0 - \hat{C}F(\cdot,w) - \hat{C}Bu|| < \epsilon.$$

We introduce the following hypothesis:

Assumption (B) For any $\varepsilon > 0$ and $p \in L^2(0,T;X)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} ||\hat{C}p - \hat{C}Bu|| < \varepsilon, \\ ||Bu||_{L^2(0,t;X)} \le q_1||p||_{L^2(0,t;X)}, \quad 0 \le t \le T. \end{cases}$$

where q_1 is a constant independent of p.

We remark that from c(11-14) of Lemma 2.1, the operator $\hat{S}$ also satisfies the condition (B), that is, for any $\varepsilon > 0$ and $p \in L^2(0,T;X)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} ||\hat{S}p - \hat{S}Bu|| < \varepsilon, \\ ||Bu||_{L^2(0,t;X)} \le q_1||p||_{L^2(0,t;X)}, \quad 0 \le t \le T. \end{cases}$$

For the sake of simplicity we assume that sine family $S(t)$ is bounded as in c(7):

$$||S(t)|| \le K(t) \quad t \ge 0.$$

Here, we may consider the following inequality:

$$K(t) \le \omega^{-1}K(e^{\omega t} - 1).$$

Lemma 3.1. *Let u_1 and u_2 be in $L^2(0,T;U)$. Then under the Assumption (G), we have*

$$||w(t;F,u_1) - w(t;F,u_2)|| \leq K(T)e^{K(T)MLT^2}\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,t;X)}.$$

for $0 \leq t \leq T$.

Proof. For $0 \leq t \leq T$, we have

$$\begin{aligned} ||w(t;F,u_1) - w(t;F,u_2)|| &\leq K(T)\sqrt{t}||Bu_1(s) - Bu_2(s)||_{L^2(0,t;X)} \\ &+K(T)MLt\int_0^t ||w(s;F,u_1) - w(s;F,u_2)||ds, \end{aligned}$$

where L is a constant in Assumption (G). Therefore, by using Gronwall's inequality this lemma follows. □

For the approximate controllability for linear equation, we know the following necessary Lemma before proving the main theorem.

Lemma 3.2. *Under Assumption (B) we have* $\overline{R_T(0)} = X^2$.

Proof. Let $\bar{x} \in D(A), \bar{y} \in E$. Putting

$$\eta_1 = \bar{x} - C(T)x_0 - S(T)y_0 \in D(A), \quad \eta_2 = \bar{y} - AS(T)x_0 - C(T)y_0 \in E,$$

then there exists some $p \in C^1([0,T] : X)$ such that

$$\eta_1 = \int_0^T S(T-t)p(t)dt, \quad \eta_2 = \int_0^T C(T-t)p(t)dt,$$

for instance, take $p(t) = \{C(t-T) + S(t-T)\}\eta_2/T$. By hypothesis (B) there exists a function $u \in L^2(0,T;U)$ such that

$$\begin{cases} ||\bar{x} - C(T)x_0 - S(T)y_0 - \hat{S}Bu|| < \epsilon, \\ ||\bar{y} - AS(T)x_0 - C(T)y_0 - \hat{C}Bu|| < \epsilon. \end{cases}$$

The denseness of the domain $D(A)$ in X implies the approximate controllability of the corresponding linear system. □

Theorem 3.1. *Under Assumptions (G), (B), the system (3.1) is approximately controllable on* $[0,T]$, $T > 0$.

Proof. We will show that $D(A) \times E \subset \overline{R_T(F)}$, i.e., for given $\varepsilon > 0$ and $(\xi_T, \tilde{\xi}_T) \in D(A) \times E$ there exists $u \in L^2(0, T; U)$ such that

$$||\xi_T - w(T; F, u)|| < \varepsilon, \tag{3.4}$$
$$||\tilde{\xi}_T - \dot{w}(T; F, u)|| < \varepsilon. \tag{3.5}$$

As $(\xi_T, \tilde{\xi}_T) \in D(A) \times E$ there exists a $p \in L^2(0, T; X)$ such that

$$\hat{S}p = \xi_T - C(T)x_0 - S(T)y_0, \quad \hat{C}p = \tilde{\xi}_T - AS(T)x_0 - C(T)y_0.$$

