creative commons
COMMONS DEED

저작자표시-비영리-변경금지 2.0 대한민국

이용자는 아래의 조건을 따르는 경우에 한하여 자유롭게

- 이 저작물을 복제, 배포, 전송, 전시, 공연 및 방송할 수 있습니다.

다음과 같은 조건을 따라야 합니다:

저작자표시. 귀하는 원저작자를 표시하여야 합니다.

비영리. 귀하는 이 저작물을 영리 목적으로 이용할 수 없습니다.

변경금지. 귀하는 이 저작물을 개작, 변형 또는 가공할 수 없습니다.

- 귀하는, 이 저작물의 재이용이나 배포의 경우, 이 저작물에 적용된 이용허락조건을 명확하게 나타내어야 합니다.
- 저작권자로부터 별도의 허가를 받으면 이러한 조건들은 적용되지 않습니다.

저작권법에 따른 이용자의 권리는 위의 내용에 의하여 영향을 받지 않습니다.

이것은 이용허락규약(Legal Code)을 이해하기 쉽게 요약한 것입니다.

Disclaimer

dCollection

Thesis for the Degree of Doctor of Philosophy

On a Decomposition of the Curvature Tensor and Variational Problems

by

Heui-Sang Son

Department of Applied Mathematics

The Graduate School

Pukyong National University

August 2015

On a Decomposition of the Curvature Tensor and Variational Problems

곡률텐서의 분해와 변분 문제에 관한 연구

Advisor : Prof. Yong-Soo Pyo

by

Heui-Sang Son

A thesis submitted in partial fulfillment of the requirement
for the degree of

Doctor of Philosophy

in the Department of Applied Mathematics, The Graduate School,
Pukyong National University

August 2015

On a Decomposition of the Curvature Tensor and Variational Problems

A dissertation

by

Huei-Sang Son

Approved by :

(Chairman) Il Yong Lee, Ph. D.

(Member) Joon-Sik Park, Ph. D.

(Member) Sung-Jin Cho, Ph. D.

(Member) Jin Han Park, Ph. D.

(Member) Yong-Soo Pyo, Ph. D.

August 21, 2015

CONTENTS

곡률텐서의 분해와 변분 문제에 관한 연구

손 희 상

부경대학교 대학원 응용수학과

요 약

본 논문에서는 불변리만계량(invariant Riemannian metric)을 갖는 등질리만다양체(homogeneous Riemannian manifold)상의 곡률텐서(curvature tensor)의 분해, 두 Lie군 사이의 조화군동형사상(harmonic group homomorphism) 및 길이적분(length integral)의 변분(variation) 등에 대해 연구하였으며, 주요 내용은 다음과 같다.

제3장에서는 $SU(3)$-불변리만계량 $g_{(\lambda_1,\lambda_2,\lambda_3)}$를 갖는 등질리만다양체 $SU(3)/T(k,l)$상의 곡률텐서를 3개의 유사곡률텐서장(curvature-like tensor fields)으로 분해하고, 그들에 대한 기하학적 성질들을 조사하였다.

제4장에서는 좌불변계량(left invariant metric) g를 갖는 폐 Lie군 (G,g)에서 좌불변계량 h를 갖는 Lie군 (H,h)에로의 조화군동형사상을 구성하고, 이들 사이의 군동형사상 ϕ가 조화사상일 필요충분조건을 구하였다. 그리고 이를 이용하여, $(SU(2),g)$에서 Heisenberg Lie군 (H,h_0)에로의 군동형사상이 조화사상일 필요충분조건도 구하였다.

그리고 제5장에서는 리만다양체상에 주어진 측지선의 미분가능 변분의 제2변분이 임계점일 필요충분조건을 상세히 증명하였다.

Chapter 1

Introduction

In this thesis, we study on a decomposition of the curvature tensor on the homogeneous Riemannian manifold $SU(3)/T(k,l)$. Specially, we evaluate curvatures such as Ricci curvature and scalar curvature on $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant metric. Roughly speaking, variational problems in the Differential Geometry are to get critical points of functionals which are defined on some proper spaces, and then to show the stability at the critical points. So, we study on harmonic homomorphisms between two Lie groups. Concisely, a harmonic mapping is a mapping which is a critical points of given energy functional. Lastly, we treat with the first and second variation of an arbitrarily given smooth variation of a geodesic on a Riemannian manifold.

In Chapter 2, we introduce briefly the fundamental definitions and concepts in the Riemannian geometry and Lie group.

In Chapter 3, using the notion of a curvature-like tensor of type $(1,3)$ on an n-dimensional real inner product space $(V, <\ ,\ >)$, we decompose the curvature tensor (field) on the homogeneous Riemannian manifold $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant Riemannian metric $g_{(\lambda_1,\lambda_2,\lambda_3)}$ into the three curvature-like tensor fields, and investigate geometric properties. Geometric properties on $SU(3)/T(k,l)$ have been studied by many mathematicians (cf. [1, 7, 10, 13, 15, 19]).

We consider two compact Riemannian manifolds (M,g) and (N,h), and let $C^\infty(M,N)$ be the set of all smooth mappings of M into N. And as the function

E on $C^\infty(M,N)$, we take

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g,$$

where $\|d\phi\|$ is the norm of the differential $d\phi$ of a mapping $\phi \in C^\infty(M,N)$ with respect to the metrics g and h. Then by definition for any deformation ϕ_t of ϕ, $-\epsilon < t < \epsilon$, $\phi_0 = \phi$, the following statements are equivalent (cf. [16, 22, 23]):

(a) ϕ is a critical point of E.

(b) $\frac{dE(\phi_t)}{dt}|_{t=0} = 0$.

(c) ϕ is a harmonic mapping, i.e., a nonlinear sigma model.

In Chapter 4, we construct group homomorphisms of a closed (compact and connected) Lie group G with a left invariant metric g into another Lie group H with a left invariant metric h which are harmonic. First of all, we get a necessary and sufficient condition for a group homomorphism ϕ of a compact Lie group G with a left invariant metric g into another Lie group H with a left invariant metric h to be a harmonic mapping. And then, using this complete condition, we obtain a necessary and sufficient condition for a group homomorphism ϕ of $SU(2)$ with a left invariant metric g into the Heisenberg Lie group (H, h_0) to be a harmonic mapping.

In Chapter 5, we obtain a necessary and sufficient condition for the second variation of an arbitrarily given smooth variation of a geodesic on a Riemannian manifold to be 0. Here, we make a minute and detailed poof of the calculus parts which are insufficient and omitted in variation problems of the length integral.

Chapter 2

Preliminaries

In this chapter, we will introduce briefly the fundamental definitions and concepts in the Riemannian geometry and Lie group.

A (connected) Hausdorff topological space M is called a *manifold modeled to a Banach space* E if for each $p \in M$ there exist an open neighborhood U_α of p and an into homeomorphism $\alpha : U_\alpha \to E$ such that $\alpha(U_\alpha) \subset E$. A pair (U_α, α) is called a *coordinate neighborhood* in M. When a collection $\{(U_\alpha, \alpha) \mid \alpha \in A\}$ satisfies the following two conditions (1) and (2), then it is called a C^k*-coordinate system*, and M is called a C^k*-manifold modeled to a Banach space* E, or simply a *Banach manifold*.

(1) $M = \cup_{\alpha \in A} U_\alpha$,

(2) for any two coordinate neighborhoods $(U_\alpha, \alpha), (U_\beta, \beta)$ with $U_\alpha \cap U_\beta \neq \emptyset$, the mapping

$$\beta \circ \alpha^{-1} : \alpha(U_\alpha \cap U_\beta) \to \beta(U_\alpha \cap U_\beta)$$

is a C^k-diffeomorphism.

In particular, if $E = \mathbb{R}^n$, then a C^k-manifold modeled to the n-dimensional Euclidean space $\mathbb{R}^n$ is called an n*-dimensional* C^k*-manifold* (cf. [22]).

We now define various tensor spaces over a fixed vector space V. For a positive integer r, we shall call $T^r = V \otimes \cdots \otimes V$ (r times tensor product) the *contravariant tensor space of degree* r. An element of T^r will be called a *contravariant tensor of degree* r. If $r = 1$, T^1 is nothing but V. By convention, we agree that T^0 is the ground field F itself. Similarly, $T_s = V^* \otimes \cdots \otimes V^*$ (s times tensor product) is called the *contravariant tensor space of degree s* and its elements *contravariant tensors of degree s*. Then $T_1 = V^*$ and, by convention, $T_0 = F$.

And we define the (mixed) *tensor space of type* (r, s), or *tensor space of contravariant degree r* and *covariant degree s*, as the tensor product

$$T^{r,s} = T^r \otimes T_s = V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$$

with V r-times and V^* s-times. In particular, $T^{r,0} = T^r$, $T^{0,s} = T_s$ and $T^{0,0} = F$. An element of $T^{r,s}$ is called a *tensor of type* (r, s), or *tensor of contravariant degree r* and *covariant degree s* (cf. [6]).

A C^k*-vector field* on a C^{k+1}-manifold M is a C^k-section X in TM, i.e., a C^k-mapping $X : M \to T(M)$ satisfying $\pi \circ X = id$, i.e., $X(p) \in T_pM$ $(p \in M)$. The value $X(p)$ of X at p is also denoted by $X_p \in T_pM$.

For $p \in M$, let $T_p^*M := L(T_pM, \mathbb{R})$, which is a Banach space with addition, scalar multiplication, and norm. T_p^*M is called the *cotangent space* of M at p. $T^*M := \cup_{p\in M} T_p^*M$, which is also a vector bundle over M, is called the *cotangent bundle.* A C^k-cross section in T^*M is called a 1-*(differential) form.*

For $s \geq 1$, an integer, we denote by $\wedge^s T_p^*M$ $(p \in M)$, the totality of all s-tuple linear mappings

$$\omega : \underbrace{T_pM \times \cdots \times T_pM}_{s} \to \mathbb{R}$$

satisfying the condition

$$\omega(u_{\sigma(1)}, \ldots, u_{\sigma(s)}) = sign(\sigma)\omega(u_1, \ldots, u_s) \quad (u_i \in T_pM, \;\; 1 \leq i \leq s)$$

for any permutation σ of $\{1, \ldots, s\}$ and $\text{sign}(\sigma)$ is its sign. Then it is a closed subspace of Banach space $L^s(T_pM; \mathbb{R})$. Moreover

$$\wedge^s T^*(M) := \cup_{p\in M} \wedge^s T_p^*M$$

is a C^k-vector bundle over M. A C^k-cross section in $\wedge^s T^*(M)$ is called a C^k-*(differential) form.*

Let $X_1, \ldots, X_s$ be s C^k-vector fields, and let ω be a C^k-s-form. Then

$$p \ \mapsto \ \omega_p(X_1(p), \ldots, X_s(p))$$

is a C^k-function on M.

