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Approaches to Generalized Hesitant Fuzzy Decision Making

by

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Approaches to Generalized Hesitant Fuzzy Decision Making
일반화된 Hesitant 퍼지 의사결정의 해결 방법

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요 약

본 논문은 일반화된 hesitant 퍼지 환경에서 의사결정문제의 해결 방법을 연구 조사한 것으로 다음과 같이 요약된다.

첫째, 일반화된 hesitant 퍼지 원(generalized hesitant fuzzy element, GHFE)에 대한 entropy, cross-entropy 및 닮음측도를 정의하여, 이 측도들을 나타내는 몇 가지 식을 제시하고, 이들 측도들 사이의 관계 및 성질을 조사한다. 또한 공리적 정의를 바탕으로 닮음측도와 entropy가 서로 변형 가능함을 보이고, 일반화된 hesitant 퍼지 entropy, cross-entropy 를 이용하여 다속성 의사결정문제의 해결 방법을 제시한다.

둘째, 일반화된 hesitant 퍼지 원에 대한 subsethood, cardinality를 정의하고, 앞서 정의한 entropy 및 닮음측도와의 관계를 조사하여, Kosko [29-32]에 의해 소개된 퍼지집합에서의 결과를 확장한다.

셋째, 일반화된 hesitant 퍼지집합의 확장인 구간값 일반화된 hesitant 퍼지집합(interval-valued generalized hesitant fuzzy set, IVGHFS)을 정의하고 그에 대한 기본 연산들이 갖는 몇 가지 성질들을 조사하고, 퍼지집합에서의 확장원리(principle extension)를 구간값 일반화된 hesitant 퍼지집합으로 일반화한다. 또한 구간값 일반화된 hesitant 퍼지 원(interval-valued generalized hesitant fuzzy element, IVGHFE)의 거리측도를 정의하고, 확장원리와 거리측도를 이용하여 구간값 일반화된 hesitant 퍼지 정보를 갖는 의사결정문제의 해결방법을 제시하고, 수치적인 예를 통하여 제안된 방법의 효용성을 보인다.

Chapter 1

Introduction

Most of real-world problems such as decision making, pattern recognition, medical diagnosis, clustering analysis, and image processing not always involve crisp data. So one cannot successfully use the traditional methods because of various types of uncertainties presented in those problems. Since Zadeh [90] introduced fuzzy sets (FSs) as a tool treating imprecision and uncertainty, many its extensions such as intuitionistic fuzzy sets (IFSs) [2], interval-valued fuzzy sets (IVFSs) [92], interval-valued intuitionistic fuzzy sets (IVIFSs) [4], hesitant fuzzy sets (HFSs) [54, 56], dual hesitant fuzzy sets (DFSs) [99], and generalized hesitant fuzzy sets (GHFSs) [47] allowed people to deal with uncertainty and information in much broader perspective.

The entropy, cross-entropy and similarity measures are three important topics in the fuzzy set theory. Entropy describes the fuzziness degree of a FS [91]. Since its appearance, entropy has received great attentions. De Luca and Termini [16] introduced some axioms which captured people's intuitive comprehension to describe the fuzziness degree of a FS, and developed several formulas based on Shannon's function. Kaufmann [28] introduced a method for measuring the fuzziness degree of a FS by the metric distance between its membership function and the membership function of its nearest crisp set. Another method proposed by Yager [79] is to view the fuzziness degree of a FS in term of a lack of distinction between the FS and its complement. Based on the axiomatic definitions, Zeng and Li [93] investigated the relationships among entropy, similarity and inclusion measures for FSs. Later on, other entropies for FSs have been given from different views [6, 18, 31, 34, 45, 49], and lots of studies of this issue have developed and extended to extended environments of FSs. Burillo and Bustince [7] introduced an entropy measure on IVFSs and IFSs. Zeng and Li [94] presented a new concept of entropy for IVFSs with a different view from [7]. Szmidt and Kacprzyk [51] introduced a non-probabilistic entropy measure for IFSs. Zhang et al. [97] proposed an entropy measure for IVIFSs by using membership interval and non-membership interval of IVIFS, which complies with the extended form of De Luca-Termini axioms for fuzzy entropy. Sen and Pal [48] proposed classes of entropy measures based on rough set theory and its certain generalizations. Xu and Xia [76] introduced an axiomatic definition of entropy for HFSs, proposed some entropy formulas for HFSs and applied them decision making.

Similarity measure and cross-entropy are mainly used to measure the discrimination information, and then it is an important measure in decision making, pattern recognition and other real-world problems. Up to now, a lot of research has been done about this issue. Vlachos and Sergiadis [57] introduced the concepts of discrimination information and cross-entropy for IFSs, and revealed the connection between the notions of entropies for FSs and IFSs in terms of fuzziness and intuitionism. Hung and Yang [27] constructed J-divergence of IFSs and introduced some useful distance and similarity measures between two IFSs, and applied them to clustering analysis and pattern recognition. Based on which, Xia and Xu [64] proposed some cross-entropy and entropy formulas for IFSs and applied them to group decision making. Ye [85] proposed a method of fault diagnosis based on the vague cross-entropy. He [86] also introduced the cross-entropy for IFSs and IVIFSs and utilized then to solve multi-criteria decision making (MCDM) problems. Wang and Li [59] provided two improved methods for solving MCDM problems, which were based on the cross-entropy for IFSs. Hung et al. [26] introduced the discrimination information and cross-entropy for IFSs and also used them to improve the fault diagnosis of turbine problems. Mao et al.

[37] introduced the cross-entropy and entropy measures for IFSs. Zang and Yu [98] constructed a series of mathematical programming models, which were based on an interval-valued intuitionistic fuzzy cross-entropy, in order to determine the criteria weights and applied them to MCDM problems. Xia and Xu [64] proposed two methods for determining the optimal weights of criteria and developed two pairs of entropy and cross-entropy measures for intuitionistic fuzzy values. The relationships among the entropy, cross-entropy and similarity measures have also attracted many attentions. For example, Liu [34] gave the axiomatic definitions of entropy, distance measure, and similarity measure of fuzzy sets and discussed their basic relations. Based on the axiomatic definitions, Zeng and Li [93] investigated the relationships among inclusion measure, similarity measure and entropy for FSs. They [94] also discussed the relationship between the similarity measure and the entropy of IVFSs. Zang and Jiang [96] discussed the relationship between the similarity measure and the entropy of IVIFSs. Zhang et al. [97] presented the cross-entropy of IVIFSs and discussed the relationship between the proposed entropy measures and the existing information measures of IVIFSs. Xu and Xia [76] analyzed the relationships among the entropy, cross-entropy and similarity measures for HFSs, and use them to develop two multi-attribute decision making methods.

