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Approaches to Group Decision Making under Dual Hesitant Fuzzy Environment

by

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쌍대 Hesitant 퍼지 환경에서 집단의사결정의 해결 방법

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by
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요 약

본 논문은 쌍대 hesitant 퍼지 집합의 거리측도, 상관계수 및 다양한 집성연산자에 기초한 집단의사결정의 해결방법을 연구한 것으로 그 주요내용은 다음과 같이 요약된다.

첫째, 쌍대 hesitant 퍼지 집합의 거리측도와 닮음측도를 정의하고, Hamming 거리, Euclidean 거리 및 Hausdorff metric에 기초한 다양한 쌍대 hesitant 거리를 소개하였다. 속성의 가중 정보를 더한 쌍대 hesitant 거리를 이용한 의사결정문제의 해결방안을 제안하고, 이 방안을 에너지 정책 채택문제에 적용하였다. 또한 쌍대 hesitant 퍼지집합의 순서가중 거리를 제안하고, 속성의 가중치를 모를 경우 가중치를 결정하는 방법을 이용하여 쌍대 hesitant 퍼지 정보를 갖는 의사결정문제의 해결방안도 제안하였다.

둘째, 쌍대 hesitant 퍼지 집합의 상관계수를 정의하고, 쌍대 hesitant 퍼지 집합들의 연산을 이용하여 다양한 상관계수 공식을 유도하고 이들의 상호관계를 조사하였다. 그 예로서 이들을 쌍대 hesitant 퍼지 환경에서의 의학진단에 적용하고, 다른 상관계수로 얻어진 결과들은 서로 다르다는 사실을 보였다.

셋째, 쌍대 hesitant 퍼지 Bonferroni 평균을 소개하고, 이 평균연산자의 멱등성, 단조성, 교환성, 유계성과 같은 성질을 조사하였다. 매개변수의 값에 따른 여러 연산자들을 유도하고, 가중된 쌍대 hesitant 퍼지 Bonferroni 평균를 소개하였다. 이 연산자를 기초한 쌍대 hesitant 퍼지환경에서의 다속성 의사결정문제의 해결방안을 제시하였다.

Chapter 1

Introduction

Hesitancy and uncertainty are usually unavoidable problems in decision making. To express decision makers' evaluation information more objectively, several tools have been proposed, such as fuzzy set (FS) [83], interval-valued fuzzy set (IVFS) [84, 41], intuitionistic fuzzy set (IFS) [1], type-2 fuzzy set (T2FS) [16, 35] fuzzy multiset (FMS) [35, 34, 71], interval-valued intuitionistic fuzzy set (IVIFS) [2, 3], hesitant fuzzy set (HFS) [45, 46] and interval-valued hesitant fuzzy set. For example, in a decision making problem, some decision makers consider as possible values for the membership degree of x into a set A a few different values 0.4, $0,5$ and 0.6, and for the nonmembership degrees 0.1, $0,2$ and 0.3 replacing just one or a tuple. Since the membership and the nonmembership can represent the opposite epistemic degrees, i.e., the membership comes to grips with epistemic certainty and the nonmembership comes to grips with epistemic uncertainty, we do not confront an interval of possibilities (IVFS or IVIFS), or some possibility distributions (T2FS) on the possible values, or multiple occurrences of an element (FMS), but several different possible values indicate the epistemic degrees whether certainty or uncertainty. To deal with this cases, Zhu et al. [86] introduced the concept of dual hesitant fuzzy set (DHFS) considered as a generalization of fuzzy set (FS). They discussed the relationships among DHFSs and other generalizations of FSs such as IFSs, T2FSs, FMSs and HFSs.

Distance and similarity are the most broadly applied indices in many fields and also important measures in data analysis and classification, pattern recognition, decision making and so on. Lots of studies have been done on these issues [8, 10, 30, 47]. As many real world data may be fuzzy, the concepts of distance and similarity have been extended to fuzzy environments, intuitionistic fuzzy environments, interval-valued fuzzy environments and hesitant fuzzy environments. For instance, Li and Cheng [31] generalized the Hamming distance and the Euclidean distance by adding a parameter and gave a similarity formula for IFSs only based on the membership degrees and nonmembership degrees. Grzegorzewski [19] defined distance measures for IVFSs and IFSs based on the Hausdorff metric. Hung and Yang [27, 28] defined similarity measures for IFSs based on Hausdorff distance and L_p metric, respectively. Xu and Chen [61] gave a comprehensive overview of distance and similarity measures for IFSs and developed several continuous distance and similarity measures for IFSs. Among them, the most used distance measures for IFSs A and B on $X = \{x_1, x_2, \ldots, x_n\}$ are the following:

- the Hamming distance:
  $d_{ih}(A, B) = \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)$;
- the normalized Hamming distance:
  $d_{inh}(A, B) = \frac{1}{n} \sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|)$;
- the Euclidean distance:
  $d_{ie}(A, B) = [\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)|^2 + |\nu_A(x_i) - \nu_B(x_i)|^2)]^{1/2}$;
- the normalized Euclidean distance:
  $d_{ine}(A, B) = \frac{1}{n} [\sum_{i=1}^{n} (|\mu_A(x_i) - \mu_B(x_i)|^2 + |\nu_A(x_i) - \nu_B(x_i)|^2)]^{1/2}$;
- the Hausdorff distance:
  $d_{ihd}(A, B) = \frac{1}{2} (\max |\mu_A(x_i) - \mu_B(x_i)| + \max |\nu_A(x_i) - \nu_B(x_i)|)$.

Because of the potential applications of distance and similarity measures, they have been further extended by Xu and Chen [61] for IVIFSs. Several new methods of deriving the distance and similarity measures for both IFSs and IVIFSs have also been proposed. Wu and Mendel [52] extended Jaccard's similarity measure for T2FSs and developed a new similarity measure for interval T2FSs. Xu and Xia [53] proposed a variety of distance measures for HFSs, based on the corresponding similarity measures can be obtained, and further developed a variety of ordered weighted distance measures for HFSs. However, the aforementioned measures cannot be used to deal with the distance and similarity between DHFSs. Due to the fact that two kinds of hesitancy (i.e., membership hesitancy and non-membership hesitancy) of decision makers' evaluation information are common problems in decision making as previously stated, it is necessary to develop some measures for DHFSs. To do this, Chapter 2 of this thesis is organized as follows. In Section 2.1, we present the axioms for distance and similarity measures, give a variety of distance measures for DHFSs and apply them to multiple attribute decision making with the known weight information on attributes. Section 2.2 propose a class of ordered weighted distance and similarity measures for DHFSs, and give several methods to determine the weighting vector associated with these distance measures. Section 2.3 ends the chapter with the concluding remarks.

Correlation indicates how well two variables move together in a linear fashion, i.e., correlation reflects a linear relationship between two variables, and then it is an important measure in data analysis [59], in particular in decision making, medical diagnosis, pattern recognition and other real-would problems [43]. Lots of studies [7, 9, 13, 18, 21, 22, 24, 25, 26, 29, 33, 36, 39, 40, 50, 57, 65, 80] on this issue have been extended to fuzzy environment and its extended environments. For instance, Hung and Wu [25] used the concept of expected value to define the correlation coefficient of fuzzy numbers, which lies in $[-1, 1]$. Hong [22] considered the fuzzy measures for correlation coefficient of fuzzy numbers under weakest t-norm-based fuzzy arithmetic operations. Gerstenkorn and Mańko [18] introduced the correlation and correlation coefficient of intuitionistic fuzzy sets (IFSs) [1]. Hung [24] and Mitchell [33] derived the correlation coefficient of IFSs from a statistical viewpoint by interpreting an IFS as an ensemble of fuzzy sets. Hung and Wu [26] proposed a method to calculate the correlation coefficient of

IFSs by means of “centroid”. This formula tells us not only the strength of the relationship between IFSs but also whether the considered IFSs are positively or negatively related. Szmidt and Kacprzyk [43] proposed a formula for measuring the correlation coefficient of IFSs adopting the concept from statistics, and showed the importance to take into account all three terms (the membership degree, nonmembership degree and hesitation margin) describing IFSs. In interval-valued intuitionistic fuzzy environments, Bustince and Burillo [9] introduced the concepts of correlation and correlation coefficient of interval-valued intuitionistic fuzzy sets (IVIFSs) [2] and gave two decomposition theorems of the correlation of IVIFSs, one in terms of the correlation of interval-valued fuzzy sets (IVFSs) [84] and the entropy of IFSs, and the other theorem in terms of the correlation of IFSs. Hong [21] generalized the concepts of correlation and correlation coefficient of IVIFSs in a general probability space and generalized the results of Bustince and Burillo [9] with remarkably simple proofs. He also introduced three more decomposition theorems of the correlation of IVIFSs in terms of the correlation of IVFSs and the entropy of IFSs. Park et al. [36], Ye [80] and Wei et al. [50] further studied the methods to calculate the correlation coefficients of IVIFSs and applied them to multiple attribute group decision making problems. Because of the potential applications of correlation coefficients, they have been further extended by Xu and Xia [66] and Chen et al. [13] for hesitant fuzzy sets (HFSs) [45, 46]. Chen et al. [13] derived some correlation coefficient formulas for HFSs and applied them to two real world examples by using clustering analysis under hesitant fuzzy environments. Xu and Xia [66] defined the correlation measures for hesitant fuzzy information and then discussed their properties. Zhu et al. [86], recently, introduced the definition of dual hesitant fuzzy set (DHFS), permitting both the membership and the nonmembership of an element, respectively, to a set having a few different values, which can arise in a group decision making problem. DHFS can reflect the human's hesitance more objectively than other extensions of fuzzy set (IFS, IVFS, IVIFS, HFS, etc.), and thus it is very necessary to develop some theories about DHFSs. However, little has been done about this issue, Huang et al. [23] studied the aggregation operators of DHFSs and applied them to decision making. In Chapter 3 of this thesis, we discuss the correlation measures of dual hesitant fuzzy information. To do this, Section 3.1 proposes the correlation measures of dual hesitant fuzzy elements, several important conclusions are obtained, and an example is given to illustrate the developed correlation measures. Finally, Section 3.2 gives the concluding remarks.

Multiple attribute group decision making is the common phenomenon in modern life, which is to select the optimal alternative(s) from several alternatives or to get their ranking by aggregating the performances of each alternative under several attributes, in which the aggregation techniques play an important role. Considering the relationships among the aggregated arguments, we can classify the aggregation techniques into two categories, the ones which consider the aggregated arguments dependently and the ones which consider the aggregated arguments independently. For the first category, the well-known ordered weighted averaging (OWA) operator [72, 73] is the representative, on the basis of which, a lot of generalizations have been developed, such as the ordered weighted geometric (OWG) operator [14, 63, 64], the ordered ordered weighted harmonic mean (OWHM) operator [11], the continuous ordered weighted averaging (C-OWA) operator [76], the continuous ordered weighted geometric (C-OWG) operator [79], and so on. The second category can reduce to two subcategories: the first subcategory focuses on changing the weight vector of the aggregation operators, such as the Choquet integral-based aggregation operators [77], in which the correlations of the aggregated arguments are measured subjectively by the decision makers, and the representatives of another subcategory are the power averaging (PA) operator [74] and the power geometric (PG) operator [68], both of which allow the aggregated arguments to support each other in aggregation process, on the basis of which the weighted vector is determined. The second subcategory focuses on the aggregated arguments such as the Bonferroni mean (BM) operator [6]. Yager [78] provided an interpretation of BM operator as involving a product of each argument with the average of the other arguments, a combined averaging and "anding" operator. Beliakov et al. [5] presented a composed aggregation technique called the generalized Bonferroni mean (GBM) operator, which models the average of the conjunctive expressions and the average of remaining. In fact, they extended the BM operator by considering the correlations of any three aggregated arguments instead of any two. However, both the BM operator and the GBM operator ignore some aggregation information and the weight vector of the aggregated arguments. To overcome this drawback, Xia et al. [56] developed the generalized weighted Bonferroni mean (GWBM) operator as the weighted version of the GBM operator. Based on the GBM operator and geometric mean operator, they also developed the generalized Bonferoni geometric mean (GWBGM) operator. The fundamental characteristic of the GWBM operator is that it focuses on the group opinions, while the GWBGM operator gives more importance to the individual opinions. Because of the usefulness of the aggregation techniques, which reflect the correlations of arguments, most of them have been extended to fuzzy, intuitionistic fuzzy or hesitant fuzzy environment [37, 44, 59, 56, 69, 82, 85]. However, how to apply the BM to deal with dual hesitant fuzzy information is new research direction, which is also the focus of this thesis. In Chapter 4 of this thesis, we investigate the BM under dual hesitant fuzzy environments. In Section 4.1 briefly reviews some basic concepts and operations related to the BM and dual hesitant fuzzy element (DHFE), which considered in the basic calculational unit of DHFS. In Section 4.2, an dual hesitant fuzzy BM (IVHFBM) is developed, and its variety of special cases are discussed. Section 4.3 introduces the weighted DHFBM (WDHFBM) and develops a procedure for multiple attribute decision making. Section 4.5 ends this chapter with some concluding remarks.

Chapter 2

Distance and similarity measures for dual hesitant fuzzy sets and their applications

2.1 Distance and similarity measures for DHFSs

Definition 2.1.1 [1] Let X be a fixed set, an intuitionistic fuzzy set (IFS) A in X is given as an object having the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{2.1}$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote, respectively, membership function and nonmembership function of A with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for any $x \in X$. Xu and Yager [67] called each pair $(\mu(x), \nu(x))$ an intuitionistic fuzzy number (IFN), and denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$.

For each IFS A in X,

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{2.2}$$

is called the hesitancy degree or intuitionistic index of x to A. Especially, if $\pi_A(x) = 0$ for all $x \in X$, then the IFS reduces to a fuzzy set. Clearly, a prominent characteristic of IFS is that it assigns to each element a membership degree, a non-membership degree and hesitation degree, and thus, IFS constitutes an extension of fuzzy set which only assigns to each element a membership degree.

The hesitant fuzzy set [45, 46], as a generalization of fuzzy set, permits the membership degree of an element to a set presented as several possible values between 0 and 1, which can better describe the situations where people have hesitancy in providing their preferences over objects in process of decision making.

Definition 2.1.2 [45, 46] Let X be a fixed set, a hesitant fuzzy set (HFS) is in terms of function that when applied to X returns a subset of $[0, 1]$, which can be represented as the following mathematical symbol:

$$E = \{\langle x, h(x)\rangle : x \in X\}, \tag{2.3}$$

where $h(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of element $x \in X$ to the set E. For convenience, Xia and Xu [53] called $h(x)$ a hesitant fuzzy element (HFE).

Zhu et al. [86] defined dual hesitant fuzzy set in terms of two functions that return two sets of membership values and nonmembership values, respectively, for each element in domain as follows.

Definition 2.1.3 [86] Let X be a fixed set, then a dual hesitant fuzzy set (DHFS) E on X is described as:

$$E = \{\langle x, h_E(x), g_E(x)\rangle | x \in X\}, \tag{2.4}$$

in which $h_E(x)$ and $g_E(x)$ are two sets of some values in $[0, 1]$, denoting the possible membership degrees and nonmembership degrees of element $x \in X$ to the set E, respectively, with the conditions:

$$0 \leq \gamma,\ \eta \leq 1,\ 0 \leq \gamma^+ + \eta^+ \leq 1, \tag{2.5}$$

where $\gamma \in h(x)$, $\eta \in g(x)$, $\gamma^+ \in h^+(x) = \max\{\gamma | \gamma \in h(x)\}$, and $\eta^+ \in g^+(x) = \max\{\eta | \eta \in g(x)\}$ for all $x \in X$. For convenience, the pair $\{h_E(x), g_E(x)\}$ is called a dual hesitant fuzzy element (DHFE), with the conditions: $\gamma \in h_E(x)$, $\eta \in g_E(x)$, $\gamma^+ = \max\{\gamma|\gamma \in h_E(x)\}$, $\eta^+ = \max\{\eta|\eta \in g_E(x)\}$, $0 \leq \gamma, \eta \leq 1$, and $\gamma^+ + \eta^+ \leq 1$.

For a given DHFS $E = \{h_E, g_E\} \neq \emptyset$, if the membership hesitancy part h_E and nonmembership hesitancy part g_E have only one value γ and η, respectively, and $\gamma + \eta \leq 1$, then the DHFS reduces an intuitionistic fuzzy set. If h_E and g_E have only one value γ and η, respectively, and $\gamma + \eta = 1$, or h_E owns one value and $g_E = \emptyset$, then the DHFS reduces to an fuzzy set (also can be regarded as hesitant fuzzy set). If $h_E \neq \emptyset$ and $g_E = \emptyset$, then the DHFS reduces the hesitant fuzzy set. Thus the definition of DHFSs encompasses these fuzzy sets above.

It is noted that the numbers of values in membership hesitancy part $h_E(x)$ and in nonmembership hesitancy part $g_E(x)$ in a DHFE $E = \{h_E(x), g_E(x)\}$, respectively, may be different, let $l(h_E(x))$ be the number of values in $h_E(x)$, and $l(g_E(x))$ be the number of values in $g_E(x)$. We arrange the elements in $h_E(x)$ and $g_E(x)$, respectively, in descending order, and let $h_E^{\sigma(j)}(x)$ be the jth largest value in $h_E(x)$, and $g_E^{\sigma(k)}(x)$ be the kth largest value in $g_E(x)$.

Because that distance and similarity measures can be applied to many areas such as pattern recognition, cluster analysis, approximate reasoning and decision making, they have attracted a lot of attention. A lot of distance and similarity measures have been developed for FSs, IFSs, FMs and DFSs as mentioned in introduction, but there is little research on DHFSs. Thus, it is very necessary to develop some distance and similarity measures under dual hesitant fuzzy environment. We first present this issue by proposing the axioms for distance and similarity measures.

Definition 2.1.4 Let M and N be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the distance measure between M and N is defined as $d(M, N)$, which satisfies the following properties:

(D1) $0 \leq d(M, N) \leq 1$;

(D2) $d(M, N) = 0$ if and only if $M = N$;

(D3) $d(M, N) = d(N, M)$.

Definition 2.1.5 Let M and N be two DHFSs on $X = \{x_1, x_2, \ldots, x_n\}$, then the similarity measure between M and N is defined as $s(M, N)$, which satisfies the following properties:

(S1) $0 \leq s(M, N) \leq 1$;

(S2) $s(M, N) = 1$ if and only if $M = N$;

(S3) $s(M, N) = s(N, M)$.

From Definitions 2.1.4 and 2.1.5, it is noted that $s(M, N) = 1 - d(M, N)$, and thus, in this paper, we only discuss the distance measures for DHFSs and the corresponding similarity measures can be obtained easily.

