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Thesis for the Degree of  
Master of Education

Iterative Algorithms for Relatively  
Non-Lipschitzian Mappings



by

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Graduate School of Education

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August 2015

Iterative Algorithms for Relatively  
Non-Lipschitzian Mappings  
(상대적인 비-Lipschitzian 사상에 대한  
반복 알고리즘)

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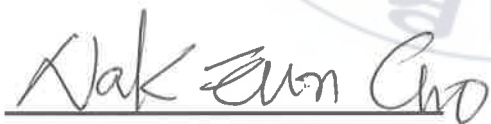
Iterative Algorithms for Relatively  
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A dissertation

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## 상대적인 비-Lipschitzian사상에 대한 반복알고리즘

### 차 해 정

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### 요 약

집합  $C$ 를 Banach 공간  $X$ 의 공집합이 아닌 닫힌볼록부분집합(closed convex subset)이라 하고, 사상  $T : C \rightarrow C$ 의 부동점들의 집합을

$$F(T) = \{x \in C : Tx = x\}$$

로 표기한다. 점  $p$ 에 약수렴하는  $C$ 내에 있는 수열  $\{x_n\}$ 이  $\|x_n - Tx_n\| \rightarrow 0$ 을 만족할 때, 점  $p \in \hat{F}(T)$ 를 점근적부동점이라 한다.  $X$ 가 smooth이면  $J$ 는  $X$ 의 정규쌍대사상으로서

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad x, y \in X$$

로 정의된 범함수  $\phi : X \times X \rightarrow \mathbb{R}$ 가 잘 정의된다. 우리는  $T : C \rightarrow C$ 가 다음 세 조건 (i)-(iii)을 만족할 때, 상대점근적비확대형사상(a mapping of relatively asymptotically nonexpansive type)라 말한다.

- (i)  $F(T) \neq \emptyset$ ,
- (ii)  $F(T) = \hat{F}(T)$ ,
- (iii) 각  $x \in C$ 에 대하여

$$\phi(p, T^n x) \leq \phi(p, x) + c_n(x), \quad \forall p \in F(T)$$

을 만족하는  $x$ 에 의존하여 0에 수렴하는 비음의 실수열  $\{c_n(x)\}$ 가 존재한다.

본 논문에서는  $X$ 가 균등볼록인 smooth한 Banach공간일 때, 먼저 한 점  $x_0 \in C$ 로부터 출발하여 다음처럼 반복적으로 정의된 새로운 혼합형 알고리즘(hybrid algorithm)

$$\begin{cases} H_n = \{z \in C : \phi(z, T^n x_n) \leq \phi(z, x_n) + c_n(x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \quad n \geq 0 \end{cases}$$

을 생각한 후,  $T : C \rightarrow C$ 가 상대점근적비확대형 균등  $k$ -Lipschitzian사상이고  $F(T) (\neq \emptyset)$ 라는 가정 하에서 위의 혼합형 알고리즘으로 정의된 수열  $\{x_n\}$ 은 항상  $F(T)$ 의 점  $\Pi_{F(T)} x_0$ 에 강수렴한다는 것을 밝혔다. 여기서  $J$ 는 정규쌍대사상(normalized duality mapping)이고  $\Pi_{F(T)}$ 는  $X$ 로부터  $F(T)$  위로의 일반화된 사영(generalized projection)이다.

# 1 Introduction

Let  $C$  be a nonempty closed convex subset of a real Banach space  $X$  and let  $T : C \rightarrow C$  be a mapping. Then  $T$  is said to be a *Lipschitzian* mapping if, for each  $n \geq 1$ , there exists a constant  $k_n > 0$  such that  $\|T^n x - T^n y\| \leq k_n \|x - y\|$  for all  $x, y \in C$ . A Lipschitzian mapping  $T$  is called *uniformly  $k$ -Lipschitzian* if  $k_n = k$  for all  $n \geq 1$ , *nonexpansive* if  $k_n = 1$  for all  $n \geq 1$ , and *asymptotically nonexpansive* [9] if  $\lim_{n \rightarrow \infty} k_n = 1$ , respectively.