From the above analysis, we can recognize that all existing entropy, cross-entropy and similarity measures are based on FSs, IFSs, IVFSs, IVIFSs, and HFSs. However, when people make a decision, they are usually hesitant and irresolute for one thing or another, which make it difficult to reach a consensus on final decision. The difficulty of establishing a common membership degree is that they have a hesitation among several possible membership degrees with uncertainties. During the evaluating process to get a more reasonable decision result, a decision organization, which contains a lot of experts, is authorized to provide the preference information about a set of alternatives. In practice, they may have several possible membership degrees take the forms of both crisp values and interval values in $[0, 1]$ when discussing the membership degree of an alternative with respect to a criterion. For example, some experts in the decision organization provide 0.4 doubtless, some provide $[0.5, 0.6]$ and the others insist on at least 0.6, and when these three parts cannot persuade each other, then these three membership degrees can be represented by a generalized hesitant fuzzy element (GHFE) $\{(0.4, 0.6), (0.5, 0.4), (0.6, 0)\}$, which is the unit of generalized hesitant fuzzy set. In such circumstances, it is not possible to solve this problem by utilizing either FSs, IFSs, IVFSs or HFSs. To deal with such cases, Qjan et al. [47] introduced the concept of generalized hesitant fuzzy sets (GHFSs) considered as a generalization of both IFSs and HFSs. GHFS can reflect the human's hesitance more objectively than other extensions of fuzzy set (IFS, IVIFS and HFS). They redefined some basic operations of GHFSs, and discussed some arithmetic operations and relationships among them. Since hesitation among several possible membership degrees with uncertainties in evaluating process is a very common problem in practical decision making, it is necessary to develop some entropy and cross-entropy measures for GHFSs.

To do this, Chapter 2 of this thesis is organized as follows. In Section 2.1, we present axiomatic definitions of entropy and similarity measure for GHFEs, and show that the entropy and the similarity measure for GHFEs can be transformed by each other based on their axiomatic definitions. Section 2.2 develops two cross-entropy formulas for GHFEs, and gives two entropy formulas based on them. In Section 2.3, we propose two approaches for solving multiple attribute decision making under generalized hesitant fuzzy environment. The first approach is based on the proposed entropy and cross-entropy measures, and second one utilizes TOPSIS method. In Section 2.4, we propose axiomatic definition of subsethood measure for GHFEs, and prove an generalized hesitant fuzzy version of the entropy-subsethood theorem [29, 30, 32], and derive entropy for GHFEs. Based on the concept of average possible cardinality, we extend the fuzzy entropy theorem [29, 30, 32] in the generalized hesitant fuzzy setting. Furthermore, we investigate the relationship between generalized hesitant fuzzy subsethood and generalized hesitant similarity measures. Finally, conclusion of Chapter 2 is given in Section 2.5.

During the evaluating process in practice, however, several possible memberships for an element to a set may be not only intuitionistic fuzzy values (IFVs), but also interval-valued IFVs. To deal with this, Chapter 3 of this thesis is organized as follows. In Section 3.1, we briefly review the concept of IVIFS and some of their operations. In Section 3.2, we extend HFSs by IVIFSs to interval-valued generalized hesitant fuzzy sets (IVGHFSs) and discuss the relationships between IVGHFSs and other types of FSs such as IFSs, IVIFSs, HFSs and GHFSs. The envelop and basic operations of interval-valued generalized hesitant fuzzy elements (IVGHFEs) are defined and then some relationships and operational laws among those operations are also discussed. We further introduce the comparison law to distinguish two IVGHFEs according to score function and consistency function. Besides, the extension principle, which enables us to employ aggregation operators of IVIFSs to aggregate IVGHFEs, are proposed. Section 3.3 develops two approaches for solving multiple attribute decision making with anonymity under interval-valued generalized hesitant fuzzy information. Two practical examples are presented to illustrate the developed approaches. Finally, we give some conclusions of Chapter 3 in Section 3.4.

Chapter 2

Generalized hesitant fuzzy entropy and cross-entropy and their use in multiple attribute decision making

In this chapter, we present the entropy, cross-entropy and similarity measure for generalized hesitant fuzzy information, and discuss their desirable properties. Some measure formulas are developed, and the relationships among them are investigated. We show that the similarity measure and entropy for generalized hesitant fuzzy information can be transformed by each other based on their axiomatic definitions. Then we develop two approaches for solving multiple attribute decision making, in which the attribute values are given in the form of generalized hesitant fuzzy elements. In first approach, the attribute weight vector is determined by the generalized hesitant fuzzy entropies, and the optimal alternative is obtained by comparing the generalized hesitant fuzzy cross-entropies between alternatives and positive-ideal or negative-ideal solutions; in second approach, the attribute weight vector is derived from the maximizing deviation method and optimal alternative is obtained by using TOPSIS method. Finally, an example is provided to illustrate the practicality and effectiveness of the developed approaches.

2.1 Entropy for generalized hesitant fuzzy elements

2.1.1 Basic concepts

Intuitionistic fuzzy set introduced by Atanassov [2] have been proven to be highly useful to deal with uncertainty and vagueness. Hesitation of which was characterized by a membership function and a nonmembership function.

Definition 2.1.1 [2] Let X be ordinary non-empty set. An intuitionistic fuzzy set (IFS) A in X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{2.1}$$

where $\mu_A, \nu_A : X \to [0,1]$ denote, respectively, the membership and nonmembership functions of A with the condition: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

For an IFS A, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ represents the degree of hesitation or intuitionistic index of x to A. For a fuzzy set, the degree of hesitation $\pi_A(x) = 0$. Thus for each x, $\mu_A(x)$ and $\nu_A(x)$ define an interval $[\mu_A(x), 1 - \nu_A(x)]$. This interval is the vague value of value set by Gau and Buethrer [21] (Bustince and Burillo [10] proved that vague sets are equivalent to IFSs). Further, the interval can also represent an interval-valued fuzzy set. Hence Xu [68] concluded that IFSs are also equivalent to interval-valued fuzzy sets, and replaced Eq. (2.1) with

$$A = \{\langle x, [\mu_A(x), 1 - \nu_A(x)]\rangle | x \in X\}. \tag{2.2}$$

The ordered pair $\alpha(x) = (\mu_\alpha(x), \nu_\alpha(x))$ is referred to an intuitionistic fuzzy value (IFV) [68], where $\mu_\alpha(x), \nu_\alpha(x) \in [0,1]$ and $\mu_\alpha(x) + \nu_\alpha(x) \leq 1$. Associated with the degree of hesitation, an IFV can also be equivalently denoted by $\alpha(x) = (\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x))$, where $\mu_\alpha(x), \nu_\alpha(x), \pi_\alpha(x) \in [0,1]$ and $\mu_\alpha(x) + \nu_\alpha(x) + \pi_\alpha(x) = 1$. In the rest of this chapter, for a certain x in X, IFV $a = (\mu, \nu, \pi)$ is abbreviated as $a = (\mu, \nu)$ when no misunderstanding raises. Since an IFV represent an interval, an interval $[\mu, 1-\nu]$ in $[0,1]$ will be directly transformed into (μ, ν).

The hesitant fuzzy set, as a generalization of FS, permits the membership degree of an element to a set presented as several possible values between 0 and 1, which can better describe the situations where people have hesitancy in providing their preferences over objects in process of decision making.

Definition 2.1.2 [54, 56] Given a fixed set X, a hesitant fuzzy set (HFS) on X in terms of function α is that when applied to X returns a subset of $[0,1]$, which can be represented as the following mathematical symbol:

$$A = \{\langle x, \alpha(x)\rangle | x \in X\}, \tag{2.3}$$

where $\alpha(x)$ is a set of the some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu [63] called $\alpha(x)$ a hesitant fuzzy element (HFE) and the set of all HFEs is denoted by HFES. Especially, it there is only one value in $\alpha(x)$, then the HFS reduces to the FS, which indicates that FSs are special type of HFSs, therefore, the theory for HFSs can also be applied to FSs.

