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Demiclosedness Principle of a Nonlipschitzian Semigroup

by

Cho Young Moon

Graduate School of Education

Pukyong National University

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Demiclosedness Principle of a Nonlipschitzian Semigroup
(어떤 비-Lipschitz반군의 반단힘원리)

Advisor : Tae Hwa Kim

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Cho Young Moon

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Cho Young Moon

Approved by :

(Chairman) Nak Eun Cho

(Member) Jin Mun Jeong

(Member) Tae Hwa Kim

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어떤 비-Lipschitz반군의 반단힘원리

문 초 영

부경대학교 교육대학원 수학교육전공

요　　약

집합 C를 실 Banach 공간 X의 볼록이고 닫힌 부분집합(*closed convex subset*)이라 할 때, 함수 $T : C \to C$의 부동점들의 집합을 $Fix(T) = \{x \in c : Tx = x\}$로 표기한다. 이산인 가산 개의 함수 $T_n : C \to C$, $n \geq 0$ 들의 족 $\Im = \{T_n : n \geq 0\}$가 점근적비확대형(*asymptotically nonexpansive type*)라 함은 0에 수렴하는 수열 $\{d_n\}$가 존재하여

$\|T_n x - T_n y\| \leq \|x - y\| + d_n$, $\forall\, x, y \in C$

을 만족하는 것을 말한다. 모든 함수 T_n이 C 상에서 연속일 때 집합족 $\Im$이 C 상에서 연속이라 부른다. 덧붙여, $\Im$가 다음 두 조건을 만족할 때 $\Im$를 C 상의 점근적비확대형반군이라 말한다.

(i) $T_0 x = x$, $\forall\, x \in C$;

(ii) $T_{n+m} x = T_n T_m x$, $\forall\, n, m \geq 0,\ x \in C$.

본 논문의 주 결과는 다음과 같다.

정리. 집합 C는 균등볼록인 Banach 공간 X의 공집합이 아닌 볼록닫힌부분집합이고 함수들의 집합족 $\Im = \{T_n : C \to C, n \geq 0\}$가 $Fix(\Im) \neq \emptyset$인 C 상의 연속인 점근적비확대형반군이라 하자. 만약 수열 $(x_n) \subset C$이 $x_n \rightharpoonup x\,(\in C)$하고

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T_k x_n\| = 0$$

을 만족하면 x는 $\Im$의 공통부동점이다. 즉, $T_n x = x$, $\forall\, n \geq 0$.

1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. Let $\{x_n\}$ be a sequence in X, $x \in X$. We denote by $x_n \to x$ the strong convergence of $\{x_n\}$ to x and by $x_n \rightharpoonup x$ the weak convergence of $\{x_n\}$ to x. Also, we denote by $\omega_w(x_n)$ the weak ω-limit set of $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : \exists\, x_{n_k} \rightharpoonup x\}.$$

Let C be a nonempty closed convex subset of X and let $T : C \to C$ be a mapping. Now let $Fix(T)$ be the fixed point set of T; namely,

$$Fix(T) := \{x \in C : Tx = x\}.$$

Recall that T is a *Lipschitzian* mapping if, for each $n \geq 1$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ (we may assume that all $k_n \geq 1$). A Lipschitzian mapping T is called *uniformly k-Lipschitzian* if $k_n = k$ for all $n \geq 1$, *nonexpansive* if $k_n = 1$ for all $n \geq 1$, and *asymptotically nonexpansive* if $\lim_{n\to\infty} k_n = 1$, respectively. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [4] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpanisve mapping $T : C \to C$ has a fixed point.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that T is a mapping of *asymptotically nonexpansive type* [7] if for each $x \in C$,

$$\limsup_{n\to\infty} \sup_{y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0 \tag{1.2}$$

and T^N is continuous for some $N \ge 1$. The other is the stronger concept due to Bruck, Kuczumov and Reich [1]. They say that T is *asymptotically nonexpansive in the intermediate sense* if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0 \tag{1.3}$$

In this case, observe that if we define

$$\delta_n := \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0, \tag{1.4}$$

(here $a \vee b := \max\{a, b\}$), then $\delta_n \ge 0$ for all $n \ge 1$, $\delta_n \to 0$ as $n \to \infty$, and thus (1.3) immediately reduces to

$$\|T^n x - T^n y\| \le \|x - y\| + \delta_n \tag{1.5}$$

for all x, $y \in C$ and $n \ge 1$.

