Thesis for the Degree of
Master of Education

Intuitionistic Fuzzy Optimized Weighted Geometric Bonferroni Means and Their Applications in Group Decision making

by

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Graduate School of Education

Pukyong National University

August 2014

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직관적인 퍼지 최적 가중 기하 Bonferroni 평균과 집단의사결정에의 응용

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A thesis submitted in partial fulfillment of the requirement for the degree of

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August 22, 2014

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요 약

기하 Bonferroni 평균 (GBM)은 집계 인수들의 상관관계를 반영하는 중요한 집성 기술이다.

본 논문에서는 GBM을 바탕으로 집계 인수의 선호도와 상호관계를 반영하는 최적 가중 기하 Bonferroni 평균 (OWGBM)과 일반화된 최적 가중 기하 Bonferroni 평균 (GOWGBM)을 개발한다. 게다가 직관적인 퍼지 최적 가중 기하 Bonferroni 평균 (IFOWGBM)과 일반화된 직관적인 퍼지 최적 가중 기하Bonferroni 평균 (GIFOWGBM)을 개발하고, 멱등성, 교환성, 단조성 및 유계성과 같은 기초적인 성질들을 조사한다. 마지막으로 IFOWGBM와 GIFOWGBM를 이용하여, 직관적인 퍼지 정보를 가지는 다중 속성 의사 결정에 접근방법을 제시하고 실제 예제를 통하여 제시된 접근방법을 설명한다.

1 Introduction

A lot of extensions of the average mean (AM) and the geometric mean (GM), which are the basic functions among the aggregation operators, have been developed. For example, Yager [29] proposed the ordered weighted averaging (OWA) operator to reordering the arguments before being aggregated, motivated by which, some authors [7, 24] investigated the ordered weighted geometric (OWA) operator. For the case that the given arguments is a continuous interval valued rather than a finite set of arguments, Yager [31] developed a continuous ordered weighted averaging (C-OWA) operator, and Yager and Xu [34] further developed the continuous ordered weighted geometric (C-OWA) operator. For the linguistic information, some aggregation operators were also developed based on the AM and the GM, such as the linguistic weighted averaging (LWA) operator [20], the linguistic ordered weighted averaging (LOWA) operator [18], the linguistic weighted geometric averaging (LWGA) operator [19] and the linguistic ordered weighted geometric averaging (LOWGA) operator [19].

It is noted that the above aggregation operators consider the aggregation arguments independent. However, the aggregated arguments are correlative, especially in multi-criteria decision making. To overcome this limitation, many aggregation operators have been developed to investigate the correlation among the arguments, Yager [30] introduced the power average (PA) to provide an aggregation operator which allows arguments values to support each other in the aggregation process, based on which, Xu and Yager [27] developed the power geometric (PG) operator and its weighted form, developed the power ordered geometric (POG) operator and the power ordered weighted geometric (POWG) operator, and studied some of their properties. Xu [23] extended the PA and applied it to aggregate intuitionistic fuzzy information. Motivated by the Choquet integral [8], Yager [32] introduced the idea of order induced aggregation to the Choquet aggregation operator and defined the induced Choquet ordered averaging operator. Xu [22], Tan and Chen [15] developed some intuitionistic fuzzy correlated operators based on Choquet integral.

The Bonferroni mean (BM) originally introduced by Bonferroni [3] and then generalized by Yager [33]. The desirable characteristic of the BM is its capability to capture the interrelationship between input arguments. Xu and Yager [28] further applied the Bonferroni mean to intuitionistic fuzzy environment and introduced the intuitionistic fuzzy Bonferroni mean (IFBM). Xia et al. [16] proposed generalized intuitionistic fuzzy Bonferroni mean. Zhou and He [35] developed some geometric Bonferroni means. Xia et al. [17] developed the geometric Bonferroni mean (GBM) based on the BM and GM and extends it to aggregate the intuitionistic fuzzy information introducing the intuitionistic fuzzy geometric Bonferroni means (IFGBM) and weighted intuitionistic fuzzy geometric Bonferroni means (WIFGBM), and proposed a method for multi-criteria decision making. However, the classical GBM and even the extended GBMs can not reflect the interrelationship between the individual criterion and other criteria. To deal with this issue, in this thesis, we developed the optimized weighted geometric Bonferroni mean (OWGBM) and the generalized optimized weighted geometric Bonferroni mean (GOWGBM), whose characteristics are to reflect the preference and interrelationship of the aggregated arguments. Furthermore, we developed the intuitionistic fuzzy optimized weighted geometric Bonferroni mean (IFOWGBM) and generalized intuitionistic fuzzy optimized weighted geometric Bonferroni mean (GIFOWGBM), and study their desirable properties such as idempotency, commutativity, monotonicity and boundedness.

The remainder of this thesis is organized as follows. In Chapter 2 we propose two GBMs inducing the OWGBM and GOWGBM. Chapter 3 extends the OWGBM and GOWGBM to aggregate the intuitionistic fuzzy information introducing the IFOWGBM and GIFOWGBM, whose properties and special cases are also studied. In Chapter 4 we develop an approach for multi-criteria decision making, and give a example to demonstrate the advantage of the presented approach. Chapter 5 ends this paper with some concluding remarks.

2 Geometric Bonferroni means

The Bonferroni mean operator was initially proposed by Bonferroni [3] and was also investigated intensively by Yager [33]:

Definition 2.1 Let $p, q \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{B}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \tag{1}$$

then $\mathrm{B}^{p,q}$ is called the Bonferroni mean (BM).

Based on the usual geometric mean (GM) and the BM, Xia et al. [17] introduced the geometric Bonferroni mean such as:

Definition 2.2 Let $p, q > 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{GB}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i + qa_j)^{\frac{1}{n(n-1)}}, \tag{2}$$

then $\mathrm{GB}^{p,q}$ is called the geometric Bonferroni mean (GBM).