During the evaluating process, several possible memberships of an alternative satisfying a certain criterion may be not only crisp values but also interval values in $[0,1]$. In order to handle this kind of assessment in decision making, Qjan et al. [47] extended HFSs by using IFSs to modify Definition 2.1.2.

Definition 2.1.3 [47] Let $([0,1]\times[0,1])^* = \{(x,y)|x,y \in [0,1], x+y \leq 1\}$. Given a fixed set X, the generalized hesitant fuzzy set (GHFS) on X is in terms of a function $\tilde{\alpha}$ that when applied to X returns a subset of $([0,1]\times[0,1])^*$, which can be represented as the following mathematical symbol:

$$A = \{\langle x, \tilde{\alpha}(x)\rangle | x \in X\}, \tag{2.4}$$

where $\tilde{\alpha}(x) = \{(\mu_{\tilde{\alpha}}(x), \nu_{\tilde{\alpha}}(x))\}$ is a set of some values in $([0,1] \times [0,1])^*$ (i.e, a set of some IFVs in $[0,1]$), denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, Qjan et al. [47] called $\tilde{\alpha}(x)$ a generalized hesitant fuzzy element (GHFE) and the set of all GHFEs is denoted by GHFES.

In particular, if there is only one value in $\tilde{\alpha}(x)$, then the GHFS reduces to the IFS; if $\mu_{\tilde{\gamma}} + \nu_{\tilde{\gamma}} = 1$ for each $\tilde{\gamma} = (\mu_{\tilde{\gamma}}, \nu_{\tilde{\gamma}}) \in \tilde{\alpha}(x)$, then the GHFS reduces to the HFS; if $\tilde{\alpha}(x)$ contains only one value $\tilde{\gamma}$ and $\mu_{\tilde{\gamma}} + \nu_{\tilde{\gamma}} = 1$, then the GHFS reduces to the FS. Thus, it indicates that FSs, IFSs and HFSs are special types of GHFSs. Some useful operations on GHFEs are as follows:

Definition 2.1.4 [47] Let $\tilde{\alpha}$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be three GHFEs and $\lambda > 0$, then

(1) $\tilde{\alpha}_1 \cup \tilde{\alpha}_2 = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{\tilde{\gamma}_1 \cup \tilde{\gamma}_2\} = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{(\max\{\mu_{\tilde{\gamma}_1}, \mu_{\tilde{\gamma}_2}\}, \min\{\nu_{\tilde{\gamma}_1}, \nu_{\tilde{\gamma}_2}\})\}$;
(2) $\tilde{\alpha}_1 \cap \tilde{\alpha}_2 = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{\tilde{\gamma}_1 \cap \tilde{\gamma}_2\} = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{(\min\{\mu_{\tilde{\gamma}_1}, \mu_{\tilde{\gamma}_2}\}, \max\{\nu_{\tilde{\gamma}_1}, \nu_{\tilde{\gamma}_2}\})\}$;
(3) $\tilde{\alpha}^c = \cup_{\tilde{\gamma} \in \tilde{\alpha}} \{\tilde{\gamma}^c\} = \cup_{\tilde{\gamma} \in \tilde{\alpha}} \{(\nu_{\tilde{\gamma}}, \mu_{\tilde{\gamma}})\}$;
(4) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{\tilde{\gamma}_1 \oplus \tilde{\gamma}_2\} = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{(\mu_{\tilde{\gamma}_1} + \mu_{\tilde{\gamma}_2} - \mu_{\tilde{\gamma}_1}\mu_{\tilde{\gamma}_2}, \nu_{\tilde{\gamma}_1}\nu_{\tilde{\gamma}_2})\}$;
(5) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{\tilde{\gamma}_1 \otimes \tilde{\gamma}_2\} = \cup_{\tilde{\gamma}_1 \in \tilde{\alpha}_1, \tilde{\gamma}_2 \in \tilde{\alpha}_2} \{(\mu_{\tilde{\gamma}_1}\mu_{\tilde{\gamma}_2}, \nu_{\tilde{\gamma}_1} + \nu_{\tilde{\gamma}_2} - \nu_{\tilde{\gamma}_1}\nu_{\tilde{\gamma}_2})\}$;
(6) $\lambda\tilde{\alpha} = \cup_{\tilde{\gamma} \in \tilde{\alpha}} \{\lambda\tilde{\gamma}\} = \cup_{\tilde{\gamma} \in \tilde{\alpha}} \{(1 - (1 - \mu_{\tilde{\gamma}})^\lambda, \nu_{\tilde{\gamma}}^\lambda)\}$;
(7) $\tilde{\alpha}^\lambda = \cup_{\tilde{\gamma} \in \tilde{\alpha}} \{\tilde{\gamma}^\lambda\} = \cup_{\tilde{\gamma} \in \tilde{\alpha}} \{(\mu_{\tilde{\gamma}}^\lambda, 1 - (1 - \nu_{\tilde{\gamma}})^\lambda)\}$.

We can obtain the following relationships among the operational laws (4)-(7):

Theorem 2.1.5 *Let $\tilde{\alpha}$, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be three GHFEs and $\lambda, \lambda_1, \lambda_2 > 0$, then*

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \tilde{\alpha}_2 \oplus \tilde{\alpha}_1$;
(2) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \tilde{\alpha}_2 \otimes \tilde{\alpha}_1$;
(3) $\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2$, $\lambda > 0$;
(4) $(\tilde{\alpha}_1 \otimes \tilde{\alpha}_2)^\lambda = \tilde{\alpha}_1^\lambda \otimes \tilde{\alpha}_2^\lambda$, $\lambda > 0$;
(5) $(\lambda_1 + \lambda_2)\tilde{\alpha} = \lambda_1\tilde{\alpha} \oplus \lambda_2\tilde{\alpha}$, $\lambda_1, \lambda_2 > 0$;
(6) $\tilde{\alpha}^{(\lambda_1 + \lambda_2)} = \tilde{\alpha}^{\lambda_1} \otimes \tilde{\alpha}^{\lambda_2}$, $\lambda_1, \lambda_2 > 0$.

Proof We prove only (5) and (6).

(5)

$$
\begin{aligned}
\lambda_1\tilde{\alpha}\oplus\lambda_2\tilde{\alpha} &= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\{\lambda_1\tilde{\gamma}\oplus\lambda_2\tilde{\gamma}\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\Big\{\Big(1-(1-\mu_{\tilde{\gamma}})^{\lambda_1}+1-(1-\mu_{\tilde{\gamma}})^{\lambda_2}\\
&\qquad -(1-(1-\mu_{\tilde{\gamma}})^{\lambda_1})(1-(1-\mu_{\tilde{\gamma}})^{\lambda_2}),\nu_{\tilde{\gamma}}^{\lambda_1}\nu_{\tilde{\gamma}}^{\lambda_2}\Big)\Big\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\Big\{\Big(1-(1-\mu_{\tilde{\gamma}})^{\lambda_1}(1-\mu_{\tilde{\gamma}})^{\lambda_2},\nu_{\tilde{\gamma}}^{\lambda_1}\nu_{\tilde{\gamma}}^{\lambda_2}\Big)\Big\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\Big\{\Big(1-(1-\mu_{\tilde{\gamma}})^{\lambda_1+\lambda_2},\nu_{\tilde{\gamma}}^{\lambda_1+\lambda_2}\Big)\Big\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\{(\lambda_1+\lambda_2)\tilde{\gamma}\}=(\lambda_1+\lambda_2)\tilde{\alpha}.
\end{aligned}
$$