In most cases of two DHFSs $M = \{h_M, g_M\}$ and $N = \{h_N, g_N\}$, we have $l(h_M(x_i)) \neq l(h_N(x_i))$ and $l(g_M(x_i)) \neq l(g_N(x_i))$, and for convenience, let $l_{h(x_i)} = \max\{l(h_M(x_i)), l(h_N(x_i))\}$ and $l_{g(x_i)} = \max\{l(g_M(x_i)), l(g_N(x_i))\}$ for each $x_i \in X$. To operate correctly, we should extend the shorter ones, respectively, until both of them in membership hesitancy part and nonmembership hesitancy part have the same length, respectively, when we compare them. To extend the shorter ones, the best way is to add the same values several times in them, respectively. In fact, we can extend the shorter ones by adding any values in them, respectively. The selection of these values mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value in membership hesitancy part and the minimum value in nonmembership hesitancy part, while pessimists expect unfavorable outcomes and may add the minimum value in membership hesitancy part and the maximum value in nonmembership hesitancy part. For example, let $\{h_M(x_i), g_M(x_i)\} = \{\{0.1, 0.2, 0.3\}, \{0.4, 0.5\}\}$, $\{h_N(x_i), g_N(x_i)\} = \{\{0.5, 0.6\}, \{0.2, 0.3, 0.4\}\}$, then we get $l(h_M(x_i)) > l(h_N(x_i))$ and $l(g_M(x_i)) < l(g_N(x_i))$. To operate correctly, we should extend $h_N(x_i)$ and $g_M(x_i)$, respectively, to $h_N(x_i) = \{0.5, 0.5, 0.6\}$ and $g_M(x_i) = \{0.4, 0.4, 0.5\}$ until they have the same lengths of $h_M(x_i)$ and $g_N(x_i)$, respectively, the optimist may extend $h_N(x_i)$ and $g_M(x_i)$ as $h_N(x_i) = \{0.5, 0.6, 0.6\}$ and $g_M(x_i) = \{0.4, 0.4, 0.5\}$, and the pessimist may extend them as $h_N(x_i) = \{0.5, 0.5, 0.6\}$ and $g_M(x_i) = \{0.4, 0.5, 0.5\}$. Although the results may be different if we extend the shorter ones, respectively, by adding different values, it is reasonable because the decision makers' risk preferences can directly influence the final decision. In this chapter, we assume that the decision makers are all pessimistic (other situation can also be studied similarly).

Based on the Hamming distance and the Euclidean distance, we define a dual hesitant normalized Hamming distance:

$$d_{dhnh}(M,N) = \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}}\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right| + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|\right] \tag{2.6}$$

and a dual hesitant normalized Euclidean distance:

$$d_{dhne}(M,N) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}}\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^2 + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^2\right)\right]^{\frac{1}{2}}, \tag{2.7}$$

where $h_M^{\sigma(j)}(x_i)$ and $h_N^{\sigma(j)}(x_i)$ are the jth largest values in $h_M(x_i)$ and $h_N(x_i)$, respectively, and $g_M^{\sigma(k)}(x_i)$ and $g_N^{\sigma(k)}(x_i)$ are the kth largest values in $g_M(x_i)$ and $g_N(x_i)$, respectively, which will be used thereafter.

We can further extend (2.6) and (2.7) into a generalized dual hesitant normalized distance:

$$d_{gdhn}(M,N) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}}\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^\lambda + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^\lambda\right)\right]^{\frac{1}{\lambda}}, \tag{2.8}$$

where $\lambda > 0$.

In particular, if $\lambda = 1$, then the generalized hesitant normalized distance reduces the dual hesitant normalized Hamming distance; if $\lambda = 2$, then the generalized hesitant normalized distance reduces the dual hesitant normalized Euclidean distance.

If we apply the Hausdorff metric to the dual hesitant distance measure, then a generalized dual hesitant normalized Hausdorff distance is given as

$$d_{gdhnh}(M,N) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^{\lambda} + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \tag{2.9}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x_i)}$ and $k = 1, 2, \ldots, l_{g(x_i)}$.

Now we discuss two special cases of the generalized dual hesitant normalized Hausdorff distance:

(1) If $\lambda = 1$, then (2.9) becomes a dual hesitant normalized Hamming-Hausdorff distance:

$$d_{dhnhh}(M,N) = \frac{1}{2n}\sum_{i=1}^{n}\left(\max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right| + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|\right). \tag{2.10}$$

(2) If $\lambda = 2$, then (2.9) becomes a dual hesitant normalized Euclidean-Hausdorff distance:

$$d_{dhneh}(M,N) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^{2} + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^{2}\right)\right]^{\frac{1}{2}}. \tag{2.11}$$

Combining (2.6)-(2.11), we define a hybrid dual hesitant normalized Hamming distance, a hybrid dual hesitant normalized Euclidean distance, and a generalized hybrid dual hesitant normalized distance as follows, respectively:

$$d_{hdhnh}(M,N) = \frac{1}{4n}\sum_{i=1}^{n}\left[\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}}\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right| + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right| + \max_j\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right| + \max_k\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|\right], \tag{2.12}$$

$$d_{hdhne}(M,N) = \left[\frac{1}{4n}\sum_{i=1}^{n}\left(\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}}\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^2 + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^2 + \max_j\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^2 + \max_k\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^2\right)\right]^{\frac{1}{2}}, \tag{2.13}$$

$$d_{ghdhn}(M,N) = \left[\frac{1}{4n}\sum_{i=1}^{n}\left(\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}}\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^\lambda + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^\lambda + \max_j\left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^\lambda + \max_k\left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^\lambda\right)\right]^{\frac{1}{\lambda}}, \tag{2.14}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x_i)}$ and $k = 1, 2, \ldots, l_{g(x_i)}$.

Usually, the weight of each element $x_i \in X$ should be taken into account, and so, we present the following weighted distance measures for DHFSs. Assume that the weight of the element $x_i \in X$ is w_i $(i = 1, 2, \ldots, n)$ with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, then we obtain a generalized dual hesitant weighted distance:

$$d_{gdhw}(M, N) = \left[\frac{1}{2}\sum_{i=1}^{n} w_i \left(\frac{1}{l_{h(x_i)}} \sum_{j=1}^{l_{h(x_i)}} \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^{\lambda} + \frac{1}{l_{g(x_i)}} \sum_{k=1}^{l_{g(x_i)}} \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}} \tag{2.15}$$

and a generalized dual hesitant weighted Hausdorff distance:

$$d_{gdhwh}(M, N) = \left[\frac{1}{2}\sum_{i=1}^{n} w_i \left(\max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^{\lambda} + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \tag{2.16}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x_i)}$ and $k = 1, 2, \ldots, l_{g(x_i)}$.

In the following, let us consider some special cases of the generalized dual hesitant weighted distance (2.15) and the generalized dual hesitant weighted Hausdorff distance (2.16), respectively, by taking different values of the parameter λ.

(1) If $\lambda = 1$, then we get a dual hesitant weighted Hamming distance:

$$d_{dhwh}(M, N) = \frac{1}{2}\sum_{i=1}^{n} w_i \left[\frac{1}{l_{h(x_i)}} \sum_{j=1}^{l_{h(x_i)}} \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right| + \frac{1}{l_{g(x_i)}} \sum_{k=1}^{l_{g(x_i)}} \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|\right] \tag{2.17}$$

and a dual hesitant weighted Hamming-Hausdorff distance:

$$d_{dhwhh}(M, N) = \frac{1}{2}\sum_{i=1}^{n} w_i \left(\max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right| + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|\right). \tag{2.18}$$

(2) If $\lambda = 2$, then we have a dual hesitant weighted Euclidean distance:

$$d_{dhwe}(M,N) = \left[\frac{1}{2}\sum_{i=1}^{n} w_i \left(\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}} \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^2 + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}} \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^2\right)\right]^{\frac{1}{2}} \tag{2.19}$$

and a dual hesitant weighted Euclidean-Hausdorff distance:

$$d_{dhweh}(M,N) = \left[\frac{1}{2}\sum_{i=1}^{n} w_i \left(\max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^2 + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^2\right)\right]^{\frac{1}{2}}. \tag{2.20}$$

Furthermore, combining the generalized dual hesitant weighted distance (2.15) and generalized dual hesitant weighted Hausdorff distance (2.16), we develop a generalized hybrid dual hesitant weighted distance as

$$d_{ghdhw}(M,N) = \left[\frac{1}{4}\sum_{i=1}^{n} w_i \left(\frac{1}{l_{h(x_i)}}\sum_{j=1}^{l_{h(x_i)}} \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^\lambda + \frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}} \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^\lambda + \max_j \left|h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i)\right|^\lambda + \max_k \left|g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i)\right|^\lambda\right)\right]^{\frac{1}{\lambda}}, \tag{2.21}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x_i)}$ and $k = 1, 2, \ldots, l_{g(x_i)}$.

In the special cases where $\lambda = 1, 2$, (2.21) reduces a hybrid dual hesitant weighted Hamming distance and a hybrid dual hesitant weighted Euclidean distance as follows, respectively:

$$d_{hdhwh}(M,N) = \frac{1}{4}\sum_{i=1}^{n} w_i \left(\frac{1}{l_{h(x_i)}} \sum_{j=1}^{l_{h(x_i)}} \left| h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i) \right| + \frac{1}{l_{g(x_i)}} \sum_{k=1}^{l_{g(x_i)}} \left| g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i) \right| + \max_j \left| h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i) \right| + \max_k \left| g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i) \right| \right), \tag{2.22}$$

$$d_{hdhwe}(M,N) = \left[\frac{1}{4}\sum_{i=1}^{n} w_i \left(\frac{1}{l_{h(x_i)}} \sum_{j=1}^{l_{h(x_i)}} \left| h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i) \right|^2 + \frac{1}{l_{g(x_i)}} \sum_{k=1}^{l_{g(x_i)}} \left| g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i) \right|^2 + \max_j \left| h_M^{\sigma(j)}(x_i) - h_N^{\sigma(j)}(x_i) \right|^2 + \max_k \left| g_M^{\sigma(k)}(x_i) - g_N^{\sigma(k)}(x_i) \right|^2 \right) \right]^{\frac{1}{2}}. \tag{2.23}$$

In aforementioned mentioned analysis, the distance measures are discrete. If both the universe of discourse and the weight of element are continuous, and the weight of $x \in X = [a,b]$ is $w(x)$, where $w(x) \in [0,1]$ and $\int_a^b w(x)dx = 1$, then we define a continuous dual hesitant weighted Hamming distance, a continuous dual hesitant weighted Euclidean distance and a generalized continuous dual hesitant weighted distance as follows, respectively:

$$d_{cdhwh}(M,N) = \frac{1}{2}\int_a^b w(x) \left(\frac{1}{l_{h(x)}} \sum_{j=1}^{l_{h(x)}} \left| h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x) \right| + \frac{1}{l_{g(x)}} \sum_{k=1}^{l_{g(x)}} \left| g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x) \right| \right) dx, \tag{2.24}$$

$$d_{cdhwe}(M,N) = \left[\frac{1}{2}\int_a^b w(x) \left(\frac{1}{l_{h(x)}} \sum_{j=1}^{l_{h(x)}} \left| h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x) \right|^2 + \frac{1}{l_{g(x)}} \sum_{k=1}^{l_{g(x)}} \left| g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x) \right|^2 \right) dx \right]^{\frac{1}{2}}, \tag{2.25}$$

$$d_{gcdhw}(M,N) = \left[\frac{1}{2}\int_a^b w(x)\left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right|^\lambda \right.\right.$$
$$\left.\left. + \frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|^\lambda\right) dx\right]^{\frac{1}{\lambda}}, \qquad (2.26)$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x)}$ and $k = 1, 2, \ldots, l_{g(x)}$.

If $w(x) = \frac{1}{b-a}$ for any $x \in [a, b]$, then the continuous dual hesitant weighted Hamming distance (2.24) reduces a continuous dual hesitant normalized Hamming distance:

$$d_{cdhnh}(M,N) = \frac{1}{2(b-a)}\int_a^b \left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right| \right.$$
$$\left. + \frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|\right) dx \qquad (2.27)$$

and the continuous dual hesitant weighted Euclidean distance (2.25) reduces a continuous dual hesitant normalized Euclidean distance:

$$d_{cdhne}(M,N) = \left[\frac{1}{2(b-a)}\int_a^b \left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right|^2 \right.\right.$$
$$\left.\left. + \frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|^2\right) dx\right]^{\frac{1}{2}} \qquad (2.28)$$

and the generalized continuous dual hesitant weighted distance (2.26) reduces a generalized continuous dual hesitant normalized distance:

$$d_{gcdhn}(M,N) = \left[\frac{1}{2(b-a)}\int_a^b \left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right|^\lambda \right.\right.$$
$$\left.\left. + \frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|^\lambda\right) dx\right]^{\frac{1}{\lambda}}, \qquad (2.29)$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x)}$ and $k = 1, 2, \ldots, l_{g(x)}$.

Similar to (2.16), if we apply the Hausdorff metric to the continuous dual hesitant weighted distance measure, then a generalized continuous dual hesitant weighted Hausdorff distance is given as

$$d_{gcdhwh}(M,N) = \left[\frac{1}{2}\int_a^b w(x)\left(\max_j \left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right|^\lambda + \max_k \left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|^\lambda\right)dx\right]^{\frac{1}{\lambda}}, \tag{2.30}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x)}$ and $k = 1, 2, \ldots, l_{g(x)}$.

In special cases where $\lambda = 1, 2$, the generalized continuous dual hesitant weighted Hausdorff distance (2.30) reduces a continuous dual hesitant weighted Hamming-Hausdorff distance:

$$d_{cdhwhh}(M,N) = \frac{1}{2}\int_a^b w(x)\left(\max_j \left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right| + \max_k \left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|\right)dx \tag{2.31}$$

and a continuous dual hesitant weighted Euclidean-Hausdorff distance:

$$d_{cdhweh}(M,N) = \left[\frac{1}{2}\int_a^b w(x)\left(\max_j \left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right|^2 + \max_k \left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|^2\right)dx\right]^{\frac{1}{2}}, \tag{2.32}$$

respectively.

In particular, if $w(x) = \frac{1}{(b-a)}$ for any $x \in [a,b]$, then the generalized continuous dual hesitant weighted Hausdorff distance (2.30) becomes a generalized continuous dual hesitant normalized Hausdorff distance:

$$d_{gcdhnh}(M,N) = \left[\frac{1}{2(b-a)}\int_a^b \left(\max_j \left|h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x)\right|^\lambda + \max_k \left|g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x)\right|^\lambda\right)dx\right]^{\frac{1}{\lambda}}, \tag{2.33}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x)}$ and $k = 1, 2, \ldots, l_{g(x)}$, while the continuous dual hesitant weighted Hamming-Hausdorff distance (2.31) becomes a continuous dual hesitant normalized Hamming-Hausdorff distance:

$$d_{cdhnhh}(M,N) = \frac{1}{2(b-a)} \int_a^b \left(\max_j \left| h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x) \right| + \max_k \left| g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x) \right| \right) dx \tag{2.34}$$

and the continuous dual hesitant weighted Euclidean-Hausdorff distance (2.32) becomes a continuous dual hesitant normalized Euclidean-Hausdorff distance:

$$d_{cdhneh}(M,N) = \left[\frac{1}{2(b-a)} \int_a^b \left(\max_j \left| h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x) \right|^2 + \max_k \left| g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x) \right|^2 \right) dx \right]^{\frac{1}{2}}. \tag{2.35}$$

Similar to (2.21), combining the generalized continuous dual hesitant weighted distance (2.26) and generalized continuous dual hesitant weighted Hausdorff distance (2.30), we develop a generalized hybrid continuous dual hesitant weighted distance as

$$\begin{aligned} d_{ghcdhw}(M,N) &= \Bigg[\frac{1}{4} \int_a^b w(x) \Bigg(\frac{1}{l_{h(x)}} \sum_{j=1}^{l_{h(x)}} \left| h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x) \right|^\lambda + \frac{1}{l_{g(x)}} \sum_{k=1}^{l_{g(x)}} \left| g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x) \right|^\lambda \\ &\quad + \max_j \left| h_M^{\sigma(j)}(x) - h_N^{\sigma(j)}(x) \right|^\lambda + \max_k \left| g_M^{\sigma(k)}(x) - g_N^{\sigma(k)}(x) \right|^\lambda \Bigg) dx \Bigg]^{\frac{1}{\lambda}}, \end{aligned} \tag{2.36}$$

where $\lambda > 0$, $j = 1, 2, \ldots, l_{h(x)}$ and $k = 1, 2, \ldots, l_{g(x)}$.

Let $\lambda = 1, 2$, then from (2.36), we get a hybrid continuous dual hesitant weighted Hamming distance and a hybrid continuous dual hesitant weighted Euclidean distance as

$d_{hcdhwh}(M,N)$

$$= \frac{1}{4}\int_a^b w(x)\left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|+\frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|\right.$$
$$\left.+\max_j\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|+\max_k\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|\right)dx \tag{2.37}$$

and

$$d_{hcdhwe}(M,N)$$
$$= \left[\frac{1}{4}\int_a^b w(x)\left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|^2+\frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|^2\right.\right.$$
$$\left.\left.+\max_j\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|^2+\max_k\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|^2\right)dx\right]^{\frac{1}{2}}, \tag{2.38}$$

respectively.

In particular, if $w(x)=\frac{1}{(b-a)}$ for any $x\in[a,b]$, then the generalized hybrid continuous dual hesitant weighted distance (2.36) becomes a generalized hybrid continuous dual hesitant normalized distance:

$$d_{ghcdhn}(M,N) = \left[\frac{1}{4(b-a)}\int_a^b\left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|^\lambda\right.\right.$$
$$+\frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|^\lambda+\max_j\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|^\lambda$$
$$\left.\left.+\max_k\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|^\lambda\right)dx\right]^{\frac{1}{\lambda}}, \tag{2.39}$$

where $\lambda>0$, $j=1,2,\ldots,l_{h(x)}$ and $k=1,2,\ldots,l_{g(x)}$, and the hybrid continuous dual hesitant weighted Hamming distance (2.37) reduces a hybrid continuous dual hesitant normalized Hamming distance:

$$d_{hcdhnh}(M,N) = \frac{1}{4(b-a)}\int_a^b\left(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|\right.$$

$$+\frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|+\max_j\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|$$

$$+\max_k\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|\Bigg)dx \tag{2.40}$$

and the hybrid continuous dual hesitant weighted Euclidean distance (2.38) becomes a hybrid continuous dual hesitant normalized Euclidean distance:

$$d_{hcdhne}(M,N)=\Bigg[\frac{1}{4(b-a)}\int_a^b\Bigg(\frac{1}{l_{h(x)}}\sum_{j=1}^{l_{h(x)}}\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|^2$$

$$+\frac{1}{l_{g(x)}}\sum_{k=1}^{l_{g(x)}}\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|^2+\max_j\left|h_M^{\sigma(j)}(x)-h_N^{\sigma(j)}(x)\right|^2$$

$$+\max_k\left|g_M^{\sigma(k)}(x)-g_N^{\sigma(k)}(x)\right|^2\Bigg)dx\Bigg]^{\frac{1}{2}}, \tag{2.41}$$

respectively.

