Regularity for semilinear hyperbolic evolution equations

(준선형 쌍곡선 발전 방정식의 규칙성)

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by
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CONTENTS

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요 약

이 논문은 코사인 족과 함께 비선형 복잡성 포함된 준선형 2계 방정식의 해의 정칙성을 다루고 주어진 방정식의 해에 대한 거듭되는 상수의 변화를 얻는다. 비선형 연산자의 Lipschitz 연속성과 주작용소의 일반적인 조건 하에 2계 쌍곡형 형태의 해에 대한 존재성과 정칙성을 고려하고, 이를 비선형방정식의 경우에 합리적이고 폭넓게 사용된다.

이 논문은 $x_0 \in D(A), y_0 \in E, k \in W^{1,2}(0,T), T > 0$ 조건 하에 $f: R \to X$ 가 연속적으로 미분가능하고 했을 때 $w \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$ 해의 존재성을 증명하였다($C(t)x$는 연속적으로 한번 미분 가능한 t의 함수이고 X상의 모든 원소 x의 부분공간의 E로서 표현한다). 새로운 기법을 사용하여 응용가능한 결과를 유도한 우수한 연구결과로 사료됨.

Regularity for semilinear hyperbolic evolution equations

Hae-Jun Hwang

Abstract

This paper deals with the regularity for solutions of semilinear second order equations contained the nonlinear convolution with cosine families and obtain a variation of constant formula for solutions of the given equations.

*Keywords:*semilinear second order equations, regularity for solutions, cosine family, sine family

AMS Classification Primary 35F25; Secondary 35K55

1 Introduction

Our objective of this paper is to investigate the regularity of solutions of the following abstract semilinear second order initial value problem

$$\begin{cases} w''(t) = Aw(t) + F(t, w) + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w'(0) = y_0 \end{cases} \tag{1.1}$$

in a Banach space X. Here, the nonlinear part is given by

$$F(t, w) = \int_0^t k(t-s)g(s, w(s))ds$$

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where k belongs to $L^2(0,T)$ and $g:[0,T]\times D(A)\to X$ is a nonlinear mapping such that $w\mapsto g(t,w)$ satisfies Lipschitz continuous. In (1.1), the principal operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t\in\mathbb{R}$.

In [1] a one-dimensional nonlinear hyperbolic equation of convolution type is considered and in [2] a hyperbolic equation of convolution type which is nonlinear in the partial differential equation part and linear in the hereditary part is treated. The existence and regularity of solutions for equations of parabolic type under some general condition of the Lipschitz continuity of the nonlinear operator is considered in [3, 4], which is reasonable and widely used in case of the nonlinear system.

In this paper, we propose an approach different from that of earlier works (see [5, 6, 7]) to study mild, strong, and classical solutions of Cauchy problems. We allow implicit arguments about L^2-regularity results for semilinear hyperbolic equations under more general hypotheses of nonlinear term F. These consequences are obtained by showing that results of the linear cases to those of [5, 8] on the L^2-regularity remain valid under the above formulation of the semilinear problem (1.1).

We will prove the existence of a solution $w\in L^2(0,T;D(A))\cap W^{1,2}(0,T;E)$ for each $T>0$ when $f:\mathbb{R}\to X$ is continuously differentiable, $x_0\in D(A), y_0\in E$, and $k\in W^{1,2}(0,T)$, $T>0$. Here, we denote by E the subspace of all $x\in X$ which $C(t)x$ is a once continuously differentiable function of t. We will make use of some of the basic ideas from cosine family referred to [9, 10] for a discussion of the results we will use. We also establish to a variation of constant formula for solutions of the second order nonlinear system (1.1) and see that the necessary assumption is more flexible than one in [11]. An example illustrated the applicability of our work is given in the last section.

2 Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas. Let X be a Banach space with norm denoted by $||\cdot||$.

Definition 2.1. *[9] A one parameter family $C(t)$, $t\in\mathbb{R}$, of bounded linear operators in X is called a strongly continuous cosine family if*

c(1) $\quad C(s+t)+C(s-t)=2C(s)C(t),\quad$ *for all* $s,\ t\in\mathbb{R}$,

c(2) $\quad C(0)=I$,

c(3) $\quad C(t)x\quad$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x\in X$.