Let $u_1 \in L^2(0, T; U)$ be arbitrary fixed. Since by the Assumption (B) there exists $u_2 \in L^2(0, T; U)$ such that

$$||\hat{S}(p - F(\,\cdot\,, w(\,\cdot\,; F, u_1))) - \hat{S}Bu_2|| < \frac{\varepsilon}{4},$$
$$||\hat{C}(p - F(\,\cdot\,, w(\,\cdot\,; F, u_1))) - \hat{C}Bu_2|| < \frac{\varepsilon}{4},$$

it follows

$$||\xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F(\,\cdot\,, w(\,\cdot\,; F, u_1)) - \hat{S}Bu_2|| < \frac{\varepsilon}{4}, \tag{3.6}$$
$$||\tilde{\xi}_T - AS(T)x_0 - C(T)y_0 - \hat{C}F(\,\cdot\,, w(\,\cdot\,; F, u_1)) - \hat{C}Bu_2|| < \frac{\varepsilon}{4}.$$

We can also choose $v_2 \in L^2(0, T; U)$ by the Assumption (B) such that

$$||\hat{S}(F(\,\cdot\,, w(\,\cdot\,; F, u_2)) - F(\,\cdot\,, w(\,\cdot\,; F, u_1))) - \hat{S}Bv_2|| < \frac{\varepsilon}{8}, \tag{3.7}$$
$$||\hat{C}(F(\,\cdot\,, w(\,\cdot\,; F, u_2)) - F(\,\cdot\,, w(\,\cdot\,; F, u_1))) - \hat{C}Bv_2|| < \frac{\varepsilon}{8}$$

and by the Assumption (B),

$$||Bv_2||_{L^2(0,t;X)} \leq q_1||F(\,\cdot\,, w(\,\cdot\,; F, u_1)) - F(\,\cdot\,, w(\,\cdot\,; F, u_2))||_{L^2(0,t;X)}$$

for $0 \leq t \leq T$. From now, we will only prove (3.4), while the proof of (3.5) is similar.

In view of Lemma 3.1 and the Assumption (B)

$$
\begin{aligned}
||Bv_2||_{L^2(0,t;X)} &\le q_1\{\int_0^t ||F(\tau, w(\tau; F, u_2)) - F(\tau, w(\tau; F, u_1))||^2 d\tau\}^{\frac{1}{2}} \\
&\le q_1 ML\{\int_0^t \int_0^\tau ||w(\tau; F, u_2) - w(\tau; F, u_1)||^2 ds d\tau\}^{\frac{1}{2}} \\
&\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}\{\int_0^t \int_0^\tau ||Bu_2 - Bu_1||^2_{L^2(0,s;X)} ds d\tau\}^{\frac{1}{2}} \\
&\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}(\int_0^t \int_0^\tau 1\, ds d\tau)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;X)} \\
&= q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}(\frac{t^2}{2})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;X)}.
\end{aligned}
$$

Put $u_3 = u_2 - v_2$. We determine v_3 such that

$$
\begin{aligned}
&||\hat{S}(F(\cdot, w(\cdot; F, u_3)) - F(\cdot, w(\cdot; F, u_2))) - \hat{S}Bv_3|| < \frac{\varepsilon}{8}, \\
&||Bv_3||_{L^2(0,t;X)} \le q_1 ||F(\cdot, w(\cdot; F, u_3)) - F(\cdot, w(\cdot; F, u_2))||_{L^2(0,t;X)}
\end{aligned}
$$