From the aforementioned analysis, it can be seen that the generalized dual hesitant weighted distance (2.15), the generalized dual hesitant weighted Hausdorff distance (2.16) and the generalized hybrid dual hesitant weighted distance (2.21) are three fundamental distance measures, based on which all of the other developed distance measures can be obtained under some special conditions.

In what follows, we apply the our distance measures for DHFSs to multiple attribute decision making under dual hesitant fuzzy environment.

Example 2.1.6 Energy is an indispensable factor for the socio-economic development of societies. Thus the correct energy policy affects economic development and environment, and so, the most appropriate energy policy selection is very important. Suppose that there are five alternatives (energy projects) G_i ($i = 1, 2, 3, 4, 5$) to be invested, and four attributes x_1 (technological), x_2 (environmental), x_3 (socio-political) and x_4 (economic) be considered (for details, see [14]). The attribute weight vector is $w = (0.15, 0.3, 0.2, 0.35)^T$. Assume that the characteristics of the alternatives G_i ($i = 1, 2, 3, 4, 5$) with respect to the attribute x_j ($j = 1, 2, 3, 4$) are represented by the DHFSs $d_{ij} = \{\{\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}, \{\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}\}$, where γ_{ij} indicates the degree that the alternative x_i satisfies the attribute y_j, η_{ij} indicates the degree that the alternative x_i does not satisfy the attribute y_j, such that $\gamma_{ij} \in [0, 1]$, $\eta_{ij} \in [0, 1]$ and $\gamma_{ij}^{+} + \eta_{ij}^{+} \leq 1$. All $d_{ij} = \{\{\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}, \{\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}\}$ ($i = 1, 2, 3, 4, 5$; $j = 1, 2, 3, 4$) are contained in the dual hesitant fuzzy decision matrix $D = (d_{ij})_{5\times4}$ (see Table 2.1).

Table 2.1: The dual hesitant fuzzy decision matrix $D = (d_{ij})_{5\times4}$

	x_1	x_2	x_3	x_4
G_1	$\{\{0.3\}, \{0.2, 0.4\}\}$	$\{\{0.7\}, \{0.2\}\}$	$\{\{0.5, 0.6\}, \{0.2, 0.3\}\}$	$\{\{0.6, 0.7\}, \{0.1, 0.2\}\}$
G_2	$\{\{0.5, 0.6\}, \{0.2\}\}$	$\{\{0.3, 0.4\}, \{0.1, 0.2\}\}$	$\{\{0.7, 0.8\}, \{0.1\}\}$	$\{\{0.4, 0.5\}, \{0.2, 0.3\}\}$
G_3	$\{\{0.3, 0.4\}, \{0.4, 0.5\}\}$	$\{\{0.7, 0.8\}, \{0.1, 0.2\}\}$	$\{\{0.4\}, \{0.3, 0.4\}\}$	$\{\{0.7, 0.8\}, \{0.1\}\}$
G_4	$\{\{0.2\}, \{0.6, 0.7\}\}$	$\{\{0.8\}, \{0.1\}\}$	$\{\{0.7, 0.8\}, \{0.2, 0.3\}\}$	$\{\{0.8\}, \{0.1, 0.2\}\}$
G_5	$\{\{0.8, 0.9\}, \{0.1\}\}$	$\{\{0.6, 0.7\}, \{0.2, 0.3\}\}$	$\{\{0.2\}, \{0.5, 0.6\}\}$	$\{\{0.5, 0.7\}, \{0.1, 0.2, 0.3\}\}$

Suppose that the ideal alternative is $A^* = \{\{1\}, \{0\}\}$ seen as a special DHFS (i.e, A^* is complete certainty), then we can calculate the distance between each alternative and the ideal alternative using our distance measures. If we use the generalized dual hesitant weighted distance (2.15), the generalized dual hesitant weighted Hausdorff distance (2.16) and the generalized hybrid dual hesitant weighted distance (2.21) to calculate the deviations between each alternative and the ideal alternative, then we obtain the rankings of these alternatives, which are listed in Tables 2.2-2.4, respectively, when some values of the parameter are given. We find that the rankings are different as the parameter λ (which can be considered as the decision maker's risk-bearing attitude) changes, and consequently, the proposed distance measures can provide the decision makers more choices as the different values of parameter are given according to the decision maker's risk-bearing attitudes.

Table 2.2: Results obtained by the generalized dual hesitant weighted distance d_{gdhw}

	G_1	G_2	G_3	G_4	G_5	Rankings
$\lambda = 1$	0.307500	0.343750	0.298750	0.265000	0.348750	$G_4 \succ G_1 \succ G_3 \succ G_2 \succ G_5$
$\lambda = 2$	0.058500	0.079063	0.061188	0.055875	0.081771	$G_4 \succ G_1 \succ G_3 \succ G_2 \succ G_5$
$\lambda = 5$	0.003353	0.005992	0.003931	0.006841	0.008499	$G_1 \succ G_3 \succ G_2 \succ G_4 \succ G_5$
$\lambda = 10$	0.000219	0.000323	0.000194	0.000934	0.001118	$G_3 \succ G_1 \succ G_2 \succ G_4 \succ G_5$

Table 2.3: Results obtained by the generalized dual hesitant weighted Hausdorff distance d_{gdhwh}

	G_1	G_2	G_3	G_4	G_5	Rankings
$\lambda = 1$	0.342500	0.385000	0.335000	0.287500	0.407500	$G_4 \succ G_3 \succ G_1 \succ G_2 \succ G_5$
$\lambda = 2$	0.056370	0.088025	0.058500	0.057288	0.082713	$G_1 \succ G_4 \succ G_3 \succ G_5 \succ G_2$
$\lambda = 5$	0.003166	0.008323	0.003353	0.007517	0.003122	$G_5 \succ G_1 \succ G_3 \succ G_4 \succ G_2$
$\lambda = 10$	0.000215	0.000537	0.000220	0.001017	0.000079	$G_5 \succ G_1 \succ G_3 \succ G_2 \succ G_4$

Table 2.4: Results obtained by the generalized hybrid dual hesitant weighted distance d_{ghdhw}

	G_1	G_2	G_3	G_4	G_5	Rankings
$\lambda = 1$	0.325000	0.364375	0.316875	0.276250	0.378125	$G_4 \succ G_3 \succ G_1 \succ G_2 \succ G_5$
$\lambda = 2$	0.057435	0.083544	0.059519	0.056581	0.082242	$G_4 \succ G_1 \succ G_3 \succ G_5 \succ G_2$
$\lambda = 5$	0.003259	0.007157	0.003642	0.007179	0.005810	$G_1 \succ G_3 \succ G_5 \succ G_2 \succ G_4$
$\lambda = 10$	0.000217	0.000430	0.000207	0.000976	0.000599	$G_3 \succ G_1 \succ G_2 \succ G_5 \succ G_4$

2.2 Ordered weighted distance measures for DHFSs

Xu and Chen [61] defined some ordered weighted distance measures, and then Yager generalized Xu and Chen's distance measures and provided a variety of ordered weighted averaging norms, based on which he proposed several similarity measures. Merigó and Gil-Lafuente [32] proposed an ordered weighted averaging distance operator and applied it in the selection of financial products. In the following, we develop several ordered distance measures for DHFSs. Motivated by the ordered weighted distance [61], we define a generalized dual hesitant ordered weighted distance:

$$d_{gdhow}(M,N)=\left[\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\frac{1}{l_{h(x_{\hat{\sigma}(i)})}}\sum_{j=1}^{l_{h(x_{\hat{\sigma}(i)})}}\left|h_M^{\sigma(j)}(x_{\hat{\sigma}(i)})-h_N^{\sigma(j)}(x_{\hat{\sigma}(i)})\right|^{\lambda}\right.\right.$$
$$\left.\left.+\frac{1}{l_{g(x_{\hat{\sigma}(i)})}}\sum_{k=1}^{l_{g(x_{\hat{\sigma}(i)})}}\left|g_M^{\sigma(k)}(x_{\hat{\sigma}(i)})-g_N^{\sigma(k)}(x_{\hat{\sigma}(i)})\right|^{\lambda}\right)\right]^{\frac{1}{\lambda}}, \quad (2.42)$$

where $\lambda>0$, $\sigma(j)$ and $\sigma(k)$ are given in Section 2.1, and $\hat{\sigma}:(1,2,\ldots,n)\rightarrow(1,2,\ldots,n)$ is a permutation satisfying the condition

$$\frac{1}{l_{h(x_{\hat{\sigma}(i+1)})}}\sum_{j=1}^{l_{h(x_{\hat{\sigma}(i+1)})}}\left|h_M^{\sigma(j)}(x_{\hat{\sigma}(i+1)})-h_N^{\sigma(j)}(x_{\hat{\sigma}(i+1)})\right|$$
$$+\frac{1}{l_{g(x_{\hat{\sigma}(i+1)})}}\sum_{k=1}^{l_{g(x_{\hat{\sigma}(i+1)})}}\left|g_M^{\sigma(k)}(x_{\hat{\sigma}(i+1)})-g_N^{\sigma(k)}(x_{\hat{\sigma}(i+1)})\right|$$
$$\geq\frac{1}{l_{h(x_{\hat{\sigma}(i)})}}\sum_{j=1}^{l_{h(x_{\hat{\sigma}(i)})}}\left|h_M^{\sigma(j)}(x_{\hat{\sigma}(i)})-h_N^{\sigma(j)}(x_{\hat{\sigma}(i)})\right|$$
$$+\frac{1}{l_{g(x_{\hat{\sigma}(i)})}}\sum_{k=1}^{l_{g(x_{\hat{\sigma}(i)})}}\left|g_M^{\sigma(k)}(x_{\hat{\sigma}(i)})-g_N^{\sigma(k)}(x_{\hat{\sigma}(i)})\right|, \quad (2.43)$$

for $i=1,2,\ldots,n-1$.

In special cases where $\lambda=1,2$, the generalized dual hesitant ordered weighted distance (2.42) reduces a dual hesitant ordered weighted Hamming distance:

$$d_{dhowh}(M,N)=\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\frac{1}{l_{h(x_{\hat{\sigma}(i)})}}\sum_{j=1}^{l_{h(x_{\hat{\sigma}(i)})}}\left|h_M^{\sigma(j)}(x_{\hat{\sigma}(i)})-h_N^{\sigma(j)}(x_{\hat{\sigma}(i)})\right|\right.$$
$$\left.+\frac{1}{l_{g(x_{\hat{\sigma}(i)})}}\sum_{k=1}^{l_{g(x_{\hat{\sigma}(i)})}}\left|g_M^{\sigma(k)}(x_{\hat{\sigma}(i)})-g_N^{\sigma(k)}(x_{\hat{\sigma}(i)})\right|\right) \quad (2.44)$$

and a dual hesitant ordered weighted Euclidean distance:

$$d_{dhowe}(M,N) = \left[\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\frac{1}{l_{h(x_{\hat{\sigma}(i)})}}\sum_{j=1}^{l_{h(x_{\hat{\sigma}(i)})}}\left|h_M^{\sigma(j)}(x_{\hat{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\hat{\sigma}(i)})\right|^2 + \frac{1}{l_{g(x_{\hat{\sigma}(i)})}}\sum_{k=1}^{l_{g(x_{\hat{\sigma}(i)})}}\left|g_M^{\sigma(k)}(x_{\hat{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\hat{\sigma}(i)})\right|^2\right)\right]^{\frac{1}{2}}, \quad (2.45)$$

respectively.

Similar to (2.16), with the Hausdorff metric, we develop a generalized dual hesitant ordered weighted Hausdorff distance as

$$d_{gdhowh}(M,N) = \left[\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\max_j\left|h_M^{\sigma(j)}(x_{\dot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\dot{\sigma}(i)})\right|^\lambda + \max_k\left|g_M^{\sigma(k)}(x_{\dot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\dot{\sigma}(i)})\right|^\lambda\right)\right]^{\frac{1}{\lambda}}, \quad (2.46)$$

where $\lambda > 0$, $\sigma(j)$ and $\sigma(k)$ are given in Section 2.1, and $\dot{\sigma} : (1, 2, \ldots, n) \to (1, 2, \ldots, n)$ is a permutation satisfying the condition

$$\max_j\left|h_M^{\sigma(j)}(x_{\dot{\sigma}(i+1)}) - h_N^{\sigma(j)}(x_{\dot{\sigma}(i+1)})\right| + \max_k\left|g_M^{\sigma(k)}(x_{\dot{\sigma}(i+1)}) - g_N^{\sigma(k)}(x_{\dot{\sigma}(i+1)})\right| \geq \max_j\left|h_M^{\sigma(j)}(x_{\dot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\dot{\sigma}(i)})\right| + \max_k\left|g_M^{\sigma(k)}(x_{\dot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\dot{\sigma}(i)})\right|, \quad i = 1, 2, \ldots, n-1. \quad (2.47)$$

In the following, we discuss two special cases of the generalized dual hesitant ordered weighted Hausdorff distance (2.46) by taking different values of the parameter λ:

(1) If $\lambda = 1$, then (2.46) becomes a dual hesitant ordered weighted Hamming-Hausdorff distance:

$$d_{dhowhh}(M,N) = \frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\max_j\left|h_M^{\sigma(j)}(x_{\dot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\dot{\sigma}(i)})\right| + \max_k\left|g_M^{\sigma(k)}(x_{\dot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\dot{\sigma}(i)})\right|\right). \quad (2.48)$$

(2) If $\lambda = 2$, then (2.46) becomes a dual hesitant ordered weighted Euclidean-Hausdorff distance:

$$d_{dhoweh}(M,N) = \left[\frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\max_j\left|h_M^{\sigma(j)}(x_{\dot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\dot{\sigma}(i)})\right|^2\right.\right.$$
$$\left.\left. + \max_k\left|g_M^{\sigma(k)}(x_{\dot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\dot{\sigma}(i)})\right|^2\right)\right]^{\frac{1}{2}}. \tag{2.49}$$

Combining (2.42) and (2.46), similar to (2.21), we develop a generalized hybrid dual hesitant ordered weighted distance as

$$d_{ghdhow}(M,N) = \left[\frac{1}{4}\sum_{i=1}^{n}\omega_i\left(\frac{1}{l_{h(x_{\ddot{\sigma}(i)})}}\sum_{j=1}^{l_{h(x_{\ddot{\sigma}(i)})}}\left|h_M^{\sigma(j)}(x_{\ddot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\ddot{\sigma}(i)})\right|^\lambda\right.\right.$$
$$+\frac{1}{l_{g(x_i)}}\sum_{k=1}^{l_{g(x_i)}}\left|g_M^{\sigma(k)}(x_{\ddot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\ddot{\sigma}(i)})\right|^\lambda + \max_j\left|h_M^{\sigma(j)}(x_{\ddot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\ddot{\sigma}(i)})\right|^\lambda$$
$$\left.\left. + \max_k\left|g_M^{\sigma(k)}(x_{\ddot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\ddot{\sigma}(i)})\right|^\lambda\right)\right]^{\frac{1}{\lambda}}, \tag{2.50}$$

where $\lambda > 0$, $\sigma(j)$ and $\sigma(k)$ are given in Section 2.1, and $\ddot{\sigma} : (1,2,\ldots,n) \to (1,2,\ldots,n)$ is a permutation satisfying the condition

$$\frac{1}{l_{h(x_{\hat{\sigma}(i+1)})}}\sum_{j=1}^{l_{h(x_{\hat{\sigma}(i+1)})}}\left|h_M^{\sigma(j)}(x_{\hat{\sigma}(i+1)}) - h_N^{\sigma(j)}(x_{\hat{\sigma}(i+1)})\right|$$
$$+\frac{1}{l_{g(x_{\ddot{\sigma}(i+1)})}}\sum_{k=1}^{l_{g(x_{\ddot{\sigma}(i+1)})}}\left|g_M^{\sigma(k)}(x_{\ddot{\sigma}(i+1)}) - g_N^{\sigma(k)}(x_{\ddot{\sigma}(i+1)})\right|$$
$$+\max_j\left|h_M^{\sigma(j)}(x_{\ddot{\sigma}(i+1)}) - h_N^{\sigma(j)}(x_{\ddot{\sigma}(i+1)})\right| + \max_k\left|g_M^{\sigma(k)}(x_{\ddot{\sigma}(i+1)}) - g_N^{\sigma(k)}(x_{\ddot{\sigma}(i+1)})\right|$$
$$\geq \frac{1}{l_{h(x_{\ddot{\sigma}(i)})}}\sum_{j=1}^{l_{h(x_{\ddot{\sigma}(i)})}}\left|h_M^{\sigma(j)}(x_{\ddot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\ddot{\sigma}(i)})\right|$$
$$+\frac{1}{l_{g(x_{\ddot{\sigma}(i)})}}\sum_{k=1}^{l_{g(x_{\ddot{\sigma}(i)})}}\left|g_M^{\sigma(k)}(x_{\ddot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\ddot{\sigma}(i)})\right|$$
$$+\max_j\left|h_M^{\sigma(j)}(x_{\ddot{\sigma}(i)}) - h_N^{\sigma(j)}(x_{\ddot{\sigma}(i)})\right| + \max_k\left|g_M^{\sigma(k)}(x_{\ddot{\sigma}(i)}) - g_N^{\sigma(k)}(x_{\ddot{\sigma}(i)})\right|,$$
$$i = 1,2,\ldots,n-1. \tag{2.51}$$

Some special cases can be obtained just as discussed in Sections 2.1 and 2.2 as the parameter and weight vector change. Furthermore, let d_o denote the ordered distance measures defined above, then the ordered similarity measures for DHFSs can be given as $s_o = 1 - d_o$.

Finally, we consider, as another important issue, the determination of the weight vectors associated with the ordered weighted distance measures. Based on the works of Xu and Chen [61] and Xu and Xia [66], we propose three methods to determine the weight vectors.