If $C(t),\ t \in \mathbb{R}$ is a strongly continuous cosine family in X , then $S(t),\ t \in \mathbb{R}$ is the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)xds, \ \ x \in X, \ \ t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $C(t),\ t \in \mathbb{R}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(0)x.$$

We endow with the domain $D(A) = \{x \in X : C(t)x$ is a twice continuously differentiable function of $t\}$ with norm

$$||x||_{D(A)} = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\} + ||Ax||.$$

We shall also make use of the set

$$E = \{x \in X : C(t)x \text{ is a once continuously differentiable function of } t\}$$

with norm

$$||x||_E = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\}.$$

It is not difficult to show that $D(A)$ and E with given norms are Banach spaces.

The following Lemma is from Proposition 2.1 and Proposition 2.2 of [1].

Lemma 2.1. *Let $C(t)(t \in \mathbb{R})$ be a strongly continuous cosine family in X. The following are true :*

c(4) $C(t) = C(-t)$ *for all* $t \in \mathbb{R}$,

c(5) $C(s), S(s), C(t)$ *and* $S(t)$ *commute for all* $s, t \in \mathbb{R}$,

c(6) $S(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$,

c(7) *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that*

$$||C(t)|| \leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R},$$

$$||S(t_1) - S(t_2)|| \leq K\left|\int_{t_2}^{t_1} e^{\omega|s|}ds\right| \text{ for all } t_1, t_2 \in \mathbb{R},$$

c(8) *if* $x \in E$, *then* $S(t)x \in D(A)$ *and*

$$\frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

c(9) *if* $x \in D(A)$, *then* $C(t)x \in D(A)$ *and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

c(10) *if* $x \in X$ *and* $r, s \in \mathbb{R}$, *then*

$$\int_r^s S(\tau)xd\tau \in D(A) \quad \text{and} \quad A(\int_r^s S(\tau)xd\tau) = C(s)x - C(r)x,$$

c(11) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$,

c(12) $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in \mathbb{R}$,

c(13) $C(s+t) = C(t)C(s) - S(t)S(s)$ *for all* $s, t \in \mathbb{R}$,

c(14) $C(s+t) - C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$.

The following Lemma is from Proposition 2.4 of [9].

Lemma 2.2. *Let* $C(t)(t \in \mathbb{R})$ *be a strongly continuous cosine family in* X *with infinitesimal generator* A. *If* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$, $y_0 \in E$, *and*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \;\; t \in \mathbb{R},$$

then $w(t) \in D(A)$ *for* $t \in \mathbb{R}$, w *is twice continuously differentiable, and* w *satisfies*

$$w^{''}(t) = Aw(t) + f(t), \;\; t \in R, \;\; w(0) = x_0, \;\; w^{'}(0) = y_0. \tag{2.2}$$

Conversely, if $f : \mathbb{R} \to X$ *is continuous,* $w(t) : \mathbb{R} \to X$ *is twice continuously differentiable,* $w(t) \in D(A)$ *for* $t \in \mathbb{R}$, *and* w *satisfies* (2.2), *then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \;\; t \in \mathbb{R}.$$

Proposition 2.1. *Let $f : R \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$. Then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}$$

is a solution of (2.2) belonging to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$. Moreover, we have that there exists a positive constant C_1 such that for any $T > 0$,

$$||w||_{L^2(0,T;D(A))} \leq C_1(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{2.3}$$

Proof. From Lemma 2.1 it follows that w satisfies (2.2), $w(t) \in D(A)$ for $t \in \mathbb{R}$ and w is twice continuously differentiable. It is easily seen that there exists a constant $C > 0$ such that

$$||w||_{L^2(0,T;X)} \leq C(||x_0||_{D(A)} + ||y_0||_E + ||f||_{L^{1,2}(0,T;X)}). \tag{2.4}$$

To prove that $w \in L^2(0,T;D(A))$, notice that using c(7) and c(9) we have

$$\int_0^T ||AC(t)x_0||^2 dt \leq K(e^{2\omega T} - 1)||x_0||^2_{D(A)}, \tag{2.5}$$

and if $y_0 \in E$, by c(8), we have

$$\int_0^T ||AS(t)y_0||^2 dt = \int_0^T ||\frac{d}{dt}C(t)y_0||^2 dt \leq T||y_0||^2_E. \tag{2.6}$$