(6)

$$
\begin{aligned}
\tilde{\alpha}^{\lambda_1}\otimes\tilde{\alpha}^{\lambda_2} &= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\{\tilde{\gamma}^{\lambda_1}\otimes\tilde{\gamma}^{\lambda_2}\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\Big\{\Big(\mu_{\tilde{\gamma}}^{\lambda_1}\mu_{\tilde{\gamma}}^{\lambda_2},1-(1-\nu_{\tilde{\gamma}})^{\lambda_1}+1-(1-\nu_{\tilde{\gamma}})^{\lambda_2}\\
&\qquad -(1-(1-\nu_{\tilde{\gamma}})^{\lambda_1})(1-(1-\nu_{\tilde{\gamma}})^{\lambda_2})\Big)\Big\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\Big\{\Big(\mu_{\tilde{\gamma}}^{\lambda_1}\mu_{\tilde{\gamma}}^{\lambda_2},1-(1-\nu_{\tilde{\gamma}})^{\lambda_1}(1-\nu_{\tilde{\gamma}})^{\lambda_2}\Big)\Big\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\Big\{\Big(\mu_{\tilde{\gamma}}^{\lambda_1+\lambda_2},1-(1-\nu_{\tilde{\gamma}})^{\lambda_1+\lambda_2}\Big)\Big\}\\
&= \cup_{\tilde{\gamma}\in\tilde{\alpha}}\{\tilde{\gamma}^{\lambda_1+\lambda_2}\}=\tilde{\alpha}^{\lambda_1+\lambda_2}.
\end{aligned}
$$

It is noted that the number of IFVs in different GHFEs may be different, let $l_{\tilde{\alpha}}$ be the number of IFVs in $\tilde{\alpha}$. By comparison method [77] of IFVs, we arrange the elements in $\tilde{\alpha}$ in decreasing order, let $\tilde{\alpha}^{\sigma(i)}=(\mu_{\tilde{\alpha}}^{\sigma(i)},\nu_{\tilde{\alpha}}^{\sigma(i)})$ $(i=1,2,\ldots,l_{\tilde{\alpha}})$ be the ith largest IFV in $\tilde{\alpha}$. To operate correctly, we assume that the GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$ should have the same length l when we compare them. If the one is shorter than the other, we should extend the shorter one until both of them have the same length. To extend the shorter one, the best way is to add the same IFVs several times in it. In fact, we can extend the shorter one by adding any IFVs in it. The selection of this IFV mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum IFV, while pessimists expect unfavorable outcomes and may add the minimum IFV.

2.1.2 Entropy measurs for GHFEs

As entropy measures have wide applications in real-world problems such as decision making, pattern recognition, clustering analysis, and image processing, it is very necessary to develop some entropy measures under generalized hesitant fuzzy environment. In what follows, we first give the axiomatic definition of entropy for GHFEs.

Definition 2.1.6 An entropy on GHFE $\tilde{\alpha}$ is a real-valued function $E : GH \to [0,1]$, satisfying the following axiomatic requirements:

(1) $E(\tilde{\alpha}) = 0$ if and only if $\tilde{\alpha} = (0,1)$ or $\tilde{\alpha} = (1,0)$;

(2) $E(\tilde{\alpha}) = 1$ if and only if $\mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $i = 1, 2, \ldots, l_{\tilde{\alpha}}$;

(3) $E(\tilde{\alpha}) \leq E(\tilde{\beta})$ if $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, or if $\mu_{\tilde{\alpha}}^{\sigma(i)} \geq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(4) $E(\tilde{\alpha}) = E(\tilde{\alpha}^c)$.

On the basis of Definition 2.1.6, we can construct some entropy formulas as follows:

$$E_1(\tilde{\alpha}) = \frac{1}{l_{\tilde{\alpha}}(\sqrt{2}-1)} \sum_{i=1}^{l_{\tilde{\alpha}}} \left(\sin \frac{\pi(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})}{4} + \sin \frac{\pi(1 - \mu_{\tilde{\alpha}}^{\sigma(i)} + \nu_{\tilde{\alpha}}^{\sigma(i)})}{4} - 1 \right) \tag{2.5}$$

$$E_2(\tilde{\alpha}) = \frac{1}{l_{\tilde{\alpha}}(\sqrt{2}-1)} \sum_{i=1}^{l_{\tilde{\alpha}}} \left(\cos \frac{\pi(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})}{4} + \cos \frac{\pi(1 - \mu_{\tilde{\alpha}}^{\sigma(i)} + \nu_{\tilde{\alpha}}^{\sigma(i)})}{4} - 1 \right) \tag{2.6}$$

$$\begin{aligned} E_3(\tilde{\alpha}) = -\frac{1}{l_{\tilde{\alpha}} \ln 2} \sum_{i=1}^{l_{\tilde{\alpha}}} \Bigg(& \frac{1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}}{2} \ln \frac{1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}}{2} \\ & + \frac{1 - \mu_{\tilde{\alpha}}^{\sigma(i)} + \nu_{\tilde{\alpha}}^{\sigma(i)}}{2} \ln \frac{1 - \mu_{\tilde{\alpha}}^{\sigma(i)} + \nu_{\tilde{\alpha}}^{\sigma(i)}}{2} \Bigg) \end{aligned} \tag{2.7}$$

Now, we give the generalized hesitant fuzzy similarity measure defined as

Definition 2.1.7 For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, the similarity measure between $\tilde{\alpha}$ and $\tilde{\beta}$, denoted as $S(\tilde{\alpha},\tilde{\beta})$, should satisfy the following properties:

(1) $S(\tilde{\alpha},\tilde{\beta})=0$ if and only if $\tilde{\alpha}=(0,1)$, $\tilde{\beta}=(1,0)$ or $\tilde{\alpha}=(1,0)$, $\tilde{\beta}=(0,1)$;

(2) $S(\tilde{\alpha},\tilde{\beta})=1$ if and only if $\tilde{\alpha}=\tilde{\beta}$, i.e. $\mu_{\tilde{\alpha}}^{\sigma(i)}=\mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)}=\nu_{\tilde{\beta}}^{\sigma(i)}$, $i=1,2,\ldots,l$;

(3) $S(\tilde{\alpha},\tilde{\gamma})\leq S(\tilde{\alpha},\tilde{\beta})$, $S(\tilde{\alpha},\tilde{\gamma})\leq S(\tilde{\beta},\tilde{\gamma})$, if $\mu_{\tilde{\alpha}}^{\sigma(i)}\leq\mu_{\tilde{\beta}}^{\sigma(i)}\leq\mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}\geq\nu_{\tilde{\beta}}^{\sigma(i)}\geq\nu_{\tilde{\gamma}}^{\sigma(i)}$ or if $\mu_{\tilde{\alpha}}^{\sigma(i)}\geq\mu_{\tilde{\beta}}^{\sigma(i)}\geq\mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}\leq\nu_{\tilde{\beta}}^{\sigma(i)}\leq\nu_{\tilde{\gamma}}^{\sigma(i)}$, $i=1,2,\ldots,l$;

(4) $S(\tilde{\alpha},\tilde{\beta})=S(\tilde{\beta},\tilde{\alpha})$.