Considering each element $\{h_M(x_{\rho(i)}), g_M(x_{\rho(i)})\}$ in M and $\{h_N(x_{\rho(i)}), g_N(x_{\rho(i)})\}$ in N, respectively, $d(\{h_M(x_{\rho(i)}), g_M(x_{\rho(i)})\}, \{h_N(x_{\rho(i)}), g_N(x_{\rho(i)})\})$ $(i = 1, 2, \ldots, n)$ as given in Section 2.1, and denoting $\hat{\sigma}$, $\dot{\sigma}$ and $\ddot{\sigma}$ as ρ, we have

(1) Let

$$\omega_i = \frac{d(\{h_M(x_{\rho(i)}), g_M(x_{\rho(i)})\}, \{h_N(x_{\rho(i)}), g_N(x_{\rho(i)})\})}{\sum_{k=1}^{n} d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\})}, \quad i = 1, 2, \ldots, n, \tag{2.52}$$

then $\omega_{i+1} \geq \omega_i \geq 0$, $i = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} \omega_i = 1$.

(2) Let

$$\omega_i = \frac{e^{-d(\{h_M(x_{\rho(i)}), g_M(x_{\rho(i)})\}, \{h_N(x_{\rho(i)}), g_N(x_{\rho(i)})\})}}{\sum_{k=1}^{n} e^{-d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\})}}, \quad i = 1, 2, \ldots, n, \tag{2.53}$$

then $0 \leq \omega_i \leq \omega_{i+1}$, $i = 1, 2, \ldots, n-1$, and $\sum_{i=1}^{n} \omega_i = 1$.

(3) Let

$$\begin{aligned}\dot{d}(\{h_M, g_M\}, \{h_N, g_N\}) \\ = \frac{1}{n} \sum_{k=1}^{n} d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\})\end{aligned} \tag{2.54}$$

and

$$\ddot{d}(d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\}), \dot{d}(\{h_M, g_M\}, \{h_N, g_N\}))$$

$$= \Big| \ d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\})$$
$$-\frac{1}{n}\sum_{k=1}^{n} d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\})\Big|, \tag{2.55}$$

then we define

$$\omega_i = \frac{1-\ddot{d}(d(a_i,b_i),\dot{d}(c,d))}{\sum_{k=1}^{n}(1-\ddot{d}(d(a_k,b_k),\dot{d}(c,d)))}, \ i = 1,2,\ldots,n, \tag{2.56}$$

where $a_i = \{h_M(x_{\rho(i)}), g_M(x_{\rho(i)})\}$, $b_i = \{h_N(x_{\rho(i)}), g_N(x_{\rho(i)})\}$, $c = \{h_M, g_M\}$, $d = \{h_N, g_N\}$, $a_k = \{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}$, $b_k = \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\}$, and so we obtain $\omega_i \geq 0$, $i = 1,2,\ldots,n$, and $\sum_{i=1}^{n}\omega_i = 1$.

From the aforementioned analysis, we know that the weight vector derived from (2.52) is monotone decreasing sequence, the weight vector derived from (2.53) is monotone increasing sequence, and the weight vector derived from (2.56) combine the above two cases, i.e., the closer the value $d(\{h_M(x_{\rho(i)}), g_M(x_{\rho(i)})\}, \{h_N(x_{\rho(i)}), g_N(x_{\rho(i)})\})$ to the mean $\frac{1}{n}\sum_{k=1}^{n} d(\{h_M(x_{\rho(k)}), g_M(x_{\rho(k)})\}, \{h_N(x_{\rho(k)}), g_N(x_{\rho(k)})\})$, the larger the weight ω_i.

Table 2.5: Results obtained by the generalized dual hesitant ordered weighted distance d_{gdhow}

	G_1	G_2	G_3	G_4	G_5	Rankings
$\lambda = 1$	0.36852	0.35096	0.42768	0.49327	0.46518	$G_2 \succ G_1 \succ G_3 \succ G_5 \succ G_4$
$\lambda = 2$	0.08509	0.08091	0.10857	0.15837	0.13647	$G_2 \succ G_1 \succ G_3 \succ G_5 \succ G_4$
$\lambda = 5$	0.00717	0.00605	0.00854	0.02519	0.01897	$G_2 \succ G_1 \succ G_3 \succ G_5 \succ G_4$
$\lambda = 10$	0.00053	0.00032	0.00045	0.00347	0.00268	$G_2 \succ G_3 \succ G_1 \succ G_5 \succ G_4$

In Example 2.1.6, if the attribute weight vector is unknown, then we can use the ordered weight distance measures for DHFSs to calculate the distance between each alternative and the ideal alternative. Without loss of generality, suppose that $d = d_{dhnh}$ in (2.52) and ρ is given as in (2.43), we use the generalized dual hesitant ordered weighted distance measure (2.42) to calculate the distance between each alternative and the ideal alternative. The derived results are shown in Table 2.5 with the different values of the parameter.

2.3 Conclusions

In this chapter, we have investigated the distance measures for DHFSs. Based on the well-known Hamming distance, the Euclidean distance, the Hausdorff metric and their generalizations, we have developed a class of dual hesitant distance measures, and discussed their properties and relations as their parameters change. We have also given a variety of ordered weighted distance measures for DHFSs in which the distances are rearranged in decreasing order, and given three ways to determine the associated weighting vectors. With the relationship between distance measures and similarity measures, the corresponding similarity measures for DHFSs have been obtained. It should be pointed out that all of the above measures are based on the assumption that if the corresponding DHFEs in DHFSs do not have same length, then the shorter one should be extended by adding the minimum value in membership hesitancy part and the maximum value in nonmembership hesitancy part in it until both the DHFEs have the same length. In fact, we can extend the shorter DHFE be adding any values in membership hesitancy part and the maximum value in nonmembership hesitancy part in it until it has the same length of the longer one according to the decision makers' preference and actual situations.

Chapter 3

Correlation measures for dual hesitant fuzzy information

3.1 Correlation measures of dual hesitant fuzzy elements

IFSs, IVFSs, IVIFSs and HFSs are the extensions of fuzzy sets [83]. However, these extensions cannot deal with the situation that people have two kinds of hesitancy in providing their preferences over objects in process of decision making, which permit the membership degrees and nonmembership degrees of an element, respectively, to a set presented as several possible values. Zhu et al. [86], recently, proposed the concept of dual hesitant fuzzy set to deal with such case.

In the following, we first introduce the concept of correlation coefficient for DHFEs and then propose several correlation coefficient formulas and discuss their properties.

Definition 3.1.1 Let M and N be two DHFEs, then the correlation coefficient of M and N is defined as $c(M, N)$, which satisfies the following properties:

(C1) $|c(M, N)| \leq 1$;

(C2) $c(M, N) = 1$ if $M = N$;

(C3) $c(M,N)=c(N,M)$.

On the basis of Definition 3.1.1, we can construct several correlation coefficients for DHFEs:

(1) $c_1(M,N)=\frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)}+g_M^{\sigma(i)}g_N^{\sigma(i)})}{[\sum_{i=1}^{l}((h_M^{\sigma(i)})^2+(g_M^{\sigma(i)})^2)\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)]^{1/2}}$;

(2) $c_2(M,N)=\frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)}+g_M^{\sigma(i)}g_N^{\sigma(i)})}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)})^2+(g_M^{\sigma(i)})^2),\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)\}}$;

(3) $c_3(M,N)=\frac{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)(h_N^{\sigma(i)}-\bar{h}_N)+(g_M^{\sigma(i)}-\bar{g}_M)(g_N^{\sigma(i)}-\bar{g}_N))}{[\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)^2+(g_M^{\sigma(i)}-\bar{g}_M)^2)\sum_{i=1}^{l}((h_N^{\sigma(i)}-\bar{h}_N)^2+(g_N^{\sigma(i)}-\bar{g}_N)^2)]^{1/2}}$,

where $\bar{h}_M=\frac{1}{l}\sum_{i=1}^{l}h_M^{\sigma(i)}$, $\bar{g}_M=\frac{1}{l}\sum_{i=1}^{l}g_M^{\sigma(i)}$, $\bar{h}_N=\frac{1}{l}\sum_{i=1}^{l}h_N^{\sigma(i)}$ and $\bar{g}_N=\frac{1}{l}\sum_{i=1}^{l}g_N^{\sigma(i)}$;

(4) $c_4(M,N)=\frac{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)(h_N^{\sigma(i)}-\bar{h}_N)+(g_M^{\sigma(i)}-\bar{g}_M)(g_N^{\sigma(i)}-\bar{g}_N))}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)^2+(g_M^{\sigma(i)}-\bar{g}_M)^2),\sum_{i=1}^{l}((h_N^{\sigma(i)}-\bar{h}_N)^2+(g_N^{\sigma(i)}-\bar{g}_N)^2)\}}$,

where $\bar{h}_M=\frac{1}{l}\sum_{i=1}^{l}h_M^{\sigma(i)}$, $\bar{g}_M=\frac{1}{l}\sum_{i=1}^{l}g_M^{\sigma(i)}$, $\bar{h}_N=\frac{1}{l}\sum_{i=1}^{l}h_N^{\sigma(i)}$ and $\bar{g}_N=\frac{1}{l}\sum_{i=1}^{l}g_N^{\sigma(i)}$.

(5) $c_5(M,N)=\frac{1}{l}\sum_{i=1}^{l}\left(\frac{\Delta\gamma_{\min}+\Delta\gamma_{\max}+\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\gamma_{\sigma(i)}+\Delta\gamma_{\max}+\Delta\eta_{\sigma(i)}+\Delta\eta_{\max}}\right)$,

where $\Delta\gamma_{\sigma(i)}=|h_M^{\sigma(i)}-h_N^{\sigma(i)}|$, $\Delta\gamma_{\min}=\min_i\{|h_M^{\sigma(i)}-h_N^{\sigma(i)}|\}$, $\Delta\gamma_{\max}=\max_i\{|h_M^{\sigma(i)}-h_N^{\sigma(i)}|\}$, $\Delta\eta_{\sigma(i)}=|g_M^{\sigma(i)}-g_N^{\sigma(i)}|$, $\Delta\eta_{\min}=\min_i\{|g_M^{\sigma(i)}-g_N^{\sigma(i)}|\}$ and $\Delta\eta_{\max}=\max_i\{|g_M^{\sigma(i)}-g_N^{\sigma(i)}|\}$.

Theorem 3.1.2 Let M and N be two DHFEs, then

(1) $c_2(M,N)\le c_1(M,N)$.

(2) $|c_4(M,N)|\le|c_3(M,N)|$.

Proof (1) Since

$$\begin{aligned}&\left(\sum_{i=1}^{l}((h_M^{\sigma(i)})^2+(g_M^{\sigma(i)})^2)\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)\right)^{\frac{1}{2}}\\ &\le\left(\left(\max\left\{\sum_{i=1}^{l}((h_M^{\sigma(i)})^2+(g_M^{\sigma(i)})^2),\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)\right\}\right)^2\right)^{\frac{1}{2}}\\ &=\max\left\{\sum_{i=1}^{l}((h_M^{\sigma(i)})^2+(g_M^{\sigma(i)})^2),\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)\right\},\end{aligned}\tag{3.1}$$

then we have

$$c_2(M,N) = \frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)} + g_M^{\sigma(i)}g_N^{\sigma(i)})}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)})^2 + (g_M^{\sigma(i)})^2), \sum_{i=1}^{l}((h_N^{\sigma(i)})^2 + (g_N^{\sigma(i)})^2)\}}$$
$$\leq \frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)} + g_M^{\sigma(i)}g_N^{\sigma(i)})}{[\sum_{i=1}^{l}((h_M^{\sigma(i)})^2 + (g_M^{\sigma(i)})^2)\sum_{i=1}^{l}((h_N^{\sigma(i)})^2 + (g_N^{\sigma(i)})^2)]^{1/2}} = c_1(M,N). \tag{3.2}$$

(2) Since

$$\left(\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2)\sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)\right)^{\frac{1}{2}}$$
$$\leq \left(\left(\max\left\{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2), \sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)\right\}\right)^2\right)^{\frac{1}{2}}$$
$$= \max\left\{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2), \sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)\right\} \tag{3.3}$$

and if $\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N)) \geq 0$, then we have

$$0 \leq c_4(M,N)$$
$$= \frac{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N))}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2), \sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)\}}$$
$$\leq \frac{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N))}{[\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2)\sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)]^{1/2}}$$
$$= c_3(M,N). \tag{3.4}$$

If $\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N)) \leq 0$, then we have

$c_3(M,N)$

$$
\begin{aligned}
&= \frac{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N))}{[\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2) \sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)]^{1/2}} \\
&\leq \frac{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N))}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2), \sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)\}} \\
&= c_4(M,N) \leq 0.
\end{aligned}
\tag{3.5}
$$

Therefore, $|c_4(M,N)| \leq |c_3(M,N)|$.

From Theorem 3.1.2, we know that (1) c_2 is always smaller than c_1, but both of them are bigger than 0, and (2) the absolute value of c_4 is always smaller than that of c_3, and their values may be smaller or bigger than 0, which not only provide us the strength of the relationship of DHFEs but also shows that the DHFEs are positively or negatively correlated.

Theorem 3.1.3 Let M and N be two DHFEs, then

(1) If $0 \leq h_M^{\sigma(i)} = kh_N^{\sigma(i)} \leq 1$, $0 \leq g_M^{\sigma(i)} = kg_N^{\sigma(i)} \leq 1$, $i = 1, 2, \ldots, l$, then $c_1(M,N) = c_3(M,N) = 1$ and

$$
c_5(M,N) = \frac{1}{l}\sum_{i=1}^{l}\left(\frac{\min_i\{h_N^{\sigma(i)}\} + \max_i\{h_N^{\sigma(i)}\} + \min_i\{g_N^{\sigma(i)}\} + \max_i\{g_N^{\sigma(i)}\}}{h_N^{\sigma(i)} + \max_i\{h_N^{\sigma(i)}\} + g_N^{\sigma(i)} + \max_i\{g_N^{\sigma(i)}\}}\right). \tag{3.6}
$$

(2) Let $0 \leq h_M^{\sigma(i)} = kh_N^{\sigma(i)} \leq 1$, $0 \leq g_M^{\sigma(i)} = kg_N^{\sigma(i)} \leq 1$, $i = 1, 2, \ldots, l$. If $k \geq 1$, then $c_2(M,N) = c_4(M,N) = \frac{1}{k}$; if $0 < k \leq 1$, then $c_2(M,N) = c_4(M,N) = k$.

(3) If $|h_M^{\sigma(i)} - h_N^{\sigma(i)}| = d$ and $|g_M^{\sigma(i)} - g_N^{\sigma(i)}| = e$, $i = 1, 2, \ldots, l$, then $c_5(M,N) = 1$.

Proof (1) If $0 \leq h_M^{\sigma(i)} = kh_N^{\sigma(i)} \leq 1$, $0 \leq g_M^{\sigma(i)} = kg_N^{\sigma(i)} \leq 1$, $i = 1, 2, \ldots, l$, and $k > 0$, then

$$
\begin{aligned}
c_1(M,N) &= \frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)} + g_M^{\sigma(i)}g_N^{\sigma(i)})}{[\sum_{i=1}^{l}((h_M^{\sigma(i)})^2 + (g_M^{\sigma(i)})^2) \sum_{i=1}^{l}((h_N^{\sigma(i)})^2 + (g_N^{\sigma(i)})^2)]^{\frac{1}{2}}} \\
&= \frac{\sum_{i=1}^{l}(k(h_N^{\sigma(i)})^2 + k(g_N^{\sigma(i)})^2)}{k\sum_{i=1}^{l}((h_N^{\sigma(i)})^2 + (g_N^{\sigma(i)})^2)} = 1,
\end{aligned}
\tag{3.7}
$$

$c_3(M,N)$

$$= \frac{\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)(h_N^{\sigma(i)} - \bar{h}_N) + (g_M^{\sigma(i)} - \bar{g}_M)(g_N^{\sigma(i)} - \bar{g}_N))}{[\sum_{i=1}^{l}((h_M^{\sigma(i)} - \bar{h}_M)^2 + (g_M^{\sigma(i)} - \bar{g}_M)^2)\sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)]^{\frac{1}{2}}}$$

$$= \frac{\sum_{i=1}^{l}(k(h_N^{\sigma(i)} - \bar{h}_N)^2 + k(g_N^{\sigma(i)} - \bar{g}_N)^2)}{k\sum_{i=1}^{l}((h_N^{\sigma(i)} - \bar{h}_N)^2 + (g_N^{\sigma(i)} - \bar{g}_N)^2)} = 1. \tag{3.8}$$

and since

$$\Delta\gamma_{\sigma(i)} = |h_M^{\sigma(i)} - h_N^{\sigma(i)}| = |1-k|h_N^{\sigma(i)},\ \Delta\eta_{\sigma(i)} = |g_M^{\sigma(i)} - g_N^{\sigma(i)}| = |1-k|g_N^{\sigma(i)}, \tag{3.9}$$

$$\begin{aligned}\Delta\gamma_{\min} &= \min_i\{|h_M^{\sigma(i)} - h_N^{\sigma(i)}|\} = |1-k|\min_i\{h_N^{\sigma(i)}\},\\ \Delta\eta_{\min} &= \min_i\{|g_M^{\sigma(i)} - g_N^{\sigma(i)}|\} = |1-k|\min_i\{g_N^{\sigma(i)}\},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\Delta\gamma_{\max} &= \max_i\{|h_M^{\sigma(i)} - h_N^{\sigma(i)}|\} = |1-k|\max_i\{h_N^{\sigma(i)}\},\\ \Delta\eta_{\max} &= \max_i\{|g_M^{\sigma(i)} - g_N^{\sigma(i)}|\} = |1-k|\max_i\{g_N^{\sigma(i)}\},\end{aligned} \tag{3.11}$$

then we have

$$\begin{aligned}c_5(M,N) &= \frac{1}{l}\sum_{i=1}^{l}\left(\frac{\Delta\gamma_{\min} + \Delta\gamma_{\max} + \Delta\eta_{\min} + \Delta\eta_{\max}}{\Delta\gamma_{\sigma(i)} + \Delta\gamma_{\max} + \Delta\eta_{\sigma(i)} + \Delta\eta_{\max}}\right)\\ &= \frac{1}{l}\sum_{i=1}^{l}\left(\frac{\min_i\{h_N^{\sigma(i)}\} + \max_i\{h_N^{\sigma(i)}\} + \min_i\{g_N^{\sigma(i)}\} + \max_i\{g_N^{\sigma(i)}\}}{h_N^{\sigma(i)} + \max_i\{h_N^{\sigma(i)}\} + g_N^{\sigma(i)} + \max_i\{g_N^{\sigma(i)}\}}\right).\end{aligned} \tag{3.12}$$