for $0 \le t \le T$. Hence, we have

$$
\begin{aligned}
&||Bv_3||_{L^2(0,t;X)} \\
&\le q_1\{\int_0^t ||F(\tau, w(\tau; F, u_3)) - F(\tau, w(\tau; F, u_2))||^2 d\tau\}^{\frac{1}{2}} \\
&\le q_1 ML\{\int_0^t \int_0^\tau ||w(s; F, u_3) - w(s; F, u_2)||^2 ds d\tau\}^{\frac{1}{2}} \\
&\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}\{\int_0^t \int_0^\tau ||Bu_3 - Bu_2||^2_{L^2(0,s:X)} ds d\tau\}^{\frac{1}{2}} \\
&\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}\{\int_0^t \int_0^\tau ||Bv_2||^2_{L^2(0,s;X)} ds d\tau\}^{\frac{1}{2}} \\
&\le (q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T})^2\{\int_0^t \int_0^\tau \frac{s^2}{2} ||Bu_2 - Bu_1||^2_{L^2(0,s;X)} ds d\tau\}^{\frac{1}{2}} \\
&\le (q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T})^2(\int_0^t \int_0^\tau \frac{s^2}{2} ds d\tau)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;X)} \\
&\le (q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T})^2(\frac{t^4}{2\cdot 4})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;X)}.
\end{aligned}
$$

By proceeding this process, and from that

$$\begin{aligned}&||B(u_n - u_{n+1})||_{L^2(0,t;X)} = ||Bv_n||_{L^2(0,t;X)}\\&\le (q_1 MLK(T)\sqrt{T}e^{K(T)MLT^2})^{n-1}(\frac{t^{2n-2}}{2\cdot 4\cdots(2n-2)})^{\frac{1}{2}}||Bu_2 - Bu_1||_{L^2(0,t;X)}\\&= (\frac{q_1 MLK(T)\sqrt{T}e^{K(T)MLT^2}t}{\sqrt{2}})^{n-1}\frac{1}{\sqrt{(n-1)!}}||Bu_2 - Bu_1||_{L^2(0,t;X)},\end{aligned}$$

it follows that

$$\begin{aligned}&\sum_{n=1}^{\infty}||Bu_{n+1} - Bu_n||_{L^2(0,T;X)}\\&\le \sum_{n=0}^{\infty}(\frac{q_1 MLK(T)\sqrt{T}e^{K(T)MLT^2}t}{\sqrt{2}})^n\frac{1}{\sqrt{n!}}||Bu_2 - Bu_1||_{L^2(0,T;X)} < \infty.\end{aligned}$$

Therefore, there exists $u^* \in L^2(0,T;X)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;X). \tag{3.8}$$

From (3.6), (3.7) it follows that

$$\begin{aligned}&||\xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F(\,\cdot\,, w(\,\cdot\, F, u_2)) - \hat{S}Bu_3||\\&= ||\xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F(\,\cdot\,, w(\,\cdot\, F, u_1)) - \hat{S}Bu_2 + \hat{S}Bv_2\\&\ - \hat{S}[F(\cdot, w(\cdot; F, u_2)) - F(\cdot, w(\cdot; F, u_1))]||\\&< (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.\end{aligned}$$

By choosing choose $v_n \in L^2(0,T;U)$ by the assumption (B) such that

$$||\hat{S}(F(\,\cdot\, w(\,\cdot\,; F, u_n)) - F(\,\cdot\, w(\,\cdot\,; F, u_{n-1})) - \hat{S}Bv_n|| < \frac{\varepsilon}{2^{n+1}},$$

putting $u_{n+1} = u_n - v_n$, we have

$$\begin{aligned}&||\xi_T - S(T)g - \hat{S}F(\cdot, z(\cdot; g, f, u_n)) - \hat{S}\Phi u_{n+1}||\\&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1\ 2, \ \cdots.\end{aligned}$$

Therefore, for $\varepsilon > 0$ there exists integer N such that

$$||\hat{S}Bu_{N+1} - \hat{S}Bu_N|| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned}
&||\xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F(\cdot, w(\cdot; F, u_N)) - \hat{S}Bu_N||\\
&\le ||\xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F(\cdot, w(\cdot; F, u_N)) - \hat{S}Bu_{N+1}||\\
&\ + ||\hat{S}Bu_{N+1} - \hat{S}Bu_N||\\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned}$$

And by the similar method, we also obtain that

$$||\tilde{\xi}_T - AS(T)x_0 - C(T)y_0 - \hat{C}F(\cdot, w(\cdot; F, u_N)) - \hat{C}Bu_N|| \le \varepsilon.$$

Thus, the system (3.1) is approximately controllable on $[0, T]$ as N tends to infinity.