Obiously, the GBM has the following properties:

(1) $\mathrm{GB}^{p,q}(0, 0, \ldots, 0) = 0$.

(2) $\mathrm{GB}^{p,q}(a, a, \ldots, a) = 0$, if $a_i = a$ for all i.

(3) $\mathrm{GB}^{p,q}(a_1, a_2, \ldots, a_n) \geq \mathrm{GB}^{p,q}(d_1, d_2, \ldots, d_n)$, i.e, $\mathrm{GB}^{p,q}$ is monotonic, if $a_i \geq d_i$ for all i.

(4) $\min_i\{a_i\} \leq \mathrm{GB}^{p,q}(a_1, a_2, \ldots, a_n) = 0 \leq \max_i\{a_i\}$.

Furthermore, if $q = 0$, then Eq. (2) reduces to the geometric mean:

$$\mathrm{GB}^{p,0}(a_1, a_2, \ldots, a_n) = \frac{1}{p} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (pa_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (a_i)^{\frac{1}{n}}. \tag{3}$$

Definition 2.3 Let $p, q, r > 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{GGB}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q+r} \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} (pa_i + qa_j + ra_k)^{\frac{1}{n(n-1)(n-2)}}, \tag{4}$$

then $\mathrm{GGB}^{p,q,r}$ is called the generalized geometric Bonferroni mean (GGBM).

The GBM and GGBM just consider the whole correlationship between the criterion a_i and all criterion and cannot reflect the interrelationship between the individual criterion a_i and other criteria a_j which is the main advantage of the GBM. To deal with these issues, in the following, we propose the optimized weighted versions of GBM and its generalized form, that is, the optimized weighted GBM (OWGBM) and the generalized optimized weighted GBM (GOWGBM). Based on the GBM, we define the following.

Definition 2.4 Let $p, q > 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector a_i $(i = 1, 2, \ldots, n)$, where w_i indicates the importance degree of a_i, satisfying $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{OWGB}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(pa_i^{w_i} + qa_j^{\frac{w_j}{1-w_i}} \right), \tag{5}$$

then $\mathrm{OWGB}^{p,q}$ is called the optimized weighted geometric Bonferroni mean (OWGBM).

Definition 2.5 Let $p, q, r > 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector a_i $(i = 1, 2, \ldots, n)$, where w_i indicates the importance degree of a_i, satisfying $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{GOWGB}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q+r} \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(pa_i^{w_i} + qa_j^{\frac{w_j}{1-w_i}} + ra_k^{\frac{w_k}{1-w_i-w_j}} \right), \tag{6}$$

then $\mathrm{GOWGB}^{p,q,r}$ is called the generalized optimized weighted geometric Bonferroni mean (GOWGBM).

Furthermore, we can transform the OWGBM and GOWGBM into the interrelationship between the OWGBM and GOWGBM forms as follows:

$$\text{OWGB}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q}\left(\prod_{i=1}^{n} pa_i^{w_i} + \prod_{\substack{j=1 \\ j\neq i}}^{n} qa_j^{\frac{w_j}{1-w_i}}\right), \tag{7}$$

$$\begin{aligned}&\text{GOWGB}^{p,q,r}(a_1, a_2, \ldots, a_n)\\ &= \frac{1}{p+q+r}\left(\prod_{i=1}^{n} pa_i^{w_i} + \prod_{\substack{j=1 \\ j\neq i}}^{n} qa_j^{\frac{w_j}{1-w_i}} + \prod_{\substack{k=1 \\ k\neq i\neq j}}^{n} ra_k^{\frac{w_k}{1-w_i-w_j}}\right).\end{aligned} \tag{8}$$

According to Eqs. (7) and (8), we see that the terms $\prod_{\substack{j=1 \\ j\neq i}}^{n} qa_j^{\frac{w_j}{1-w_i}}$ and $\prod_{\substack{j,k=1 \\ k\neq j\neq i}}^{n} \left(qa_j^{\frac{w_j}{1-w_i}} + ra_j^{\frac{w_k}{1-w_i-w_j}}\right)$, respectively, is the weighted power geometric satisfaction of all criteria except a_i, which represents the interrelationship between the individual criterion a_i and other criteria a_j $(j \neq i)$, and $\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{w_j}{1-w_i} = 1$ and $\sum_{\substack{k=1 \\ k\neq j\neq i}}^{n} \frac{w_k}{1-w_i-w_j} = 1$. If we, respectively, denote the above terms as qu_i and $qu_i + rv_i$, then

$$\text{OWGB}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q}\prod_{i=1}^{n}(pa_i^{w_i} + qu_i), \tag{9}$$

$$\text{GOWGB}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{p+q+r}\prod_{i=1}^{n}(pa_i^{w_i} + qu_i + rv_i). \tag{10}$$

3 Intuitionistic fuzzy GBMs based on OWGBM and GOWGBM

The Intuitionistic fuzzy sets (IFS) [1, 2] $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ on the set X with the condition that $\mu_A(x)$, $\nu_A(x) \geq 0$ and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ is a useful tool to express the fuzziness and uncertainty, because that it contains three parts: the membership function $\mu_A(x)$, the non-membership function $\nu_A(x)$ and the hesitant function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, which can reflect the decision makers' preference more objectively. It is noted that the hesitant function $\pi_A(x)$ is determined by the membership function $\mu_A(x)$ and the non-membership function $\nu_A(x)$, therefore, we only consider $\mu_A(x)$ and $\nu_A(x)$ in this paper. If the aggregation information in OWGBM and GOWGBM are replaced by intuitionistic fuzzy numbers (IFNs), which is the basic element of IFS and denoted by $\alpha = (\mu_\alpha, \nu_\alpha)$, where $\mu_\alpha, \nu_\alpha \geq 0$, $\mu_\alpha + \nu_\alpha \leq 1$, then we introduce two new aggregation operators in this section. Before doing this, we first introduce some basic operational laws for IFNs.:

Definition 3.1 Let $\alpha_i = (\mu_{a_i}, \nu_{a_i})$ $(i = 1, 2)$ and $\alpha = (\mu_a, \nu_a)$ be three IFNs, then we have

(1) $\alpha_1 \oplus \alpha_2 = (\mu_{a_1} + \mu_{a_2} - \mu_{a_1}\mu_{a_2}, \nu_{a_1}\nu_{a_2})$.