Based on Definition 2.1.7, some generalized hesitant fuzzy similarity measures can be constructed as:

$$S_1(\tilde{\alpha},\tilde{\beta})=1-\frac{1}{2l}\sum_{i=1}^{l}\left(|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\right)\tag{2.8}$$

$$S_2(\tilde{\alpha},\tilde{\beta})=1-\sqrt{\frac{1}{2l}\sum_{i=1}^{l}\left((\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)})^2+(\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)})^2\right)}\tag{2.9}$$

$$S_3(\tilde{\alpha},\tilde{\beta})=1-\sqrt[p]{\frac{1}{2l}\sum_{i=1}^{l}\left(|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|^p+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|^p\right)}\tag{2.10}$$

$$S_4(\tilde{\alpha},\tilde{\beta})=1-\frac{1}{2}\left(\max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|\}+\max_i\{|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\}\right)\tag{2.11}$$

$$S_5(\tilde{\alpha},\tilde{\beta})=1-\frac{1}{2}\left(\max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|^2\}+\max_i\{|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|^2\}\right)\tag{2.12}$$

$$S_6(\tilde{\alpha},\tilde{\beta})=1-\frac{1}{2}\left(\max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|^p\}+\max_i\{|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|^p\}\right)\tag{2.13}$$

$$S_7(\tilde{\alpha},\tilde{\beta}) = 1 - \frac{1}{4}\Bigg(\frac{1}{l}\sum_{i=1}^{l}\left(|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}| + |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\right) + \max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|\} + \max_i\{|\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\}\Bigg) \quad (2.14)$$

$$S_8(\tilde{\alpha},\tilde{\beta}) = 1 - \frac{1}{2}\Bigg(\sqrt{\frac{1}{2l}\sum_{i=1}^{l}\left((\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)})^2 + (\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)})^2\right)} + \frac{1}{2}\left(\max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|^2\} + \max_i\{|\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|^2\}\right)\Bigg) \quad (2.15)$$

$$S_9(\tilde{\alpha},\tilde{\beta}) = 1 - \frac{1}{2}\Bigg(\sqrt[p]{\frac{1}{2l}\sum_{i=1}^{l}\left(|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|^p + |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|^p\right)} + \frac{1}{2}\left(\max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|^p\} + \max_i\{|\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|^p\}\right)\Bigg) \quad (2.16)$$

From analyzing these similarity measures, we can find that S_1 and S_2 are based on the Hamming distance and the Euclidean distance, respectively; S_4 and S_5 apply the Hausdorff metric to S_1 and S_2; S_7 (resp. S_8) combines S_1 (resp. S_2) and S_4 (resp. S_5); S_3, S_6 and S_9 are further generalizations of S_1 and S_2, S_4 and S_5, and S_7 and S_8, respectively; if $p = 1$, then S_3 reduces to S_1, S_6 reduces to S_4, and S_9 becomes S_7; if $p = 2$, then S_3 reduces to S_2, S_6 reduces to S_5, and S_9 becomes S_8.

The relationships between similarity measures and entropy formulas have been studied by many authors under different environments, such as interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets and hesitant fuzzy sets. In the following, we investigate the relationships between generalized hesitant fuzzy similarity measures and generalized hesitant fuzzy entropy formulas:

Theorem 2.1.8 *Let $\tilde{\alpha}$ be a GHFE, then $S(\tilde{\alpha},\tilde{\alpha}^c)$ is an entropy for $\tilde{\alpha}$.*

Proof (1) $S(\tilde{\alpha}, \tilde{\alpha}^c) = 0 \Leftrightarrow \tilde{\alpha} = (0,1)$ and $\tilde{\alpha}^c = (1,0)$ or $\tilde{\alpha} = (1,0)$ and $\tilde{\alpha}^c = (0,1)$;

(2) $S(\tilde{\alpha}, \tilde{\alpha}^c) = 1 \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \mu_{\tilde{\alpha}^c}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}^c}^{\sigma(i)}$, $i = 1, 2, \ldots, l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(3) Suppose that $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, then $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\alpha}}^{\sigma(i)}$. Therefore, by the definition of similarity measure of GHFEs, $S(\tilde{\alpha}, \tilde{\alpha}^c) \leq S(\tilde{\beta}, \tilde{\alpha}^c) \leq S(\tilde{\beta}, \tilde{\beta}^c)$. With the same reason, when $\mu_{\tilde{\alpha}}^{\sigma(i)} \geq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, we can prove $S(\tilde{\alpha}, \tilde{\alpha}^c) \leq S(\tilde{\beta}, \tilde{\beta}^c)$;

(4) $S(\tilde{\alpha}, \tilde{\alpha}^c) = S(\tilde{\alpha}^c, \tilde{\alpha})$.

Example 2.1.9 For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, we can construct the following entropy formulas based on the similarity measures S_k $(1, 2, \ldots, 9)$:

$$S_1(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \frac{1}{l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} |\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| \tag{2.17}$$

$$S_2(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \sqrt{\frac{1}{l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2} \tag{2.18}$$

$$S_3(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \sqrt[p]{\frac{1}{l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} |\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^p} \tag{2.19}$$

$$S_4(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \max_i \{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|\} \tag{2.20}$$

$$S_5(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \max_i \{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^2\} \tag{2.21}$$

$$S_6(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^p\} \tag{2.22}$$

$$S_7(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \frac{1}{2}\left(\frac{1}{l_{\tilde{\alpha}}}\sum_{i=1}^{l_{\tilde{\alpha}}}|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| + \max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|\}\right) \tag{2.23}$$

$$S_8(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \frac{1}{2}\left(\sqrt{\frac{1}{l_{\tilde{\alpha}}}\sum_{i=1}^{l_{\tilde{\alpha}}}|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^2} + \max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^2\}\right) \tag{2.24}$$

$$S_9(\tilde{\alpha}, \tilde{\alpha}^c) = 1 - \frac{1}{2}\left(\sqrt[p]{\frac{1}{l_{\tilde{\alpha}}}\sum_{i=1}^{l_{\tilde{\alpha}}}|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^p} + \max_i\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|^p\}\right) \tag{2.25}$$

In the following, we propose a transform method of setting up generalized hesitant fuzzy similarity measure based on generalized hesitant fuzzy entropy.

Theorem 2.1.10 *For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, let $|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}| \leq |\mu_{\tilde{\alpha}}^{\sigma(i+1)} - \mu_{\tilde{\beta}}^{\sigma(i+1)}|$, $|\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}| \geq |\nu_{\tilde{\alpha}}^{\sigma(i+1)} - \nu_{\tilde{\beta}}^{\sigma(i+1)}|$, $i = 1, 2, \ldots, l-1$, and we define a GHFE $f(\tilde{\alpha}, \tilde{\beta})$ as follows:*

$$\begin{aligned} f(\tilde{\alpha}, \tilde{\beta}) = \Bigg\{ & \left(\frac{1 + |\mu_{\tilde{\alpha}}^{\sigma(1)} - \mu_{\tilde{\beta}}^{\sigma(1)}|}{2}, \frac{1 - |\nu_{\tilde{\alpha}}^{\sigma(1)} - \nu_{\tilde{\beta}}^{\sigma(1)}|}{2}\right), \\ & \left(\frac{1 + |\mu_{\tilde{\alpha}}^{\sigma(2)} - \mu_{\tilde{\beta}}^{\sigma(2)}|}{2}, \frac{1 - |\nu_{\tilde{\alpha}}^{\sigma(2)} - \nu_{\tilde{\beta}}^{\sigma(2)}|}{2}\right), \\ & \ldots, \left(\frac{1 + |\mu_{\tilde{\alpha}}^{\sigma(l)} - \mu_{\tilde{\beta}}^{\sigma(l)}|}{2}, \frac{1 - |\nu_{\tilde{\alpha}}^{\sigma(l)} - \nu_{\tilde{\beta}}^{\sigma(l)}|}{2}\right)\Bigg\}, \end{aligned} \tag{2.26}$$

then $E(f(\tilde{\alpha}, \tilde{\beta}))$ is the similarity measure of $\tilde{\alpha}$ and $\tilde{\beta}$.