(2) Let $0 \leq h_M^{\sigma(i)} = kh_N^{\sigma(i)} \leq 1$, $0 \leq g_M^{\sigma(i)} = kg_N^{\sigma(i)} \leq 1$, $i = 1, 2, \ldots, l$. If $k \geq 1$, then

$$\begin{aligned}c_2(M,N) &= \frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)} + g_M^{\sigma(i)}g_N^{\sigma(i)})}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)})^2 + (g_M^{\sigma(i)})^2), \sum_{i=1}^{l}((h_N^{\sigma(i)})^2 + (g_N^{\sigma(i)})^2)\}}\\ &= \frac{\sum_{i=1}^{l}k((h_N^{\sigma(i)})^2 + (g_N^{\sigma(i)})^2)}{\sum_{i=1}^{l}(k^2(h_N^{\sigma(i)})^2 + k^2(g_N^{\sigma(i)})^2)} = \frac{1}{k},\end{aligned} \tag{3.13}$$

and

$$
\begin{aligned}
&c_4(M,N)\\
&=\frac{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)(h_N^{\sigma(i)}-\bar{h}_N)+(g_M^{\sigma(i)}-\bar{g}_M)(g_N^{\sigma(i)}-\bar{g}_N))}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)^2+(g_M^{\sigma(i)}-\bar{g}_M)^2),\sum_{i=1}^{l}((h_N^{\sigma(i)}-\bar{h}_N)^2+(g_N^{\sigma(i)}-\bar{g}_N)^2)\}}\\
&=\frac{\sum_{i=1}^{l}(k(h_N^{\sigma(i)}-\bar{h}_N)^2+k(g_N^{\sigma(i)}-\bar{g}_N)^2)}{k^2\sum_{i=1}^{l}((h_N^{\sigma(i)}-\bar{h}_N)^2+(g_N^{\sigma(i)}-\bar{g}_N)^2)}=\frac{1}{k}. \qquad (3.14)
\end{aligned}
$$

If $0 \leq k \leq 1$, then

$$
\begin{aligned}
c_2(M,N)&=\frac{\sum_{i=1}^{l}(h_M^{\sigma(i)}h_N^{\sigma(i)}+g_M^{\sigma(i)}g_N^{\sigma(i)})}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)})^2+(g_M^{\sigma(i)})^2),\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)\}}\\
&=\frac{\sum_{i=1}^{l}k((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)}{\sum_{i=1}^{l}((h_N^{\sigma(i)})^2+(g_N^{\sigma(i)})^2)}=k, \qquad (3.15)
\end{aligned}
$$

and

$$
\begin{aligned}
&c_4(M,N)\\
&=\frac{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)(h_N^{\sigma(i)}-\bar{h}_N)+(g_M^{\sigma(i)}-\bar{g}_M)(g_N^{\sigma(i)}-\bar{g}_N))}{\max\{\sum_{i=1}^{l}((h_M^{\sigma(i)}-\bar{h}_M)^2+(g_M^{\sigma(i)}-\bar{g}_M)^2)\sum_{i=1}^{l}((h_N^{\sigma(i)}-\bar{h}_N)^2+(g_N^{\sigma(i)}-\bar{g}_N)^2)\}}\\
&=\frac{\sum_{i=1}^{l}(k(h_N^{\sigma(i)}-\bar{h}_N)^2+k(g_N^{\sigma(i)}-\bar{g}_N)^2)}{\sum_{i=1}^{l}((h_N^{\sigma(i)}-\bar{h}_N)^2+(g_N^{\sigma(i)}-\bar{g}_N)^2)}=k. \qquad (3.16)
\end{aligned}
$$

(3) If $|h_M^{\sigma(i)}-h_N^{\sigma(i)}|=d$ and $|g_M^{\sigma(i)}-g_N^{\sigma(i)}|=e$, $i=1,2,\ldots,l$, then

$$
\begin{aligned}
\Delta\gamma_{\sigma(i)}&=|h_M^{\sigma(i)}-h_N^{\sigma(i)}|=\Delta\gamma_{\min}=\min_i\{|h_M^{\sigma(i)}-h_N^{\sigma(i)}|\}\\
&=\Delta\gamma_{\max}=\max_i\{|h_M^{\sigma(i)}-h_N^{\sigma(i)}|\}=d, \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\eta_{\sigma(i)}&=|g_M^{\sigma(i)}-g_N^{\sigma(i)}|=\Delta\eta_{\min}=\min_i\{|g_M^{\sigma(i)}-g_N^{\sigma(i)}|\}\\
&=\Delta\eta_{\max}=\max_i\{|g_M^{\sigma(i)}-g_N^{\sigma(i)}|\}=e, \qquad (3.18)
\end{aligned}
$$

and thus

$$
c_5(M,N)=\frac{1}{l}\sum_{i=1}^{l}\left(\frac{\Delta\gamma_{\min}+\Delta\gamma_{\max}+\Delta\eta_{\min}+\Delta\eta_{\max}}{\Delta\gamma_{\sigma(i)}+\Delta\gamma_{\max}+\Delta\eta_{\sigma(i)}+\Delta\eta_{\max}}\right)=1. \qquad (3.19)
$$

Theorem 3.1.3 tells us that (1) if the values $h_M^{\sigma(i)}$ and $g_M^{\sigma(i)}$ in DHFE M, respectively, are k times the values of $h_N^{\sigma(i)}$ and $g_N^{\sigma(i)}$ in DHFE N, then the correlation coefficients c_1 and c_3 are 1, c_2 and c_4 are $\frac{1}{k}$ $(k \geq 1)$ or k $(0 < k \leq 1)$ and (2) if $|h_M^{\sigma(i)} - h_N^{\sigma(i)}| = d$ and $|g_M^{\sigma(i)} - g_N^{\sigma(i)}| = e$, then c_5 is 1. This indicates that these five correlation coefficient formulas reflect different relationships between two DHFEs M and N, and therefore they may produce different results for the same two DHFEs, which is reasonable.

In the following, we use an example to illustrate the proposed correlation coefficient formulas.

Example 3.1.4 [42] To make a proper diagnosis $D = \{$Viral fever, Malaria, Typhoid, Stomach problem, Chest problem$\}$ for a patient with the given values of the symptoms, $S = \{$Temperature, headache, cough, stomach pain, chest pain$\}$, a medical knowledge base is necessary that involves elements described in terms of dual hesitant fuzzy sets. The data are given in Table 3.1, and each symptom is described by a DHFE. The set of patients is $P = \{$Al, Bob, Joe, Ted$\}$. The symptoms are given in Table 3.2. We need to seek a diagnosis for each patient.

Table 3.1: Symptoms characteristic for the considered diagnoses

	Temperature	Headache	Cough
Viral fever	$\{\{0.6, 0.4, 0.3\}, \{0.4, 0.3, 0.2\}\}$	$\{\{0.7, 0.5\}, \{0.2, 0.1\}\}$	$\{\{0.6, 0.5\}, \{0.3, 0.2\}\}$
Malaria	$\{\{0.8, 0.7, 0.6\}, \{0.2, 0.15, 0.1\}\}$	$\{\{0.5, 0.3\}, \{0.3, 0.2\}\}$	$\{\{0.3, 0.1\}, \{0.7, 0.5\}\}$
Typhoid	$\{\{0.6, 0.3, 0.1\}, \{0.4, 0.2, 0.1\}\}$	$\{\{0.8, 0.7\}, \{0.2, 0.1\}\}$	$\{\{0.5, 0.3\}, \{0.4, 0.2\}\}$
Stomach problem	$\{\{0.5, 0.4, 0.2\}, \{0.4, 0.3, 0.1\}\}$	$\{\{0.4, 0.3\}, \{0.5, 0.4\}\}$	$\{\{0.4, 0.3\}, \{0.4, 0.3\}\}$
Chest problem	$\{\{0.3, 0.2, 0.1\}, \{0.6, 0.5, 0.4\}\}$	$\{\{0.5, 0.3\}, \{0.3, 0.1\}\}$	$\{\{0.3, 0.2\}, \{0.5, 0.4\}\}$

	Stomach pain	Chest pain
Viral fever	$\{\{0.5, 0.4, 0.3\}, \{0.4, 0.3, 0.2\}\}$	$\{\{0.5, 0.4, 0.2, 0.1\}, \{0.5, 0.3, 0.2, 0.1\}\}$
Malaria	$\{\{0.6, 0.3, 0.2\}, \{0.3, 0.2, 0.1\}\}$	$\{\{0.4, 0.3, 0.2, 0.1\}, \{0.5, 0.4, 0.3, 0.2\}\}$
Typhoid	$\{\{0.5, 0.4, 0.2\}, \{0.4, 0.3, 0.2\}\}$	$\{\{0.6, 0.4, 0.3, 0.1\}, \{0.4, 0.3, 0.2, 0.1\}\}$
Stomach problem	$\{\{0.8, 0.7, 0.65\}, \{0.2, 0.15, 0.1\}\}$	$\{\{0.5, 0.4, 0.2, 0.1\}, \{0.4, 0.3, 0.2, 0.1\}\}$
Chest problem	$\{\{0.6, 0.5, 0.3\}, \{0.4, 0.3, 0.1\}\}$	$\{\{0.8, 0.7, 0.6, 0.5\}, \{0.2, 0.15, 0.1, 0.05\}\}$

Table 3.2: Symptoms characteristic for the considered patients

	Temperature	Headache	Cough
Al	$\{ 0.6, 0.5, 0.2\}, \{0.4, 0.3, 0.2\}\}$	$\{\{0.7, 0.3\}, \{0.2, 0.1\}\}$	$\{\{0.6, 0.4\}, \{0.3, 0.2\}\}$
Bob	$\{\{0.8, 0.6, 0.5\}, \{0.2, 0.15, 0.1\}\}$	$\{\{0.5, 0.4\}, \{0.3, 0.1\}\}$	$\{\{0.3, 0.25\}, \{0.7, 0.5\}\}$
Joe	$\{\{0.6, 0.4, 0.2\}, \{0.4, 0.2, 0.1\}\}$	$\{\{0.3, 0.15\}, \{0.3, 0.2\}\}$	$\{\{0.5, 0.3\}, \{0.4, 0.3\}\}$
Ted	$\{\{0.5, 0.4, 0.2\}, \{0.4, 0.3, 0.1\}\}$	$\{\{0.4, 0.35\}, \{0.5, 0.1\}\}$	$\{\{0.4, 0.3\}, \{0.5, 0.2\}\}$

	Stomach pain	Chest pain
Al	$\{\{0.5, 0.4, 0.1\}, \{0.4, 0.3, 0.1\}\}$	$\{\{0.5, 0.4, 0.3, 0.1\}, \{0.5, 0.3, 0.2, 0.1\}\}$
Bob	$\{\{0.6, 0.3, 0.1\}, \{0.3, 0.2, 0.1\}\}$	$\{\{0.4, 0.3, 0.2, 0.1\}, \{0.5, 0.4, 0.3, 0.1\}\}$
Joe	$\{\{0.5, 0.4, 0.3\}, \{0.4, 0.3, 0.1\}\}$	$\{\{0.6, 0.4, 0.3, 0.2\}, \{0.4, 0.3, 0.2, 0.1\}\}$
Ted	$\{\{0.8, 0.7, 0.3\}, \{0.2, 0.15, 0.1\}\}$	$\{\{0.5, 0.3, 0.2, 0.1\}, \{0.4, 0.3, 0.2, 0.1\}\}$

We utilize the correlation coefficient c_1 to derive a diagnosis for each patient. All the results for the considered patients are listed in Table 3.3. From the arguments in Table 3.3, we can conclude that Al suffers from viral fever, Bob from malaria, and Joe and Ted from stomach problem.

Table 3.3: Values of c_1 for each patient to the considered set of possible diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	**0.9851**	0.9010	0.9752	0.9117	0.8902
Bob	0.9168	**0.9901**	0.9261	0.9104	0.8421
Joe	0.9491	0.9299	0.9513	**0.9733**	0.9125
Ted	0.9300	0.9189	0.9399	**0.9721**	0.9052

Table 3.4: Values of c_2 for each patient to the considered set of possible diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	**0.9279**	0.8034	0.8661	0.7836	0.7601
Bob	0.8034	**0.9521**	0.7394	0.7198	0.6872
Joe	0.8251	0.7296	**0.8368**	0.7935	0.7298
Ted	0.7891	0.7065	0.8135	**0.9129**	0.7428

If we utilize the correlation coefficient formulas c_2, c_3, c_4 and c_5 to derive a diagnosis, then the results are listed in Tables 3.4-3.7, respectively. From Tables 3.4-3.7, we know that the results obtained by different correlation coefficient formulas are different. This is because these correlation coefficient formulas are based on different relationships and may produce different results.

Table 3.5: Values of c_3 for each patient to the considered set of possible diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	**0.9545**	0.9485	0.9371	0.9235	0.9286
Bob	0.8973	0.9196	**0.9281**	0.9128	0.8918
Joe	0.9708	0.9458	0.9609	0.9509	**0.9711**
Ted	0.8243	0.8373	**0.9011**	0.8917	0.8860

Table 3.6: Values of c_4 for each patient to the considered set of possible diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.6640	0.6532	**0.6839**	0.5960	0.6266
Bob	0.7058	**0.7559**	0.6392	0.6086	0.6834
Joe	0.7374	0.6954	**0.8253**	0.7137	0.6507
Ted	0.5196	0.6092	**0.6733**	0.5422	0.5577

Table 3.7: Values of c_5 for each patient to the considered set of possible diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.7750	0.8206	0.8192	0.8265	**0.8761**
Bob	0.8565	0.7950	**0.8657**	0.7997	0.7838
Joe	0.8311	**0.8986**	0.8212	0.7970	0.8245
Ted	0.8029	**0.8704**	0.7569	0.8583	0.7955

3.2 Conclusions

Dual hesitant fuzzy set, as an extension of fuzzy set, can be describe the situation that people have hesitancy when they make a decision more objectively than other extensions of fuzzy set (intuitionistic fuzzy set, interval-valued fuzzy set, interval-valued intuitionistic fuzzy set, hesitant fuzzy set). In this chapter, the correlation coefficients for DHFEs have been studied. To operate correctly, we have assumed that the two DHFEs have the same length for membership and nonmembership parts and their values are arranged in decreasing order when we compare them. Their properties have been discussed, and the differences and correlations among them have been investigated in detail. One example is employed to illustrate that the results obtained by different correlation coefficient formulas, based on different linear relationships, are different.

Chapter 4

Dual hesitant fuzzy Bonferroni means and their applications in group decision making

4.1 Basic concepts and operations

4.1.1 Bonferroni means

The Bonferroni mean operator was initially proposed by Bonferroni [6] and was also investigated intensively by Yager [78]:

Definition 4.1.1 Let $p, q \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{B}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \tag{4.1}$$

then $\mathrm{B}^{p,q}$ is called the Bonferroni mean (BM) operator.

Obviously, the BM has the following properties.

1) $\mathrm{B}^{p,q}(0,0,\ldots,0)=0$.

2) $\mathrm{B}^{p,q}(a,a,\ldots,a)=a$, if $a_i=a$ for all i.

3) $\mathrm{B}^{p,q}(a_1,a_2,\ldots,a_n)\geq \mathrm{B}^{p,q}(b_1,b_2,\ldots,b_n)$, i.e., $\mathrm{B}^{p,q}$ is monotonic, if $a_i\geq b_i$ for all i.

4) $\min_i\{a_i\}\leq \mathrm{B}^{p,q}(a_1,a_2,\ldots,a_n)\leq \max_i\{a_i\}$.

Furthermore, if $q=0$, then, by (4.1), it follows that

$$\mathrm{B}^{p,0}(a_1,a_2,\ldots,a_n)=\left(\frac{1}{n}\sum_{i=1}^{n}a_i^p\left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n}a_j^0\right)\right)^{\frac{1}{p}}=\left(\frac{1}{n}\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}} \tag{4.2}$$

which is a generalized mean operator [17], in particular, the following cases hold.

1) If $p=2$ and $q=0$, then (4.2) reduces to the square mean

$$\mathrm{B}^{2,0}(a_1,a_2,\ldots,a_n)=\left(\frac{1}{n}\sum_{i=1}^{n}a_i^2\right)^{\frac{1}{2}}. \tag{4.3}$$

2) If $p=1$ and $q=0$, then (4.2) reduces to the usual average mean

$$\mathrm{B}^{1,0}(a_1,a_2,\ldots,a_n)=\frac{1}{n}\sum_{i=1}^{n}a_i. \tag{4.4}$$

3) If $p\to+\infty$ and $q=0$, then (4.2) reduces to the max operator

$$\lim_{p\to+\infty}\mathrm{B}^{p,0}(a_1,a_2,\ldots,a_n)=\max_i\{a_i\}. \tag{4.5}$$

4) If $p\to 0$ and $q=0$, then (4.2) reduces to the geometric mean

$$\lim_{p\to 0}\mathrm{B}^{p,0}(a_1,a_2,\ldots,a_n)=\left(\prod_{i=1}^{n}a_i\right)^{\frac{1}{n}}. \tag{4.6}$$

4.1.2 Dual hesitant fuzzy elements

Zhu et al. [86] defined the some operations on DHFEs. For a DHFE $d=\{h_d,g_d\}$, the corresponding lower and upper bounds to h_d and g_d are h_d^-, h_d^+, g_d^- and g_d^+, respectively, where $h_d^-=\min\{\gamma|\gamma\in h_d\}$, $h_d^+=\max\{\gamma|\gamma\in h_d\}$, $g_d^-=\min\{\eta|\eta\in g_d\}$ and $g_d^+=\max\{\eta|\eta\in g_d\}$ represent this group notations and no confusion will arise in the rest of the paper.