(2) $\alpha_1 \otimes \alpha_2 = (\mu_{a_1}\mu_{a_2}, \nu_{a_1} + \nu_{a_2} - \nu_{a_1}\nu_{a_2})$.

(3) $\lambda\alpha = (1 - (1 - \mu_a)^\lambda, \nu_a^\lambda), \lambda > 0$.

(4) $\alpha^\lambda = (\mu_a^\lambda, 1 - (1 - \nu_a)^\lambda), \lambda > 0$.

Moreover, the relations of these operational laws are given as:

(5) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$.

(6) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$.

(7) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$.

(8) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda$.

(9) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha$.

(10) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1 + \lambda_2}$.

To Rank any two IFNs $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1, 2)$, Xu and Yager [26] gave a straightforward method:

Definition 3.2 Let $s_{\alpha_i} = \mu_{\alpha_i} - \nu_{\alpha_i}$ $(i = 1, 2)$ be the scores of α_i $(i = 1, 2)$ respectively, and $h_{\alpha_i} = \mu_{\alpha_i} + \nu_{\alpha_i}$ $(i = 1, 2)$ be the accuracy degrees of α_i $(i = 1, 2)$ respectively, then

- If $s_{\alpha_1} > s_{\alpha_2}$, then α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$;
- If $s_{\alpha_1} = s_{\alpha_2}$, then

  1) if $h_{\alpha_1} = h_{\alpha_2}$, then α_1 and α_2 represent the same information, i.e., $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$, denoted by $\alpha_1 = \alpha_2$;

  2) if $h_{\alpha_1} < h_{\alpha_2}$, then α_1 is smaller than α_2, denoted by $\alpha_1 < \alpha_2$.

To aggregate the intuitionistic fuzzy correlated information, based on the OWGBM and GOWGBM, respectively, we develop two intuitionistic fuzzy GBM operators:

Definition 3.3 Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector α_i $(i = 1, 2 \ldots, n)$, where w_i indicates the importance degree of α_i, satisfying $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$.
For $p, q > 0$, if

$$\text{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{p+q} \otimes_{\substack{i,j=1 \\ i \neq j}}^{n} \left(p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}} \right), \tag{11}$$

then $\text{IFOWGB}^{p,q}$ is called the intuitionistic fuzzy optimized weighted geometric Bonferroni mean (IFOWGBM).

Based on the operational laws of IFNs in Definition 3.1, we can derive the following theorem:

Theorem 3.4 Let $p, q > 0$, $\alpha_i = (\mu_{a_i}, \nu_{a_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector α_i $(i = 1, 2, \ldots, n)$ such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregated value by using the IFOWGBM is also an IFN, and

$$\text{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

$$= \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}},\right.$$
$$\left.\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}}\right). \tag{12}$$

Proof By the operational laws (1), (3) and (4) described in Definition 3.1, we have

$$p\alpha_i^{w_i} = \left(1 - (1 - \mu_{a_i}^{w_i})^p, (1 - (1 - \nu_{\alpha_i})^{w_i})^p\right),$$
$$q\alpha_j^{\frac{w_j}{1-w_i}} = \left(1 - (1 - \mu_{\alpha_j}^{\frac{w_j}{1-w_i}})^q, (1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right) \tag{13}$$

and then

$$p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}}$$
$$= \left(1 - (1 - \mu_{a_i}^{w_i})^p + 1 - (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q - (1 - (1 - \mu_{a_i}^{w_i})^p)(1 - (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q),\right.$$
$$\left.(1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right)$$
$$= \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q, (1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right). \tag{14}$$

Let $\beta_{ij} = (\mu_{\beta_{ij}}, \nu_{\beta_{ij}})$

$$= p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}}$$
$$= \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q, (1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right),$$

then

$$\text{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{p+q} \otimes_{\substack{i,j=1 \\ i \neq j}}^{n} \left(p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}}\right)$$
$$= \frac{1}{p+q} \otimes_{\substack{i,j=1 \\ i \neq j}}^{n} \beta_{ij}. \tag{15}$$

Since

$$\otimes_{\substack{i,j=1\\ i\neq j}}^{n}\beta_{ij}=\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\mu_{\beta_{ij}},1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\nu_{\beta_{ij}})\right), \tag{16}$$

which has been proven in [26], then we replace β_{ij}, $\mu_{\beta_{ij}}$ and $\nu_{\beta_{ij}}$ by $p\alpha_i^{w_i}\oplus q\alpha_j^{\frac{w_j}{1-w_i}}$, $1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q$ and $(1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q$ in Eq. (16), respectively:

$$\begin{aligned}\otimes_{\substack{i,j=1\\ i\neq j}}^{n}\beta_{ij}=&\left(\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q\right),\right.\\ &\left.1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right)\right)\end{aligned} \tag{17}$$

and then by Eq. (17) and the operational law (3), it yields

$$\begin{aligned}&\mathrm{IFOWGB}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\ &=\frac{1}{p+q}\otimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(p\alpha_i^{w_i}\oplus q\alpha_j^{\frac{w_j}{1-w_i}}\right)\\ &=\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}},\right.\\ &\quad\left.\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_i})^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}}\right),\end{aligned} \tag{18}$$

i.e.,Eq (12) holds. In addition, since

$$0\leq\ 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}}\leq 1 \tag{19}$$

and

$$0 \leq \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_i})^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}} \leq 1, \tag{20}$$

then we have

$$\begin{aligned}
&1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_j}})^q\right)\right)^{\frac{1}{p+q}} \\
&\quad + \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_i})^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}} \\
&\quad \leq 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}} \\
&\qquad + \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}} \\
&= 1
\end{aligned} \tag{21}$$

which completes the proof of Theorem 1.