Proof (1) $E(f(\tilde{\alpha},\tilde{\beta}))=0 \Leftrightarrow f(\tilde{\alpha},\tilde{\beta})=(1,0)$ or $f(\tilde{\alpha},\tilde{\beta})=(0,1) \Leftrightarrow \tilde{\alpha}=(0,1)$, $\tilde{\beta}=(1,0)$ or $\tilde{\alpha}=(1,0)$, $\tilde{\beta}=(0,1)$;

(2) $E(f(\tilde{\alpha},\tilde{\beta}))=1 \Leftrightarrow \frac{1+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|}{2}=\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{2}$, $i=1,2,\ldots,l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)}=\mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}=\nu_{\tilde{\beta}}^{\sigma(i)}$, $i=1,2,\ldots,l$;

(3) Since $\mu_{\tilde{\alpha}}^{\sigma(i)}\leq\mu_{\tilde{\beta}}^{\sigma(i)}\leq\mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}\geq\nu_{\tilde{\beta}}^{\sigma(i)}\geq\nu_{\tilde{\gamma}}^{\sigma(i)}$, $i=1,2,\ldots,l$, then we obtain $\frac{1+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|}{2}\leq\frac{1+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\gamma}}^{\sigma(i)}|}{2}$ and $\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{2}\geq\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\gamma}}^{\sigma(i)}|}{2}$, $i=1,2,\ldots,l$. Hence $\mu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}\leq\mu_{f(\tilde{\alpha},\tilde{\gamma})}^{\sigma(i)}$ and $\nu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}\geq\nu_{f(\tilde{\alpha},\tilde{\gamma})}^{\sigma(i)}$, $i=1,2,\ldots,l$. From the definition of $f(\tilde{\alpha},\tilde{\beta})$, we know that $\mu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}\geq\nu_{f(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}$, $i=1,2,\ldots,l$, and thus $E(f(\tilde{\alpha},\tilde{\gamma}))\leq E(f(\tilde{\alpha},\tilde{\beta}))$. With the same reason, we can prove that it is also true for $\mu_{\tilde{\alpha}}^{\sigma(i)}\geq\mu_{\tilde{\beta}}^{\sigma(i)}\geq\mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}\leq\nu_{\tilde{\beta}}^{\sigma(i)}\leq\nu_{\tilde{\gamma}}^{\sigma(i)}$, $i=1,2,\ldots,l$;

(4) $E(f(\tilde{\alpha},\tilde{\beta}))=E(f(\tilde{\beta},\tilde{\alpha}))$.

Example 2.1.11 For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, we get

$$E_1(f(\tilde{\alpha},\tilde{\beta}))=\frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left(\sin\frac{\pi(2+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8}\right.$$
$$\left.+\sin\frac{\pi(2-|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8}-1\right) \quad (2.27)$$

$$E_2(f(\tilde{\alpha},\tilde{\beta}))=\frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\left(\cos\frac{\pi(2+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8}\right.$$
$$\left.+\cos\frac{\pi(2-|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|)}{8}-1\right) \quad (2.28)$$

$$E_3(f(\tilde{\alpha},\tilde{\beta}))=-\frac{1}{l\ln 2}\sum_{i=1}^{l}\left(\frac{2+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{4}\right.$$
$$\times\ln\frac{2+|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|+|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{4}$$

$$+\frac{2-|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{4}$$
$$\left.\times\ln\frac{2-|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|-|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|}{4}\right) \quad (2.29)$$

Corollary 2.1.12 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, and E be the entropy of GHFE, then $E((f(\tilde{\alpha},\tilde{\beta}))^c)$ is the similarity measure of the GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$.

Corollary 2.1.13 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, $|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}| \leq |\mu_{\tilde{\alpha}}^{\sigma(i+1)}-\mu_{\tilde{\beta}}^{\sigma(i+1)}|$, $|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}| \geq |\nu_{\tilde{\alpha}}^{\sigma(i+1)}-\nu_{\tilde{\beta}}^{\sigma(i+1)}|$, $i=1,2,\ldots,l-1$, and we define a GHFE $g(\tilde{\alpha},\tilde{\beta})$ as follows:

$$g(\tilde{\alpha},\tilde{\beta})=\left\{\left(\frac{1+|\mu_{\tilde{\alpha}}^{\sigma(1)}-\mu_{\tilde{\beta}}^{\sigma(1)}|^p}{2},\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(1)}-\nu_{\tilde{\beta}}^{\sigma(1)}|^p}{2}\right),\right.$$
$$\left(\frac{1+|\mu_{\tilde{\alpha}}^{\sigma(2)}-\mu_{\tilde{\beta}}^{\sigma(2)}|^p}{2},\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(2)}-\nu_{\tilde{\beta}}^{\sigma(2)}|^p}{2}\right),\ldots,$$
$$\left.\left(\frac{1+|\mu_{\tilde{\alpha}}^{\sigma(l)}-\mu_{\tilde{\beta}}^{\sigma(l)}|^p}{2},\frac{1-|\nu_{\tilde{\alpha}}^{\sigma(l)}-\nu_{\tilde{\beta}}^{\sigma(l)}|^p}{2}\right)\right\},\ p>0, \quad (2.30)$$

then $E(g(\tilde{\alpha},\tilde{\beta}))$ is the similarity measure of $\tilde{\alpha}$ and $\tilde{\beta}$.

Theorem 2.1.14 *For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, let $|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}| \leq |\mu_{\tilde{\alpha}}^{\sigma(i+1)}-\mu_{\tilde{\beta}}^{\sigma(i+1)}|$, $|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}| \geq |\nu_{\tilde{\alpha}}^{\sigma(i+1)}-\nu_{\tilde{\beta}}^{\sigma(i+1)}|$, $i=1,2,\ldots,l-1$, and we define a GHFE $h(\tilde{\alpha},\tilde{\beta})$ as follows:*

$$h(\tilde{\alpha},\tilde{\beta})$$
$$=\left\{\left(\frac{1+\min\left\{\begin{array}{c}|\mu_{\tilde{\alpha}}^{\sigma(1)}-\mu_{\tilde{\beta}}^{\sigma(1)}|,\\ |\nu_{\tilde{\beta}}^{\sigma(1)}-\nu_{\tilde{\beta}}^{\sigma(1)}|\end{array}\right\}}{2},\frac{1-\max\left\{\begin{array}{c}|\mu_{\tilde{\alpha}}^{\sigma(1)}-\mu_{\tilde{\beta}}^{\sigma(1)}|,\\ |\nu_{\tilde{\beta}}^{\sigma(1)}-\nu_{\tilde{\beta}}^{\sigma(1)}|\end{array}\right\}}{2}\right),\right.$$