In general, considering the tensor space

$$T_p^{r,s}M := \bigotimes{}^r \, T_pM \otimes \bigotimes{}^s \, T_p^*M \quad (p \in M),$$

we get the tensor bundle $T^{r,s}M$ whose C^k-section is called a C^k-*tensor field of type* (r, s) (cf. [22]).

Let M be a C^{k+1}-manifold, and let $(H, < \ , \ >)$ be a separable Hilbert space. If M is a C^{k+1}-manifold modeled to $(H, < \ , \ >)$, then M is called a *Hilbert manifold*. Since the Euclidean space $(\mathbb{R}^n, < \ , \ >)$ is an n-dimensional Hilbert space, any n-dimensional manifold is a Hilbert manifold. If a C^k-tensor g of type (0,2) on M satisfies

(1) $g_p(u, v) = g_p(v, u) \quad (u, v \in T_pM, \ p \in M)$,

and

(2) $0 \leq g_p(u, u)$, and the equality holds if and only if $u = 0$,

then we call g a C^k-*Riemannian metric* on M, and (M, g) is called a C^k-*Riemannian manifold* (cf. [22]).

Let M be an n-dimensional C^∞-manifold, and let $(x_1, \ldots, x_n)$ be the standard coordinates of $\mathbb{R}^n$. For any local coordinate neighborhood (U_α, α) of M, where $\alpha : U_\alpha \to \mathbb{R}^n$, define a *local coordinate* $(x_1^\alpha, \ldots, x_n^\alpha)$ by

$$x_i^\alpha := x_i \circ \alpha : U_\alpha \to \mathbb{R} \quad (i = 1, \ldots, n).$$

Then each point of U_α can be uniquely expressed by the coordinate $(x_1^\alpha, \ldots, x_n^\alpha)$. We often simply write U, $(x_1, \ldots, x_n)$ for the coordinate neighborhood and its local coordinate, omitting α.

Let U_α, $(x_1^\alpha, \ldots, x_n^\alpha)$, be the local coordinate of M, and then for $p \in U_\alpha$ we denote by

$$(x_1^\alpha(p), \ldots, x_n^\alpha(p)) = (a_1, \ldots, a_n).$$

Then we consider a C^1-curve c_i through p defined by

$$c_i(t) := (a_1, \ldots, a_{i-1}, a_i + t, a_{i+1}, \ldots, a_n),$$

and we denote by $\left(\frac{\partial}{\partial x_i^\alpha}\right)_p$, its tangent vector $c_i'(0)$ at p. Then

$$\left\{\left(\frac{\partial}{\partial x_1^\alpha}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n^\alpha}\right)_p\right\}$$

is a basis of the tangent space T_pM of M at p. A C^∞-vector field X on M is written on U_α as

$$X_p = \sum_{i=1}^n \xi_i^\alpha(p) \left(\frac{\partial}{\partial x_i^\alpha}\right)_p \qquad (p \in U_\alpha),$$

where ξ_i^α, $i = 1, \ldots, n$, are in $C^\infty(U_\alpha)$. Moreover, if we take another coordinate neighborhood U_β, $(x_1^\beta, \ldots, x_n^\beta)$ and denote also on U_β,

$$X = \sum_{i=1}^n \xi_i^\beta \left(\frac{\partial}{\partial x_i^\beta}\right),$$

then both $(\xi_1^\alpha, \ldots, \xi_n^\alpha)$, $(\xi_1^\beta, \ldots, \xi_n^\beta)$ satisfy

$$\xi_i^\beta = \sum_{j=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha} \xi_j^\alpha$$

on $U_\alpha \cap U_\beta$. We denote by $\mathfrak{X}(M)$, the totality of all C^∞-vector fields on M (cf. [22]).

A *connection* (*covariant differentiation*) ∇ on a C^∞-manifold M is a mapping

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \ni (X, Y) \ \mapsto \ \nabla_X Y \in \mathfrak{X}(M)$$

satisfying the following conditions:

(1) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,

(2) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$,

(3) $\nabla_{fX} Y = f\nabla_X Y$,

(4) $\nabla_X(fY) = (Xf)Y + f\nabla_X Z$

for $f \in C^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Due to (3), it turns out that the value $(\nabla_X Y)_p \in T_pM$ of $\nabla_X Y$ at $p \in M$ depends only on $X_p \in T_pM$ and Y.

Let (M, g) be an n-dimensional C^∞-Riemannian manifold. Then a connection ∇ (called the *Levi-Civita connection*) can be given as follows (cf. [22]);

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z])$$

for $X, Y, Z \in \mathfrak{X}(M)$. Moreover, the connection ∇ satisfies

(1) $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,

and

(2) $\nabla_X Y - \nabla_Y X - [X,Y] = 0$.

Let G be a C^∞-manifold with a countable base. If G is a group, and if the mapping $(x, y) \mapsto xy$ from the product manifold $G \times G$ to G and the mapping $x \mapsto x^{-1}$ from G to G are both differentiable, then G is called a *Lie group*.

If a vector field X on a Lie group G satisfies

$$(L_g)_* X = X$$

for all $g \in G$, then X is called a *left invariant vector filed*. If X satisfies

$$(R_g)_* X = X$$

for all $g \in G$, then X is called a *right invariant vector field*.

The Lie algebra $\mathfrak{g}$ formed by the set of all left invariant vector fields on G is called the *Lie algebra of the Lie group* G (cf. [8]).

For $X \in \mathfrak{g}$, set

$$\exp tX = (\mathrm{Exp}\ tX)(e).$$

The mapping $X \mapsto \exp X$ is a mapping from $\mathfrak{g}$ to G, and is called the *exponential mapping*. $\exp tX$ is a one-parameter subgroup of G, and we have

$$\exp(t+s)X = \exp(tX)\exp(sX), \quad R_{\exp tX} = \mathrm{Exp}\ tX.$$

We have

$$[X,Y]_g = \lim_{t\to 0}\frac{1}{t}\{Y_g - ((R_{\exp tX})_* Y)_g\} \quad (X, Y \in \mathfrak{g}).$$

Since, for an arbitrary element g of G, we have $A_g = R_{g^{-1}}L_g$, it follows that

$$A_{g*}Y = R_{g^{-1}} * Y$$

for $Y \in \mathfrak{g}$. However, since L_h and $R_{g^{-1}}$ commute, we have

$$L_{h*}(R_{g^{-1}*}Y) = R_{g^{-1}*}(L_{h*}Y) = R_{g^{-1}*}Y,$$

so that $R_{g^{-1}*}Y \in \mathfrak{g}$. Hence $A_{g*}Y \in \mathfrak{g}$ for $g \in \mathfrak{g}$. The mapping $Y \mapsto A_{g*}Y$ is a linear transformation of the vector space $\mathfrak{g}$, and we denote this linear transformation by $\mathrm{Ad}(g)$. That is,

$$\mathrm{Ad}(g)Y = A_{g*}Y = R_{g^{-1}*}Y \quad (g \in G,\ Y \in \mathfrak{g}).$$

Furthermore, since $A_{gh} = A_g A_h$, we have

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h)$$

for any two elements g, h of G. In particular, it is clear from the definition that $\mathrm{Ad}(e)$ is the identity transformation 1 of the vector space $\mathfrak{g}$. Hence we have $\mathrm{Ad}(g^{-1})\mathrm{Ad}(g) = 1$. Hence $\mathrm{Ad}(g)$ is a nonsingular linear transformation of $\mathfrak{g}$, and

$$\mathrm{Ad}(g)^{-1} = \mathrm{Ad}(g^{-1})$$

holds. The mapping $g \mapsto \mathrm{Ad}(g)$ is called the *adjoint representation* of the Lie group G (cf. [8]).

Let us denote by $M(n,\mathbb{R})$ the totality of all $n \times n$ real matrices, and let

$$GL(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) \mid \det A \neq 0\},$$

where $\det A$ is the determinant of A. Then $GL(n,\mathbb{R})$ is an open submanifold of $M(n,\mathbb{R}) = \mathbb{R}^{n^2}$. It can be shown that $GL(n,\mathbb{R})$ is a Lie group, i.e., the mapping defined by

$$GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \ni (A,B) \mapsto AB \in GL(n,\mathbb{R}),$$

$$GL(n,\mathbb{R}) \ni A \mapsto A^{-1} \in GL(n,\mathbb{R})$$

are both C^∞ mappings. The mappings $GL(n,\mathbb{R}) \ni A = (a_{ij}) \mapsto a_{ij}$ $(1 \leq i,j \leq n)$, give the coordinates of $GL(n,\mathbb{R})$. The groups

$$O(n) := \{A \in M(n,\mathbb{R}) \mid {}^tAA = A{}^tA = I\}$$

and

$$SO(n) := \{A \in O(n) \mid \det A = 1\}$$

are compact closed Lie subgroups, called the *orthogonal*, the *special orthogonal groups*, respectively. Here we denote by ${}^tA$ the transposed matrix, and I is the unit matrix. The Lie algebra of both $O(n)$ and $SO(n)$ is

$$\mathfrak{so}(n) := \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid {}^tA + A = O\},$$

where $\mathfrak{gl}(n,\mathbb{R})$ is the Lie algebra of $GL(n,\mathbb{R})$.

And, we denote by $M(n,\mathbb{C})$ the totality of all $n \times n$ complex matrices. Then

$$U(n) := \{Z \in M(n,\mathbb{C}) \mid {}^t\bar{Z}Z = Z{}^t\bar{Z} = I\}$$

and

$$SU(n) := \{Z \in U(n) \mid \det Z = 1\}$$

are both compact Lie groups, called the *unitary* and the *special unitary groups*, respectively. Here $\bar{Z}$ implies the complex conjugate of $Z \in M(n, \mathbb{C})$. The Lie algebras of $U(n)$ and $SU(n)$ are

$$\mathfrak{u}(n) := \{Z \in M(n, \mathbb{C}) \mid {}^t\bar{Z} + Z = O\}$$

and

$$\mathfrak{su}(n) := \{Z \in \mathfrak{u}(n) \mid \mathrm{tr} Z = 0\},$$

where $\mathrm{tr}Z$ is the trace of $Z \in M(n, \mathbb{C})$ (cf. [22]).

Chapter 3

A decomposition of the curvature tensor

In this chapter, we decompose the curvature tensor (field) on the homogeneous Riemannian manifold $SU(3)/T(k,l)$ with an arbitrarily given $SU(3)$-invariant Riemannian metric into three curvature-like tensor fields, and investigate geometric properties.