On the other hand, as the classes of non-Lipschitzian mappings, there appear in the literature two definitions, one is due to Kirk who says that  $T$  is a mapping of *asymptotically nonexpansive type* [15] if for each  $x \in C$ ,

$$\limsup_{n \rightarrow \infty} \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \quad (1.1)$$

and  $T^N$  is continuous for some  $N \geq 1$ . The other is the stronger concept due to Bruck, Kuczumov and Reich [3]. They say that  $T$  is *asymptotically nonexpansive in the intermediate sense* if  $T$  is uniformly continuous and

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \quad (1.2)$$

*Remark 1.1.* In the case of (1.1), for each  $x \in C$ , if we define

$$c_n(x) := \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \vee 0,$$

(here  $a \vee b := \max\{a, b\}$ ), then  $c_n(x) \geq 0$  for all  $n \geq 1$ ,  $c_n(x) \rightarrow 0$  as  $n \rightarrow \infty$  for each  $x \in C$ , and thus (1.1) immediately reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n(x) \quad (1.3)$$

for all  $y \in C$  and  $n \geq 1$ . Observe that the converse always remains true, namely, (1.3) also implies (1.1). Indeed, (1.3) implies

$$\|T^n x - T^n y\| - \|x - y\| \leq c_n(x), \quad y \in C,$$

i.e.,  $c_n(x)$  is upper bound of  $\{\|T^n x - T^n y\| - \|x - y\| : y \in C\}$  and thus

$$\sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq c_n(x),$$

which implies (1.1) since  $c_n(x) \rightarrow 0$ . Similarly, if we define

$$c_n := \sup_{x \in C} c_n(x)$$

for each  $n \geq 1$ , note that (1.2) is equivalent to the following (1.4)

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \quad (1.4)$$

for all  $x, y \in C$  and  $n \geq 1$ , where  $\{c_n\}$  is a sequence of nonnegative real numbers such that  $c_n \rightarrow 0$  as  $n \rightarrow \infty$ .

A point  $x \in C$  is a *fixed point* of  $T$  provided  $Tx = x$ . Denote by  $F(T)$  the set of fixed points of  $T$ ; that is,  $F(T) = \{x \in C : Tx = x\}$ . A point  $p$  in  $C$  is said to be an *asymptotic fixed point* of  $T$  [24] if  $C$  contains a sequence  $\{x_n\}$  which converges weakly to  $p$  such that the strong  $\lim_{n \rightarrow \infty} (x_n - Tx_n) = 0$ . The set of asymptotic fixed points of  $T$  will be denoted by  $\hat{F}(T)$ .

Let  $X$  be a smooth Banach space and let  $X^*$  be the dual of  $X$ . The function  $\phi : X \times X \rightarrow \mathbb{R}$  is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all  $x, y \in X$ , where  $J$  is the normalized duality mapping from  $X$  to  $X^*$ . Recall that a mapping  $T : C \rightarrow C$  is *relatively asymptotically nonexpansive* (denoted simply by *RAN*) [14] if  $F(T)$  is nonempty,  $\hat{F}(T) = F(T)$  and, for each  $n \geq 1$  there exists a constant  $k_n > 0$  such that

$$\phi(p, T^n x) \leq k_n^2 \phi(p, x) \tag{1.5}$$

for  $x \in C$  and  $p \in F(T)$ , where  $\lim_{n \rightarrow \infty} k_n = 1$ . In particular,  $T$  is called *relatively nonexpansive* [19] if  $k_n = 1$  for all  $n$ ; see also [3,4,5].

Motivated and initiated by (1.3) and (1.5), we say that  $T : C \rightarrow C$  is a mapping of *relatively asymptotically nonexpansive type* (denoted

simply by *RANT*) if  $F(T)$  is nonempty,  $\hat{F}(T) = F(T)$  and, for each  $x \in C$  there exists a sequence  $\{c_n(x)\}$  of nonnegative real numbers,  $c_n(x) \rightarrow 0$ , such that

$$\phi(p, T^n x) \leq \phi(p, x) + c_n(x) \quad (1.6)$$

for all  $p \in F(T)$  and  $n \geq 1$ .

*Remark 1.2.* Observe that if  $T : C \rightarrow C$  is *RAN* and  $F(T)$  is bounded, then it is clearly a mapping of *RANT* by taking

$$c_n(x) = (k_n^2 - 1) \sup_{p \in F(T)} \phi(p, x) \rightarrow 0$$

as  $n \rightarrow \infty$  for each  $x \in C$ .