Definition 4.1.2 Let $d = \{h_d, g_d\}$, $d_1 = \{h_{d_1}, g_{d_1}\}$ and $d_2 = \{h_{d_2}, g_{d_2}\}$ be three DHFEs, then some useful operations on DHFEs is defined as follows [86]:

(1) $d_1 \oplus d_2 = \{h_{d_1} \oplus h_{d_2}, g_{d_1} \otimes g_{d_2}\} = \{\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2 | \gamma_1 \in h_{d_1}, \gamma_2 \in h_{d_2}\}, \{\eta_1\eta_2 | \eta_1 \in g_{d_1}, \eta_2 \in g_{d_2}\}\}$;

(2) $d_1 \otimes d_2 = \{h_{d_1} \otimes h_{d_2}, g_{d_1} \oplus g_{d_2}\} = \{\{\gamma_1\gamma_2 | \gamma_1 \in h_{d_1}, \gamma_2 \in h_{d_2}\}, \{\eta_1 + \eta_2 - \eta_1\eta_2 | \eta_1 \in g_{d_1}, \eta_2 \in g_{d_2}\}\}$;

(3) $\lambda d = \{\lambda h, \lambda g\} = \{\{1 - (1-\gamma)^\lambda | \gamma \in h_d\}, \{\eta^\lambda | \eta \in g_d\}\}$, $\lambda > 0$;

(4) $d^\lambda = \{h^\lambda, g^\lambda\} = \{\{\gamma^\lambda | \gamma \in h_d\}, \{1 - (1-\eta)^\lambda | \eta \in g_d\}\}$, $\lambda > 0$;

(5) $d^c = \begin{cases} \{\{\eta | \eta \in g_d\}, \{\gamma | \gamma \in h_d\}\}, & \text{if } g \neq \emptyset, h \neq \emptyset; \\ \{\{1-\gamma | \gamma \in h_d\}, \{\emptyset\}\}, & \text{if } g = \emptyset, h \neq \emptyset; \ ; \\ \{\{\emptyset\}, \{1 - \eta | \eta \in g_d\}\}, & \text{if } g \neq \emptyset, h = \emptyset. \end{cases}$

(6) $d_1 \cup d_2 = \{\{\gamma \in (h_{d_1} \cup h_{d_2}) | \gamma \geq \max(h^-_{d_1}, h^-_{d_2})\}, \{\eta \in (g_{d_1} \cap g_{d_2}) | \eta \leq \min(g^+_{d_1}, g^+_{d_2})\}\}$;

(7) $d_1 \cap d_2 = \{\{\gamma \in (h_{d_1} \cap h_{d_2}) | \gamma \leq \min(h^+_{d_1}, h^+_{d_2})\}, \{\eta \in (g_{d_1} \cup g_{d_2}) | \eta \geq \max(g^-_{d_1}, g^-_{d_2})\}\}$.

We can easily prove the following relationships among the operations (1)-(4):

Theorem 4.1.3 Let d, d_1 and d_2 be any three DHFEs, then

(1) $d_1 \oplus d_2 = d_2 \oplus d_1$;

(2) $d_1 \otimes d_2 = d_2 \otimes d_1$;

(3) $\lambda(d_1 \oplus d_2) = \lambda d_1 \oplus \lambda d_2$, $\lambda > 0$;

(4) $(d_1 \otimes d_2)^\lambda = d_1^\lambda \otimes d_2^\lambda$, $\lambda > 0$;

(5) $(\lambda_1 + \lambda_2)d = \lambda_1 d \oplus \lambda_2 d$, $\lambda_1, \lambda_2 > 0$;

(6) $d^{(\lambda_1+\lambda_2)} = d^{\lambda_1} \otimes d^{\lambda_2}$, $\lambda_1, \lambda_2 > 0$.

Proof Since (1)-(4) can be proven easily, we prove (5) and (6).

(5) By the operations (1) and (3) in Definition 2.1.3, we have

$$\begin{aligned} \lambda_1 d \oplus \lambda_2 d &= \{\{1 - (1-\gamma)^{\lambda_1} | \gamma \in h_d\}, \{\eta^{\lambda_1} | \eta \in g_d\}\} \oplus \\ &\qquad \{\{1 - (1-\gamma)^{\lambda_2} | \gamma \in h_d\}, \{\eta^{\lambda_2} | \eta \in g_d\}\} \\ &= \{\{1 - (1-\gamma)^{\lambda_1}(1-\gamma)^{\lambda_2} | \gamma \in h_d\}, \{\eta^{\lambda_1}\eta^{\lambda_2} | \eta \in g_d\}\} \\ &= \{\{1 - (1-\gamma)^{\lambda_1+\lambda_2} | \gamma \in h_d\}, \{\eta^{\lambda_1+\lambda_2} | \eta \in g_d\}\} = (\lambda_1 + \lambda_2)d. \end{aligned}$$

(6) By the operations (2) and (4) in Definition 2.1.3, we have

$$
\begin{aligned}
d^{\lambda_1} \otimes d^{\lambda_2} &= \{\{\gamma^{\lambda_1} | \gamma \in h_d\}, \{1-(1-\eta)^{\lambda_1} | \eta \in g_d\}\} \otimes \\
&\qquad \{\{\gamma^{\lambda_2} | \gamma \in h_d\}, \{1-(1-\eta)^{\lambda_2} | \eta \in g_d\}\} \\
&= \{\{\gamma^{\lambda_1}\gamma^{\lambda_2} | \gamma \in h_d\}, \{1-(1-\eta)^{\lambda_1}(1-\eta)^{\lambda_2} | \eta \in g_d\}\} \\
&= \{\{\gamma^{\lambda_1+\lambda_2} | \gamma \in h_d\}, \{1-(1-\eta)^{\lambda_1+\lambda_2} | \eta \in g_d\}\} = d^{(\lambda_1+\lambda_2)}.
\end{aligned}
$$

To compare the DHFEs, based on the comparing methods [53, 67] of HFEs and IFNs, we give the following comparison laws.

Definition 4.1.4 For a DHFE $d = \{h_d, g_d\}$, $s(d) = \frac{1}{\#h_d}\sum_{\gamma \in h_d}\gamma - \frac{1}{\#g_d}\sum_{\eta \in g_d}\eta$ is called the score function of d, and $p(d) = \frac{1}{\#h_d}\sum_{\gamma \in h_d}\gamma + \frac{1}{\#g_d}\sum_{\eta \in g_d}\eta$ is called the accuracy function of d, where $\#h_d$ and $\#g_d$ are the numbers of the elements in h_d and g_d, respectively. Let $d_1 = \{h_{d_1}, g_{d_1}\}$ and $d_2 = \{h_{d_2}, g_{d_2}\}$ be two DHFEs, then

(1) if $s(d_1) > s(d_2)$, then d_1 is superior to d_2, denoted by $d_1 \succ d_2$;

(2) if $s(d_1) = s(d_2)$, then

  (i) if $p(d_1) = p(d_2)$, then d_1 is equivalent to d_2, denoted by $d_1 \sim d_2$;

  (ii) if $p(d_1) > p(d_2)$, then d_1 is superior than d_2, denoted by $d_1 \succ d_2$.

Now, we define the hesitancy degree of the DHFE $d = \{h_d, g_d\}$ as follows

$$\pi(d) = 1 - \frac{1}{\#h_d}\sum_{\gamma \in h_d}\gamma + \frac{1}{\#g_d}\sum_{\eta \in g_d}\eta, \tag{4.7}$$

where $\#h_d$ and $\#g_d$ are the numbers of the elements in h_d and g_d, respectively. Then we get the relation between the hesitancy degree and the accuracy degree of the DHFE d

$$\pi(d) = 1 - \frac{1}{\#h_d}\sum_{\gamma \in h_d}\gamma + \frac{1}{\#g_d}\sum_{\eta \in g_d}\eta = 1 - p(d),$$

i.e.,

$$\pi(d) + p(d) = 1. \tag{4.8}$$

From (4.8), we know that the higher the accuracy degree $p(d)$, the lower the hesitancy degree $\pi(d)$.

4.2 Dual hesitant fuzzy Bonferroni means

In this section, we shall investigate the BM under dual hesitant fuzzy environments. Based on (4.1), we give the definition of DHFBM as follows.

Definition 4.2.1 Let $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of DHFEs. For any $p, q > 0$, if

$$\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) = \left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(d_i^p \otimes d_j^q)\right)\right)^{\frac{1}{p+q}} \tag{4.9}$$

then $\mathrm{DHFB}^{p,q}$ is called the dual hesitant fuzzy Bonferroni mean (DHFBM).

Based on operations (1)-(4) of DHFEs described in Definition 4.1.2, we can derive the following result.

Theorem 4.2.2 Let $p, q > 0$, and $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of DHFEs. Then, the aggregated value, by using the DHFBM, is also a DHFE, and

$$\begin{aligned}
&\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n)\\
&= \left\{\left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_{d_i}^p \otimes h_{d_j}^q)\right)\right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}(g_{d_i}^p \oplus g_{d_j}^q)\right)\right)^{\frac{1}{p+q}}\right\}\\
&= \left\{\left\{\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1 - \gamma_i^p \gamma_j^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \middle| \gamma_i \in h_{d_i}, \gamma_j \in h_{d_j}\right\},\right.
\end{aligned}$$

$$\left\{1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\eta_i)^p(1-\eta_j)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\Big|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\right\}\Bigg\}. \tag{4.10}$$

Proof By operations (2) and (4) described in Definition 4.1.2, we have

$$\begin{aligned} d_i^p &= \{h_{d_i}^p, g_{d_i}^p\} = \{\{\gamma_i^p|\gamma_i\in h_{d_i}\},\{1-(1-\eta_i)^p|\eta_i\in g_{d_i}\}\},\\ d_j^q &= \{h_{d_j}^q, g_{d_j}^q\} = \{\{\gamma_j^q|\gamma_j\in h_{d_j}\},\{1-(1-\eta_j)^q|\eta_j\in g_{d_j}\}\} \end{aligned} \tag{4.11}$$

and then

$$\begin{aligned} d_i^p\otimes d_j^q &= \{h_{d_i}^p\otimes h_{d_j}^q, g_{d_i}^p\oplus g_{d_j}^q\}\\ &= \left\{\{\gamma_i^p\gamma_j^q|\gamma_i\in h_{d_i},\gamma_j\in h_{d_j}\},\{1-(1-\eta_i)^p(1-\eta_j)^q|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\}\right\}. \end{aligned} \tag{4.12}$$

In what follows, we first prove that

$$\begin{aligned} \oplus_{\substack{i,j=1\\i\neq j}}^{n}(d_i^p\otimes d_j^q) &= \left\{\oplus_{\substack{i,j=1\\i\neq j}}^{n}(h_{d_i}^p\otimes h_{d_j}^q),\otimes_{\substack{i,j=1\\i\neq j}}^{n}(g_{d_i}^p\oplus g_{d_j}^q)\right\}\\ &= \left\{\left\{1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\gamma_i^p\gamma_j^q\right)\Big|\gamma_i\in h_{d_i},\gamma_j\in h_{d_j}\right\},\right.\\ &\qquad \left.\left\{\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\eta_i)^p(1-\eta_j)^q\right)\Big|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\right\}\right\}. \end{aligned} \tag{4.13}$$

by using mathematical induction on n as follows.

1) For $n=2$, we have

$$\begin{aligned} \oplus_{\substack{i,j=1\\i\neq j}}^{2}(d_i^p\otimes d_j^q) &= (d_1^p\otimes d_2^q)\oplus(d_2^p\otimes d_1^q)\\ &= \left\{\left\{1-\left(1-\gamma_1^p\gamma_2^q\right)\left(1-\gamma_2^p\gamma_1^q\right)\Big|\gamma_1\in h_{d_1},\gamma_2\in h_{d_2}\right\},\right.\\ &\qquad\left.\left\{\left(1-(1-\eta_1)^p(1-\eta_2)^q\right)\left(1-(1-\eta_2)^p(1-\eta_1)^q\right)\Big|\eta_1\in g_{d_1},\eta_2\in g_{d_2}\right\}\right\}. \end{aligned} \tag{4.14}$$

2) If (4.13) holds for $n = k$, i.e.,

$$\begin{aligned}\oplus_{\substack{i,j=1\\i\neq j}}^{k}(d_i^p \otimes d_j^q) &= \left\{\oplus_{\substack{i,j=1\\i\neq j}}^{k}(h_{d_i}^p \otimes h_{d_j}^q), \otimes_{\substack{i,j=1\\i\neq j}}^{k}(g_{d_i}^p \oplus g_{d_j}^q)\right\} \\ &= \left\{\left\{1 - \prod_{\substack{i,j=1\\i\neq j}}^{k}\left(1 - \gamma_i^p \gamma_j^q\right) \middle| \gamma_i \in h_{d_i}, \gamma_j \in h_{d_j}\right\}, \right. \\ &\left. \left\{\prod_{\substack{i,j=1\\i\neq j}}^{k}(1 - (1-\eta_i)^p (1-\eta_j)^q) \middle| \eta_i \in g_{d_i}, \eta_j \in g_{d_j}\right\}\right\}, \end{aligned} \tag{4.15}$$

then, when $n = k+1$, by operations (1)-(3) given in Definition 4.1.2, we have

$$\begin{aligned}\oplus_{\substack{i,j=1\\i\neq j}}^{k+1}(d_i^p \otimes d_j^q) &= \left(\oplus_{\substack{i,j=1\\i\neq j}}^{k}(d_i^p \otimes d_j^q)\right) \oplus \left(\oplus_{i=1}^{k}(d_i^p \otimes d_{k+1}^q)\right) \\ &\oplus \left(\oplus_{j=1}^{k}(d_{k+1}^p \otimes d_j^q)\right). \end{aligned} \tag{4.16}$$

Now, we prove that

$$\begin{aligned}\oplus_{i=1}^{k}(d_i^p \otimes d_{k+1}^q) &= \left\{\oplus_{i=1}^{k}(h_{d_i}^p \otimes h_{d_{k+1}}^q), \otimes_{i=1}^{k}(g_{d_i}^p \oplus g_{d_{k+1}}^q)\right\} \\ &= \left\{\left\{1 - \prod_{i=1}^{k}\left(1 - \gamma_i^p \gamma_{k+1}^q\right) \middle| \gamma_i \in h_{d_i}, \gamma_{k+1} \in h_{d_{k+1}}\right\}, \right. \\ &\left. \left\{\prod_{i=1}^{k}(1 - (1-\eta_i)^p (1-\eta_{k+1})^q) \middle| \eta_i \in g_{d_i}, \eta_{k+1} \in g_{d_{k+1}}\right\}\right\} \end{aligned} \tag{4.17}$$

by using mathematical induction on k as follows.

1) For $k = 2$, then by (4.12), we have

$$\begin{aligned}d_i^p \otimes d_{2+1}^q &= \{h_{d_i}^p \otimes h_{d_{2+1}}^q, g_{d_i}^p \oplus g_{d_{2+1}}^q\} \\ &= \left\{\{\gamma_i^p \gamma_{2+1}^q | \gamma_i \in h_{d_i}, \gamma_{2+1} \in h_{d_{2+1}}\}, \right. \\ &\left. \{1 - (1-\eta_i)^p (1-\eta_{2+1})^q | \eta_i \in g_{d_i}, \eta_{2+1} \in g_{d_{2+1}}\}\right\},\ i = 1, 2 \end{aligned} \tag{4.18}$$

and thus

$$\oplus_{i=1}^{2}(d_i^p \otimes d_{2+1}^q) = \left\{\oplus_{i=1}^{2}(h_{d_i}^p \otimes h_{d_{2+1}}^q), \otimes_{i=1}^{2}(g_{d_i}^p \oplus g_{d_{2+1}}^q)\right\}$$

$$= \left\{ \left\{ 1 - \prod_{i=1}^{2} (1 - \gamma_i^p \gamma_3^q) | \gamma_i \in h_{d_i}, \gamma_3 \in h_{d_3} \right\}, \right.$$

$$\left. \left\{ \prod_{i=1}^{2} (1 - (1 - \eta_i)^p (1 - \eta_3)^q) | \eta_i \in g_{d_i}, \eta_3 \in g_{d_3} \right\} \right\}. \quad (4.19)$$

2) If (4.17) holds for $k = k_0$, i.e.,

$$\oplus_{i=1}^{k_0} (d_i^p \otimes d_{k_0+1}^q) = \left\{ \oplus_{i=1}^{k_0} (h_{d_i}^p \otimes h_{d_{k_0+1}}^q), \otimes_{i=1}^{k_0} (g_{d_i}^p \oplus g_{d_{k_0 51}}^q) \right\}$$

$$= \left\{ \left\{ 1 - \prod_{i=1}^{k_0} (1 - \gamma_i^p \gamma_{k_0+1}^q) | \gamma_i \in h_{d_i}, \gamma_{k_0+1} \in h_{d_{k_0+1}} \right\}, \right.$$

$$\left. \left\{ \prod_{i=1}^{k_0} (1 - (1 - \eta_i)^p (1 - \eta_{k_0+1})^q) | \eta_i \in g_{d_i}, \eta_{k_0+1} \in g_{d_{k_0+1}} \right\} \right\}, \quad (4.20)$$

then, when $k = k_0 + 1$, by (4.12) and operations (1) and (2) given in Definition 4.1.2, we have

$$\oplus_{i=1}^{k_0+1} (d_i^p \otimes d_{k_0+2}^q) = \oplus_{i=1}^{k_0} (d_i^p \otimes d_{k_0+2}^q) \oplus (d_{k_0+1}^p \otimes d_{k_0+2}^q)$$

$$= \left\{ \oplus_{i=1}^{k_0+1} (h_{d_i}^p \otimes h_{d_{k_0+2}}^q) \oplus (h_{d_{k_0+1}}^p \otimes h_{d_{k_0+2}}^q), \right.$$

$$\left. \otimes_{i=1}^{k_0} (g_{d_i}^p \oplus g_{d_{k_0+1}}^q) \otimes (g_{d_{k_0+1}}^p \oplus g_{d_{k_0+2}}^q) \right\}$$

$$= \left\{ \left\{ 1 - \prod_{i=1}^{k_0+1} (1 - \gamma_i^p \gamma_{k_0+2}^q) | \gamma_i \in h_{d_i}, \gamma_{k_0+2} \in h_{d_{k_0+2}} \right\}, \right.$$

$$\left. \left\{ \prod_{i=1}^{k_0+1} (1 - (1 - \eta_i)^p (1 - \eta_{k_0+2})^q) | \eta_i \in g_{d_i}, \eta_{k_0+2} \in g_{d_{k_0+2}} \right\} \right\}, \quad (4.21)$$

i.e., (4.17) holds for $k = k_0 + 1$. Thus (4.17) holds for all k.