3.1 Introduction

Let $(V, <\ ,\ >)$ be an n-dimensional real inner product space. In this chapter, we use the notion of a curvature-like tensor of type $(1,3)$ on $(V, <\ ,\ >)$ (cf. (3.2.1)). We put

$$\mathfrak{L}(V) := \{L \mid L \text{ is a curvature-like tensor on } (V, <\ ,\ >)\},$$

$$\mathfrak{L}_1(V) := \{L \in \mathfrak{L}(V) \mid L(u,v) = c\ u \wedge v \text{ for } u, v \in V \text{ and some } \ c \in \mathbb{R}\},$$

$$\mathfrak{L}_\omega(V) := \{L \in \mathfrak{L}(V) \mid \text{the Ricci tensor } Ric_L \text{ of } L \text{ is zero}\},$$

$$\mathfrak{L}_2(V) := \{L \in \mathfrak{L}_1(V)^\perp \mid\ < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_\omega(V)\}.$$

Then $\mathfrak{L}(V)$ is decomposed into the orthogonal direct sum $\mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V)$. Let $L = L_1 + L_\omega + L_2$ $(L \in \mathfrak{L}(V))$ be the decomposition corresponding to $\mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V)$. The component L_ω of $L \in \mathfrak{L}(V)$ is said to be the *Weyl tensor* of L. The curvature-like tensors L_1, L_ω, L_2 of $L = L_1 + L_\omega + L_2 \in \mathfrak{L}(V)$ are given in terms of the Ricci tensor Ric_L and the scalar curvature S_L of L (cf. Lemma 3.2.1).

In this chapter, using Lemma 3.2.1 we decompose the curvature tensor (field) on the homogeneous Riemannian manifold $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ into three curvature-like tensor fields. On the manifold $SU(3)/T(k,l)$, we deal with an arbitrary $SU(3)$-invariant Riemannian metric $g = g_{(\lambda_1,\lambda_2,\lambda_3)}$.

Now, let R be the curvature tensor (field) on the homogeneous manifold $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$, and $R = R^{(1)} + R^{\omega} + R^{(2)}$ the orthogonal decomposition of the curvature tensor R corresponding to

$$\mathfrak{L}(T_o(G/H)) = \mathfrak{L}_1(T_o(G/H)) \oplus \mathfrak{L}_\omega(T_o(G/H)) \oplus \mathfrak{L}_2(T_o(G/H))$$

(cf. Lemma 3.2.1), where $G := SU(3)$, $H := T(k,l)$ and $O := \{T(k,l)\}$.

Let $\mathfrak{m}$ be the subspace of $\mathfrak{su}(3)$ such that

$$B(\mathfrak{m}, \mathfrak{t}(k,l)) = 0 \text{ and } \mathrm{Ad}(h)\mathfrak{m} \subset \mathfrak{m} \quad (h \in T(k,l)),$$

where $\mathfrak{su}(3)$ is the Lie algebra of $SU(3)$, B is the negative of the Killing form of $\mathfrak{su}(3)$, $\mathfrak{t}(k,l)$ is the Lie algebra of $T(k,l)$, and Ad is the adjoint representation of $SU(3)$ on $\mathfrak{su}(3)$.

In this chapter, we represent the curvature-like tensors $R^{(1)}$, R^{ω} and $R^{(2)}$ in the orthogonal decomposition $R = R^{(1)} + R^{\omega} + R^{(2)}$ $(\in \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V))$ of the curvature tensor R on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ for $(k,l) \in D$, where

$$D := \mathbb{Z}^2 \setminus \{(0,t), (t,0), (t,t), (t,-t), (t,-2t), (2t,-t) \mid t \in \mathbb{R}\}$$

(cf. Theorem 3.4.3). And then, under the condition $(k,l) \in D \subset \mathbb{Z}^2$, we obtain the Ricci tensor $Ric^{(2)}$ of the component $R^{(2)}$ of the curvature $R = R^{(1)} + R^{\omega} + R^{(2)}$ on the homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ (cf. Corollary 3.4.4). Furthermore, we estimate the Ricci curvature $r^{(2)}$ of the curvature-like tensor $R^{(2)}$ (cf. Proposition 3.4.5).

3.2 Preliminaries

Let $(V, < \ , \ >)$ be an n-dimensional real inner product space and $\mathfrak{gl}(V)$ the vector space of all endomorphisms of V. We denote by $\mathfrak{L}(V)$ the vector space of all tensors of type $(1, 3)$ on V which satisfy the following properties:

$$L : V \times V \ \rightarrow \ \mathfrak{gl}(V)$$

is an $\mathbb{R}$-bilinear mapping such that, for all $v_1, v_2, v_3, v_4 \in V$,

$$(3.2.1) \qquad \begin{aligned} &< L(v_1, v_2)v_3, v_4 >= - < L(v_2, v_1)v_3, v_4 >= - < L(v_1, v_2)v_4, v_3 >, \\ &< L(v_1, v_2)v_3, v_4 > + < L(v_2, v_3)v_1, v_4 > + < L(v_3, v_1)v_2, v_4 >= 0. \end{aligned}$$

A tensor $L \in \mathfrak{L}(V)$ (of type $(1, 3)$ on $(V, < \ , \ >)$ which satisfies the condition (3.2.1)) is called a *curvature-like tensor* (cf. [3, 4]). If $L \in \mathfrak{L}(V)$, then we get from (3.2.1)

$$(3.2.2) \qquad < L(v_1, v_2)v_3, v_4 >=< L(v_3, v_4)v_1, v_2 > \quad (v_1, v_2, v_3, v_4 \in V).$$

From now on, let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of $(V, < \ , \ >)$. The *Ricci tensor* Ric_L of type $(0, 2)$ with respect to a curvature-like tensor L on V is defined by

$$(3.2.3) \qquad Ric_L(v, w) := \sum_{i=1}^{n} < L(e_i, v)w, e_i > \quad (v, w \in V).$$

The *Ricci tensor* Ric_L of type $(1, 1)$ with respect to $L \in \mathfrak{L}(V)$ is defined by

$$(3.2.4) \qquad < \mathrm{Ric}_L(v), w >= Ric_L(v, w) \quad (v, w \in V).$$

For $L \in \mathfrak{L}(V)$, we obtain from (3.2.1) $\sim$ (3.2.4)

$$Ric_L(v, w) =< \mathrm{Ric}_L(v), w >= Ric_L(w, v) =< \mathrm{Ric}_L(w), v >$$

for $v, w \in V$.

The trace of Ric_L for $L \in \mathfrak{L}(V)$

$$(3.2.5) \qquad S_L := \sum_{i=1}^{n} < \mathrm{Ric}_L(e_i), e_i >= \sum_{i,j=1}^{n} < L(e_j, e_i)e_i, e_j >$$

is called the *scalar curvature* with respect to $L \in \mathfrak{L}(V)$. The *sectional curvature* $K_L(\sigma)$ $(L \in \mathfrak{L}(V))$ for each plane $\sigma = \{v, w\}_{\mathbb{R}}$ $(\subset V)$ is defined by

$$K_L(\sigma) = \frac{< L(v,w)w, v >}{< v, v >< w, w > - < v, w >^2}.$$

In general, the inner product $< \, , \, >$ on $\mathfrak{L}(V)$ is defined by

$$(3.2.6) \qquad < L, L' >= \sum_{i,j,k,l=1}^{n} {L_{ijk}}^{l} \cdot {L'_{ijk}}^{l},$$

where ${L_{ijk}}^{l} =< L(e_i, e_j)e_k, e_l >$.

Let $\mathfrak{L}_1(V)$ be the subspace of $\mathfrak{L}(V)$ which consists of all elements $L \in \mathfrak{L}(V)$ such that

$$L(v, w) = c \; v \wedge w \text{ for } v, w \in V \text{ and some } c \in \mathbb{R}.$$

Here $v \wedge w$ is an element of $\mathfrak{gl}(V)$ which is defined by

$$(v \wedge w)(z) =< w, z > v- < v, z > w.$$

We put

$$\mathfrak{L}_1(V)^{\perp} := \{L \in \mathfrak{L}(V) \mid \; < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_1(V)\}.$$

Then $\mathfrak{L}_1(V)^{\perp} = \{L \in \mathfrak{L}(V) \mid S_L = 0\}$. In fact, for $L \in \mathfrak{L}(V)$ and $L' \in \mathfrak{L}_1(V)$, we get from (3.2.5) and (3.2.6), and the definition of $\mathfrak{L}_1(V)$

$$(3.2.7) \qquad < L, L' >= 2c \; S_L,$$

where $L'(v, w) = c \; v \wedge w$ for some $c \in \mathbb{R}$. From (3.2.7), we obtain the following;

$$\begin{aligned} < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_1(V) &\iff 2c \; S_L = 0 \text{ for all } c \in \mathbb{R} \\ &\iff S_L = 0. \end{aligned}$$

Putting

$$\{L \in \mathfrak{L}_1(V)^\perp \mid \mathrm{Ric}_L = 0\} =: \mathfrak{L}_\omega(V)$$

and

$$\{L \in \mathfrak{L}_1(V)^\perp \mid \ < L, L' >= 0 \text{ for all } L' \in \mathfrak{L}_\omega(V)\} =: \mathfrak{L}_2(V),$$

we get the orthogonal direct sum decomposition of $\mathfrak{L}(V)$ as follows:

$$\mathfrak{L}(V) = \mathfrak{L}_1(V) \oplus \mathfrak{L}_\omega(V) \oplus \mathfrak{L}_2(V).$$

Putting together the results above, we obtain the following (cf. [6])

Lemma 3.2.1. *Let V be an $n(\geq 3)$-dimensional real inner product space and $L \in \mathfrak{L}(V)$. Then components $L_1 \in \mathfrak{L}_1(V)$, $L_\omega \in \mathfrak{L}_\omega(V)$ and $L_2 \in \mathfrak{L}_2(V)$ of $L(= L_1 + L_\omega + L_2)$ are given as follows*:

$$\begin{aligned} L_1(u,v) &= \frac{S_L}{n(n-1)}\, u \wedge v, \\ L_\omega(u,v) &= L(u.v) \\ &\quad - \frac{1}{n-2}\left\{\mathrm{Ric}_L(u) \wedge v + u \wedge \mathrm{Ric}_L(v) - \frac{S_L}{n-1}\, u \wedge v\right\}, \\ L_2(u,v) &= \frac{1}{n-2}\left\{\mathrm{Ric}_L(u) \wedge v + u \wedge \mathrm{Ric}_L(v) - \frac{2S_L}{n}\, u \wedge v\right\}. \end{aligned} \tag{3.2.8}$$

Proof. The fact that L_1, L_2, L_ω appeared in (3.2.8) belong to $\mathfrak{L}(V)$ is easily verified. And, $L = L_1 + L_\omega + L_2$. Moreover from straightforward computations we get

$$S_{L_2} = 0, \quad \mathrm{Ric}_{L_\omega} = 0, \quad < L_2, L_\omega >= 0.$$

Thus the proof of Lemma 3.2.1 is completed.