In what follows, we investigate some desirable properties of IFOWGBM:

(1) **(Idempotency)** If all α_i $(i = 1, 2, \ldots, n)$ are eqaul, i.e, $\alpha_i = \alpha = (\mu_\alpha, \nu_\alpha)$, for all i, then

$$\begin{aligned}
\text{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \text{IFOWGB}^{p,q}(\alpha, \alpha, \ldots, \alpha) \\
&= \frac{1}{p+q} \otimes_{\substack{i,j=1 \\ i \neq j}}^{n} (p\alpha^{w_i} \oplus q\alpha^{\frac{w_j}{1-w_i}}) \\
&= \frac{1}{p+q} \left((\otimes_{\substack{i=1 \\ i \neq j}}^{n} (p\alpha^{w_i})) \oplus (\otimes_{\substack{j=1 \\ j \neq i}}^{n} (q\alpha^{\frac{w_j}{1-w_i}}))\right) \\
&= \frac{1}{p+q}(p\alpha \oplus q\alpha) = \alpha.
\end{aligned} \tag{22}$$

(2) **(Commutativity)** Let $\alpha_i = (\mu_{a_i}, \nu_{a_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs. Then

$$\mathrm{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{IFOWGB}^{p,q}(\dot{\alpha_1}, \dot{\alpha_2}, \ldots, \dot{\alpha_n}), \tag{23}$$

where $(\dot{\alpha_1}, \dot{\alpha_2}, \ldots, \dot{\alpha_n})$ is any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Proof Since $(\dot{\alpha_1}, \dot{\alpha_2}, \ldots, \dot{\alpha_n})$ is any permutation of $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, then

$$\begin{aligned}\mathrm{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \frac{1}{p+q} \otimes_{\substack{i,j=1\\ i\neq j}}^{n} \left(p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}} \right) \\ &= \frac{1}{p+q} \otimes_{\substack{i,j=1\\ i\neq j}}^{n} \left(p\dot{\alpha_i}^{w_i} \oplus q\dot{\alpha_j}^{\frac{w_j}{1-w_i}} \right) \\ &= \mathrm{IFOWGB}^{p,q}(\dot{\alpha_1}, \dot{\alpha_2}, \ldots, \dot{\alpha_n}). \end{aligned} \tag{24}$$

(3) **(Monotonicity)** Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ and $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})$ $(i = 1, 2, \ldots, n)$ be two collections of IFNs. If $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ for all i, then

$$\mathrm{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathrm{IFOWGB}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n). \tag{25}$$

Proof Since $\mu_{\alpha_i} \leq \mu_{\beta_i}$ and $\nu_{\alpha_i} \geq \nu_{\beta_i}$ for all i, then

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \mu_{\alpha_i}^{w_i})^p (1 - \mu_{\alpha_j}^{\frac{w_j}{1-w_i}})^q \right) \leq \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \mu_{\beta_i}^{w_i})^p (1 - \mu_{\beta_j}^{\frac{w_j}{1-w_i}})^q \right), \tag{26}$$

$$\begin{aligned} &1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \mu_{\alpha_i}^{w_i})^p (1 - \mu_{\alpha_j}^{\frac{w_j}{1-w_i}})^q \right) \right)^{\frac{1}{p+q}} \\ &\leq 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1 - \mu_{\beta_i}^{w_i})^p (1 - \mu_{\beta_j}^{\frac{w_j}{1-w_i}})^q \right) \right)^{\frac{1}{p+q}} \end{aligned} \tag{27}$$

Similarly, we obtain

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}}$$
$$\geq\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-\nu_{\beta_i})^{w_i})^p(1-(1-\nu_{\beta_j})^{\frac{w_j}{1-w_i}})^q\right)\right)^{\frac{1}{p+q}}. \quad (28)$$

Let $\alpha = \text{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = \text{IFOWGB}^{p,q}(\beta_1, \beta_2, \ldots, \beta_n)$, and let s_α and s_β be the scores of α and β, repectively. By Eqs. (27) and (28), and Definition 3.2, we have $s_\alpha \leq s_\beta$ and thus it clear that Eq. (25) holds.

(4) **(Boundedness)** Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs, and let $\alpha^- = (\min_i\{\mu_{\alpha_i}\}, \max_i\{\nu_{\alpha_i}\})$ and $\alpha^+ = (\max_i\{\mu_{\alpha_i}\}, \min_i\{\nu_{\alpha_i}\})$, then

$$\alpha^- \leq \text{IFOWGB}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+, \quad (29)$$

which can be obtained easily by the monotonicity.

If the valued of the parameters p and q change in the IFOWGBM, then some special cases can bo obtained as follows:

Case 1. If $q \to 0$, then by Eq. (12), we have

$$\begin{aligned}\text{IFOWGB}^{p,0}(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \frac{1}{p+q} \otimes_{\substack{i,j=1\\ i\neq j}}^{n} \left(p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}}\right)\\ &= \frac{1}{p} \otimes_{i=1}^{n} (p\alpha_i^{w_i})\\ &= \left(1-\left(1-\prod_{i=1}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p\right)\right)^{\frac{1}{p}},\right.\\ &\qquad \left.\left(1-\prod_{i=1}^{n}(1-(1-(1-\nu_{\alpha_i})^{w_i})^p)\right)^{\frac{1}{p}}\right)\\ &= \text{IFOWGB}^{p,0}(\alpha_1, \alpha_2, \ldots, \alpha_n) \end{aligned} \quad (30)$$

which we call the generalized intuitionistic fuzzy weighted geometric mean (GIFWGM).