It is proved in Proposition 2.4 of [9] that

$$A\int_0^t S(t-s)f(s)ds = C(t)f(0) - f(0) + \int_0^t (C(t-s) - I)f'(s)ds.$$

So, noting that

$$\begin{aligned}\int_0^T ||\int_0^t C(t-s)f'(s)ds||^2 dt &\leq (Ke^{\omega T} + 1)^2 \int_0^T (\int_0^t ||f'(s)||ds)^2 dt \\ &\leq (Ke^{\omega T} + 1)^2 \int_0^T t \int_0^t ||f'(s)||^2 ds dt \\ &\leq (Ke^{\omega T} + 1)^2 \frac{T^2}{2} \int_0^T ||f'(s)||^2 ds,\end{aligned}$$

it holds that

$$\int_0^T ||A\int_0^t S(t-s)f(s)ds||^2 dt \le \int_0^T ||C(t)f(0)||^2 dt \tag{2.7}$$
$$+ T||f(0)||^2 + \int_0^T || \int_0^t C(t-s)f^{'}(s)ds||^2 dt$$
$$\le K^2 e^{2\omega T} T||f(0)||^2 + T||f(0)||^2$$
$$+ (Ke^{\omega T}+1)^2 \frac{T^2}{2}\int_0^T ||f^{'}(s)||^2 ds.$$

Noting that from c(8)

$$\frac{d}{dt}C(t)\int_0^t S(t-s)f(s)ds \tag{2.8}$$
$$= AS(t)\int_0^t S(t-s)f(s)ds = S(t)A\int_0^t S(t-s)f(s)ds,$$

we can obtain the relation of (2.3) from (2.4)-(2.8). Combing (2.1) and c(7), we also obtain that an analogous estimate to (2.3) holds for $w \in W^{1,2}(0,T;E)$. □

If f is continuously differentiable and $(x_0, y_0) \in D(A)\times E$, it is easily shown that w is continuously differentiable and satisfies

$$w^{'}(t) = AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)f(s)ds, \quad t \in \mathbb{R}.$$

Let us remark that if w is a solution of (2.2) in an interval $[0, t_1+t_2]$ with $t_1, t_2 > 0$. Then when $t \in [0, t_1+t_2]$, from c(11)-c(14), we have

$$\begin{aligned} w(t) &= C(t-t_1)w(t_1) + S(t-t_1)w^{'}(t_1) + \int_{t_1}^t S(t-s)f(s)ds \\ &= C(t-t_1)\{C(t_1)x_0 + S(t_1)y_0 + \int_0^{t_1} S(t_1-s)f(s)ds\} \\ &\quad + S(t-t_1)\{AS(t_1)x_0 + C(t_1)y_0 + \int_0^{t_1} C(t_1-s)f(s)ds\} \\ &\quad + \int_{t_1}^t S(t-s)f(s)ds \\ &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \end{aligned}$$

here, we used the relation

$$S(t)AS(s) = AS(t)S(s) = \frac{1}{2}C(t+s) - \frac{1}{2}C(t-s) = C(t+s) - C(t)C(s)$$

for all $s, t \in \mathbb{R}$. This mean the mapping $t \mapsto w(t_1 + t)$ is a solution of (2.2) in $[0, t_1 + t_2]$ with initial data $(w(t_1), w^{'}(t_1)) \in D(A) \times E$.

3 Nonlinear equations

This section is to investigate the regularity of solutions of abstract semilinear second order initial value problem

$$\begin{cases} w^{''}(t) = Aw(t) + F(t, w) + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w^{'}(0) = y_0, \end{cases} \tag{3.1}$$

in a Banach space X. Let $g : [0, T] \times D(A) \to X$ be a nonlinear mapping such that $t \mapsto g(t, w)$ is measurable and

(g1) $||g(t, w_1) - g(t, w_2)||_{D(A)} \le L||w_1 - w_2||$,

(g2) $g(t, 0) = 0$

for a positive constant L.

For $w \in L^2(0, T; D(A))$, we set

$$F(t, w) = \int_0^t k(t-s)g(s, w(s))ds$$

where k belongs to $L^2(0, T)$. Then, we will seek a mild solution of (3.1), that is, a solution of the integral equation

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s, w) + f(s)\}ds. \tag{3.2}$$

Remark 3.1. *If $g : [0, T] \times X \to X$ is a nonlinear mapping satisfying*

$$||g(t, w_1) - g(t, w_2)|| \le L||w_1 - w_2||$$

for a positive constant L, then our results can be obtained immediately.