Let C be a nonempty closed convex subset of a real Banach space X, and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \ne \emptyset$. Recall that the following Mann [8] iterative method is extensively used for solving a fixed point equation of the form $Tx = x$:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \ge 0, \tag{1.6}$$

where $\{a_n\}$ is a sequence in $[0, 1]$ and $x_0 \in C$ is arbitrarily chosen. In infinite-dimensional spaces, Mann's algorithm has generally only weak convergence. In fact, it is known [10] that if the sequence $\{\alpha_n\}$ is such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then Mann's algorithm (1.6) converges weakly to a fixed point of T provided the underlying space is a Hilbert space or more general, a uniformly convex Banach space which has a Fréchet differentiable norm or satisfies Opial's property. Furthermore, Mann's algorithm (1.6) also converges weakly to a fixed point of T if X is a uniformly convex Banach space such that its dual X^* enjoys the *Kadec-Klee property* (*KK-property*, in brief), i.e., $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\| \quad \Rightarrow \quad x_n \to x$. It is well known [2] that the duals of reflexive Banach spaces with a Frechet differentiable norms have the KK-property. There exists uniformly convex spaces which have neither a Fréchet differentiable norm nor the Opial property but their duals do have the KK-property; see Example 3.1 of [3].

Let C be a nonempty closed convex subset of a real Banach space X. Recall also that a discrete family $\Im = \{T_n : C \to C\}$ is said to be *asymptotically nonexpansive type* (in briefly, ANT) on C if there exists a nonnegative real sequence $\{d_n\}$, $n \geq 1$ with $d_n \to 0$ such that

$$\|T_n x - T_n y\| \leq \|x - y\| + d_n, \tag{1.7}$$

for all $x, y \in C$ and $n \geq 1$. Furthermore, we say that $\Im$ is *continuous* on C provided each $T_n \in \Im$ is continuous on C. In particular, we say that $\Im = \{T_n : C \to C\}$ is simply ANT when $C = X$.

Let C be a nonempty closed convex subset of a real Banach space X. In this paper, we firstly consider a discrete ANT semigroup $\Im = \{T_n : C \to C, n \geq 0\}$ on C, namely, a ANT family equipped with the following semigroup properties:

(i) $T_0 x = x, \ x \in C$,

(ii) $T_{n+m} x = T_n T_m x, \ n, m \geq 0, \ x \in C$.

In section 3, we give the demiclosedness of $I - \Im$ at zero of such a continuous ANT semigroup $\Im$ in the sense that if $\{x_n\}$ is a sequence in C converging weakly to $x \in C$ and

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\|x_n - T_k x_n\| = 0, \tag{1.8}$$

then $x \in Fix(\Im) := \cap_{n=1}^{\infty} Fix(T_n)$, the set of common fixed points of $\Im$.

2 An example

The following example is a special case of Examples 1.7 in [6]. For the sake of convenience we introduce its proof.

Example 2.1. ([6]) *Let C be a nonempty closed convex subset of a real Banach space X. Let $\Im = \{T_n : C \to C\}$ be a continuous ANT family on C with a control sequence d_n such that $\sum_{n=1}^{\infty} d_n < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$. Then the family $\mathcal{S} = \{S_n : C \to C\}$ defined by*

$$S_n = (1-\alpha_n)I + \alpha_n T_n[(1-\beta_n)I + \beta_n T_n]$$

is also continuous ANT on C, i.e., there exist $\{\tilde{d}_n\}$ converging to zero, $\sum_{n=1}^{\infty}\tilde{d}_n < \infty$, such that

$$\|S_n x - S_n y\| \leq \|x-y\| + \tilde{d}_n, \quad x, y \in C.$$

Proof. Putting $U_n := (1-\beta_n)I + \beta_n T_n$ and using (1.7) yield

$$\begin{aligned} \|U_n x - U_n y\| &\leq (1-\beta_n)\|x-y\| + \beta_n\|T_n^{(2)}x - T_n^{(2)}y\| \\ &\leq (1-\beta_n)\|x-y\| + \beta_n(\|x-y\| + d_n) \\ &\leq \|x-y\| + d_n \end{aligned}$$