Case 2. If $p = 2$ and $q \to 0$, then by Eq. (12) is transformed as:

$$\text{IFOWGB}^{2,0}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{2} \otimes_{i=1}^{n} (2\alpha_i^{w_i})$$
$$= \left(1 - \left(1 - \prod_{i=1}^{n}\left(1 - (1 - \mu_{a_i}^{w_i})^2\right)\right)^{\frac{1}{2}},\right.$$
$$\left.\left(1 - \prod_{i=1}^{n}\left(1 - (1 - (1 - \nu_{\alpha_i})^{w_i})^2\right)\right)^{\frac{1}{2}}\right) \tag{31}$$

which we call the intuitionistic fuzzy weighted square geometric mean (IFWSGM).

Case 3. If $p = 1$ and $q \to 0$, then by Eq. (12) reduces to intuitionistic fuzzy weighted geometric mean (IFWGM) [26]:

$$\text{IFOWGB}^{1,0}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \otimes_{i=1}^{n} (\alpha_i^{w_i})$$
$$= \left(\prod_{i=1}^{n} \mu_{a_i}^{w_i}, \left(1 - \prod_{i=1}^{n}(1 - \nu_{\alpha_i})^{w_i}\right)\right). \tag{32}$$

Case 4. If $p = q = 1$, then by Eq. (12) reduces to the following:

$$\text{IFOWGB}^{1,1}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{2} \otimes_{i=1}^{n} \left(\alpha_i^{w_i} \oplus \alpha_j^{\frac{w_j}{1-w_i}}\right)$$
$$= \left(1 - \left(1 - \prod_{i=1}^{n}\left(1 - (1 - \mu_{a_i}^{w_i})(1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})\right)\right)^{\frac{1}{2}},\right.$$
$$\left.\left(1 - \prod_{i=1}^{n}\left(1 - (1 - (1 - \nu_{\alpha_i})^{w_i})(1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})\right)\right)^{\frac{1}{2}}\right). \tag{33}$$

which we call the intuitionistic fuzzy interrelated weighted square geometric mean (IFIWSGM).

The IFOWGBM operator, however, can only deal with the situation that there are correlations between any two aggregated arguments, but not the situation that there exist connections among any three aggregated arguments. To solve this issue, and motivated by Definition 2.5, we define the following:

Definition 3.5 Let $\alpha_i = (\mu_{a_i}, \nu_{a_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector α_i $(i = 1, 2 \ldots, n)$, where w_i indicates the importance degree of α_i, satisfying $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$.
For $p, q, r > 0$, if

$$\begin{aligned} &\mathrm{GIFOWGB}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ &\quad = \frac{1}{p+q+r} \otimes_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}} \oplus r\alpha_k^{\frac{w_k}{1-w_i-w_j}} \right), \end{aligned} \tag{34}$$

then $\mathrm{GIFOWGB}^{p,q,r}$ is called the generalized intuitionistic fuzzy optimized weig hted geometric Bonferroni mean (GIFOWGBM).

Similar to Theorem 3.4, we can derive the following theorem:

Theorem 3.6 Let $p, q, r > 0$, $\alpha_i = (\mu_{a_i}, \nu_{a_i})$ $(i = 1, 2, \ldots, n)$ be a collection of IFNs and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector α_i $(i = 1, 2, \ldots, n)$ such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$. Then the aggregated value by using the GIFOWGBM is also an IFN, and

$$\begin{aligned} &\mathrm{GIFOWGB}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ &= \left(1 - \left(1 - \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(1 - (1 - \mu_{a_i}^{w_i})^p (1 - \mu_{a_j}^{\frac{w_j}{1-w_i}})^q (1 - \mu_{a_k}^{\frac{w_k}{1-w_i-w_j}})^r \right) \right)^{\frac{1}{p+q+r}}, \right. \\ &\quad \left(1 - \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(1 - (1 - (1 - \nu_{\alpha_i})^{w_i})^p (1 - (1 - \nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q \right. \right. \\ &\quad \left. \left. \left. \times (1 - (1 - \nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}})^r \right) \right)^{\frac{1}{p+q+r}} \right). \end{aligned} \tag{35}$$

Proof By the operational laws (1), (3) and (4) described in Definition 3.1, we have

$$p\alpha_i^{w_i} = \left(1 - (1 - \mu_{a_i}^{w_i})^p, (1 - (1 - \nu_{\alpha_i})^{w_i})^p \right),$$

$$q\alpha_j^{\frac{w_j}{1-w_i}} = \left(1-(1-\mu_{\alpha_j}^{\frac{w_j}{1-w_i}})^q, (1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right)$$

$$r\alpha_k^{\frac{w_k}{1-w_i-w_j}} = \left(1-(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r, (1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}})^r\right) \quad (36)$$

and then

$$\begin{aligned}
&p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}} \oplus r\alpha_k^{\frac{w_k}{1-w_i-w_j}} \\
&= \left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q, (1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\right) \\
&\quad \oplus\left(1-(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r, (1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}})^r\right) \\
&= \left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r,\right. \\
&\quad \left.(1-(1-\nu_{\alpha_i})^{w_i})^p\,(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q(1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}})^r\right). \quad (37)
\end{aligned}$$