3.3 Inequivalent isotropy irreducible representations in $SU(3)/T(k,l)$

3.3.1 Isotropy irreducible representations

Let G be a compact connected semisimple Lie group and H a closed subgroup of G. The homogeneous space G/H is *reductive*, that is, in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sun of vector subspaces) and $\mathrm{Ad}(h)\ \mathfrak{m} \subset \mathfrak{m}$ for all $h \in H$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ corresponding to the identity component H_o of H and $\mathrm{Ad}(h)$ denotes the adjoint representation of H in $\mathfrak{m}$.

Let τ_x $(x \in G)$ be the transformation of G/H which is induced by x. Taking differentials of τ_x at $p_o := \{H\}$ $(\in G/H)$, we obtain the fact that the tangent space $T_{p_o}(G/H) = \mathfrak{m}$ is $\mathrm{Ad}(H)$-invariant. The homogeneous space G/H is said to be *isotropy irreducible* if $(T_{p_o}(G/H), \mathrm{Ad}(H))$ is an irreducible representation.

3.3.2 Inequivalent isotropy irreducible summands in $SU(3)/T(k,l)$

Here and from now on, without further specification, we use the following notations:

$$
\begin{aligned}
&G := SU(3), \quad \mathfrak{g} : \text{the Lie algebra of } SU(3), \quad i = \sqrt{-1}, \\
&H := T(k,l) = \{diag[e^{2\pi i k\theta}, e^{2\pi i l\theta}, e^{-2\pi i(k+l)\theta} \mid \theta \in \mathbb{R}\} \text{ for } (k,l) \in \mathbb{Z}^2 \\
&\qquad\qquad \text{and } |k| + |l| \neq 0, \\
&\mathfrak{t}(k,l) : \text{the Lie algebra of } T(k,l), \quad \gamma = k^2 + kl + l^2, \\
&(X,Y)_0 = B(X,Y) = -6\ \mathrm{tr}(XY),\ X, Y \in \mathfrak{g} : \text{the negative of the} \\
&\qquad\qquad \text{Killing form of } \mathfrak{g}.
\end{aligned}
$$

Let E_{ij} be a real 3×3 matrix with 1 on entry (i,j) and 0 elsewhere. And we put

$$
\begin{aligned}
X_1 &= \frac{1}{\sqrt{12}}(E_{12} - E_{21}), \qquad X_2 = \frac{i}{\sqrt{12}}(E_{12} + E_{21}), \\
X_3 &= \frac{1}{\sqrt{12}}(E_{13} - E_{31}), \qquad X_4 = \frac{i}{\sqrt{12}}(E_{13} + E_{31}), \\
X_5 &= \frac{1}{\sqrt{12}}(E_{23} - E_{32}), \qquad X_6 = \frac{i}{\sqrt{12}}(E_{23} + E_{32}), \\
X_7 &= \frac{i}{\sqrt{36\gamma}}\ diag[(k+2l), -(2k+l), (k-l)], \\
X_8 &= \frac{i}{\sqrt{12\gamma}}\ diag[k, l, -(k+l)].
\end{aligned} \tag{3.3.1}
$$

Then

$$\{X_1, \ldots, X_7\} \quad (\text{resp. } \{X_8\})$$

is an orthonormal basis of $\mathfrak{m}$ (resp. $\mathfrak{t}(k,l)$) with respect to $(\cdot\ ,\ \cdot)_0$ such that

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{t}(k,l) \text{ and } (\mathfrak{m}, \mathfrak{t}(k,l))_0 = 0.$$

If we put $\{X_1, X_2\}_{\mathbb{R}} = \mathfrak{m}_1$, $\{X_3, X_4\}_{\mathbb{R}} = \mathfrak{m}_2$, $\{X_5, X_6\}_{\mathbb{R}} = \mathfrak{m}_3$, and $\{X_7\}_{\mathbb{R}} = \mathfrak{m}_4$, then $\mathfrak{m}_i$ are irreducible $\mathrm{Ad}(T)$-representation spaces.

In general, two representations (μ_1, V_1) and (μ_2, V_2) of a Lie group G are called *equivalent* if there exists a linear isomorphism ρ of V_1 onto V_2 such that $\rho \circ \mu_1(x) = \mu_2(x) \circ \rho$ for all $x \in G$.

Park (cf. [13]) obtained the following

Theorem 3.3.1. *Assume that* $|k| + |l| \neq 0$ $(k, l \in \mathbb{Z})$. *Then a necessary and sufficient condition for* $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l)))$ $(i = 1,2,3,4)$ *to be mutually inequivalent is*

$$k \neq 0, \quad l \neq 0, \quad k \neq \pm l, \quad k \neq -2l \quad \textit{and} \quad l \neq -2k.$$

3.4 A decomposition of the curvature tensor on $SU(3)/T(k,l)$

3.4.1 The curvature tensor field on a homogeneous Riemannian space

Let G be a compact connected semisimple Lie group and H a closed subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras of G and H, respectively. Let B be the negative of the Killing form of $\mathfrak{g}$. We consider the $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $B(\mathfrak{h}, \mathfrak{m}) = 0$. Then the set of G-invariant symmetric covariant 2-tensor fields on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant Riemannian metrics on G/H is identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$ (cf. [2, 6, 10, 13]).

Let $< \, , \, >$ be an inner product which is invariant with respect to $\mathrm{Ad}(H)$ on $\mathfrak{m}$, where Ad denotes the adjoint representation of H in $\mathfrak{g}$. This inner product $< \, , \, >$ determines a G-invariant Riemannian metric $g_{<,>}$ on G/H.

For the sake of the calculus, we take a neighborhood V of the identity element e in G and a subset N (resp. N_H) of G (resp. H) in such a way that

(i) $N = V \cap \exp(\mathfrak{m})$, $\;N_H = V \cap \exp(\mathfrak{h})$,

(ii) the mapping $N \times N_H \ni (c, h) \mapsto ch \in N \cdot N_H$ is a diffeomorphism,

(iii) the projection π of G onto G/H is a diffeomorphism of N onto a neighborhood $\pi(N)$ of the origin $\{H\}$ in G/H. Here, $\{\exp(tX) \mid t \in \mathbb{R}\}$ for $X \in \mathfrak{g}$ is a 1-parameter subgroup of G.

Now for an element $X \in \mathfrak{m}$, we define a vector field X^* on the neighborhood $\pi(N)$ of $\{H\}$ in G/H by

$$X^*_{\pi(c)} := (\tau_c)_* X_{\{H\}} \in T_{\pi(c)} G/H \quad (c \in N),$$

where τ_c denotes the transformation of G/H which is induced by c. Let $\{X_i\}_i$ be an orthonormal basis of the inner product space $(\mathfrak{m}, < \ , \ >)$. Then $\{X_i\}_i$ is an orthonormal frame on $\pi(N)(\subset G/H)$.

On the other hand, the connection function α (cf. [9]) on $\mathfrak{m} \times \mathfrak{m}$ corresponding to the invariant Riemannian connection of $(G/H, g_{< \ , \ >})$ is given as follows:

$$\alpha(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y) \quad (X, Y \in \mathfrak{m}),$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > = < [Z,X]_{\mathfrak{m}}, Y > + < X, [Z,Y]_{\mathfrak{m}} >$$

for $X, Y, Z \in \mathfrak{m}$, and $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Let ∇ be the Levi-Civita connection on the Riemannian manifold $(G/H, g_{< \ , \ >})$. Then on $\pi(N)$ $(\nabla_{X^*} Y^*)_{\{H\}} = \alpha(X,Y)$ $(X, Y \in \mathfrak{m})$. Moreover, the expression for the value at $p_o := \{H\}(\in G/H)$ of the curvature tensor field is as follows (cf. [9]):

$$\begin{aligned} R(X,Y)Z &= \alpha(X, \alpha(Y,Z)) - \alpha(Y, \alpha(X,Z)) \\ &\qquad - \alpha([X,Y]_{\mathfrak{m}}, Z) - [[X,Y]_{\mathfrak{h}}, Z] \quad (X, Y, Z \in \mathfrak{m}), \end{aligned} \tag{3.4.1}$$

where $X_{\mathfrak{m}}$ (resp. $X_{\mathfrak{h}}$) denotes the $\mathfrak{m}$ - component (resp. $\mathfrak{h}$ -component) of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

In general, the Ricci tensor field Ric of type (0,2) on a Riemannian manifold (M, g) is defined by

$$Ric(Y,Z) = \mathrm{tr}\{X \mapsto R(X,Y)Z\} \quad (X, Y, Z \in \mathfrak{X}(M)). \tag{3.4.2}$$

Let $\{Y_j\}_j$ be an orthonormal basis of the inner product $(\mathfrak{m}, < \ , \ >)$. Since the group G is unimodular, we obtain the fact (cf. [2]) that

$$\sum_j U(Y_j, Y_j) = 0. \tag{3.4.3}$$

Using (3.4.1), (3.4.2) and (3.4.3), we obtain the following expression (cf. [2]) for the value at p_o of the Ricci tensor field Ric on $(G/H, g_{< \, , \, >})$:

$$\begin{aligned} Ric(Y,Y) = &-\frac{1}{2}\sum_j < [Y,Y_j]_{\mathfrak{m}}, [Y,Y_j]_{\mathfrak{m}} > + \frac{1}{2}B(Y,Y) \\ &+ \frac{1}{4}\sum_{i,j} < [Y_i,Y_j]_{\mathfrak{m}}, Y >^2 \end{aligned} \tag{3.4.4}$$

for $Y \in \mathfrak{m}$, where B is the negative of the Killing form of the Lie algebra $\mathfrak{g}$.

3.4.2 Ricci tensor fields on inequivalent isotropy irreducible homogeneous spaces

We retain the notation as in Subsection 3.4.1. The set of G-invariant symmetric tensor fields of type $(0,2)$ on G/H can be identified with the set of $\mathrm{Ad}(H)$-invariant symmetric bilinear forms on $\mathfrak{m}$. In particular, the set of G-invariant metrics on G/H is identified with the set of $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{m}$.

Let $(\, , \,)_o$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ such that $(\mathfrak{m}, \mathfrak{h})_o = 0$. For the sake of simplicity, we put $(\, , \,)_o =: B$. Let $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_q$ be an orthogonal $\mathrm{Ad}(H)$-invariant decomposition of the space $(\mathfrak{m}, B)$ such that $\mathrm{Ad}(H)_{\mathfrak{m}_i}$ is irreducible for $i = 1, \ldots, q$, and assume that $(\mathfrak{m}_i, \mathrm{Ad}(H))$ are mutually inequivalent irreducible representations. Then, the space of G-invariant symmetric tensor fields of type $(0,2)$ on G/H is given by

$$\{\lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_q B|_{\mathfrak{m}_q} \mid \lambda_1, \ldots, \lambda_q \in \mathbb{R}\},$$

and the space of G-invariant Riemannian metrics on G/H is given by

$$\{\lambda_1 B|_{\mathfrak{m}_1} + \cdots + \lambda_q B|_{\mathfrak{m}_q} \mid \lambda_1 > 0, \ldots, \lambda_q > 0\}. \tag{3.4.5}$$

In fact, for an arbitrarily given $\mathrm{Ad}(H)$-invariant inner product $< \, , \, >$ on $\mathfrak{m}$, we have $< \, , \, > |_{\mathfrak{m}_i} = \lambda_i B|_{\mathfrak{m}_i}$ on each $\mathfrak{m}_i$ by the help of Shur's lemma (cf. [23, 24]), and $< \mathfrak{m}_i, \mathfrak{m}_j >= 0$ for i, j $(i \neq j)$ since $(\mathfrak{m}_i, \mathrm{Ad}(H))$ are mutually inequivalent (cf. [10, 13, 20]).