Similarly, we can prove that

$$\oplus_{j=1}^{k} (d_{k+1}^p \otimes d_j^q) = \left\{ \oplus_{j=1}^{k} (h_{d_{k+1}}^p \otimes h_{d_j}^q), \otimes_{j=1}^{k} (g_{d_{k+1}}^p \oplus g_{d_j}^q) \right\}$$

$$= \left\{ \left\{ 1 - \prod_{j=1}^{k} \left(1 - \gamma_{k+1}^p \gamma_j^q\right) \middle| \gamma_{k+1} \in h_{d_{k+1}}, \gamma_j \in h_{d_j} \right\}, \right.$$

$$\left. \left\{ \prod_{j=1}^{k} (1 - (1 - \eta_{k+1})^p (1 - \eta_j)^q) \middle| \eta_{k+1} \in g_{d_{k+1}}, \eta_j \in g_{d_j} \right\} \right\}. \quad (4.22)$$

Thus, by (4.15), (4.17) and (4.22), we further transform (4.16) as

$$\oplus_{\substack{i,j=1\\ i\neq j}}^{k+1}(d_i^p\otimes d_j^q)=\left\{\oplus_{\substack{i,j=1\\ i\neq j}}^{k}(h_{d_i}^p\otimes h_{d_j}^q),\otimes_{\substack{i,j=1\\ i\neq j}}^{k}(g_{d_i}^p\oplus g_{d_j}^q)\right\}$$
$$\oplus\left\{\oplus_{i=1}^{k}(h_{d_i}^p\otimes h_{d_{k+1}}^q),\otimes_{i=1}^{k}(g_{d_i}^p\oplus g_{d_{k+1}}^q)\right\}$$
$$\oplus\left\{\oplus_{j=1}^{k}(h_{d_{k+1}}^p\otimes h_{d_j}^q),\otimes_{j=1}^{k}(g_{d_{k+1}}^p\oplus g_{d_j}^q)\right\}$$
$$=\left\{\left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-\gamma_i^p\gamma_j^q\right)\Big|\gamma_i\in h_{d_i},\gamma_j\in h_{d_j}\right\},\right.$$
$$\left.\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{k}(1-(1-\eta_i)^p(1-\eta_j)^q)\Big|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\right\}\right\}$$
$$\oplus\left\{\left\{1-\prod_{i=1}^{k}\left(1-\gamma_i^p\gamma_{k+1}^q\right)\Big|\gamma_i\in h_{d_i},\gamma_{k+1}\in h_{d_{k+1}}\right\},\right.$$
$$\left.\left\{\prod_{i=1}^{k}(1-(1-\eta_i)^p(1-\eta_{k+1})^q)\Big|\eta_i\in g_{d_i},\eta_{k+1}\in g_{d_{k+1}}\right\}\right\}$$
$$\oplus\left\{\left\{1-\prod_{j=1}^{k}\left(1-\gamma_{k+1}^p\gamma_j^q\right)\Big|\gamma_{k+1}\in h_{d_{k+1}},\gamma_j\in h_{d_j}\right\},\right.$$
$$\left.\left\{\prod_{j=1}^{k}(1-(1-\eta_{k+1})^p(1-\eta_j)^q)\Big|\eta_{k+1}\in g_{d_{k+1}},\eta_j\in g_{d_j}\right\}\right\}$$
$$=\left\{\left\{1-\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(1-\gamma_i^p\gamma_j^q\right)\Big|\gamma_i\in h_{d_i},\gamma_j\in h_{d_j}\right\},\right.$$
$$\left.\left\{\prod_{\substack{i,j=1\\ i\neq j}}^{k+1}(1-(1-\eta_i)^p(1-\eta_j)^q)\Big|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\right\}\right\},\quad (4.23)$$

i.e., (4.13) holds for $n=k+1$. Thus, (4.13) holds for all n. Then, by (4.13) and operation (3) described in Definition 4.1.2, we obtain

$$\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(d_i^p\otimes d_j^q)\right)$$

$$= \left\{ \frac{1}{n(n-1)} \left(\oplus_{\substack{i,j=1 \\ i \neq j}}^{n} (h_{d_i}^{p} \otimes h_{d_j}^{q}) \right), \frac{1}{n(n-1)} \left(\otimes_{\substack{i,j=1 \\ i \neq j}}^{n} (g_{d_i}^{p} \oplus g_{d_j}^{q}) \right) \right\}$$

$$= \left\{ \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \gamma_i^p \gamma_j^q \right) \right)^{\frac{1}{n(n-1)}} \middle| \gamma_i \in h_{d_i}, \gamma_j \in h_{d_j} \right\}, \left\{ \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (1-\eta_i)^p (1-\eta_j)^q)^{\frac{1}{n(n-1)}} \middle| \eta_i \in g_{d_i}, \eta_j \in g_{d_j} \right\} \right\} \tag{4.24}$$

and then, by (4.24) and operation (4), it yields

$$\text{DHFB}^{p,q}(d_1, d_2, \ldots, d_n)$$

$$= \left\{ \left(\frac{1}{n(n-1)} \left(\oplus_{\substack{i,j=1 \\ i \neq j}}^{n} (h_{d_i}^{p} \otimes h_{d_j}^{q}) \right) \right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)} \left(\otimes_{\substack{i,j=1 \\ i \neq j}}^{n} (g_{d_i}^{p} \oplus g_{d_j}^{q}) \right) \right)^{\frac{1}{p+q}} \right\}$$

$$= \left\{ \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \gamma_i^p \gamma_j^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \middle| \gamma_i \in h_{d_i}, \gamma_j \in h_{d_j} \right\}, \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (1-\eta_i)^p (1-\eta_j)^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \middle| \eta_i \in g_{d_i}, \eta_j \in g_{d_j} \right\} \right\}, \tag{4.25}$$

i.e., (4.10) holds. In addition, let $d = \text{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) = \{h_d, g_d\}$, $\gamma^+ \in h_d^+$ and $\eta^+ \in g_d^+$, since

$$0 \leq \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\gamma_i^+)^p (\gamma_j^+)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq 1,$$

$$0 \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1-\eta_i^+)^p (1-\eta_j^+)^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leq 1 \tag{4.26}$$

and by (2.5), we have

$$\gamma^{+}+\eta^{+}=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(\gamma_i^{+})^{p}(\gamma_j^{+})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$+1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\eta_i^{+})^{p}(1-\eta_j^{+})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\leq 1+\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\eta_i^{+})^{p}(1-\eta_j^{+})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\eta_i^{+})^{p}(1-\eta_j^{+})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}=1 \tag{4.27}$$

which completes the proof of Theorem 4.2.2.

Now, let us look at some desirable properties of the DHFBM.

1) **Idempotency:** If all d_i's ($i=1,2,\ldots,n$) are equal, i.e., $d_i=d=\{h_d,g_d\}$, for all i, then

$$\mathrm{DHFB}^{p,q}(d_1,d_2,\ldots,d_n)=\mathrm{DHFB}^{p,q}(d,d,\ldots,d)$$
$$=\left\{\left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_d^{p}\otimes h_d^{q})\right)\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}(g_d^{p}\oplus g_d^{q})\right)\right)^{\frac{1}{p+q}}\right\}$$
$$=\left\{\left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\gamma^{p}\gamma^{q})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\middle|\gamma\in h_d\right\},\right.$$
$$\left.\left\{1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\eta)^{p}(1-\eta)^{q})^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\middle|\eta\in g_d\right\}\right\}$$
$$=\left\{\left\{\left(1-\left(1-\gamma^{p+q}\right)\right)^{\frac{1}{p+q}}|\gamma\in h_d\right\},\left\{1-\left(1-\left(1-(1-\eta)^{p+q}\right)\right)^{\frac{1}{p+q}}|\eta\in g_d\right\}\right\}$$

$$
\begin{aligned}
&= \left\{ \left\{ \left(\gamma^{p+q}\right)^{\frac{1}{p+q}} | \gamma \in h_d \right\}, \left\{ 1 - \left((1-\eta)^{p+q} \right)^{\frac{1}{p+q}} | \eta \in g_d \right\} \right\} \\
&= \left\{ \{\gamma | \gamma \in h_d\}, \{\eta | \eta \in g_d\} \right\} \\
&= \{h_d, g_d\} = d.
\end{aligned} \tag{4.28}
$$

In particular, if $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ is a collection of the smallest DHFEs, i.e., $d_i = d_* = \{\{0\}, \{1\}\}$, for all i, then

$$
\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) = \mathrm{DHFB}^{p,q}(d_*, d_*, \ldots, d_*) = \{\{0\}, \{1\}\} \tag{4.29}
$$

which is also the smallest DHFE. If $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ is a collection of the largest DHFEs, i.e., $d_i = d^* = \{\{1\}, \{0\}\}$, for all i, then

$$
\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) = \mathrm{DHFB}^{p,q}(d^*, d^*, \ldots, d^*) = \{\{1\}, \{0\}\} \tag{4.30}
$$

which is also the largest DHFE.

2) **Monotonicity:** Let $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ and $d'_i = \{h_{d'_i}, g_{d'_i}\}$ $(i = 1, 2, \ldots, n)$ be two collections of DHFEs. If $\#h_{d_i} = \#h_{d'_i}$, $\#g_{d_i} = \#g_{d'_i}$, $\gamma_i \leq \gamma'_i$ and $\eta_i \geq \eta'_i$ for all $\gamma_i \in h_{d_i}$, $\gamma'_i \in h_{d'_i}$, $\eta_i \in g_{d_i}$, $\eta'_i \in g_{d'_i}$, then

$$
\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) \leq \mathrm{DHFB}^{p,q}(d'_1, d'_2, \ldots, d'_n). \tag{4.31}
$$

Proof Since $\gamma_i \leq \gamma'_i$ and $\eta_i \geq \eta'_i$ for all $\gamma_i \in h_{d_i}$, $\gamma'_i \in h_{d'_i}$, $\eta_i \in g_{d_i}$, $\eta'_i \in g_{d'_i}$, then $\gamma_i^p \gamma_j^q \leq (\gamma'_i)^p (\gamma'_i)^q$ and $(1-\eta_i)^p (1-\eta_j)^q \leq (1-(\eta'_i))^p (1-(\eta'_j))^q$ for all $\gamma_i \in h_{d_i}$, $\gamma'_i \in h_{d'_i}$, $\eta_i \in g_{d_i}$, $\eta'_i \in g_{d'_i}$, $\gamma_j \in h_{d_j}$, $\gamma'_j \in h_{d'_j}$, $\eta_j \in g_{d_j}$, $\eta'_j \in g_{d'_j}$. Then we have

$$
\prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \gamma_i^p \gamma_j^q)^{\frac{1}{n(n-1)}} \geq \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (\gamma'_i)^p (\gamma'_j)^q)^{\frac{1}{n(n-1)}}, \tag{4.32}
$$

$$
\prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (1-\eta_i)^p (1-\eta_j)^q)^{\frac{1}{n(n-1)}} \geq \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1-\gamma'_i)^p (1-\gamma'_j)^q\right)^{\frac{1}{n(n-1)}} \tag{4.33}
$$

and hence

$$
\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \gamma_i^p \gamma_j^q)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}
$$

$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(\gamma_i')^p(\gamma_j')^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, \tag{4.34}$$

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\geq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\gamma_i')^p(1-\gamma_j')^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}. \tag{4.35}$$

Thus we obtain

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\gamma_i^p\gamma_j^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$
$$\leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(\gamma_i')^p(\gamma_j')^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\gamma_i')^p(1-\gamma_j')^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right). \tag{4.36}$$

Let $d=\{h_d,g_d\}=\mathrm{DHFB}^{p,q}(d_1,d_2,\ldots,d_n)$ and $d'=\{h_{d'},g_{d'}\}=\mathrm{DHFB}^{p,q}(d_1', d_2',\ldots,d_n')$, and let $s(d)$ and $s(d')$ be the scores of d and d', respectively. Then since $\#h_{d_i}=\#h_{d_i'}$ and $\#g_{d_i}=\#g_{d_i'}$, (4.36) is equivalent to $s(d)\leq s(d')$. Now, we discuss the following cases.

Case 1. If $s(d)<s(d')$, then, by Definition 4.1.4, we obtain

$$\mathrm{DHFB}^{p,q}(d_1,d_2,\ldots,d_n)<\mathrm{DHFB}^{p,q}(d_1',d_2',\ldots,d_n'). \tag{4.37}$$

Case 2. If $s(d) = s(d')$, then, since $\#h_d = (\#h_{d_1} \times \cdots \times \#h_{d_n}) \times n!$, $\#g_d = (\#g_{d_1} \times \cdots \times \#g_{d_n}) \times n!$, $\#h_{d'} = (\#h_{d'_1} \times \cdots \times \#h_{d'_n}) \times n!$ and $\#g_{d'} = (\#g_{d'_1} \times \cdots \times \#g_{d'_n}) \times n!$, we have

$$\frac{1}{\#h_d} \sum_{\#h_d} \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \gamma_i^p \gamma_j^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$- \frac{1}{\#g_d} \sum_{\#g_d} \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (1-\eta_i)^p (1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$
$$= \frac{1}{\#h_{d'}} \sum_{\#h_{d'}} \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (\gamma'_i)^p (\gamma'_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$- \frac{1}{\#g_{d'}} \sum_{\#g_{d'}} \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1-\gamma'_i)^p (1-\gamma'_j)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right). \quad (4.38)$$

Since $\gamma_i \leq \gamma'_i$ and $\eta_i \geq \eta'_i$, for all $\gamma_i \in h_{d_i}$, $\gamma'_i \in h_{d'_i}$, $\eta_i \in g_{d_i}$, $\eta'_i \in g_{d'_i}$, then

$$\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \gamma_i^p \gamma_j^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} = \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (\gamma'_i)^p (\gamma'_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},$$

$$1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - (1-\eta_i)^p (1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$= 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1-\gamma'_i)^p (1-\gamma'_j)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \quad (4.39)$$

and thus

$$h(d) = \frac{1}{\#h_d} \sum_{\#h_d} \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \gamma_i^p \gamma_j^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$+\frac{1}{\#g_d}\sum_{\#g_d}\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$=\frac{1}{\#h_{d'}}\sum_{\#h_{d'}}\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(\gamma_i')^p(\gamma_j')^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$+\frac{1}{\#g_{d'}}\sum_{\#g_{d'}}\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\gamma_i')^p(1-\gamma_j')^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$=h(d'). \tag{4.40}$$

Then, by Definition 4.1.4, we get

$$\mathrm{DHFB}^{p,q}(d_1,d_2,\ldots,d_n)=\mathrm{DHFB}^{p,q}(d_1',d_2',\ldots,d_n') \tag{4.41}$$

and hence, (4.37) and (4.41) indicate that (4.31) holds.

3) **Commutativity:** Let $d_i=\{h_{d_i},g_{d_i}\}$ $(i=1,2,\ldots,n)$ be a collection of DHFEs. Then

$$\mathrm{DHFB}^{p,q}(d_1,d_2,\ldots,d_n)=\mathrm{DHFB}^{p,q}(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_n), \tag{4.42}$$

where $(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_n)$ is any permutation of $(d_1,d_2,\ldots,d_n)$.

Proof Since $(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_n)$ is any permutation of $(d_1,d_2,\ldots,d_n)$, then

$$\mathrm{DHFB}^{p,q}(d_1,d_2,\ldots,d_n)$$

$$=\left\{\left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_d^p\otimes h_d^q)\right)\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}(g_d^p\oplus g_d^q)\right)\right)^{\frac{1}{p+q}}\right\}$$

$$=\left\{\left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_{\dot{d}}^p\otimes h_{\dot{d}}^q)\right)\right)^{\frac{1}{p+q}},\left(\frac{1}{n(n-1)}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}(g_{\dot{d}}^p\oplus g_{\dot{d}}^q)\right)\right)^{\frac{1}{p+q}}\right\}$$

$$=\mathrm{DHFB}^{p,q}(\dot{d}_1,\dot{d}_2,\ldots,\dot{d}_n). \tag{4.43}$$

4) **Boundedness:** Let $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of DHFEs, and let

$$d^+ = \left\{\max_i\{h^+_{d_i}\}, \min_i\{g^-_{d_i}\}\right\},$$
$$d^- = \left\{\min_i\{h^-_{d_i}\}, \max_i\{g^+_{d_i}\}\right\}, \tag{4.44}$$

where $h^+_{d_i} = \max_{\gamma_i \in h_{d_i}}\{\gamma_i\}$, $h^-_{d_i} = \min_{\gamma_i \in h_{d_i}}\{\gamma_i\}$, $g^+_{d_i} = \max_{\eta_i \in g_{d_i}}\{\eta_i\}$ and $g^-_{d_i} = \min_{\eta_i \in g_{d_i}}\{\eta_i\}$. Then

$$d^- \leq \mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) \leq d^+. \tag{4.45}$$

Proof Since $\gamma_i^- \leq \gamma_i \leq \gamma_i^+$ and $\eta_i^- \leq \eta_i \leq \eta_i^+$, for all $\gamma^- \in \min_i\{h^-_{d_i}\}$, $\gamma_i \in h_{d_i}$, $\gamma^+ \in \max_i\{h^-_{d_i}\}$, $\eta^- \in \min_i\{g^-_{d_i}\}$, $\eta_i \in g_{d_i}$ and $\eta^+ \in \max_i\{g^-_{d_i}\}$, then

$$\left(\gamma^-\right)^{p+q} \leq \gamma_i^p \gamma_j^q \leq \left(\gamma^+\right)^{p+q},$$
$$\left(1-\eta^+\right)^{p+q} \leq (1-\eta_i)^p(1-\eta_j)^q \leq \left(1-\eta^-\right)^{p+q} \tag{4.46}$$

and thus

$$\gamma^- = \left(1-\left(1-\left(\gamma^-\right)^{p+q}\right)\right)^{\frac{1}{p+q}} \leq \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\gamma_i^p\gamma_j^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\leq \left(1-\left(1-\left(\gamma^+\right)^{p+q}\right)\right)^{\frac{1}{p+q}} = \gamma^+, \tag{4.47}$$

$$\eta^- = 1-\left(1-\left(1-\left(1-\eta^-\right)^{p+q}\right)\right)^{\frac{1}{p+q}}$$
$$\leq 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\eta_i)^p(1-\eta_j)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$
$$\leq 1-\left(1-\left(1-\left(1-\eta^+\right)^{p+q}\right)\right)^{\frac{1}{p+q}} = \eta^+. \tag{4.48}$$

Let $d = \text{DHFB}^{p,q}(d_1, d_2, \dots, d_n) = \{h_d, g_d\}$. Then we have

$$s(d) = \frac{1}{\#h_d} \sum_{\gamma \in h_d} \gamma - \frac{1}{\#g_d} \sum_{\eta \in g_d} \eta$$
$$\leq \frac{1}{\#\max_i h^+_{d_i}} \sum_{\gamma^+ \in \max_i h^+_{d_i}} \gamma^+ - \frac{1}{\#\min_i g^-_{d_i}} \sum_{\eta^- \in \min_i g^-_{d_i}} \eta^- = s(d^+), \tag{4.49}$$

$$s(d) = \frac{1}{\#h_d} \sum_{\gamma \in h_d} \gamma - \frac{1}{\#g_d} \sum_{\eta \in g_d} \eta$$
$$\geq \frac{1}{\#\min_i h^-_{d_i}} \sum_{\gamma^- \in \min_i h^-_{d_i}} \gamma^+ - \frac{1}{\#\max_i g^+_{d_i}} \sum_{\eta^+ \in \max_i g^+_{d_i}} \eta^+ = s(d^-). \tag{4.50}$$

In what follows, three cases need to be discussed.