Construction of approximating fixed points of nonexpansive mappings is an important subject in the theory of nonexpansive mappings and its applications in a number of applied areas, in particular, in image recovery and signal processing. However, the sequence  $\{T^n x\}$  of iterates of the mapping  $T$  at a point  $x \in C$  may not converge even in the weak topology. Thus three averaged iteration methods often prevail to approximate a fixed point of a nonexpansive mapping  $T$ . The first one is introduced by Halpern [10] and is defined as follows: Take an initial guess  $x_0 \in C$  arbitrarily and

define  $\{x_n\}$  recursively by

$$x_{n+1} = t_n x_0 + (1 - t_n) T x_n, \quad n \geq 0, \quad (1.7)$$

where  $\{t_n\}$  is a sequence in the interval  $[0, 1]$ .

The second iteration process is now known as Mann's iteration process [17] which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \quad (1.8)$$

where the initial guess  $x_0$  is taken in  $C$  arbitrarily and the sequence  $\{\alpha_n\}$  is in the interval  $[0, 1]$ .

The third iteration process is referred to as Ishikawa's iteration process [11] which is defined recursively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T y_n, \end{cases} \quad n \geq 0, \quad (1.9)$$

where the initial guess  $x_0$  is taken in  $C$  arbitrarily and  $\{\alpha_n\}$  and  $\{\beta_n\}$  are sequences in the interval  $[0, 1]$ . By taking  $\beta_n = 1$  for all  $n \geq 0$  in (1.9), Ishikawa's iteration process reduces to the Mann's iteration process (1.8). It is known in [6] that the process (1.8) may fail to converge while the process (1.9) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space.

In general, the iteration process (1.7) has been proved to be strongly convergent in both Hilbert spaces [10, 16, 28] and uniformly smooth Banach spaces [22, 25, 30], while Mann's iteration (1.8) has only weak convergence even in a Hilbert space [8].

Attempts to modify the Mann iteration method (1.8) or the Ishikawa iteration method (1.9) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [21] proposed the following modification of Mann's iteration process (1.8) for a single nonexpansive mapping  $T$  with  $F(T) \neq \emptyset$  in a Hilbert space  $H$ :

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \quad (1.10)$$

where  $P_K$  denotes the metric projection from  $H$  onto a closed convex subset  $K$  of  $H$ . They proved that if the sequence  $\{\alpha_n\}$  is bounded above from one, then the sequence  $\{x_n\}$  generated by (1.10) converges strongly to  $P_{F(T)}x_0$ . As a special case, taking  $\alpha_n = 0$  for all  $n$ , the above iteration scheme (1.10) reduces to the following:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ C_n = \{z \in C : \|Tx_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \quad (1.11)$$

On the other hand, Kamimura and Takahashi [12] considered the problem of finding an element  $u$  of a Banach space  $X$  satisfying  $0 \in Au$ , where  $A \subset X \times X^*$  is a maximal monotone operator and  $X^*$  is the dual space of  $X$ . They studied the following algorithm:

$$\begin{cases} x_0 \in X \text{ chosen arbitrarily,} \\ 0 = v_n + \frac{1}{r_n}(Jy_n - Jx_n), \ v_n \in Ay_n, \\ H_n = \{z \in X : \langle y_n - z, v \rangle \geq 0\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases} \quad (1.12)$$

where  $J$  is the duality mapping on  $X$ ,  $\{r_n\}$  is a sequence of positive real numbers and  $\Pi_K$  denotes the generalized projection from  $X$  onto a closed convex subset  $K$  of  $X$ ; see the section 2 for more details. They proved that if  $A^{-1}0 \neq \emptyset$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ , then the sequence  $\{x_n\}$  generated by (1.12) converges strongly to an element of  $A^{-1}0$ . This generalizes the result due to Solodov and Svaiter [26] in a Hilbert space.

Recently, Matsushita and Takahashi [19] extended Nakajo and Takahashi's iteration process (1.10) to the following modification of Mann's iteration process (1.8) using the hybrid method in mathematical programming for a relatively nonexpansive mapping  $T : C \rightarrow C$  in a uniformly convex and uniformly smooth Banach space  $X$ :

$$\left\{ \begin{array}{l} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JT x_n), \\ H_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{array} \right. \quad (1.13)$$