Note that the Ricci tensor field Ric of a G-invariant Riemannian metric on G/H is a G-invariant symmetric tensor field of type $(0, 2)$ on G/H, and we identify Ric with an $\mathrm{Ad}(H)$-invariant symmetric bilinear form on $\mathfrak{m}$. Thus, if $(\mathfrak{m}_i, \mathrm{Ad}(H))$ are mutually inequivalent irreducible representations, then Ric is written as

$$Ric = y_1 B|_{\mathfrak{m}_i} + \cdots + y_q B|_{\mathfrak{m}_q} \tag{3.4.6}$$

for some $y_1, \ldots, y_q \in \mathbb{R}$.

3.4.3 The Ricci tensor field and the scalar curvature on $SU(3)/T(k,l)$ with a $SU(3)$-invariant metric

We retain the notation as in Subsection 3.4.2. In this subsection, we assume that the isotropy irreducible representations $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l)))$ $(i = 1, 2, 3, 4;\ k, l \in \mathbb{Z})$ are mutually inequivalent. For the sake of simplicity, we put

$$D := \mathbb{Z}^2 \setminus \{(0,t), (t,0), (t,t), (t,-t), (t,-2t), (2t,-t) \mid t \in \mathbb{Z}\}.$$

Let $(\ ,\ )_0$ be the negative of the Killing form of $\mathfrak{su}(3)$, and $<\ ,\ >$ an arbitrarily given $\mathrm{Ad}(T(k,l))$-invariant inner product on $\mathfrak{m}$. By Theorem 3.3.1, we obtain the fact that the isotropy irreducible representations $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l))$ $(i = 1, 2, 3, 4;\ k, l \in \mathbb{Z})$ are mutually inequivalent if and only if (k, l) in $T(k, l)$ belongs to D. Since $(\mathfrak{m}_i, \mathrm{Ad}(T(k,l))$ are mutually inequivalent, for the inner product $<\ ,\ >$ on $\mathfrak{m}$ there are corresponding positive numbers λ_1, λ_2, λ_3 and λ_4 such that

$$\begin{aligned} \{&X_1/\sqrt{\lambda_1} =: Y_1, \quad X_2/\sqrt{\lambda_1} =: Y_2, \quad X_3/\sqrt{\lambda_2} =: Y_3, \\ &X_4/\sqrt{\lambda_2} =: Y_4, \quad X_5/\sqrt{\lambda_3} =: Y_5, \quad X_6/\sqrt{\lambda_3} =: Y_6, \\ &X_7/\sqrt{\lambda_4} =: Y_7\} \end{aligned} \tag{3.4.7}$$

is an orthonormal basis of $\mathfrak{m}$ with respect to the inner product $< , >$, by virtue of (3.3.1), Theorem 3.3.1 and (3.4.5). This inner product $< , >$ determines a $SU(3)$-invariant Riemannian metric $g_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)}$ on $SU(3)/T(k,l)$.

Now we normalize $SU(3)$-invariant Riemannian metrics on $SU(3)/T(k,l)$ by putting $\lambda_4 = 1$, and denote by $g_{(\lambda_1,\lambda_2,\lambda_3)}$ the metric defined by

$$\lambda_1 B|_{\mathfrak{m}_1} + \lambda_2 B|_{\mathfrak{m}_2} + \lambda_3 B|_{\mathfrak{m}_3} + B|_{\mathfrak{m}_4}.$$

By virtue of (3.3.1), (3.4.4), (3.4.6) and (3.4.7), we obtain the following result (cf. [13]).

Lemma 3.4.1. *Assume that $(k,l) \in D$. Then the Ricci tensor Ric on the Riemannian homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ is given as follows*:

$$\begin{aligned}
&Ric(Y_i, Y_j) = 0 \quad (i \neq j),\\
&Ric(Y_1, Y_1) = Ric(Y_2, Y_2) = \frac{{\lambda_1}^2 - {\lambda_2}^2 - {\lambda_3}^2 + 6\lambda_2\lambda_3}{12\lambda_1\lambda_2\lambda_3} - \frac{(k+l)^2}{8\gamma{\lambda_1}^2},\\
&Ric(Y_3, Y_3) = Ric(Y_4, Y_4) = \frac{{\lambda_2}^2 - {\lambda_3}^2 - {\lambda_1}^2 + 6\lambda_3\lambda_1}{12\lambda_1\lambda_2\lambda_3} - \frac{l^2}{8\gamma{\lambda_2}^2},\\
&Ric(Y_5, Y_5) = Ric(Y_6, Y_6) = \frac{{\lambda_3}^2 - {\lambda_1}^2 - {\lambda_2}^2 + 6\lambda_1\lambda_2}{12\lambda_1\lambda_2\lambda_3} - \frac{k^2}{8\gamma{\lambda_3}^2},\\
&Ric(Y_7, Y_7) = \frac{1}{8\gamma}\left\{\frac{(k+l)^2}{{\lambda_1}^2} + \frac{l^2}{{\lambda_2}^2} + \frac{k^2}{{\lambda_3}^2}\right\},
\end{aligned}$$

where $\gamma := k^2 + kl + l^2$.

The trace of the Ricci tensor *Ric* of a Riemannian manifold (M, g), (i.e., $\sum_j Ric(e_j, e_j)$, where $\{e_j\}_j$ is a (locally defined) orthonormal frame on (M, g)), is called the *scalar curvature* of (M, g).

By virtue of Lemma 3.4.1, we get (cf. [13])

Lemma 3.4.2. *The scalar curvature* $S_{(\lambda_1,\lambda_2,\lambda_3)}$ *of the Riemannian homogeneous space* $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$, $(k,l) \in D$, *is given as follows*:

$$\begin{aligned} S_{(\lambda_1,\lambda_2,\lambda_3)} =& \frac{-({\lambda_1}^2+{\lambda_2}^2+{\lambda_3}^2)+6(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)}{6\lambda_1\lambda_2\lambda_3} \\ &- \frac{1}{8\gamma}\left\{\frac{(k+l)^2}{{\lambda_1}^2}+\frac{l^2}{{\lambda_2}^2}+\frac{k^2}{{\lambda_3}^2}\right\}, \end{aligned}$$

where $\gamma := k^2 + kl + l^2$.

3.4.4 A decomposition of the curvature tensor field on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$

We retain the notation as in Section 3.1 and Subsection 3.4.3. Let ∇ be the Levi-Civita connection on the homogeneous space $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ and ${}^{\nabla}R$ the curvature tensor field with respect to ∇.

For the sake of convenience, we use the following notations:

$$V := T_{\{T(k,l)\}}(SU(3)/T(k,l)), \quad (V, <\ ,\ >) := (V, g_{(\lambda_1,\lambda_2,\lambda_3)}|_V), \quad R := {}^{\nabla}R.$$

Then, the curvature tensor R at $p_o(=\{T(k,l)\})$ of $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$ is uniquely decomposed as

$$\begin{aligned} R =& R^{(1)} + R^{\omega} + R^{(2)} \\ &(R^{(1)} \in \mathfrak{L}_1(V),\ R^{\omega} \in \mathfrak{L}_{\omega}(V),\ R^{(2)} \in \mathfrak{L}_2(V)). \end{aligned} \tag{3.4.8}$$

The curvature-like tensor R^{ω} appeared in (3.4.8) is said to be the *Weyl tensor* (*field*) of the curvature tensor field R on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$.

Then, by virtue of (3.2.8), Lemmas 3.4.1 and 3.4.2, we obtain

Theorem 3.4.3. *Let* $R^{(1)}$, R^{ω} *and* $R^{(2)}$ *be the the curvature-like tensors appeared in the curvature tensor* $R = R^{(1)} + R^{\omega} + R^{(2)}$ $(\in \mathfrak{L}_1(V) \oplus \mathfrak{L}_{\omega}(V) \oplus \mathfrak{L}_2(V))$

on $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$. *If* $(k,l) \in D$, *then*

$$R^{(1)}(Y_i,Y_j) = \frac{1}{42} S_{(\lambda_1,\lambda_2,\lambda_3)} Y_i \wedge Y_j,$$

$$R^{\omega}(Y_i,Y_j) = R(Y_i,Y_j) - \frac{1}{5}\{\mathrm{Ric}(Y_i)\wedge Y_j + Y_i \wedge \mathrm{Ric}(Y_j)\} + \frac{1}{30} S_{(\lambda_1,\lambda_2,\lambda_3)} Y_i \wedge Y_j,$$

$$R^{(2)}(Y_i,Y_j) = \frac{1}{5}\{\mathrm{Ric}(Y_i)\wedge Y_j + Y_i \wedge \mathrm{Ric}(Y_j)\} - \frac{2}{35} S_{(\lambda_1,\lambda_2,\lambda_3)} Y_i \wedge Y_j,$$

where $\{Y_i\}_{i=1}^7$ *is an orthonormal basis on* $(\mathfrak{m}, < \ , \ >)$ *and* $S_{(\lambda_1,\lambda_2,\lambda_3)}$ *is the scalar curvature of* $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$.

In general, the *Ricci curvature* r of a Riemannian manifold (M,g) with respect to a nonzero vector $v \in TM$ is defined by

$$r(v) = \frac{Ric(v,v)}{\|v\|_g^2}.$$

From Theorem 3.4.3, we get

Corollary 3.4.4. *Let* $R^{(2)}$ *be the curvature-like tensor appeared in the curvature tensor* $R = R^{(1)} + R^{\omega} + R^{(2)}$ *on* $(SU(3)/T(k,l), g_{(\lambda_1,\lambda_2,\lambda_3)})$, *where* $(k,l) \in D$. *Then the Ricci tensor of* $R^{(2)}$ *is given as follows*:

$$Ric^{(2)}(Y_i,Y_j) = -\frac{1}{7} S_{(\lambda_1,\lambda_2,\lambda_3)}\, \delta_{ij} + Ric(Y_i,Y_j).$$

By the help of Lemma 3.4.1 and Corollary 3.4.4, we obtain

Proposition 3.4.5. *Assume that* $(k,l) \in D$, $k > l > 0$, *and*

$$\lambda \le \frac{3l^2}{10(k^2+kl+l^2)}$$

in $(SU(3)/T(k,l), g_{(\lambda,\lambda,\lambda)})$, $\lambda > 0$. *Then the Ricci curvature* $r^{(2)}$ *of the curvature-like tensor* $R^{(2)}$ *in the curvature tensor* $R = R^{(1)} + R^{\omega} + R^{(2)}$ *on* $(SU(3)/T(k,l),$ $g_{(\lambda,\lambda,\lambda)})$ *is estimated as follows:*

$$r^{(2)}(Y_1) = r^{(2)}(Y_2) \le r^{(2)} \le r^{(2)}(Y_7),$$

where $r^{(2)}(Y_i) = Ric^{(2)}(Y_i,Y_i)$ *for* $i = 1,2,\ldots,7$.