Lemma 3.1. *Let $w \in L^2(0,T;D(A))$, $T>0$. Then $F(\cdot,w) \in L^2(0,T;X)$ and*

$$||F(\cdot,w)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w||_{L^2(0,T;D(A))}.$$

Moreover if $w_1,\ w_2 \in L^2(0,T;D(A))$, then

$$||F(\cdot,w_1)-F(\cdot,w_2)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w_1-w_2||_{L^2(0,T;D(A))}.$$

Proof. From (g1), (g2) and using the Hölder inequality, it is easily seen that

$$\begin{aligned} ||F(\cdot,w)||^2_{L^2(0,T;X)} &\le \int_0^T || \int_0^t k(t-s)g(s,w(s))ds||^2 dt \\ &\le ||k||^2_{L^2(0,T)} \int_0^T \int_0^t L^2||w(s)||^2 ds dt \\ &\le L^2||k||^2_{L^2(0,T)} T||w||^2_{L^2(0,T;D(A))}. \end{aligned}$$

The proof of the second paragraph is similar. □

Lemma 3.2. *If $k \in W^{1,2}(0,T)$, $T>0$, then*

$$\begin{aligned} A\int_0^t S(t-s)F(s,w)ds = &-F(t,w) \\ &+ \int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds \\ &+ \int_0^t (C(t-s)-I)k(0)g(s,w(s))ds. \end{aligned} \tag{3.3}$$

Proof. From the fact that

$$A\int_0^t S(t-s)F(s,w)ds = -F(t,w) + \int_0^t (C(t-s)-I)\frac{d}{ds}F(s,w)ds$$

and

$$\frac{d}{ds}F(s,w) = \int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau + k(0)g(s,w(s)),$$

it follows (3.3). □

Now we are ready to give the following result on a local solvability of (3.1).

Theorem 3.1. *Suppose that the assumptions (g1), and (g2) are satisfied. If* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$, $y_0 \in E$, *and* $k \in W^{1,2}(0,T)$, $T > 0$, *then there exists a time* $T \geq T_0 > 0$ *such that the functional differential equation (3.1) admits a unique solution* w *in* $L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)$.

Proof. Let us fix $T_0 > 0$ so that

$$\begin{aligned} C_2 \equiv &\omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)} \\ &+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}L||k||_{L^2(0,T_0)}\sqrt{T_0} \\ &+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||||k||_{W^{1,2}(0,T_0)} \\ &+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||||k(0)|| < 1 \end{aligned} \tag{3.4}$$

where K and L are constants in c(7) and (g2), respectively. Invoking Proposition 2.1, for any $v \in L^2(0,T_0;D(A))$ we obtain the equation

$$\begin{cases} w''(t) = Aw(t) + F(t,v) + f(t), & 0 < t \leq T_0, \\ w(0) = x_0, \quad w'(0) = y_0 \end{cases} \tag{3.5}$$

has a unique solution $w \in L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)$. Let $w_1,\ w_2$ be the solutions of (3.5) with v replaced by $v_1,\ v_2 \in L^2(0,T_0;D(A))$, respectively, that is

$$\begin{cases} (w_1-w_2)''(t) = A(w_1-w_2)(t) + F(t,v_1) - F(t,v_2), & 0 < t, \\ (w_1-w_2)(0) = 0, \quad (w_1-w_2)'(0) = 0. \end{cases}$$

Put

$$J(w)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s,v)+f(s)\}ds.$$

Then

$$J(w_1)(t) - J(w_2)(t) = \int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds,$$

so, from Lemmas 3.1, 3.2, it follows that for $0 \leq t \leq T_0$,

$$\begin{aligned} &||\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \\ &\leq \omega^{-1}KLT_0(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}, \end{aligned} \tag{3.6}$$

and

$$||A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \tag{3.7}$$
$$\begin{aligned}
&\le ||F(t,v_1)-F(s,v_2)|| \\
&\quad + ||\int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)(g(\tau,v_1(\tau))-g(\tau,v_2(\tau))))d\tau\, ds|| \\
&\quad + ||\int_0^t (C(t-s)-I)k(0)(g(s,v_1(s))-g(s,v_2(s)))ds|| \\
&\le ||F(t,v_1)-F(s,v_2)|| \\
&\quad + tL||Ke^{\omega t}+1||||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\
&\quad + \sqrt{t}L||Ke^{\omega t}+1||||k(0)||||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}$$