where  $J$  is the normalized duality mapping. Then they proved that if the sequence  $\{\alpha_n\}$  is a sequence in  $[0, 1)$  and  $\limsup_{n \rightarrow \infty} \alpha_n < 1$ , then the sequence  $\{x_n\}$  generated by (1.13) converges strongly to  $\Pi_{F(T)} x_0$ , where  $\Pi_K$  denotes the generalized projection from  $X$  onto a closed convex subset  $K$  of  $X$ . As a special case, taking  $\alpha_n = 0$  for all  $n$  in (1.13), the iteration scheme reduces to the following:

$$\left\{ \begin{array}{l} x_0 \in C \text{ chosen arbitrarily,} \\ H_n = \{z \in C : \phi(z, Tx_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{array} \right. \quad (1.14)$$

which generalizes the iteration scheme (1.11) in a Hilbert spaces. Very recently, they also established that even though the condition of uniformly smooth of  $X$  is only weakened by the smooth condition of  $X$ , the sequence  $\{x_n\}$  generated by (1.14) still converges strongly to  $\Pi_{F(T)}x_0$ .

The purpose of this paper is to employ the idea due to Matsushita and Takahashi [20] to prove some strong convergence theorems for uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive in uniformly convex and smooth Banach spaces. The paper is organized as follows. In the next section we introduce some lemmas and propositions studied recently in [12] and [13, 14] which play crucial roles for our argument. In Section 3, motivated by [20], we extend Matsushita and Takahashi's iteration process (1.14) to the wider class of uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive and discuss some applications relating to our main result.

## 2 Preliminaries

Let  $X$  be a real Banach space with norm  $\|\cdot\|$  and let  $X^*$  be the dual of  $X$ . Denote by  $\langle \cdot, \cdot \rangle$  the duality product. When  $\{x_n\}$  is a

sequence in  $X$ , we denote the strong convergence of  $\{x_n\}$  to  $x \in X$  by  $x_n \rightarrow x$  and the weak convergence by  $x_n \rightharpoonup x$ . We also denote the weak  $\omega$ -limit set of  $\{x_n\}$  by  $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ . The normalized duality mapping  $J$  from  $X$  to  $X^*$  is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for  $x \in X$ .

A Banach space  $X$  is said to be *strictly convex* if  $\|(x+y)/2\| < 1$  for all  $x, y \in X$  with  $\|x\| = \|y\| = 1$  and  $x \neq y$ . It is also said to be *uniformly convex* if  $\|x_n - y_n\| \rightarrow 0$  for any two sequences  $\{x_n\}, \{y_n\}$  in  $X$  such that  $\|x_n\| = \|y_n\| = 1$  and  $\|(x_n + y_n)/2\| \rightarrow 1$ .

Let  $U = \{x \in X : \|x\| = 1\}$  be the unit sphere of  $X$ . Then the Banach space  $X$  is said to be *smooth* provided

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for each  $x, y \in U$ . It is also known that if  $X$  is uniformly smooth, then  $J$  is uniformly norm-to-norm continuous on each bounded subset of  $X$ . Some properties of the duality mapping have been given in [7, 23, 27]. A Banach space  $X$  is said to have the *Kadec-Klee* property if a sequence  $\{x_n\}$  of  $X$  satisfying that  $x_n \rightharpoonup x \in X$  and  $\|x_n\| \rightarrow \|x\|$ , then  $x_n \rightarrow x$ . It is known that if

$X$  is uniformly convex, then  $X$  has the Kadec-Klee property; see [7, 27] for more details.

Let  $X$  be a smooth Banach space. Recall that the function  $\phi : X \times X \rightarrow \mathbb{R}$  is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all  $x, y \in X$ . It is obvious from the definition of  $\phi$  that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad (2.2)$$

for all  $x, y \in X$ . Further, we have that for any  $x, y, z \in X$ ,

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, J(z) - J(y) \rangle.$$

In particular, it is easy to see that if  $X$  is strictly convex, for  $x, y \in X$ ,  $\phi(y, x) = 0$  if and only if  $y = x$  (see, for example, Remark 2.1 of [19]).

Let  $X$  be a reflexive, strictly convex and smooth Banach space and let  $C$  be a nonempty closed convex subset of  $X$ . Then, for any  $x \in X$ , there exists a unique element  $\tilde{x} \in C$  such that

$$\phi(\tilde{x}, x) = \inf_{z \in C} \phi(z, x).$$

Then a mapping  $\Pi_C : X \rightarrow C$  defined by  $\Pi_C x = \tilde{x}$  is called the *generalized projection* (see [1, 2, 12]). In Hilbert spaces, notice that

the generalized projection is clearly coincident with the metric projection.