Chapter 4

Harmonic homomorphisms between two Lie groups

In this chapter, we get a complete condition for a group homomorphism of a compact Lie group with an arbitrarily given left invariant Riemannian metric into another Lie group with a left invariant metric to be a harmonic mapping, and then obtain a necessary and sufficient condition for a group homomorphism of $(SU(2), g)$ with a left invariant metric g into the Heisenberg group (H, h_0) to be a harmonic mapping.

4.1 Introduction

Harmonic mappings of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) are the extrema of the energy functional (cf. [22])

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g,$$

where $\|d\phi\|$ is the norm of the differential $d\phi$ of a mapping $\phi \in C^\infty(M, N)$ with respect to the metrics g, h.

In this chapter, we construct group homomorphisms of a closed (compact and connected) Lie group G with a left invariant metric g into another Lie group H with a left invariant metric h which are harmonic.

It is well known that every inner automorphism of a Lie group G into itself is both isomorphic and harmonic with respect to a bi-invariant Riemannian metric g_0 on G.

However, we here deal with a group homomorphism between two Lie groups with arbitrarily given left invariant metrics.

First of all, we get a necessary and sufficient condition (cf. Proposition 4.2.1) for a group homomorphism ϕ of a compact Lie group G with a left invariant metric g into another Lie group H with a left invariant metric h to be a harmonic mapping.

And then, using this complete condition, we obtain a necessary and sufficient condition for a group homomorphism ϕ of $SU(2)$ $(= G)$ with a left invariant metric g into the Heisenberg Lie group (H, h_0) (cf. [12, 14]) to be a harmonic mapping.

4.2 Harmonic group homomorphisms

Let (M, g), (N, h) be two Riemannian manifolds of dimension n, m, respectively. Let $\phi : M \to N$ be a smooth mapping and let $E := \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N. We denote by $\Gamma(E)$, the space of all sections V of E, that is, $V \in \Gamma(E)$ implies that V is a mapping of M into E such that $V_x \in T_{\phi(x)}N$ for all $x \in M$. For $X \in \Gamma(TM)$, we define $\phi_* X \in \Gamma(E)$ by $(\phi_* X)_x := \phi_{*x} X_x \in T_{\phi(x)}N$ $(x \in M)$, where ϕ_{*x} is the differential of ϕ at x. For $Y \in \Gamma(TN)$, we also define $\tilde{Y} \in \Gamma(E)$ by $\tilde{Y}_x := Y_{\phi(x)}$ $(x \in M)$.

We denote ∇, ${}^N\nabla$ the Levi-Civita connections of (M, g), (N, h), respectively. Then we give the induced connection $\tilde{\nabla}$ on E (cf. [5, 6]) by

$$(\tilde{\nabla}_X V)_x := \frac{d}{dt}\, {}^N P_{\phi(\gamma(t))}^{-1}\, V_{\gamma(t)}|_{t=o} \quad (X \in \Gamma(TM),\ V \in \Gamma(E)),$$

where $x \in M$, $\gamma(t)$ is a curve through x at $t = 0$ whose tangent vector at x is X_x, and ${}^N P_{\phi(\gamma(t))} : T_{\phi(x)}N \to T_{\phi(\gamma(t))}N$ is the parallel displacement along a curve $\phi(\gamma(s))$ $(0 \le s \le t)$ given by the Levi-Civita connection ${}^N\nabla$ of (N, h)).

We define a tension field $\tau(\phi) \in \Gamma(E)$ of ϕ by

$$\tau(\phi) := \sum_{i=1}^{n} \left(\tilde{\nabla}_{\mathbf{e}_i} \phi_* \mathbf{e}_i - \phi_* \nabla_{\mathbf{e}_i} \mathbf{e}_i \right), \tag{4.2.1}$$

where $\{\mathbf{e}_i\}_{i=1}^n$ is a (locally defined) orthonormal frame field on M. We call ϕ to be a *harmonic mapping* if $\tau(\phi) = 0$ on M.

Let G be an n-dimensional closed (compact and connected) Lie group with an arbitrarily given left invariant metric g, and H an m-dimensional Lie group with a left invariant metric h. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of all left invariant vector fields on G (resp. H). Let $\phi : G \to H$ be a group homomorphism, $\{\mathbf{e}_i\}_{i=1}^n$ (resp. $\{\mathbf{d}_a\}_{a=1}^m$) an orthonormal basis of $(\mathfrak{g}, g)$ (resp. $(\mathfrak{h}, h)$). We use the following notations:

$$
\begin{aligned}
(d\phi)(\mathbf{e}_i) &=: \sum_{a=1}^{m} \phi_i{}^a \ \mathbf{d}_a, \\
{}^g\nabla_{\mathbf{e}_i}\mathbf{e}_j &=: D_{\mathbf{e}_i}\mathbf{e}_j =: \sum_{k=1}^{n} \alpha_{ij}{}^k \ \mathbf{e}_k, \\
{}^h\nabla_{\mathbf{d}_a}\mathbf{d}_b &=: \nabla_{\mathbf{d}_a}\mathbf{d}_b =: \sum_{c=1}^{m} \beta_{ab}{}^c \ \mathbf{d}_c.
\end{aligned} \tag{4.2.2}
$$

Here D (resp. ∇) is the Levi-Civita connection on (G, g) (resp. (H, h)), and $d\phi$ $(= \phi_*)$ is the differential of the group homomorphism ϕ. From (4.2.2) we obtain

$$
\begin{aligned}
\tilde{\nabla}_{\mathbf{e}_i}\phi_*\mathbf{e}_i &= \sum_{a,b,c=1}^{m} \phi_i{}^a \ \phi_i{}^b \ \beta_{ab}{}^c \ \mathbf{d}_c \\
\phi_*(D_{\mathbf{e}_i}\mathbf{e}_i) &= \sum_{j=1}^{n}\sum_{a=1}^{m} \alpha_{ii}{}^j \ \phi_j{}^a \ \mathbf{d}_a
\end{aligned} \tag{4.2.3}
$$

since $\alpha_{ij}{}^k$ and $\beta_{ab}{}^c$ are constants. By the help of (4.2.1), (4.2.3) and the definition of harmonic mapping, we obtain the following proposition.

Proposition 4.2.1. *Let (G, g) be an n-dimensional closed Lie group with an arbitrarily given left invariant metric g, (H, h) an m-dimensional Lie group with an arbitrarily given left invariant metric h. Then a group homomorphism $\phi : (G, g) \to (H, h)$ is a harmonic mapping if and only if*

$$
\sum_{i=1}^{n}\left(\sum_{a,b=1}^{m} \phi_i{}^a \ \phi_i{}^b \ \beta_{ab}{}^c - \sum_{j=1}^{n} \alpha_{ii}{}^j \ \phi_j{}^c\right) = 0 \tag{4.2.4}
$$

for all $c = 1, 2, \ldots, m$.

4.3 Harmonic group homomorphism between $SU(2)$ and the Heisenberg group

In this section, we will construct harmonic group homomorphisms of $(SU(2), g)$ into the Heisenberg Riemannian Lie group (H, h_0).

Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. The Killing form B of $\mathfrak{su}(2)$ satisfies

$$B(X, Y) = 4\ \mathrm{tr}(XY) \quad (X, Y \in \mathfrak{su}(2)).$$

We define an inner product $(\ ,\ )_0$ on $\mathfrak{su}(2)$ by

$$(X, Y)_0 := -B(X, Y) \quad (X, Y \in \mathfrak{su}(2)).$$

Here and from now on, let g be an arbitrarily given left invariant Riemannian metric on $SU(2)$. The following lemma is known (cf. [5, 11, 18]).

Lemma 4.3.1. *Let g be a left invariant Riemannian metric on $SU(2)$. Let $<\ ,\ >$ be an inner product on $\mathfrak{su}(2)$ defined by $< X, Y > := g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{su}(2)$ and e is the identity matrix of $SU(2)$. Then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $(\ ,\ )_0$ $(= -B)$ such that*

$$\begin{aligned} &[X_1, X_2] = (1/\sqrt{2})X_3, \qquad &&[X_2, X_3] = (1/\sqrt{2})X_1, \\ &[X_3, X_1] = (1/\sqrt{2})X_2, \qquad &&< X_i, X_j >= \delta_{ij}{a_i}^2, \end{aligned} \tag{4.3.1}$$

where a_i $(i = 1, 2, 3)$ are positive constants determined by the given left invariant Riemannian metric g on $SU(2)$.

Let H be the Heisenberg group (cf. [12, 14]), that is,

$$H = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid a_{12}, a_{23}, a_{13} \in \mathbb{R} \right\}.$$

Denote by $x,\ y,\ z$ coordinates on H, say for $A \in H$, $x(A) = a_{12}$, $y(A) = a_{23}$, $z(A) = a_{13}$. If L_B is the left translation by an element $B \in H$, we have

$$L_B^*\, dx = dx, \quad L_B^*\, dy = dy, \quad L_B^*\,(dz - xdy) = dz - xdy.$$

On H, the vector fields

$$\mathbf{d}_1 := \frac{\partial}{\partial x}, \quad \mathbf{d}_2 := \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{d}_3 := \frac{\partial}{\partial z} \tag{4.3.2}$$

are dual to $dx,\ dy,\ dz - xdy$, and are left invariant. Moreover, $\{\mathbf{d}_a\}_{a=1}^3$ is orthonormal with respect to the left invariant metric h_0 on H given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$

The Riemannian manifold (H, h_0) is referred to as the *Heisenberg Riemannian Lie group.*

We retain the notations as in Sections 4.2. In general, the Riemannian connection ∇ on a Riemannian manifold (M, g) is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) \\ &\quad - g([Y,Z],X) \qquad (X,Y,Z \in \mathfrak{X}(M)). \end{aligned} \tag{4.3.3}$$

We fix an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $(\ ,\ )_0$ satisfying (4.3.1) in Lemma 4.3.1 and denote by $g_{(a_1,a_2,a_3)}$ the left invariant Riemannian metric on $SU(2)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma 4.3.1. Moreover, we normalize left invariant Riemannian metrics on $SU(2)$ by putting $a_3 = 1$. We denote by $g_{(a_1,a_2,1)}$, or simply by $g_{(a_1,a_2)}$, the left invariant Riemannian metric which is determined by positive real numbers $a_3 = 1, a_1, a_2$.