Let

$$\begin{aligned}
\beta_{ijk} &= (\mu_{\beta_{ijk}}, \nu_{\beta_{ijk}}) \\
&= p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}} \oplus r\alpha_k^{\frac{w_k}{1-w_i-w_j}} \\
&= \left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r,\right. \\
&\qquad \left.(1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q(1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}})^r\right),
\end{aligned}$$

then we have

$$\begin{aligned}
&\mathrm{GIFOWGB}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \frac{1}{p+q+r} \otimes_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} \left(p\alpha_i^{w_i} \oplus q\alpha_j^{\frac{w_j}{1-w_i}} \oplus r\alpha_k^{\frac{w_k}{1-w_i-w_j}}\right) \\
&= \frac{1}{p+q+r} \otimes_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} \beta_{ijk}. \quad (38)
\end{aligned}$$

Since

$$\begin{aligned}
\otimes_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} \beta_{ijk} &= \left(\prod_{\substack{i,j,k=1 \\ i\neq j\neq j}}^{n} \mu_{\beta_{ijk}}, 1-\prod_{\substack{i,j.k=1 \\ i\neq j\neq j}}^{n} (1-\nu_{\beta_{ijk}})\right) \\
&= \left(\prod_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{n} \left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r\right)\right.,
\end{aligned}$$

$$1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(\left(1-(1-\nu_{\alpha_i})^{w_i}\right)^p\left(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}}\right)^q\left(1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}}\right)^r\right)\Bigg), \tag{39}$$

then by Eq. (39) and the operational law (3), it yields

$$\begin{aligned}
&\mathrm{GIFOWGB}^{p,q,r}(\alpha_1,\alpha_2,\ldots,\alpha_n)\\
&\quad=\frac{1}{p+q+r}\otimes^{n}_{\substack{i,j,k=1\\ i\neq j\neq k}}\left(p\alpha_i^{w_i}\oplus q\alpha_j^{\frac{w_j}{1-w_i}}\oplus r\alpha_k^{\frac{w_k}{1-w_i-w_j}}\right)\\
&\quad=\left(1-\left(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r\right)\right)^{\frac{1}{p+q+r}},\right.\\
&\qquad\left(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-\left(1-(1-\nu_{\alpha_i})^{w_i}\right)^p\left(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}}\right)^q\right.\right.\\
&\qquad\qquad\left.\left.\left.\times\left(1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}}\right)^r\right)\right)^{\frac{1}{p+q+r}}\right),
\end{aligned}\tag{40}$$

i.e, Eq. (35) holds. In addition, since

$$0\leq 1-\left(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r\right)\right)^{\frac{1}{p+q+r}}\leq 1 \tag{41}$$

and

$$\begin{aligned}
&\left(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-\left(1-(1-\nu_{\alpha_i})^{w_i}\right)^p\left(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}}\right)^q\right.\right.\\
&\qquad\left.\left.\times\left(1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}}\right)^r\right)\right)^{\frac{1}{p+q+r}}\leq 1,
\end{aligned}\tag{42}$$

then we have

$$1-\left(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\left(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r\right)\right)^{\frac{1}{p+q+r}}$$

$$
\begin{aligned}
&+\Bigg(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\Big(1-(1-(1-\nu_{\alpha_i})^{w_i})^p(1-(1-\nu_{\alpha_j})^{\frac{w_j}{1-w_i}})^q\Big)\\
&\qquad\qquad \times(1-(1-\nu_{\alpha_k})^{\frac{w_k}{1-w_i-w_j}})^r\Big)\Bigg)^{\frac{1}{p+q+r}}\\
\leq\ & 1-\Bigg(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\Big(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r\Big)\Bigg)^{\frac{1}{p+q+r}}\\
&+\Bigg(1-\prod_{\substack{i,j,k=1\\ i\neq j\neq k}}^{n}\Big(1-(1-\mu_{a_i}^{w_i})^p(1-\mu_{a_j}^{\frac{w_j}{1-w_i}})^q(1-\mu_{\alpha_k}^{\frac{w_k}{1-w_i-w_j}})^r\Big)\Bigg)^{\frac{1}{p+q+r}}\\
&=1
\end{aligned}
\tag{43}
$$

which completes the proof of Theorem 3.6.

Similar to properties of IFOWGBM, we obtain desirable properties of GIFOWGBM as follows:

(1) **(Idempotency)** If all α_i $(i=1,2,\ldots,n)$ are eqaul, i.e, $\alpha_i=\alpha=(\mu_\alpha,\nu_\alpha)$, for all$i$, then

$$\mathrm{GIFOWGB}^{p,q,r}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\mathrm{GIFOWGB}^{p,q,r}(\alpha,\alpha,\ldots,\alpha)=\alpha. \tag{44}$$

(2) **(Commutativity)** Let $\alpha_i=(\mu_{a_i},\nu_{a_i})$ $(i=1,2,\ldots,n)$ be a collection of IFNs. Then

$$\mathrm{GIFOWGB}^{p,q,r}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\mathrm{GIFOWGB}^{p,q,r}(\dot{\alpha_1},\dot{\alpha_2},\ldots,\dot{\alpha_n}), \tag{45}$$

where $(\dot{\alpha_1},\dot{\alpha_2},\ldots,\dot{\alpha_n})$ is any permutation of $(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

(3) **(Monotonicity)** Let α_i=$(\mu_{\alpha_i},\nu_{\alpha_i})$ $(i=1,2,\ldots,n)$ and $\beta_i=(\mu_{\beta_i},\nu_{\beta_i})$ $(i=1,2,\ldots,n)$ be two collections of IFNs. If $\mu_{\alpha_i}\leq\mu_{\beta_i}$ and $\nu_{\alpha_i}\geq\nu_{\beta_i}$ for all i, then

$$\mathrm{GIFOWGB}^{p,q,r}(\alpha_1,\alpha_2,\ldots,\alpha_n)\leq\mathrm{GIFOWGB}^{p,q,r}(\beta_1,\beta_2,\ldots,\beta_n). \tag{46}$$

(4) **(Boundedness)** Let $\alpha_i=(\mu_{\alpha_i},\nu_{\alpha_i})$ $(i=1,2,\ldots,n)$ be a collection of IFNs, and let $\alpha^-=(\min_i\{\mu_{\alpha_i}\},\max_i\{\nu_{\alpha_i}\})$ and $\alpha^+=(\max_i\{\mu_{\alpha_i}\},\min_i\{\nu_{\alpha_i}\})$,

then

$$\alpha^- \le \mathrm{GIFOWGB}^{p,q,r}(\alpha_1, \alpha_2, \ldots, \alpha_n) \le \alpha^+. \tag{47}$$

4 Decision making based on intuitionistic fuzzy information

In this section, we apply the IFOWGBM or the GIIFOWGBM to multi-criteria decision making under intuitionistic fuzzy environment, which involves the following steps.