Case 1. If $s(d) < s(d^+)$ and $s(d) > s(d^-)$, then, from Definition 4.1.4, it follows that

$$d^- < \text{DHFB}^{p,q}(d_1, d_2, \dots, d_n) < d^+. \tag{4.51}$$

Case 2. If $s(d) = s(d^+)$, then, by (4.47) and (4.48), we have

$$\frac{1}{\#h_d} \sum_{\gamma \in h_d} \gamma = \frac{1}{\#\max_i h^+_{d_i}} \sum_{\gamma^+ \in \max_i h^+_{d_i}} \gamma^+, \quad \frac{1}{\#g_d} \sum_{\eta \in g_d} \eta = \frac{1}{\#\min_i g^-_{d_i}} \sum_{\eta^- \in \min_i g^-_{d_i}} \eta^- \tag{4.52}$$

and thus,

$$h(d) = \frac{1}{\#h_d} \sum_{\gamma \in h_d} \gamma + \frac{1}{\#g_d} \sum_{\eta \in g_d} \eta$$
$$= \frac{1}{\#\max_i h^+_{d_i}} \sum_{\gamma^+ \in \max_i h^+_{d_i}} \gamma^+ + \frac{1}{\#\min_i g^-_{d_i}} \sum_{\eta^- \in \min_i g^-_{d_i}} \eta^- = h(d^+). \tag{4.53}$$

Hence, by Definition 4.1.4, we get

$$\text{DHFB}^{p,q}(d_1, d_2, \dots, d_n) = d^+. \tag{4.54}$$

Case 3. $s(d) = s(d^-)$, then, from (4.47) and (4.48), it can be obtained that $\frac{1}{\#h_d}\sum_{\gamma\in h_d}\gamma = \frac{1}{\#\min_i h^-_{d_i}}\sum_{\gamma^-\in\max_i h^-_{d_i}}\gamma^-$ and $\frac{1}{\#g_d}\sum_{\eta\in g_d}\eta = \frac{1}{\#\max_i g^+_{d_i}}\sum_{\eta^+\in\max_i g^+_{d_i}}\eta^+$. Consequently, we have

$$\hbar(d) = \frac{1}{\#h_d}\sum_{\gamma\in h_d}\gamma + \frac{1}{\#g_d}\sum_{\eta\in g_d}$$
$$= \frac{1}{\#\min_i h^-_{d_i}}\sum_{\gamma^-\in\min_i h^-_{d_i}}\gamma^- + \frac{1}{\#\max_i g^+_{d_i}}\sum_{\eta^+\in\max_i g^+_{d_i}}\eta^+ = h(d^-). \quad (4.55)$$

Hence, by Definition 4.1.4, we get

$$\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) = d^-. \quad (4.56)$$

Therefore, from all the above-mentioned cases, it is clear that (4.45) holds.

In the following, let us consider some special cases of the DHFBM by taking different values of the parameters p and q.

Case 1. If $q \to 0$, then, by (4.10), we have

$$\lim_{q\to 0}\mathrm{DHFB}^{p,q}(d_1, d_2, \ldots, d_n) = \lim_{q\to 0}\left(\frac{1}{n(n-1)}\left(\oplus^n_{\substack{i,j=1\\ i\neq j}}(d_i^p \otimes d_j^q)\right)\right)^{\frac{1}{p+q}}$$
$$= \lim_{q\to 0}\left\{\left(\frac{1}{n(n-1)}\left(\oplus^n_{\substack{i,j=1\\ i\neq j}}(h^p_{d_i} \otimes h^q_{d_j})\right)\right)^{\frac{1}{p+q}},\right.$$
$$\left.\left(\frac{1}{n(n-1)}\left(\otimes^n_{\substack{i,j=1\\ i\neq j}}(g^p_{d_i} \oplus g^q_{d_j})\right)\right)^{\frac{1}{p+q}}\right\}$$
$$= \lim_{q\to 0}\left(\left\{\left\{\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^n\left(1-\gamma_i^p\gamma_j^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\middle|\gamma_i \in h_{d_i}, \gamma_j \in h_{d_j}\right\},\right.\right.$$
$$\left.\left.\left\{1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^n(1-(1-\eta_i)^p(1-\eta_j)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\middle|\eta_i \in g_{d_i}, \eta_j \in g_{d_j}\right\}\right\}\right)$$
$$= \left\{\left\{\left(1 - \prod_{i=1}^n(1-\gamma_i^p)^{\frac{n-1}{n(n-1)}}\right)^{\frac{1}{p}}\middle|\gamma_i \in h_{d_i}\right\},\right.$$

$$\left\{1-\left(1-\prod_{i=1}^{n}\left(1-(1-\eta_i)^p\right)^{\frac{n-1}{n(n-1)}}\right)^{\frac{1}{p}} \middle| \eta_i \in g_{d_i}\right\}\right\}$$

$$= \left\{\left\{\left(1-\prod_{i=1}^{n}\left(1-\gamma_i^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}} \middle| \gamma_i \in h_{d_i}\right\},\right.$$

$$\left.\left\{1-\left(1-\prod_{i=1}^{n}\left(1-(1-\eta_i)^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}} \middle| \eta_i \in g_{d_i}\right\}\right\}$$

$$= \left\{\left(\frac{1}{n}\left(\oplus_{i=1}^{n} h_{d_i}^p\right)\right)^{\frac{1}{p}}, \left(\frac{1}{n}\left(\otimes_{i=1}^{n} g_{d_i}^p\right)\right)^{\frac{1}{p}}\right\}$$

$$= \left(\frac{1}{n}\left(\oplus_{i=1}^{n} d_i^p\right)\right)^{\frac{1}{p}}$$

$$= \mathrm{DHFB}^{p,0}(d_1, d_2, \ldots, d_n) \tag{4.57}$$

which we call the generalized dual hesitant fuzzy mean.

Case 2. If $p = 1$ and $q \to 0$, then (4.10) is transformed as

$$\mathrm{DHFB}^{1,0}(d_1, d_2, \ldots, d_n) = \frac{1}{n}\left(\oplus_{i=1}^{n} d_i\right)$$

$$= \left\{\frac{1}{n}\left(\oplus_{i=1}^{n} h_{d_i}\right), \frac{1}{n}\left(\otimes_{i=1}^{n} g_{d_i}\right)\right\}$$

$$= \left\{\left\{1-\left(\prod_{i=1}^{n}(1-\gamma_i)\right)^{\frac{1}{n}} \middle| \gamma_i \in h_{d_i}\right\}, \left\{\left(\prod_{i=1}^{n}\eta_i\right)^{\frac{1}{n}} \middle| \eta_i \in g_{d_i}\right\}\right\} \tag{4.58}$$

which we call the dual hesitant fuzzy average.

Case 3. If $p = q = 1$, then (4.10) reduces to the following:

$$\mathrm{DHFB}^{1,1}(d_1, d_2, \ldots, d_n) = \left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(d_i \otimes d_j)\right)\right)^{\frac{1}{2}}$$

$$= \left\{\left(\frac{1}{n(n-1)}\left(\oplus_{\substack{i,j=1\\ i\neq j}}^{n}(h_{d_i} \otimes h_{d_j})\right)\right)^{\frac{1}{2}}, \left(\frac{1}{n(n-1)}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}(g_{d_i} \oplus g_{d_j})\right)\right)^{\frac{1}{2}}\right\}$$

$$= \left\{\left\{\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\gamma_i\gamma_j\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}} \middle| \gamma_i \in h_{d_i}, \gamma_j \in h_{d_j}\right\},\right.$$

$$\left\{1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\eta_i)(1-\eta_j)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\middle|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\right\}\right\} \quad (4.59)$$

which we call the dual hesitant fuzzy interrelated square mean.

Example 4.2.3 Assume that we have three DHFEs: $d_1 = \{\{0.2, 0.4\}, \{0.3\}\}$, $d_2 = \{\{0.5\}, \{0.1, 0.3\}\}$, and $d_3 = \{\{0.7, 0.9\}, \{0.1\}\}$. Here, we use the DHFBM to fuse these dual hesitant fuzzy data. Without the loss of generality, we let $p = q = 1$. Then

$$d_1 \otimes d_2 = d_2 \otimes d_1 = \{\{0.10, 0.20\}, \{0.37, 0.51\}\},$$
$$d_1 \otimes d_3 = d_3 \otimes d_1 = \{\{0.14, 0.18, 0.28, 0.36\}, \{0.37\}\},$$
$$d_2 \otimes d_3 = d_3 \otimes d_2 = \{\{0.35, 0.45\}, \{0.19, 0.37\}\}$$

and thus, by (4.10), we get

$$\begin{aligned}\text{DHFB}^{1,1}(d_1,d_2,d_3) &= \left(\frac{1}{6}\left(\oplus_{\substack{i,j=1\\i\neq j}}^{3}(d_i\otimes d_j)\right)\right)^{\frac{1}{2}}\\ &= \left\{\left\{\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{3}(1-\gamma_i\gamma_j)^{\frac{1}{6}}\right)^{\frac{1}{2}}\middle|\gamma_i\in h_{d_i},\gamma_j\in h_{d_j}\right\},\right.\\ &\quad\left.\left\{1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{3}\left(1-(1-\eta_i)(1-\eta_j)\right)^{\frac{1}{6}}\right)^{\frac{1}{2}}\middle|\eta_i\in g_{d_i},\eta_j\in g_{d_j}\right\}\right\}\\ &= \{\{0.4524, 0.5095, 0.5285, 0.5870\}, \{0.1611, 0.2230\}\}.\end{aligned}$$

If we use the dual hesitant fuzzy average (4.57) to aggregate the DHFEs d_i $(i = 1, 2, 3)$, then we have

$$\begin{aligned}\text{DHFB}^{1,0}(d_1,d_2,d_3) &= \frac{1}{3}\left(\oplus_{i=1}^{3}d_i\right)\\ &= \left\{\left\{1-\left(\prod_{i=1}^{3}(1-\gamma_i)\right)^{\frac{1}{3}}\middle|\gamma_i\in h_{d_i}\right\},\left\{\left(\prod_{i=1}^{3}\eta_i\right)^{\frac{1}{3}}\middle|\eta_i\in g_{d_i}\right\}\right\}\\ &= \{\{0.5068, 0.6580, 0.5519, 0.6893\}, \{0.1442, 0.2080\}\}.\end{aligned}$$

Based on the above-mentioned computational analysis, it can be seen that the dual hesitant fuzzy average is simpler than the DHFBM from the computational point of view, but the DHFBM can capture the interrelationship of the given arguments and thus can take much information into account than the former one.

4.3 Weighted dual hesitant fuzzy Bonferroni means

In the above-mentioned analysis, only the input data and their interrelationships are involved in aggregation process, but the importance of each datum is not emphasized. However, in many practical situations, the weights of the data should be taken into account. For example, in multiple attribute decision making, the considered attribute usually have different importance and thus need to be assigned different weights. Now, we define an weighted DHFBM.

Definition 4.3.1 Let $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of DHFEs, let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\tilde{h}_i$ $(i = 1, 2, \ldots, n)$, where w_i indicates the importance degree of d_i $(i = 1, 2, \ldots, n)$, satisfying $w_i > 0$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$. For any $p, q > 0$, if

$$\mathrm{DHFB}_w^{p,q}(d_1, d_2, \ldots, d_n) = \left(\frac{1}{n(n-1)} \left(\oplus_{\substack{i,j=1 \\ i \neq j}}^{n} ((w_i d_i)^p \otimes (w_j d_j)^q) \right) \right)^{\frac{1}{p+q}} \quad (4.60)$$

then $\mathrm{DHFB}_w^{p,q}$ is called the weighted dual hesitant fuzzy Bonferroni mean (WDHFBM).

Similar to Theorem 4.2.2, we have the following theorem.

Theorem 4.3.2 Let $d_i = \{h_{d_i}, g_{d_i}\}$ $(i = 1, 2, \ldots, n)$ be a collection of DHFEs, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, which satyisfies $w_i > 0$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$, and let $p, q > 0$. Then the aggregated value, by using the WDHFBM (4.60), is also a DHFE, and

$$\begin{aligned}
&\mathrm{DHFB}_w^{p,q}(d_1, d_2, \ldots, d_n) \\
&= \left\{ \left(\frac{1}{n(n-1)} \left(\oplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left((w_i h_{d_i})^p \otimes (w_j h_{d_j})^q \right) \right) \right)^{\frac{1}{p+q}}, \right. \\
&\qquad\qquad \left. \left(\frac{1}{n(n-1)} \left(\otimes_{\substack{i,j=1 \\ i \neq j}}^{n} \left((w_i g_{d_i})^p \oplus (w_j g_{d_j})^q \right) \right) \right)^{\frac{1}{p+q}} \right\} \\
&= \left\{ \left\{ \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1-\gamma_i)^{w_i})^p (1 - (1-\gamma_j)^{w_j})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right| \right. \\
&\qquad\qquad \left. \gamma_i \in h_{d_i}, \gamma_j \in h_{d_j} \right\}, \\
&\quad \left. \left\{ 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \eta_i^{w_i})^p (1 - \eta_j^{w_j})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \middle| \eta_i \in g_{d_i}, \eta_j \in g_{d_j} \right\} \right\}.
\end{aligned} \tag{4.61}$$

In what follows, we apply the WDHFBM to multiple attribute decision making under dual hesitant fuzzy environment, which involves the following steps.

Step 1. For a multiple attribute decision making problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of m alternatives, and $Y = \{y_1, y_2, \ldots, y_n\}$ be a set of n attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, satisfying $w_j > 0$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$, where w_j denotes the importance degree of the attribute y_j. The performance of the alternative x_i with respect to the attribute y_j is measured by a DHFE $d_{ij} = \{\{\gamma_{ij} | \gamma_{ij} \in h_{d_{ij}}\}, \{\eta_{ij} | \eta_{ij} \in g_{d_{ij}}\}\}$, where γ_{ij} indicates the degree that the alternative x_i satisfies the attribute y_j, η_{ij} indicates the degree that the alternative x_i does not satisfy the attribute y_j, such that $\gamma_{ij} \in [0, 1]$, $\eta_{ij} \in [0, 1]$, $\gamma_{ij}^+ + \eta_{ij}^+ \leq 1$. All $d_{ij} = \{\{\gamma_{ij} | \gamma_{ij} \in h_{d_{ij}}\}, \{\eta_{ij} | \eta_{ij} \in g_{d_{ij}}\}\}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are contained in a dual hesitant fuzzy decision matrix $D = (d_{ij})_{m \times n}$ (see Table 4.1).

If all attribute y_j $(j = 1, 2, \ldots, n)$ are of the same type, then the performance values do not need normalization. Whereas there are, generally, benefit attribute

Table 4.1: The dual hesitant fuzzy decision matrix

	y_1	y_2	$\cdots$	y_n
x_1	d_{11}	d_{12}	$\cdots$	d_{in}
x_2	d_{21}	d_{22}	$\cdots$	d_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	d_{m1}	d_{m2}	$\cdots$	d_{mn}

(the bigger the performance the better) and cost attribute (the smaller the performance values the better) in multiple attribute decision making, in such cases, we may transform the performance values of the cost type into the performance values of the benefit type. Then, $D = (d_{ij})_{m\times n}$ can be transformed into the matrix $E = (e_{ij})_{m\times n}$, where

$$\begin{aligned} e_{ij} &= \{h_{e_{ij}}, g_{e_{ij}}\} \\ &= \begin{cases} \{\{\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}, \{\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}\}, \\ \text{(for benefit attribute } y_j) \\ \begin{cases} \{\{\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}, \{\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}\}, & \text{if } h_{d_{ij}} \neq \emptyset, g_{d_{ij}} \neq \emptyset, \\ \{\{1-\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}, \{\emptyset\}\}, & \text{if } h_{d_{ij}} \neq \emptyset, g_{d_{ij}} = \emptyset, \\ \{\{\emptyset\}, \{1-\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}\}, & \text{if } h_{d_{ij}} = \emptyset, g_{d_{ij}} \neq \emptyset, \end{cases} \\ \text{(for cost attribute } y_j) \end{cases} \\ & i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n, \end{aligned} \tag{4.62}$$

and $d^c_{ij} = \begin{cases} \{\{\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}, \{\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}\}, & \text{if } h_{d_{ij}} \neq \emptyset, g_{d_{ij}} \neq \emptyset, \\ \{\{1-\gamma_{ij}|\gamma_{ij} \in h_{d_{ij}}\}, \{\emptyset\}\}, & \text{if } h_{d_{ij}} \neq \emptyset, g_{d_{ij}} = \emptyset, \\ \{\{\emptyset\}, \{1-\eta_{ij}|\eta_{ij} \in g_{d_{ij}}\}\}, & \text{if } h_{d_{ij}} = \emptyset, g_{d_{ij}} \neq \emptyset, \end{cases}$
is the complement of d_{ij}.

Step 2. Utilize the WDHFBM (in general, we can take $p = q = 1$)

$$e_i = \{h_{e_i}, g_{e_i}\} = \text{DHFB}^{p,q}_w(e_{i1}, e_{i2}, \ldots, e_{in}) \tag{4.63}$$

to aggregate all the performance values e_{ij} ($j = 1, 2, \ldots, n$) of the ith line and get the overall performance value e_i corresponding to the alternative x_i.

Step 3. Utilize the method in Definition 4.1.4 to rank the overall performance values e_i $(i = 1, 2, \ldots, m)$.

Step 4. Rank all alternatives x_i $(i = 1, 2, \ldots, m)$ in accordance with e_i $(i = 1, 2, \ldots, m)$ in descending order, and then, select the most desirable alternative with the largest overall performance value.

In the above-mentioned procedure, we utilized the WDHFBM to aggregate the performance values of each alternative with respect to a collection of the pregiven attributes, so as to rank and select the alternatives. The desirable characteristic of the WDHFBM is that it can not only consider the importance of each attribute but also reflect the interrelationship of the individual attributes and thus takes the decision information into account as much as possible.

4.4 Conclusions

The BM is a traditional mean-type aggregation operator and is generally used to aggregate the crisp numerical values rather than any other types of data. In this chapter, we have extended the BM to accommodate dual hesitant fuzzy environments. We have developed some new dual hesitant fuzzy aggregation operators, including the DHFBM, the WDHFBM, and the various special cases of the DHFBM. Then, we have applied the WDHFBM to multiple attribute decision making with dual hesitant fuzzy information. The main advantage of the WDHFBM in multiple attribute decision making is that it can not only consider the importance of each attribute but also reflect the interrelationship of the individual attributes and thus takes the decision information into account as much as possible. The applications of these operators in many actual fields, such as pattern recognition, medical diagnosis and clustering analysis, are open questions for future research.

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