We also obtain that

$$\begin{aligned}
&||\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \\
&\le ||AS(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \\
&= ||S(t)A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||.
\end{aligned} \tag{3.8}$$

Thus, from (3.6),(3.7), and (3.8), we conclude that

$$||J(w_1)-J(w_2)||_{L^2(0,T_0;D(A))} \tag{3.9}$$
$$\begin{aligned}
&\le \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\
&\quad + \{\omega^{-1}K(e^{\omega T_0}-1)+1\}L||k||_{L^2(0,T_0)}\sqrt{T_0}||v_1-v_2||_{L^2(0,T_0;D(A))} \\
&\quad + \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2|| \\
&\quad + \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||.
\end{aligned}$$

So by virtue of the condition (3.4) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $[0,T_0]$. $\square$

Now, we give a norm estimation of the solution of (3.1) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.2. *Suppose that the assumptions (g1), and (g2) are satisfied. If* $f:\mathbb{R}\to X$ *is continuously differentiable,* $x_0\in D(A), y_0\in E$*, and* $k\in W^{1,2}(0,T)$*,* $T>0$*,* *then the solution w of (3.1) exists and is unique in $L^2(0,T;D(A))\cap W^{1,2}(0,T;E)$, and there exists a constant C_3 depending on T such that*

$$||w||_{L^2(0,T;D(A))} \le C_3(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T;X)}). \tag{3.10}$$

Proof. Let $w(\cdot)$ be the solution of (3.1) in the interval $[0,T_0]$ where T_0 is a constant in (3.4) and $v(\cdot)$ be the solution of the following equation

$$v^{''}(t) = Av(t)+f(t),\ \ 0<t,$$
$$v(0)=x_0,\ \ v^{'}(0)=y_0.$$

Then

$$(w-v)(t)=\int_0^t S(t-s)F(s,w)ds,$$

and in view of (3.9)

$$||w-v||_{L^2(0,T_0;D(A))} \le C_2||w||_{L^2(0,T_0;D(A))}, \tag{3.11}$$

that is, combining (3.11) with Proposition 2.1 we have

$$\begin{aligned} ||w||_{L^2(0,T_0;D(A))} &\le \frac{1}{1-C_2}||v||_{L^2(0,T_0;D(A))} \\ &\le \frac{C_1}{1-C_2}(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T_0;X)}). \end{aligned} \tag{3.12}$$

Now from

$$\begin{aligned} &A\int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds \\ &= C(T_0)f(0)-f(T_0)+\int_0^{T_0}(C(T_0-s)-I)f^{'}(s)ds \\ &\quad - F(T_0,w)+\int_0^{T_0}(C(T_0-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds \\ &\quad + \int_0^{T_0}(C(T_0-s)-I)k(0)g(s,w(s))ds \end{aligned}$$

and since

$$\begin{aligned} &\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,v)+f(s)\}ds \\ &= S(t)A\int_0^t S(t-s)\{F(s,v)+f(s)\}ds, \end{aligned}$$

we have

$$
\begin{aligned}
&||w(T_0)||_{D(A)} = ||C(T_0)x_0 + S(T_0)y_0 + \int_0^{T_0} S(T_0 - s)\{F(s,w) + f(s)\}ds||_{D(A)} \\
&\le (\omega^{-1}K(e^{\omega T_0} - 1) + 1)\{Ke^{\omega T_0}||x_0||_{D(A)} + \omega^{-1}K(e^{\omega T_0} - 1)||y_0||_E \\
&\quad + ||Ke^{\omega T_0} f(0)|| + ||f(T_0)|| + ||K(e^{\omega T_0} + 1)\sqrt{T_0}||f||_{W^{1,2}(0,T;X)} \\
&\quad + L||k||_{L^2(0,T_0)}||w||_{L^2(0,T_0;D(A))} \\
&\quad + tL||Ke^{\omega t} + 1||\ ||k||_{W^{1,2}(0,T_0)}||w||_{L^2(0,T_0;D(A))} \\
&\quad + \sqrt{t}L||Ke^{\omega t} + 1||\ ||k(0)||||w||_{L^2(0,T_0;D(A))}\}.
\end{aligned}
$$

Hence, from (3.12), there exists a positive constant $C > 0$ such that

$$||w(T_0)||_{D(A)} \le C(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T_0;X)}).$$