The following result is well known (see, for example, [1, 2, 12]).

**Proposition 2.1.** ([1, 2, 12]) *Let  $K$  be a nonempty closed convex subset of a real Banach space  $X$  and let  $x \in X$ .*

(a) *If  $X$  is smooth, then,  $\tilde{x} = \Pi_K x$  if and only if  $\langle \tilde{x} - y, Jx - J\tilde{x} \rangle \geq 0$  for  $y \in K$ .*

(b) *If  $X$  is reflexive, strictly convex and smooth, then*

$$\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x)$$

*for all  $y \in K$ .*

The following subsequent two lemmas are motivated by Lemmas 1.3 and 1.5 of Martinez-Yanes and Xu [18] in Hilbert spaces, respectively; for detailed proofs, see [13].

**Lemma 2.2.** ([13]) *Let  $C$  be a nonempty closed convex subset of a smooth Banach space  $X$ ,  $x, y, z \in X$  and  $\lambda \in [0, 1]$ . Given also a real number  $a \in \mathbb{R}$ , the set*

$$D := \{v \in C : \phi(v, z) \leq \lambda\phi(v, x) + (1 - \lambda)\phi(v, y) + a\}$$

*is closed and convex.*

**Lemma 2.3.** ([13]) *Let  $X$  be a reflexive, strictly convex and smooth Banach space with the Kadec-Klee property, and let  $K$  be a nonempty closed convex subset of  $X$ . Let  $x_0 \in X$  and  $q := \Pi_K x_0$ , where  $\Pi_K$  denotes the generalized projection from  $X$  onto  $K$ . If  $\{x_n\}$  is a sequence in  $X$  such that  $\omega_w(x_n) \subset K$  and satisfies the condition*

$$\phi(x_n, x_0) \leq \phi(q, x_0)$$

*for all  $n$ . Then  $x_n \rightarrow q (= \Pi_K x_0)$ .*

Recently, Kamimura and Takahashi [12] proved the following result, which plays a crucial role in our discussion.

**Proposition 2.4.** ([12]) *Let  $X$  be a uniformly convex and smooth Banach space and let  $\{y_n\}, \{z_n\}$  be two sequences of  $X$ . If  $\phi(y_n, z_n) \rightarrow 0$  and either  $\{y_n\}$  or  $\{z_n\}$  is bounded, then  $y_n - z_n \rightarrow 0$ .*

Finally, concerning the set of fixed points of a mapping of RANT, we shall prove the following result.

**Proposition 2.5.** *Let  $X$  be a uniformly convex and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $X$ , and let  $T : C \rightarrow C$  be a continuous mapping of RANT. Then  $F(T)$  is closed and convex.*

*Proof.* First, we show that  $F(T)$  is closed. Let  $\{x_n\}$  be a sequence of  $F(T)$  such that  $x_n \rightarrow x \in C$ . Since  $T$  is a mapping of RANT, we have that

$$\phi(x_n, T^m x) \leq \phi(x_n, x) + c_m(x)$$

for each  $n, m \geq 1$ . Fix  $m \geq 1$ . Firstly taking the limit on both sides as  $n \rightarrow \infty$ , we have

$$\begin{aligned} \phi(x, T^m x) &= \lim_{n \rightarrow \infty} \phi(x_n, T^m x) \leq \lim_{n \rightarrow \infty} [\phi(x_n, x) + c_m(x)] \\ &= \phi(x, x) + c_m(x) = c_m(x). \end{aligned}$$

As taking the limit on both sides as  $m \rightarrow \infty$ , since  $c_m(x) \rightarrow 0$  as  $m \rightarrow \infty$ . It follows from Proposition 2.4 that  $T^m z \rightarrow z$  as  $m \rightarrow \infty$  and hence  $z \in F(T)$  by the continuity of  $T$ . Next, we show that  $F(T)$  is convex. For  $x, y \in F(T)$  and  $\lambda \in (0, 1)$ , put  $z = \lambda x + (1 - \lambda)y$ . It suffices to show that  $z \in F(T)$ . Indeed, as in [19], we have that for  $n \geq 1$ ,

$$\begin{aligned} \phi(z, T^n z) &= \|z\|^2 - 2\langle z, JT^n z \rangle + \|T^n z\|^2 \\ &= \|z\|^2 - 2\langle \lambda x + (1 - \lambda)y, JT^n z \rangle + \|T^n z\|^2 \\ &= \|z\|^2 - 2\lambda\langle x, JT^n z \rangle - 2(1 - \lambda)\langle y, JT^n z \rangle + \|T^n z\|^2 \\ &= \|z\|^2 + \lambda\phi(x, T^n z) + (1 - \lambda)\phi(y, T^n z) - \lambda\|x\|^2 - (1 - \lambda)\|y\|^2 \\ &\leq \|z\|^2 + [\lambda\phi(x, z) + (1 - \lambda)\phi(y, z)] + 2c_n(z) - \lambda\|x\|^2 - (1 - \lambda)\|y\|^2. \end{aligned}$$