for all $x, y \in C$. Then, we can also compute

$$\begin{aligned}\|S_n x - S_n y\| &\leq (1-\alpha_n)\|x-y\| + \alpha_n\|T_n(U_n x) - T_n(U_n y)\| \\ &\leq (1-\alpha_n)\|x-y\| + \alpha_n(\|U_n x - U_n y\| + d_n) \\ &\leq \|x-y\| + 2d_n\end{aligned}$$

Therefore, the family $\mathcal{S} = \{S_n : C \to C\}$ is continuous TAN on C with $\tilde{d}_n = 2d_n$.

□

3 Demiclosedness principle of ANT semigroups

Here we summarize the notations used in the sequel. The convex hull of a subset A of a real Banach space X is denoted by $co\,A$, and the closed convex hull by $\overline{co}\,A$. We put

$$\triangle^{n-1} = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) : \lambda_i \geq 0\,(i = 1, 2, \cdots, n) \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}$$

and for $r > 0$

$$B_r = \{x \in X : \|x\| \leq r\}.$$

Now let us begin with the following slight modification of Lemma 2.1 in [9].

Lemma 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let a family $\Im = \{T_n : C \to C\}$ be ANT on C with $Fix(\Im) = \cap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Let K be a bounded closed convex subset of C containing x^* for some $x^* \in Fix(\Im)$. Then, for $\epsilon > 0$ there exists an integers $N_\epsilon \geq 1$ and $\delta_{2,\epsilon}$ with $0 < \delta_{2,\epsilon} \leq \epsilon$ such that $k \geq N_\epsilon$, $x_1, x_2 \in K$ and if $\|x_1 - x_2\| - \|T_k x_1 - T_k x_2\| \leq \delta_{2,\epsilon}$, then*

$$\|T_k(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 T_k x_1 - \lambda_2 T_k x_2\| < \epsilon$$

for all $\lambda = (\lambda_1,\, \lambda_2) \in \triangle^1$.

Proof. We employ the method of the proof in [9]. Since X is uniformly convex, the modulus of convexity δ is a continuous and strictly increasing function on $[0, 2]$ (see [5] for more details). Then the function $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$F(x) = \begin{cases} \frac{1}{2}\int_0^x \delta(t)dt, & \text{if } 0 \le x \le 2; \\ \frac{1}{2}(x-2) + F(2), & \text{if } x > 2. \end{cases}$$

is clearly strictly increasing, continuous and convex on $\mathbb{R}^+$. Obviously, since $F(x) \le \delta(x)\,(0 \le x \le 2)$, the uniform convexity of X implies that

$$2\lambda_1\lambda_2 F(\|x - y\|) \le 1 - \|\lambda_1 x + \lambda_2 y\| \tag{3.1}$$

for $\lambda = (\lambda_1, \lambda_2) \in \triangle^1$, $\|x\| \le 1$ and $\|y\| \le 1$.

If either λ_1 or λ_2 is 1 or 0, our conclusion is clearly satisfied. So assume that $0 < \lambda_1,\, \lambda_2 < 1$ and let $\epsilon > 0$ be arbitrary given. Set

$$M := \operatorname{diam} K < \infty.$$

Choose $d_\epsilon > 0$ such that $\frac{M}{2}F^{-1}\left(\frac{2d_\epsilon}{M}\right) < \epsilon$ and put $\delta_{2,\epsilon} = \min\left\{\epsilon,\, d_\epsilon,\, \frac{M}{4}\right\}$. For $\bar{\delta}_{2,\epsilon} = \min\{\lambda_i \delta_{2,\epsilon} : i = 1, 2\} > 0$, since $d_n \to 0$, there exists an integer $N_\epsilon \ge 1$ (depending on the set K) such that if $k \ge N_\epsilon$,