Step 1. For a multi-criteria decision making problem, let $X = \{x_1, x_2, \ldots, x_m\}$ be a set of m alternatives, and $Y = \{y_1, y_2, \ldots, y_n\}$ be a set of n criteria, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$, satisfying $w_j > 0$ $(j = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_j = 1$, where w_j denotes the importance degree of criterion y_j. The performance of the alternative x_i with respect to the criterion y_j is measured by an IFN $\alpha_{ij} = (\mu_{ij}, \nu_{ij})$, where μ_{ij} indicates the degree that the alternative x_i satisfies the criterion y_j and ν_{ij} indicates the degree that the alternative x_i does not satisfies the criterion y_j, such that $0 \leq \mu_{ij}, \nu_{ij} \leq 1$ and $\mu_{ij} + \nu_{ij} \leq 1$. All $\alpha_{ij} = (\mu_{ij}, \nu_{ij})$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$ are contained in an intuitionistic fuzzy decision matrix $A = (\alpha_{ij})_{m \times n}$ (see Table 1).

Table 1: Intuitionistic fuzzy decision matrix A

	y_1	y_2	$\cdots$	y_n
x_1	(μ_{11}, ν_{11})	(μ_{12}, ν_{12})	$\cdots$	(μ_{1n}, ν_{1n})
x_2	(μ_{21}, ν_{21})	(μ_{22}, ν_{22})	$\cdots$	(μ_{2n}, ν_{2n})
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	(μ_{m1}, ν_{m1})	(μ_{m2}, ν_{m2})	$\cdots$	(μ_{mn}, ν_{mn})

If all the criteria y_j $(j = 1, 2, \ldots, n)$ are of the same type, then the performance values do not need normalization. Whereas there are, generally, benefit criteria (the bigger the performance values the better) and cost criteria (the smaller the performance values the better) in multi-criteria decision making, in such case, we may transform the performances values of the cost type into the performance values of benefit type. Then, $A = (\alpha_{ij})_{m \times n}$ can be transformed into the matrix

$B = (\beta_{ij})_{m\times n}$, where

$$\beta_{ij} = (t_{ij}, f_{ij}) = \begin{cases} \alpha_{ij}, & \text{for benefit criterion } y_j; \\ \bar{\alpha}_{ij}, & \text{for cost criterion } y_j, \end{cases} \quad i = 1,2,\dots,m; j = 1,2,\dots,n, \tag{48}$$

where $\bar{\alpha}_{ij}$ is the complement of α_{ij} such that $\bar{\alpha}_{ij} = (\nu_{ij}, \mu_{ij})$.

Step 2. Utilize the IFOWGBM (in general, we can take $p \neq 0$ and $q \neq 0$):

$$\beta_i = (t_i, f_i) = \text{IFOWGB}^{p,q}(\beta_{i1}, \beta_{i2}, \dots, \beta_{in}) \tag{49}$$

or the GIFOWGBM (in general, we can take $p \neq 0$, $q \neq 0$ and $r \neq 0$):

$$\beta_i = (t_i, f_i) = \text{GIFOWGB}^{p,q,r}(\beta_{i1}, \beta_{i2}, \dots, \beta_{in}) \tag{50}$$

to aggregate all the performance value β_{ij} $(j = 1,2,\dots,n)$ of the ith line and get the overall performance values β_i corresponding to the alternatives x_i.

Step 3. Utilize the method in Definition 3.2 to rank the overall performance value β_i $(i = 1,2,\dots,m)$.

Step 4. Rank all the alternatives x_i $(i = 1,2,\dots,m)$ in accordance with β_i $(i = 1,2,\dots,m)$ in descending order, and then, select the most desirable alternative with the largest overall performance value.

Especially, if we do not consider the non-membership information in intuitionistic fuzzy decision making, then the usual fuzzy decision making method can be obtained as follows:

The performance of the alternative x_i with respect to the criterion y_j is measured by a usual fuzzy number α_{ij}, where $0 \leq \alpha_{ij} \leq 1$, and all the values, α_{ij} $(i = 1,2,\dots,m;\ j = 1,2,\dots,n)$, are contained in the fuzzy decision matrix $A = (\alpha_{ij})_{m\times n}$. Then we can use the OWGBM or the GOWGBM to solve this problem:

Step 1′. Transform the decision matrix $A = (\alpha_{ij})_{m\times n}$ into the normalized decision matrix $B = (\beta_{ij})_{m\times n}$, where

$$\beta_{ij} = \begin{cases} \alpha_{ij}, & \text{for benefit criterion } y_j; \\ 1-\alpha_{ij}, & \text{for cost criterion } y_j, \end{cases} \quad i = 1,2,\dots,m; j = 1,2,\dots,n,$$

(51)

Step 2′. Aggregate all the performance values β_{ij} $(j = 1, 2, \ldots, n)$ of the $i$the line, and get the overall performance values β_i corresponding to the alternative x_i by the OWGBM or the GOWGBM:

$$\beta_i = \mathrm{OWGB}^{p,q}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{in}) = \frac{1}{p+q} \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(pa_i^{w_i} + qa_j^{\frac{w_j}{1-w_i}} \right), p, q > 0 \quad (52)$$

or

$$\begin{aligned} \beta_i &= \mathrm{GOWGB}^{p,q,r}(\beta_{i1}, \beta_{i2}, \ldots, \beta_{in}) \\ &= \frac{1}{p+q+r} \prod_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} \left(pa_i^{w_i} + qa_j^{\frac{w_j}{1-w_i}} + ra_k^{\frac{w_k}{1-w_i-w_j}} \right), p, q, r > 0 \end{aligned} \quad (53)$$

Step 3′. Rank the overall performance values β_i $(i = 1, 2, \ldots, m)$ and obtain the priority of the alternatives x_i $(i = 1, 2, \ldots, m)$.