$$\left.\left(\frac{1+\min\left\{\begin{array}{c}|\mu_{\tilde{\alpha}}^{\sigma(2)}-\mu_{\tilde{\beta}}^{\sigma(2)}|,\\|\nu_{\tilde{\beta}}^{\sigma(2)}-\nu_{\tilde{\beta}}^{\sigma(2)}|\end{array}\right\}}{2},\frac{1-\max\left\{\begin{array}{c}|\mu_{\tilde{\alpha}}^{\sigma(2)}-\mu_{\tilde{\beta}}^{\sigma(2)}|,\\|\nu_{\tilde{\beta}}^{\sigma(2)}-\nu_{\tilde{\beta}}^{\sigma(2)}|\end{array}\right\}}{2}\right),\ldots,\left(\frac{1+\min\left\{\begin{array}{c}|\mu_{\tilde{\alpha}}^{\sigma(l)}-\mu_{\tilde{\beta}}^{\sigma(l)}|,\\|\nu_{\tilde{\beta}}^{\sigma(l)}-\nu_{\tilde{\beta}}^{\sigma(l)}|\end{array}\right\}}{2},\frac{1-\max\left\{\begin{array}{c}|\mu_{\tilde{\alpha}}^{\sigma(l)}-\mu_{\tilde{\beta}}^{\sigma(l)}|,\\|\nu_{\tilde{\beta}}^{\sigma(l)}-\nu_{\tilde{\beta}}^{\sigma(l)}|\end{array}\right\}}{2}\right)\right\}, \quad (2.31)$$

then $E(h(\tilde{\alpha},\tilde{\beta}))$ is the similarity measure of $\tilde{\alpha}$ and $\tilde{\beta}$.

Proof (1) $E(h(\tilde{\alpha},\tilde{\beta}))=0 \Leftrightarrow h(\tilde{\alpha},\tilde{\beta})=(1,0)$ or $h(\tilde{\alpha},\tilde{\beta})=(0,1) \Leftrightarrow \tilde{\alpha}=(0,1)$, $\tilde{\beta}=(1,0)$ or $\tilde{\alpha}=(1,0)$, $\tilde{\beta}=(0,1)$;

(2) $E(h(\tilde{\alpha},\tilde{\beta}))=1 \Leftrightarrow \frac{1+\min\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\}}{2}=\frac{1-\max\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\}}{2}$, $i=1,2,\ldots,l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)}=\mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}=\nu_{\tilde{\beta}}^{\sigma(i)}$, $i=1,2,\ldots,l$;

(3)According to the assumption, $\mu_{\tilde{\alpha}}^{\sigma(i)}\le\mu_{\tilde{\beta}}^{\sigma(i)}\le\mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}\ge\nu_{\tilde{\beta}}^{\sigma(i)}\ge\nu_{\tilde{\gamma}}^{\sigma(i)}$, $i=1,2,\ldots,l$, then we have $\frac{1+\min\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\}}{2}\le\frac{1+\min\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\gamma}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\gamma}}^{\sigma(i)}|\}}{2}$, $\frac{1-\max\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\}}{2}\ge\frac{1-\max\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\gamma}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\gamma}}^{\sigma(i)}|\}}{2}$, $i=1,2,\ldots,l$, which implies $\mu_{h(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}\le\mu_{h(\tilde{\alpha},\tilde{\gamma})}^{\sigma(i)}$ and $\nu_{h(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}\ge\nu_{h(\tilde{\alpha},\tilde{\gamma})}^{\sigma(i)}$, $i=1,2,\ldots,l$. From the definition of $h(\tilde{\alpha},\tilde{\beta})$, we know that $\mu_{h(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}\ge\nu_{h(\tilde{\alpha},\tilde{\beta})}^{\sigma(i)}$, $i=1,2,\ldots,l$, and thus $E(h(\tilde{\alpha},\tilde{\gamma}))\le E(h(\tilde{\alpha},\tilde{\beta}))$. With the same reason, we can prove that it is also true for $\mu_{\tilde{\alpha}}^{\sigma(i)}\ge\mu_{\tilde{\beta}}^{\sigma(i)}\ge\mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}\le\nu_{\tilde{\beta}}^{\sigma(i)}\le\nu_{\tilde{\gamma}}^{\sigma(i)}$, $i=1,2,\ldots,l$;

(4) $E(h(\tilde{\alpha},\tilde{\beta}))=E(h(\tilde{\beta},\tilde{\alpha}))$.

Example 2.1.15 For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, we get

$$E_1(f(\tilde{\alpha},\tilde{\beta}))$$

$$= \frac{1}{l(\sqrt{2}-1)} \sum_{i=1}^{l} \left(\sin \frac{\pi \begin{pmatrix} 2 + \min\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ + \max\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{8} \right.$$

$$\left. + \sin \frac{\pi \begin{pmatrix} 2 - \min\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ - \max\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{8} - 1 \right) \tag{2.32}$$

$$E_2(f(\tilde{\alpha}, \tilde{\beta}))$$

$$= \frac{1}{l(\sqrt{2}-1)} \sum_{i=1}^{l} \left(\cos \frac{\pi \begin{pmatrix} 2 + \min\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ + \max\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{8} \right.$$

$$\left. + \cos \frac{\pi \begin{pmatrix} 2 - \min\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ - \max\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{8} - 1 \right) \tag{2.33}$$

$$E_3(f(\tilde{\alpha}, \tilde{\beta})) = -\frac{1}{l \ln 2} \sum_{i=1}^{l} \left(\frac{\begin{pmatrix} 2 + \min\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ + \max\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{4} \right.$$

$$\times \ln \frac{\begin{pmatrix} 2 + \min\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ + \max\{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}|, |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{4}$$

$$+\frac{\begin{pmatrix} 2-\min\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ -\max\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{4} \times \ln\frac{\begin{pmatrix} 2-\min\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\} \\ -\max\{|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}|,|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}|\} \end{pmatrix}}{4}\Bigg) \quad (2.34)$$

Corollary 2.1.16 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, and E be the entropy of GHFE, then $E((h(\tilde{\alpha},\tilde{\beta}))^c)$ is the similarity measure of the GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$.