Example. Let $X = L^2([0, \pi]; \mathbb{R})$. We consider the following partial differential equation

$$\begin{cases} \frac{d^2 w(t,x)}{dt^2} = Aw(t,x) + F(t,w) + Bu(t), \quad 0 < t, \quad 0 < x < \pi, \\ w(t,0) = w(t,\pi) = 0, \quad t \in \mathbb{R} \\ w(0,x) = x_0(x), \quad \frac{d}{dt}w(0,x) = y_0(x), \quad 0 < x < \pi \end{cases} \tag{3.9}$$

Let $e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$. Then $\{e_n : n = 1, \cdots\}$ is an orthonormal base for X. Let $A : X \to X$ be defined by

$$Aw(x) = w''(x),$$

where $D(A) = \{w \in X : w, \dot{w}$ are absolutely continuous, $\ddot{w} \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2 (w, e_n) e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt (w, e_n) e_n, \quad w \in X.$$

Let $g_1(t, x, w, p)$, $\quad p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, r) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \le \gamma$ such that

(f1) $g_1(t, x, 0, 0) = 0$,

(f2) $|g_1(t,x,w,p) - g_1(t,x,w,q)| \le \rho(t,|w|)|p-q|$,
(f3) $|g_1(t,x,w_1,p) - g_1(t,x,w_2,p)| \le \rho(t,|w_1|+|w_2|)|w_1-w_2|$.

Let

$$g(t,w)x = g_1(t,x,w,Dw).$$

Then noting that

$$\begin{aligned}||g(t,w_1) - g(t,w_2)||_{0,2}^2 \le 2\int_\Omega |g_1(t,x,w_1,Dw_1) - g_1(tx,w_2,DW_2q)|^2du\\ + 2\int_\Omega |g_1(t,u,w_1,q) - g_1(t,u,w_2,q)|^2du,\end{aligned}$$

it follows from (f1), (f2) and (f3) that

$$||g(t,w_1) - g(t,w_2)||_{0,2}^2 \le L(||w_1||_{D(A)},||y||_{D(A)})||w_1-w_2||_{D(A)}$$

where $L(||w_1||_{D(A)},||w_2||_{D(A)})$ is a constant depending on $||w_1||_{D(A)}$ and $||w_2||_{D(A)}$. We set

$$F(t,w) = \int_0^t k(t-s)g(s,w(s))ds$$

where k belongs to $L^2(0,T)$.

Let $U = X$, $0 < \alpha < T$ and define the intercept controller operator B_α on $L^2(0,T;X)$ by

$$B_\alpha u(t) = \begin{cases} 0, & 0 \le t < \alpha, \\ u(t), & \alpha \le t \le T \end{cases}$$

for $u \in L^2(0,T;X)$. For a given $p \in L^2(0,T;X)$ let us choose a control function u satisfying

$$u(t) = \begin{cases} 0, & 0 \le t < \alpha, \\ p(t) + \frac{\alpha}{T-\alpha}C(t - \frac{\alpha}{T-\alpha}(t-\alpha))p(\frac{\alpha}{T-\alpha}(t-\alpha)), & \alpha \le t \le T. \end{cases}$$

Then $u \in L^2(0,T;X)$ and $\hat{S}p = \hat{S}B_\alpha u$

From the following:

$$\begin{aligned}||B_\alpha u||_{L^2(0,T;X)} &= ||u||_{L^2(\alpha,T;X)}\\ &\le ||p||_{L^2(\alpha,T;X)} + Ke^{\omega T}||p(\frac{\alpha}{T-\alpha}(\cdot-\alpha))||_{L^2(\alpha,T;X)}\\ &\le (1 + Ke^{\omega T}\sqrt{\frac{T-\alpha}{\alpha}})||p||_{L^2(0,T;X)}\end{aligned}$$

it follows that the controller B_α satisfies Assumption (B). Therefore, from Theorem 3.1, we have that the nonlinear system given by (3.9) approximate controllable on $[0,T]$ □

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