$$d_k < \bar{\delta}_{2,\epsilon}.$$

Then, by (1.7), we have

$$\|T_k x - T_k y\| \le \|x - y\| + d_k \le \|x - y\| + \bar{\delta}_{2,\epsilon} \tag{3.2}$$

for all $k \ge N_\epsilon$, $x, y \in K$. Now let $k \ge N_\epsilon$ and let $x_1, x_2 \in K$ with $\|x_1 - x_2\| - \|T_k x_1 - T_k x_2\| \le \delta_{2,\epsilon}$. On letting

$$x := \frac{T_k x_2 - T_k(\lambda_1 x_1 + \lambda_2 x_2)}{\lambda_1(\|x_1 - x_2\| + \delta_{2,\epsilon})} \quad \text{and} \quad y := \frac{T_k(\lambda_1 x_1 + \lambda_2 x_2) - T_k x_1}{\lambda_2(\|x_1 - x_2\| + \delta_{2,\epsilon})},$$

we have $\|x\| \le 1,\ \|y\| \le 1$ by with help of (3.2) and

$$\lambda_1 x + \lambda_2 y = \frac{T_k x_2 - T_k x_1}{\|x_1 - x_2\| + \delta_{2,\epsilon}}. \tag{3.3}$$

On letting $0 < t := \frac{2}{M}\lambda_1\lambda_2(\|x_1 - x_2\| + \delta_{2,\epsilon}) \le \frac{2}{M}\frac{1}{4}\left(M + \frac{M}{4}\right) < 1$, we observe that

$$\frac{2}{M}\|\lambda_1 T_k x_1 + \lambda_2 T_k x_2 - T_k(\lambda_1 x_1 + \lambda_2 x_2)\| = t\|x - y\| \tag{3.4}$$

and

$$\begin{aligned} \frac{1}{2\lambda_1\lambda_2}(1 - \|\lambda_1 x + \lambda_2 y\|) &= \frac{\|x_1 - x_2\| - \|T_k x_1 - T_k x_2\| + \delta_{2,\epsilon}}{2\lambda_1\lambda_2(\|x_1 - x_2\| + \delta_{2,\epsilon})} \\ &\le \frac{2\delta_{2,\epsilon}}{t\,M}. \end{aligned} \tag{3.5}$$

Using (3.1), (3.4), (3.5) and the convexity of F with $F(0) = 0$, we have

$$\begin{aligned} & F\left(\frac{2}{M}\|\lambda_1 T_k x_1 + \lambda_2 T_k x_2 - T_k(\lambda_1 x_1 + \lambda_2 x_2)\|\right) \\ = \ & F(t\|x - y\|) = F\big(t\|x - y\| + (1 - t)0\big) \\ \le \ & tF(\|x - y\|) + (1 - t)F(0) \\ \le \ & \frac{t}{2\lambda_1\lambda_2}(1 - \|\lambda_1 x + \lambda_2 y\|) \le \frac{2\delta_{2,\epsilon}}{M} \le \frac{2d_\epsilon}{M} \end{aligned}$$

and so we have

$$\|\lambda_1 T_k x_1 + \lambda_2 T_k x_2 - T_k(\lambda_1 x_1 + \lambda_2 x_2)\| \le \frac{M}{2}F^{-1}\left(\frac{2d_\epsilon}{M}\right) < \epsilon$$

from the choice of d_ϵ and the proof is complete. □

Now on mimicking Lemma 2.2 and 2.3 in [9] we have the following result.

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let a family $\Im = \{T_n : C \to C\}$ be ANT on C*

with $Fix(\Im) \neq \emptyset$. Let K be a bounded closed convex subset of C containing x^* for some $x^* \in Fix(\Im)$. Then, for $\epsilon > 0$ there exists an integers $N_\epsilon \geq 1$ and δ_ϵ with $0 < \delta_\epsilon \leq \epsilon$ such that $k \geq N_\epsilon$, $x_1, x_2, \cdots, x_n \in K$ and if $\|x_i - x_j\| - \|T_k x_i - T_k x_j\| \leq \delta_\epsilon$ for $1 \leq i, j \leq n$, then

$$\left\| T_k \left(\sum_{i=1}^{n} \lambda_i x_i \right) - \sum_{i=1}^{n} \lambda_i T_k x_i \right\| < \epsilon$$

for all $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \triangle^{n-1}$.