For the orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $-B =: (\ ,\ )_0$ in Lemma 4.3.1, if we put

$$\mathbf{e}_1 := \frac{1}{a_1} X_1, \quad \mathbf{e}_2 := \frac{1}{a_2} X_2, \quad \mathbf{e}_3 := X_3,$$

then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal frame basis of $(SU(2), g_{(a_1,a_2)})$. From (4.3.1), we have

$$[\mathbf{e}_1, \mathbf{e}_2] = \frac{1}{\sqrt{2}\ a_1 a_2} \mathbf{e}_3,\ [\mathbf{e}_2, \mathbf{e}_3] = \frac{a_1}{\sqrt{2}\ a_2} \mathbf{e}_1,\ [\mathbf{e}_3, \mathbf{e}_1] = \frac{a_2}{\sqrt{2}\ a_1} \mathbf{e}_2. \tag{4.3.4}$$

By virtue of (4.3.3) and (4.3.4), we get

$$\begin{aligned} D_{\mathbf{e}_1} \mathbf{e}_2 &= \frac{1 - (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2} \mathbf{e}_3, \quad D_{\mathbf{e}_2} \mathbf{e}_3 = \frac{1 + (a_1)^2 - (a_2)^2}{2\sqrt{2}\ a_1 a_2} \mathbf{e}_1, \\ D_{\mathbf{e}_3} \mathbf{e}_1 &= \frac{-1 + (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2} \mathbf{e}_2, \quad D_{\mathbf{e}_i} \mathbf{e}_i = 0 \ \ (i = 1, 2, 3). \end{aligned} \tag{4.3.5}$$

Using (4.2.2), (4.3.4) and (4.3.5), we have

$$\begin{aligned} {\alpha_{12}}^3 &= -{\alpha_{13}}^2 = \frac{1 - (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{23}}^1 &= -{\alpha_{21}}^3 = \frac{1 + (a_1)^2 - (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{31}}^2 &= -{\alpha_{32}}^1 = \frac{-1 + (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{ij}}^k &= 0 \ \ \text{otherwise}. \end{aligned} \tag{4.3.6}$$

Moreover, by virtue of (4.3.2) and (4.3.3), we get

$$\begin{aligned} &[\mathbf{d}_1, \mathbf{d}_2] = \mathbf{d}_3, \quad [\mathbf{d}_2, \mathbf{d}_3] = [\mathbf{d}_3, \mathbf{d}_1] = 0, \\ &\nabla_{\mathbf{d}_i} \mathbf{d}_i = 0 \ \ (i = 1, 2, 3), \quad \nabla_{\mathbf{d}_1} \mathbf{d}_2 = -\nabla_{\mathbf{d}_2} \mathbf{d}_1 = \frac{1}{2} \mathbf{d}_3, \\ &\nabla_{\mathbf{d}_2} \mathbf{d}_3 = \nabla_{\mathbf{d}_3} \mathbf{d}_2 = \frac{1}{2} \mathbf{d}_1, \quad \nabla_{\mathbf{d}_3} \mathbf{d}_1 = \nabla_{\mathbf{d}_1} \mathbf{d}_3 = -\frac{1}{2} \mathbf{d}_2. \end{aligned} \tag{4.3.7}$$

From (4.2.2) and (4.3.7), we have

$$\begin{aligned} &{\beta_{12}}^3 = -{\beta_{21}}^3 = \frac{1}{2}, \quad {\beta_{23}}^1 = {\beta_{32}}^1 = \frac{1}{2}, \\ &{\beta_{31}}^2 = {\beta_{13}}^2 = -\frac{1}{2}, \quad {\beta_{bc}}^a = 0 \ \text{ otherwise.} \end{aligned} \tag{4.3.8}$$

By virtue of (4.2.4) in Proposition 4.2.1, (4.3.6) and (4.3.8), we obtain a group homomorphism $\phi : (SU(2), g_{(a_1,a_2)}) \to (H, h_0)$ is a harmonic mapping if and only if

$$\sum_{i=1}^{3} {\phi_i}^2 {\phi_i}^3 = 0 \ \text{ and } \sum_{i=1}^{3} {\phi_i}^3 {\phi_i}^1 = 0.$$

Hence, we have the following theorem.

Theorem 4.3.2. *A group homomorphism ϕ of $(SU(2), g_{(a_1,a_2)})$ into the Heisenberg group (H, h_0) is a harmonic mapping if and only if*

$$\sum_{i=1}^{3} h_0(\phi_* \mathbf{e}_i, \mathbf{d}_2) \cdot h_0(\phi_* \mathbf{e}_i, \mathbf{d}_3) = 0$$

and

$$\sum_{i=1}^{3} h_0(\phi_* \mathbf{e}_i, \mathbf{d}_3) \cdot h_0(\phi_* \mathbf{e}_i, \mathbf{d}_1) = 0.$$

Chapter 5

Variations of the length integral

In this chapter, we obtain a necessary and sufficient condition for the second variation of an arbitrarily given smooth variation of a geodesic on a Riemannian manifold to be 0.

5.1 Introduction

Let $\tau_s : [0,1] \to M$ $(-\varepsilon < s < \varepsilon)$ be a smooth variation of a geodesic $\tau_0 = \tau = x_t$ $(0 \leq t \leq 1)$ on a Riemannian manifold (M, g) such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$. We put

$$L(s) := L(\tau_s) := \int_0^1 \| \frac{d\tau_s}{dt} \| \, dt,$$

and calculate the second variation $(d^2L(s)/ds^2)_{s=0}$. The second variation is expressed in terms of the sectional curvature (cf. [13, 15, 17, 21]) and the variation vector field along the curve $\tau = x_t$ $(0 \leq t \leq 1)$. And then, we get a necessary and sufficient condition for the second variation $(d^2L(s)/ds^2)_{s=0}$ to be 0.

These calculus methods in variation problems are very useful in the study on natural sciences. But, in most of references, the calculus of variations is insufficient and omitted. In this chapter, we make a minute and detailed poof of the calculus parts which are insufficient and omitted in variation problems of the length integral.

5.2 The first and second variations of the length integral

Let (M, g) be a complete Riemannian manifold and ∇ the Levi-Civita connection for the Riemannian metric g. For a C^∞-curve $\tau = x_t$ $(0 \le t \le 1)$ on M, let

$$\phi : (t, s) \in [0, 1] \times (-\varepsilon, \varepsilon) \ \mapsto \ \phi(t, s) \in M$$

be a C^∞-mapping which satisfies

$$\phi(t, 0) = x_t.$$

Such a mapping ϕ is called a *variation* of $\tau = x_t$ $(0 \le t \le 1)$. From now on, we assume that $\tau = x_t$ $(0 \le t \le 1)$ is parametrized by its arc length. Let $\tau_s : [0, 1] \to M$ $(-\varepsilon < s < \varepsilon)$ be a C^∞-mapping which is defined by $\tau_s(t) := \phi(t, s)$. The length $L(s)$ of τ_s is given by

$$L(s) := L(\tau_s) = \int_0^1 \sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}\, dt. \tag{5.2.1}$$

Let $\phi_*(\partial/\partial t)_{(t,s)}$ and $\phi_*(\partial/\partial s)_{(t,s)}$ are defined on $[0, 1] \times (-\varepsilon, \varepsilon)$. Moreover, we define a vector field X_t along the curve $\tau = x_t$ $(0 \le t \le 1)$ by

$$X_t := \phi_*(\partial/\partial s)_{(t,0)} \quad (0 \le t \le 1). \tag{5.2.2}$$

Such a vector field X_t $(0 \le t \le 1)$ along the curve $\tau = x_t$ $(0 \le t \le 1)$ is called the *variation vector field* along ϕ. In this chapter, we assume that $\phi(0, s) = x_0$ and $\phi(1, s) = x_1$, $s \in (-\varepsilon, \varepsilon)$, for any variation ϕ of the curve $\tau = x_t$ $(0 \le t \le 1)$. Then we have

$$X_0 = 0 \in T_{x_0}(M), \quad X_1 = 0 \in T_{x_1}(M). \tag{5.2.3}$$

Now we calculate the first variation $(dL/ds)_{s=0}$ of $L(s)$. First of all, we get

$$\begin{aligned} &\frac{\partial}{\partial s}\left(\sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}\,\right) \\ &\qquad = \frac{g(\nabla_s \phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}{\sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}}. \end{aligned} \tag{5.2.4}$$

Since

$$\begin{aligned} &\nabla_s \phi_*(\partial/\partial t)_{(t,s)} - \nabla_t \phi_*(\partial/\partial s)_{(t,s)} - \phi_*[(\partial/\partial s)_{(t,s)}, (\partial/\partial t)_{(t,s)}] \\ &\qquad = T^{\nabla}(\phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) \end{aligned}$$

and $T^{\nabla} = 0$ (cf. [21]), we obtain on $[0,1] \times (-\varepsilon, \varepsilon)$

$$\nabla_s \phi_*(\partial/\partial t)_{(t,s)} = \nabla_t \phi_*(\partial/\partial s)_{(t,s)}. \tag{5.2.5}$$

Here, T^{∇} is the torsion of ∇. We get from (5.2.2) and (5.2.5)

$$\phi_*(\partial/\partial t)_{(t,0)} = x'_t, \quad (\nabla_s \phi_*(\partial/\partial t)_{(t,s)})_{s=0} = \nabla_t X_t. \tag{5.2.6}$$

From (5.2.1), (5.2.4) and (5.2.6), we obtain

$$\left(\frac{dL(s)}{ds}\right)_{s=0} = \int_0^1 g(\nabla_t X_t, x'_t)\, dt. \tag{5.2.7}$$

Moreover, we get

$$g(\nabla_t X_t, x'_t) = \frac{d}{dt}(g(X_t, x'_t)) - g(X_t, \nabla_t x'_t). \tag{5.2.8}$$

By the help of (5.2.7) and (5.2.8), we have

$$\begin{aligned} \left(\frac{dL(s)}{ds}\right)_{s=0} &= \int_0^1 \left\{\frac{d}{dt}(g(X_t, x'_t)) - g(X_t, \nabla_t x'_t)\right\} dt \\ &= [g(X_t, x'_t)]_0^1 - \int_0^1 g(X_t, \nabla_t x'_t)\, dt. \end{aligned} \tag{5.2.9}$$

Since $X_0 = 0 \in T_{x_0}(M)$ and $X_1 = 0 \in T_{x_1}(M)$ from (5.2.3), we get from (5.2.9)

$$\left(\frac{dL(s)}{ds}\right)_{s=0} = -\int_0^1 g(X_t, \nabla_t x'_t)\, dt.$$

Hence we have the following theorem.