Next, we give an example to illustrate the proposed method:

Example 1. A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library (adapted from Ref. [28]). The contractor offers five feasible alternatives x_i $(i = 1, 2, 3, 4, 5)$, which might be adapted to the physical structure of the library. Suppose that three criteria: (1) y_1: economic (2) y_2: functional (3) y_3: operational, are taken into consideration in the installation problem, the weight vector of the criteria y_j $(i = 1, 2, 3)$ is $w = (0.3, 0.5, 0.2)^T$. Assume that the characteristics of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ with respect to the criteria y_j $(j = 1, 2, 3)$ are represented by IFNs $\alpha_{ij} = (\mu_{ij}, \nu_{ij})$, and all α_{ij} $(i = 1, 2, 3, 4, 5; j = 1, 2, 3)$ are contained in the intuitionistic fuzzy decision matrix $A = (\alpha_{ij})_{5\times3}$ (see Table2).

Table 2: Intuitionistic fuzzy decision matrix A

	y_1	y_2	y_3
x_1	$(0.3, 0.4)$	$(0.7, 0.2)$	$(0.5, 0.3)$
x_2	$(0.5, 0.2)$	$(0.4, 0.1)$	$(0.7, 0.1)$
x_3	$(0.4, 0.5)$	$(0.7, 0.2)$	$(0.4, 0.4)$
x_4	$(0.2, 0.6)$	$(0.8, 0.1)$	$(0.8, 0.2)$
x_5	$(0.9, 0.1)$	$(0.6, 0.3)$	$(0.2, 0.5)$

Step 1. Considering all criteria y_j $(j = 1, 2, 3)$ are the benefit criteria, the performance values of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ do not need normalization.

Step 2. Utilize the IFOWGBM (let $p = q = 1$) to aggregate all the performance values α_{ij} $(j = 1, 2, 3)$ of the ith line, and get the overall performance value α_i corresponding to the alternative x_i $(i = 1, 2, 3, 4, 5)$:

$$\alpha_1 = (0.4704, 0.3063), \quad \alpha_2 = (0.4612, 0.1343), \quad \alpha_3 = (0.4771, 0.3678)$$
$$\alpha_4 = (0.5897, 0.2678), \quad \alpha_5 = (0.5406, 0.2999).$$

Step 3. Calculate the scores of all the alternatives:

$$s_{\alpha_1} = 0.1642, \quad s_{\alpha_2} = 0.3269, \quad s_{\alpha_3} = 0.1092,$$
$$s_{\alpha_4} = 0.3219, \quad s_{\alpha_5} = 0.2406.$$

Since $s_{\alpha_2} > s_{\alpha_4} > s_{\alpha_5} > s_{\alpha_1} > s_{\alpha_3}$, then the ranking of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ is:

$$x_2 \succ x_4 \succ x_5 \succ x_1 \succ x_3.$$

If we utilize the GIFOWGBM (let $p = q = r = 1$), then by $\alpha_i = \text{GIFOWGB}^{1,1,1}(\alpha_{i1}, \alpha_{i2}, \alpha_{i3})$, we get

$$\alpha_1 = (0.6215, 0.1693), \quad \alpha_2 = (0.6559, 0.0465), \quad \alpha_3 = (0.6182, 0.2236),$$
$$\alpha_4 = (0.7092, 0.1216), \quad \alpha_5 = (0.7881, 0.1345).$$

Then we calculate the scores of all the alternatives:

$$s_{\alpha_1} = 0.4522, \quad s_{\alpha_1} = 0.6095, \quad s_{\alpha_1} = 0.3945,$$
$$s_{\alpha_1} = 0.6686, \quad s_{\alpha_1} = 0.6536.$$

Since $s_{\alpha_4} > s_{\alpha_5} > s_{\alpha_2} > s_{\alpha_1} > s_{\alpha_3}$, then the ranking of the alternatives x_i $(i = 1, 2, 3, 4, 5)$ is:

$$x_4 \succ x_5 \succ x_2 \succ x_1 \succ x_3.$$

5 Conclusions

To further develop the GBM, in this paper, we have developed the optimized weighted geometric Bonferroni mean (OWGBM) and generalized optimized weig hted geometric Bonferroni mean (GOWGBM). Then, we developed the new GBMs under the intuitionistic fuzzy environment, that is, the intuitionistic fuzzy optimized weighted geometric Bonferroni means (IFOWGBM) and the generalized intuitionistic fuzzy optimized weighted geometric Bonferroni means (GI-FOWGBM). The new GBMs can reflect the preference and interrelationship of the aggregated arguments and can satisfy the basic properties of the aggregation techniques. Some desirable properties of the IFOWGBM and GIFOWGBM are investigated. Based on the IFOWGBM and GIFOWGBM, we have proposed an approach to multi-criteria decision making with intuitionistic fuzzy information, and have also applied the proposed approach to the problem of determining what kind of air-conditioning systems should be installed in the library. The merit of the proposed approach is that it is more flexible than the classical ones because it can provide the decision makers more choices as parameters are assigned different values.

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