Since  $c_n(z) \rightarrow 0$ , the right hand side of the above inequality converges to 0 because

$$\begin{aligned}
& \|z\|^2 + \lambda\phi(x, z) + (1 - \lambda)\phi(y, z) - \lambda\|x\|^2 - (1 - \lambda)\|y\|^2. \\
= & \|z\|^2 - 2\langle \lambda x + (1 - \lambda)y, Jz \rangle + \|z\|^2 \\
= & \|z\|^2 - 2\langle z, Jz \rangle + \|z\|^2 = 0
\end{aligned}$$

By Proposition 2.4 again, we have  $T^n z \rightarrow z$  and hence  $z \in F(T)$  by the continuity of  $T$ .  $\square$

### 3 Strong convergence theorems

In this section we first propose an iteration process, motivated by the idea due to [20], to have strong convergence for uniformly Lipschitzian mappings which are relatively asymptotically nonexpansive in the intermediate sense in uniformly convex and smooth Banach spaces.

**Theorem 3.1.** *Let  $X$  be a uniformly convex and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $X$  and let  $T : C \rightarrow C$  be a uniformly  $k$ -Lipschitzian mapping of RANT. As-*

sume that  $F(T) \neq \emptyset$ . Define a sequence  $\{x_n\}$  in  $C$  by the algorithm:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ H_n = \{v \in C : \phi(v, T^n x_n) \leq \phi(v, x_n) + c_n(x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0. \end{cases}$$

Then the sequence  $\{x_n\}$  converges in norm to  $\Pi_{F(T)} x_0$ .

*Proof.* First, observe that  $H_n$  is closed and convex by Lemma 2.2, and that  $W_n$  is obviously closed and convex for each  $n \geq 0$ . Next we show that  $F(T) \subset H_n$  for all  $n$ . Indeed, for all  $p \in F(T)$ , Since  $T$  is a mapping of RANT, we get

$$\phi(p, T^n x_n) \leq \phi(p, x_n) + c_n(x_n)$$

and so  $p \in H_n$ ; hence  $F(T) \subset H_n$  for all  $n \geq 0$ . Moreover, we show that

$$F(T) \subset H_n \cap W_n \tag{3.1}$$

for all  $n \geq 0$ . It suffices to show that  $F(T) \subset W_n$  for all  $n \geq 0$ . We prove this by induction. For  $n = 0$ , we have  $F(T) \subset C = W_0$ . Assume that  $F(T) \subset W_k$  for some  $k \geq 1$ . Since  $x_{k+1}$  is the generalized projection of  $x_0$  onto  $H_k \cap W_k$ , by Proposition 2.1 (a) we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1} \rangle \geq 0$$

for all  $z \in H_k \cap W_k$ . As  $F(T) \subset H_k \cap W_k$ , the last inequality holds, in particular, for all  $z \in F(T)$ . This together with the definition of  $W_{k+1}$  implies that  $F(T) \subset W_{k+1}$ . Hence (3.1) holds for all  $n \geq 0$ . So,  $\{x_n\}$  is well defined. Obviously, since  $x_n = \Pi_{W_n} x_0$  by the definition of  $W_n$  and Proposition 2.1 (a), and  $F(T) \subset W_n$ , we have  $\phi(x_n, x_0) \leq \phi(p, x_0)$  for all  $p \in F(T)$ . In particular, we obtain, for all  $n \geq 0$ ,

$$\phi(x_n, x_0) \leq \phi(q, x_0), \quad \text{where } q := \Pi_{F(T)} x_0. \quad (3.2)$$

Therefore,  $\{\phi(x_n, x_0)\}$  is bounded; so is  $\{x_n\}$  by (2.2).