Theorem 5.2.1. *Let $\tau_s : [0,1] \to M$ $(-\varepsilon < s < \varepsilon)$ be an arbitrarily given smooth variation of $\tau = x_t$ $(0 \le t \le 1)$ such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$. Then, $(dL(s)/ds)_{s=0} = 0$ if and only if $\nabla_t x'_t = 0$ for every $t \in (0,1)$, that is, $\tau = x_t$ $(0 \le t \le 1)$ is a geodesic in the Riemannian manifold (M, g).*

Next, we calculate the second variation $(d^2L(s)/ds^2)_{s=0}$ of the geodesic $\tau = x_t$ $(0 \le t \le 1)$. From (5.2.4) and (5.2.5), we get

$$\begin{aligned}
&\frac{\partial^2}{\partial s^2}\left(\sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}\right) \\
&\quad = \frac{g(\nabla_s \nabla_t \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) + \|\nabla_t \phi_*(\partial/\partial s)_{(t,s)}\|^2}{\{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\}^{\frac{1}{2}}} \\
&\quad\quad - \frac{\{g(\nabla_t \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\}^2}{\{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\}^{\frac{3}{2}}}.
\end{aligned} \tag{5.2.10}$$

Moreover, the following is well known (cf. [17]):

$$\begin{aligned}
&\nabla_s \nabla_t \phi_*(\partial/\partial s)_{(t,s)} - \nabla_t \nabla_s \phi_*(\partial/\partial s)_{(t,s)} \\
&\quad = R(\phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\phi_*(\partial/\partial s)_{(t,s)}.
\end{aligned} \tag{5.2.11}$$

Here, R is the curvature tensor field on (M, g). Furthermore,

$$\begin{aligned}
&g(\nabla_t \nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) \\
&\quad = \frac{d}{dt}\{g(\nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\} \\
&\quad\quad - g(\nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \nabla_t \phi_*(\partial/\partial t)_{(t,s)}).
\end{aligned} \tag{5.2.12}$$

By the help of (5.2.6), (5.2.10), (5.2.11) and (5.2.12), we obtain

$$\begin{aligned}
\left(\frac{d^2L(s)}{ds^2}\right)_{s=0} &= \left[(g(\nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}))_{s=0}\right]_{t=0}^{1} \\
&\quad + \int_0^1 \{g(R(X_t, x'_t)X_t, x'_t) + \|\nabla_t X_t\|^2 \\
&\quad\quad - (g(\nabla_t X_t, x'_t))^2\}dt.
\end{aligned} \tag{5.2.13}$$

Let $X_t^{\perp}$ $(0 \le t \le 1)$ be the component of X_t perpendicular to the geodesic $\tau = x_t$ $(0 \le t \le 1)$, that is,

$$X_t^{\perp} = X_t - g(X_t, x_t')\ x_t'. \tag{5.2.14}$$

Since $\tau_s(0) = \phi(0, s) = x_0$ and $\tau_s(1) = \phi(1, s) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$, we get

$$\nabla_s \phi_*(\partial/\partial s)_{(0,s)} = 0, \quad \nabla_s \phi_*(\partial/\partial s)_{(1,s)} = 0. \tag{5.2.15}$$

From (5.2.14), we obtain

$$\begin{aligned} \nabla_t X_t &= \nabla_t X_t^{\perp} + \frac{d(g(X_t, x_t'))}{dt} x_t', \\ 0 &= \frac{d(g(X_t^{\perp}, x_t'))}{dt} = g(\nabla_t X_t^{\perp}, x_t'), \end{aligned} \tag{5.2.16}$$

because x_t $(0 \le t \le 1)$ is a geodesic in the Riemannian manifold (M, g). From (5.2.16), we get

$$\|\nabla_t X_t\|^2 - (g(\nabla_t X_t, x_t'))^2 = \|\nabla_t X_t^{\perp}\|^2. \tag{5.2.17}$$

We obtain from (5.2.13), (5.2.14), (5.2.15) and (5.2.17)

$$\left(\frac{d^2 L(s)}{ds^2}\right)_{s=0} = \int_0^1 \{\|\nabla_t X_t^{\perp}\|^2 - g(R(X_t^{\perp}, x_t')x_t', X_t^{\perp})\} dt.$$

Thus, we get the following theorem.

Theorem 5.2.2. *Let $\tau_s : [0,1] \to M$ $(-\varepsilon < s < \varepsilon)$ be an arbitrarily given variation of a geodesic $\tau_0 = \tau = x_t$ $(0 \le t \le 1)$ on (M, g) such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$. Then the second variation $(d^2L/ds^2)_{s=0}$ is given as follows:*

$$\begin{aligned} \left(\frac{d^2 L(s)}{ds^2}\right)_{s=0} &= \int_0^1 \{\|\nabla_t X_t^{\perp}\|^2 - g(R(X_t^{\perp}, x_t')x_t', X_t^{\perp})\} dt \\ &= \int_0^1 \{\|\nabla_t X_t^{\perp}\|^2 - \|X_t^{\perp}\|^2\ \sigma(X_t^{\perp}, x_t')\} dt, \end{aligned}$$

where $\sigma(X_t^{\perp}, x'_t)$ *is the sectional curvature determined by* $\{X_t^{\perp}, x'_t\}$.

By virtue of Theorem 5.2.2, we obtain the following corollary.

Corollary 5.2.3. *Assume that* (M, g) *is a space of constant negative. For an arbitrarily given variation* $\tau_s : [0,1] \to M$ $(-\varepsilon < s < \varepsilon)$ *of a geodesic* $\tau_0 = \tau = x_t$ $(0 \leq t \leq 1)$ *on* (M, g) *such that* $\tau_s(0) = x_0$ *and* $\tau_s(1) = x_1$ *for every* $s \in (-\varepsilon, \varepsilon)$*, a necessary and sufficient condition for the second variation* $(d^2L/ds^2)_{s=0}$ *to be* 0 *is* $X_t^{\perp} = 0$ $(0 \leq t \leq 1)$.

References

[1] S. Aloff and N. R. Wallach, *An infinite family of distinct 7-manifolds admitting positively Riemannian metrics,* Bull. Amer. Math. Soc. 81 (1975), 93-97.

[2] A. L. Besse, *Einstein Manifolds,* Springer Verlag, 1987.

[3] P.-Y. Kim, J.-S. Park and Y.-S. Pyo, *Harmonic maps between the group of automorphisms of the quaternion algebra,* J. Chungcheong Math. Soc. 25(2) (2012), 331-339.

[4] H. W. Kim, J.-S. Park and Y.-S. Pyo, *Torsion tensor forms on induced bundles,* J. Chungcheong Math. Soc. 26(4) (2013), 793-798.

[5] H. W. Kim, Y.-S. Pyo and H.-J. Shin, *Ricci and scalar curvatures on* $SU(3)$*,* Honam Math. J. 34(2) (2012), 231-239.

[6] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,* Vol. I, 1963; Vol. II, 1969, John Wiley and Sons, New York.

[7] M. Kreck and S. Stolz, *Some nondiffeomorphic homogeneous 7-manifolds with positive sectional curvature,* J. Differential Geom. 33 (1991), 465-486.

[8] Y. Matsushima, *Differentiable Manifolds,* Marcel Dekker, New York, 1972.

[9] K. Nomizu, *Invariant affine connections on homogeneous spaces,* Amer. J. Math. 76 (1954), 33-65.

[10] J.-S. Park, *Stability of the identity map of* $SU(3)/T(k,l)$*,* Tokyo J. Math. 17(2) (1994), 281-289.

[11] J.-S. Park, *Yang-Mills connections in orthonormal frame bundles over* $SU(2)$*,* Tsukuba J. Math., 18 (1994), 203-206.

[12] J.-S. Park, *Critical homogeneous metrics on the Heisenberg manifold,* Int. Inform. Sci., 11 (2005), 31-34.

[13] J.-S. Park, *Curvatures on* $SU(3)/T(k,l)$, Kyushu J. Math. 67 (2013), 55-65.

[14] J.-S. Park and W. T. Oh, *The Abbena-Thurston manifold as a critical point,* Can. Math. Bull. 39 (1996), 352-359.

[15] Y.-S. Pyo, H.-J. Shin and J.-S. Park, *Scalar curvatures on* $SU(3)/T(k,l)$, Honam Math. J. 33(4) (2011), 547-556.

[16] R. T. Smith, *The second variation formula for harmonic mappings,* Proc. Amer. Math. Soc. 47 (1975), 229-236.

[17] M. Spivak, *A Comprehensive Introduction to Differential Geometry II*, Publish or Perish, Inc. Berkeley, 1979.

[18] K. Sugahara, *The sectional curvature and the diameter estimate for the left invariant metrics on* $SU(2,\mathbb{C})$ *and* $SO(3,\mathbb{R})$, Math. Japonica 26 (1981), 153-159.

[19] H. Urakawa, *Numerical computation of the spectra of the Laplacian on 7-dimensional homogeneous manifolds* $SU(3)/T_{k,l}$, SIAM J. Math. Anal. 15 (1984), 979-987.

[20] H. Urakawa, *The first eigenvalue of the Laplacian for a positively curved homogeneous Riemannian manifold,* Compositio Math. 59 (1986), 57-71.

[21] H. Urakawa, *Calculus of Variations and Harmonic Maps (in Japanese)*, Shokabo Publ., 1990.

[22] H. Urakawa, *Calculus of variations and harmonic Maps,* Amer. Math. Soc., Providence, Rhode Island, 1993.

[23] N. Wallach, *Harmonic Analysis on Homogeneous Spaces,* Marcel Dekker, New York, 1973.

[24] M. Y. Wang, *Some examples of homogeneous Einstein manifolds in dimension seven,* Duke Math. J. 49 (1982), 23-28.

PUKYONG NATIONAL UNIVERSITY
부경대학교

Acknowledgments

First of all, I am deeply indebted to my supervisor, Yong-Soo Pyo. I would like to express my sincere appreciation to him for directing this research, for his patience, academical assistance, especially valuable advice, warmhearted encouragement, his concerns during the work of dissertation, and graduate course works. Also, I have to appreciate Dr. Joon-Sik Park, Professor at Pusan University of Foreign Studies, for his critical support and comments about various aspects of this dissertation. If it had not been for his practical and critical assistance, I would have been unable to finish this thesis. I am thankful to Professor Il Yong Lee's leadership as a chairman of my doctoral committee and also grateful to my other committee members, Professor, Sung-Jin Cho and Jin Han Park for their serious reading of the manuscript of this dissertation and for their helpful comments and suggestion.

I also would like to thank faculty members of the Department of Applied Mathematics for their aids and comments through my study. And I thank my colleague of our laboratory, Dr. Hyun Woong Kim for his real help and fellowship.

And I am very grateful to my parents and sisters for their sacrifice and love. I especially thank my wife, Mi Jin Yeo, who supported me sincerely by using all of her holidays. Lastly, I would like to tell my two children, Young Hyun and Young Che, that I love them.