Since the condition (3.4) is independent of initial values, the solution of (3.1) can be extended to the interval $[0, nT_0]$ for every natural number n. An analogous estimate to (3.12) holds for the solution in $[0, nT_0]$, and hence for the initial value $(w(nT_0), w'(nT_0)) \in D(A) \times E$ in the interval $[nT_0, (n+1)T_0]$. $\square$

Example. Let $X = L^2([0,\pi];\mathbb{R})$. We consider the following partial differential equation

$$
\begin{cases}
w''(t,x) = Aw(t,x) + F(t,w) + f(t), \quad 0 < t, \quad 0 < x < \pi, \\
w(t,0) = w(t,\pi) = 0, \quad t \in \mathbb{R} \\
w(0,x) = x_0(x), \quad w'(0,x) = y_0(x), \quad 0 < x < \pi.
\end{cases} \tag{E}
$$

Let $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$. Then $\{e_n : n = 1, \cdots\}$ is an orthonormal base for X. Let $A : X \to X$ be defined by

$$Aw(x) = w''(x),$$

where $D(A) = \{w \in X : w,\ w'$ are absolutely continuous, $w'' \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2(w,e_n)e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w,e_n)e_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(w, e_n)e_n, \quad w \in X.$$

Let $g_1(t, x, w, p), \quad p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, r) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \leq \gamma$ such that

(f1) $g_1(t, x, 0, 0) = 0$,

(f2) $|g_1(t, x, w, p) - g_1(t, x, w, q)| \leq \rho(t, |w|)|p - q|$,

(f3) $|g_1(t, x, w_1, p) - g_1(t, x, w_2, p)| \leq \rho(t, |w_1| + |w_2|)|w_1 - w_2|$.

Let

$$g(t, w)x = g_1(t, x, w, Dw, D^2w).$$

Then noting that

$$\begin{aligned} ||g(t, w_1) - g(t, w_2)||_{0,2}^2 \leq 2 &\int_\Omega |g_1(t, w_1, p) - g_1(t, w_2, q)|^2 du \\ + 2 &\int_\Omega |g_1(t, u, w_1, q) - g_1(t, u, w_2, q)|^2 du \end{aligned}$$

where $p = (Dw_1, D^2w_1)$ and $q = (Dw_2, D^2w_2)$, it follows from (f1), (f2) and (f3) that

$$||g(t, x) - g(t, y)||_{0,2}^2 \leq L(||x||_{D(A)}, ||y||_{D(A)})||x - y||_{D(A)}$$

where $L(||w_1||_{D(A)}, ||w_2||_{D(A)})$ is a constant depending on $||w_1||_{D(A)}$ and $||w_2||_{D(A)}$. We set

$$F(t, w) = \int_0^t k(t - s)g(s, w(s))ds$$

where k belongs to $L^2(0, T)$. Then, from the results in section 3, the solution w of (E) exists and is unique in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$, and there exists a constant C_3 depending on T such that

$$||w||_{L^2(0,T;D(A))} \leq C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}).$$

References

[1] R. C. Maccamy, *A model for one-dimensional, nonlinear viscoelasticity*, Quart. Appl. Math. 35(1977), 21-33.

[2] R. C. Maccamy, *An integro-differential equation with applications in heat flow*, Ibid. 35(1977), 1-19.

[3] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Regularity for solutions of nonlinear evolution equations with nonlinear perturbations*, Comput. Math. Appl. 43(2002), 1231-1237.

[4] J. M. Jeong, Y. C. Kwun and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynam. Control Systems, l5(3)(1998), 329-346.

[5] M. Ikawa, Mixed problems for hyperbolic equations of second order, *J. Math. Soc. Japan,* **20(4)** (1968), 580–608.

[6] G. Webb, Continuous nonlinear perturbations of linear accretive operators in Banach spaces, *J. Fun. Anal.,* **10** (1972), 191–203.

[7] M. L. Heard, An abstract parabolic Volterra integro-differential equation, *J. Appl. Math.,* **17** (1981), 175–202.

[8] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Academic Press Limited, 1993.

[9] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta. Math. 32(1978), 75-96.

[10] C. C. Travis and G. F. Webb, *An abstract second order semilinear Volterra integrodifferential equation*, SIAM J. Math. Anal. 10(2)(1979), 412-423.

[11] J. Y. Park, H. K. Han and Y. C. Kwun, *Approximate controllability of second order integrodifferntial systems*, Indian J. pure appl. Math. 29(9)(1998), 941-950.