Noticing that  $x_n = \Pi_{W_n} x_0$  again and the fact that  $x_{n+1} \in H_n \cap W_n \subset W_n$ , we get

$$\phi(x_n, x_0) = \min_{z \in W_n} \phi(z, x_0) \leq \phi(x_{n+1}, x_0),$$

which shows that the sequence  $\{\phi(x_n, x_0)\}$  is increasing and so the  $\lim_{n \rightarrow \infty} \phi(x_n, x_0)$  exists. Simultaneously, from Proposition 2.1 (b), we have

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{W_n} x_0) \leq \phi(x_{n+1}, x_0) - \phi(\Pi_{W_n} x_0, x_0) \\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \rightarrow 0. \end{aligned} \quad (3.3)$$

By Proposition 2.4, we have

$$\|x_{n+1} - x_n\| \rightarrow 0. \quad (3.4)$$

Now since  $x_{n+1} \in H_n$ , it follows from the definition of  $H_n$ , (3.3) and  $c_n(x_n) \rightarrow 0$  by virtue of Cantor's diagonal process that

$$\phi(x_{n+1}, T^n x_n) \leq \phi(x_{n+1}, x_n) + c_n(x_n) \rightarrow 0.$$

Using Proposition 2.4 again yields

$$\|x_{n+1} - T^n x_n\| \rightarrow 0$$

and this combined with (3.4) gives

$$\|x_n - T^n x_n\| \rightarrow 0. \quad (3.5)$$

Since  $T$  is uniformly  $k$ -Lipschitzian, it follows from (3.4) and (3.5) that

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + \|T^{n+1} x_{n+1} - T^{n+1} x_n\| + \|T^{n+1} x_n - T x_n\| \\ &\leq (1+k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1} x_{n+1}\| \\ &\quad + k\|T^n x_n - x_n\| \rightarrow 0. \end{aligned} \quad (3.6)$$

By (3.6),  $\omega_w(x_n) \subset \hat{F}(T) = F(T)$ . This, combined with (3.2) and Lemma 2.3 (with  $K = F(T)$ ), guarantees that  $x_n \rightarrow q = \Pi_{F(T)} x_0$ . The proof is complete.  $\square$

As a direct consequence of Remark 1.2 and Theorem 3.1 we have the following

**Corollary 3.2.** *Let  $X$  be a uniformly convex and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $X$  and let  $T : C \rightarrow C$  be a uniformly  $k$ -Lipschitzian mapping of RAN. Assume that  $F(T)$  is a nonempty bounded subset of  $C$ . Define a sequence  $\{x_n\}$  in  $C$  by the algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ H_n = \{v \in C : \phi(v, T^n x_n) \leq \phi(v, x_n) + c_n(x_n)\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases}$$

where  $c_n(x_n) = (k_n^2 - 1) \cdot \sup\{\phi(p, x_n) : p \in F(T)\}$ . Then the sequence  $\{x_n\}$  converges in norm to  $\Pi_{F(T)} x_0$ .

Let  $C$  be a closed convex subset of a Hilbert space  $H$  and let  $T : C \rightarrow C$  be an asymptotically nonexpansive mapping. Then, after noticing that  $\phi(x, y) = \|x - y\|^2$  for all  $x, y \in H$ , we see that  $\|T^n x - T^n y\| \leq k_n \|x - y\|$  is equivalent to  $\phi(T^n x, T^n y) \leq k_n^2 \phi(x, y)$ . It is therefore easy to show that every asymptotically nonexpansive mapping is both uniformly  $k$ -Lipschitzian and RAN. In fact, it suffices to show that  $\hat{F}(T) \subset F(T)$ . The inclusion follows easily from the well-known demiclosedness at zero of  $I - T$  (c.f., [29]), where  $I$  denotes the identity operator. Thus we have the following Hilbert

space's version of Corollary 3.2.

**Corollary 3.3.** *Let  $X$  be a Hilbert space, let  $C$  be a nonempty closed convex subset of  $X$  and let  $T : C \rightarrow C$  be an asymptotically nonexpansive mapping. Assume that  $F(T)$  is a nonempty bounded subset of  $C$ . Define a sequence  $\{x_n\}$  in  $C$  by the algorithm:*

$$\left\{ \begin{array}{l} x_0 \in C \text{ chosen arbitrarily,} \\ C_n = \{v \in C : \|v - T^n x_n\|^2 \leq \|v - x_n\|^2 + \eta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{array} \right.$$

where  $\eta_n = (k_n^2 - 1) \cdot \sup\{\|p - x_n\|^2 : p \in F(T)\}$ . Then the sequence  $\{x_n\}$  converges in norm to  $P_{F(T)} x_0$ , where  $P_{F(T)}$  is the metric projection from  $X$  onto  $F(T)$ .