As a direct application of Lemma 3.2, we have the following demiclosedness principle for continuous ANT semigroups.

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\Im = \{T_n : C \to C, n \geq 0\}$ be a continuous ANT semigroup on C with $Fix(\Im) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup x \,(\in C)$ and it satisfies (1.8), namely,*

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

Then $x \in Fix(\Im)$.

Proof. First, we claim that $\lim_{k\to\infty} T_k x = x$. For this end, fix $p \in Fix(\Im)$. Since $\{x_n\}$ is bounded in C, take K in Lemma 3.2 by the closed convex hull of $\{p\} \cup \{x_n : n \geq 1\}$. For $\epsilon > 0$, take $N_\epsilon \geq 1$ and δ_ϵ with $0 < \delta_\epsilon \leq \epsilon$ as in Lemma 3.2. From (1.8), there exists an integer $k_0 \,(\geq N_\epsilon)$ such that

$$\limsup_{n\to\infty} \|x_n - T_k x_n\| < \delta_\epsilon / 2$$

for all $k \geq k_0$. Also, we can choose an integer $n_0 \,(\geq k_0)$ such that

$$\|x_n - T_k x_n\| \leq \delta_\epsilon / 2 \qquad (k, n \geq n_0). \tag{3.6}$$

Since $x_n \rightharpoonup x$ and $x \in \overline{co}\{x_i : i \geq n\}$ for each $n \geq 1$, we can choose for each $n \geq 1$ a convex combination

$$y_n = \sum_{i=1}^{m(n)} \lambda_i^{(n)} x_{i+n}, \quad \text{where } \lambda^{(n)} = (\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_{m(n)}^{(n)}) \in \triangle^{m(n)-1}$$

such that $\|y_n - x\| \to 0$. Let $k, n \geq n_0$. Then it follows from (3.6) that, for $1 \leq i,\, j \leq m(n)$,

$$\begin{aligned} & \|x_{i+n} - x_{j+n}\| - \|T_k x_{i+n} - T_k x_{j+n}\| \\ \leq\ & \|x_{i+n} - T_k x_{i+n}\| + \|x_{j+n} - T_k x_{j+n}\| \\ \leq\ & \delta_\epsilon/2 + \delta_\epsilon/2 = \delta_\epsilon \end{aligned}$$

and so applying Lemma 3.2 yields

$$\left\| T_k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T_k x_{i+n} \right\| < \epsilon.$$

and hence

$$\begin{aligned} \|T_k y_n - y_n\| &\leq \left\| T_k y_n - \sum_{i=1}^{m(n)} \lambda_i^{(n)} T_k x_{i+n} \right\| + \left\| \sum_{i=1}^{m(n)} \lambda_i^{(n)} (T_k x_{i+n} - x_{i+n}) \right\| \\ &< \epsilon + \delta_\epsilon/2 \leq (3/2)\epsilon \end{aligned}$$

for $k, n \geq n_0$. Since $\Im = \{T_n : C \to C\}$ is ANT on C, this implies that, for $k, n \geq n_0$,

$$\begin{aligned} \|T_k x - x\| &\leq \|T_k x - T_k y_n\| + \|T_k y_n - y_n\| + \|y_n - x\| \\ &\leq \|x - y_n\| + d_k + (3/2)\epsilon + \|y_n - x\| \\ &= 2\|y_n - x\| + d_k + (3/2)\epsilon. \end{aligned} \tag{3.7}$$

Taking the lim sup as $n \to \infty$ at first and next the lim sup as $k \to \infty$ in both sides of (3.7), we have $\limsup_{k\to\infty} \|T_k x - x\| \leq (3/2)\epsilon$ and since ϵ is arbitrary given, $T_k x \to x$. Then we easily obtain that $x \in Fix(\Im)$ because, for each fixed $n \geq 1$, we get

$$T_n x = T_n(\lim_{k\to\infty} T_k x) = \lim_{k\to\infty} T_{n+k} x = x$$

using continuity of T_n and semigroup property of $\Im$. The proof is complete. □

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