Finally, as a slight modification of Theorem 3.1, we propose another iteration process to have strong convergence for uniformly Lipschitzian mappings of RANT in uniformly convex and smooth Banach spaces.

**Theorem 3.4.** *Let  $X$  be a uniformly convex and smooth Banach space, let  $C$  be a nonempty closed convex subset of  $X$  and let  $T : C \rightarrow C$  be a uniformly  $k$ -Lipschitzian mapping of RANT. Assume that  $F(T)$  is nonempty. Define a sequence  $\{x_n\}$  in  $C$  by the*

*algorithm:*

$$\left\{ \begin{array}{l} x_0 \in C \text{ chosen arbitrarily,} \\ H_n = \{v \in C : \limsup_{i \rightarrow \infty} [\phi(v, T^i x_n) - \phi(v, x_n)] \leq 0\}, \\ W_n = \{v \in C : \langle x_n - v, Jx_n - Jx_0 \rangle \leq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0. \end{array} \right.$$

Then  $\{x_n\}$  converges in norm to  $\Pi_{F(T)} x_0$ .

*Proof.* We first show that  $H_n$  is closed and convex. In fact, the closedness of  $H_n$  is obvious from the continuity of  $\phi(\cdot, x)$  for  $x \in X$ . Let us show that  $H_n$  is convex. As a matter of fact, the defining inequality in  $H_n$  is equivalent to the inequality

$$\limsup_{i \rightarrow \infty} [2\langle v, Jx_n - JT^i x_n \rangle + \|T^i x_n\|^2 - \|x_n\|^2] \leq 0.$$

Thus,  $H_n$  is clearly convex.

Next we show that  $F(T) \subset H_n$  for all  $n$ . Indeed, for all  $p \in F(T)$ , Since  $T$  is a mapping of RANT,

$$\phi(p, T^i x_n) - \phi(p, x_n) \leq c_i(x_n)$$

and taking the  $\limsup$  on the both sides as  $i \rightarrow \infty$ , the right side converges to 0 for each  $n \geq 1$  and so  $p \in H_n$ ; hence  $F(T) \subset H_n$  for all  $n \geq 0$ . Moreover, using the same processes of the proof of

Theorem 3.1, we can show that

$$F(T) \subset H_n \cap W_n \quad (3.7)$$

for all  $n \geq 0$ , and furthermore (3.2)-(3.4). Now since  $x_{n+1} \in H_n$ , from the definition of  $H_n$ , we have

$$\limsup_{i \rightarrow \infty} [\phi(x_{n+1}, T^i x_n) - \phi(x_{n+1}, x_n)] \leq 0,$$

and so

$$\lim_{n \rightarrow \infty} \limsup_{i \rightarrow \infty} \phi(x_{n+1}, T^i x_n) = 0.$$

Then it is not hard to see that there exists a  $j \in \mathbb{N} \cup \{0\}$  such that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, T^{n+j} x_n) = 0.$$

Using Proposition 2.4 again yielding

$$\|x_{n+1} - T^{n+j} x_n\| \rightarrow 0$$

and this combined with (3.5) gives

$$\|x_n - T^{n+j} x_n\| \rightarrow 0. \quad (3.8)$$

Since  $T$  is uniformly  $k$ -Lipschitzian, it follows from (3.4) and (3.8)

that

$$\begin{aligned}
\|x_n - Tx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+j+1}x_{n+1}\| \\
&\quad + \|T^{n+j+1}x_{n+1} - T^{n+j+1}x_n\| + \|T^{n+j+1}x_n - Tx_n\| \\
&\leq (1+k)\|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1+j}x_{n+1}\| \\
&\quad + k\|T^{n+j}x_n - x_n\| \rightarrow 0.
\end{aligned} \tag{3.9}$$

By (3.9),  $\omega_w(x_n) \subset \hat{F}(T) = F(T)$ . This, combined with (3.2) and Lemma 2.3 (with  $K = F(T)$ ), guarantees that  $x_n \rightarrow q = \Pi_{F(T)}x_0$ . The proof is complete.  $\square$

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