Corollary 2.1.17 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, $|\mu_{\tilde{\alpha}}^{\sigma(i)}-\mu_{\tilde{\beta}}^{\sigma(i)}| \leq |\mu_{\tilde{\alpha}}^{\sigma(i+1)}-\mu_{\tilde{\beta}}^{\sigma(i+1)}|$, $|\nu_{\tilde{\alpha}}^{\sigma(i)}-\nu_{\tilde{\beta}}^{\sigma(i)}| \geq |\nu_{\tilde{\alpha}}^{\sigma(i+1)}-\nu_{\tilde{\beta}}^{\sigma(i+1)}|$, $i=1,2,\ldots,l-1$, and we define a GHFE $k(\tilde{\alpha},\tilde{\beta})$ as follows:

$$\begin{aligned} & k(\tilde{\alpha},\tilde{\beta}) \\ & = \left\{ \left(\frac{1+\min\left\{ \begin{array}{c} |\mu_{\tilde{\alpha}}^{\sigma(1)}-\mu_{\tilde{\beta}}^{\sigma(1)}|^p, \\ |\nu_{\tilde{\beta}}^{\sigma(1)}-\nu_{\tilde{\beta}}^{\sigma(1)}|^p \end{array} \right\}}{2}, \frac{1-\max\left\{ \begin{array}{c} |\mu_{\tilde{\alpha}}^{\sigma(1)}-\mu_{\tilde{\beta}}^{\sigma(1)}|^p, \\ |\nu_{\tilde{\beta}}^{\sigma(1)}-\nu_{\tilde{\beta}}^{\sigma(1)}|^p \end{array} \right\}}{2} \right), \right. \\ & \left(\frac{1+\min\left\{ \begin{array}{c} |\mu_{\tilde{\alpha}}^{\sigma(2)}-\mu_{\tilde{\beta}}^{\sigma(2)}|^p, \\ |\nu_{\tilde{\beta}}^{\sigma(2)}-\nu_{\tilde{\beta}}^{\sigma(2)}|^p \end{array} \right\}}{2}, \frac{1-\max\left\{ \begin{array}{c} |\mu_{\tilde{\alpha}}^{\sigma(2)}-\mu_{\tilde{\beta}}^{\sigma(2)}|^p, \\ |\nu_{\tilde{\beta}}^{\sigma(2)}-\nu_{\tilde{\beta}}^{\sigma(2)}|^p \end{array} \right\}}{2} \right), \ldots, \\ & \left. \left(\frac{1+\min\left\{ \begin{array}{c} |\mu_{\tilde{\alpha}}^{\sigma(l)}-\mu_{\tilde{\beta}}^{\sigma(l)}|^p, \\ |\nu_{\tilde{\beta}}^{\sigma(l)}-\nu_{\tilde{\beta}}^{\sigma(l)}|^p \end{array} \right\}}{2}, \frac{1-\max\left\{ \begin{array}{c} |\mu_{\tilde{\alpha}}^{\sigma(l)}-\mu_{\tilde{\beta}}^{\sigma(l)}|^p, \\ |\nu_{\tilde{\beta}}^{\sigma(l)}-\nu_{\tilde{\beta}}^{\sigma(l)}|^p \end{array} \right\}}{2} \right) \right\}, p>0, \quad (2.35) \end{aligned}$$

then $E(k(\tilde{\alpha},\tilde{\beta}))$ is the similarity measure of $\tilde{\alpha}$ and $\tilde{\beta}$.

Theorem 2.1.18 *For a GHFE $\tilde{\alpha}$, let $|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| \leq |\mu_{\tilde{\alpha}}^{\sigma(i+1)} - \nu_{\tilde{\alpha}}^{\sigma(i+1)}|$, $i = 1, 2, \ldots, l-1$, and we define two GHFEs $m(\tilde{\alpha})$ and $n(\tilde{\alpha})$ as follows:*

$$
\begin{aligned}
m(\tilde{\alpha}) = \Bigg\{ & \left(\frac{1 + (\mu_{\tilde{\alpha}}^{\sigma(1)} - \nu_{\tilde{\alpha}}^{\sigma(1)})^4}{2}, \frac{1 - |\mu_{\tilde{\alpha}}^{\sigma(1)} - \nu_{\tilde{\alpha}}^{\sigma(1)}|}{2} \right), \\
& \left(\frac{1 + (\mu_{\tilde{\alpha}}^{\sigma(2)} - \nu_{\tilde{\alpha}}^{\sigma(2)})^4}{2}, \frac{1 - |\mu_{\tilde{\alpha}}^{\sigma(2)} - \nu_{\tilde{\alpha}}^{\sigma(2)}|}{2} \right), \\
& \ldots, \left(\frac{1 + (\mu_{\tilde{\alpha}}^{\sigma(l)} - \nu_{\tilde{\alpha}}^{\sigma(l)})^4}{2}, \frac{1 - |\mu_{\tilde{\alpha}}^{\sigma(l)} - \nu_{\tilde{\alpha}}^{\sigma(l)}|}{2} \right) \Bigg\}
\end{aligned} \tag{2.36}
$$

$$
\begin{aligned}
n(\tilde{\alpha}) = \Bigg\{ & \left(\frac{1 - |\mu_{\tilde{\alpha}}^{\sigma(1)} - \nu_{\tilde{\alpha}}^{\sigma(1)}|}{2}, \frac{1 + (\mu_{\tilde{\alpha}}^{\sigma(1)} - \nu_{\tilde{\alpha}}^{\sigma(1)})^2}{2} \right), \\
& \left(\frac{1 - |\mu_{\tilde{\alpha}}^{\sigma(2)} - \nu_{\tilde{\alpha}}^{\sigma(2)}|}{2}, \frac{1 + (\mu_{\tilde{\alpha}}^{\sigma(2)} - \nu_{\tilde{\alpha}}^{\sigma(2)})^2}{2} \right), \\
& \ldots, \left(\frac{1 - |\mu_{\tilde{\alpha}}^{\sigma(l)} - \nu_{\tilde{\alpha}}^{\sigma(l)}|}{2}, \frac{1 + (\mu_{\tilde{\alpha}}^{\sigma(l)} - \nu_{\tilde{\alpha}}^{\sigma(l)})^2}{2} \right) \Bigg\},
\end{aligned} \tag{2.37}
$$

then $S(m(\tilde{\alpha}), n(\tilde{\alpha}))$ is the entropy of $\tilde{\alpha}$.

Proof (1) $S(m(\tilde{\alpha}), n(\tilde{\alpha})) = 0 \Leftrightarrow m(\tilde{\alpha}) = (1, 0)$, $n(\tilde{\alpha}) = (0, 1)$ or $m(\tilde{\alpha}) = (0, 1)$, $n(\tilde{\alpha}) = (1, 0) \Leftrightarrow \tilde{\alpha} = (1, 0)$ or $\tilde{\alpha} = (0, 1)$;

(2) $S(m(\tilde{\alpha}), n(\tilde{\alpha})) = 1 \Leftrightarrow \frac{1+(\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^4}{2} = \frac{1-|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|}{2}$ and $\frac{1-|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|}{2} = \frac{1+(\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2}{2}$ for $i = 1, 2, \ldots, l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $i = 1, 2, \ldots, l$;

(3) Since $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\alpha}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, which implies $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, we have $\frac{|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|}{2} \geq \frac{|\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|}{2}$, $i = 1, 2, \ldots, l$, which means that $\frac{1+(\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^4}{2} \geq \frac{1+(\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)})^4}{2} \geq \frac{1-|\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|}{2} \geq \frac{1-|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|}{2}$ and $\frac{1-|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}|}{2} \leq \frac{1-|\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|}{2} \leq \frac{1+(\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)})^2}{2} \leq \frac{1+(\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2}{2}$, $i = 1, 2, \ldots, l$. Therefore, from the definition of similarity measure of GHFEs, we have $S(m(\tilde{\alpha}), n(\tilde{\alpha})) \leq S(m(\tilde{\beta}), n(\tilde{\alpha})) \leq S(m(\tilde{\beta}), n(\tilde{\beta}))$. With the

same reason, when $\mu_{\tilde{\alpha}}^{\sigma(i)} \geq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\beta}}^{\sigma(i)} \geq \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, we can also prove $S(m(\tilde{\alpha}), n(\tilde{\alpha})) \leq S(m(\tilde{\beta}), n(\tilde{\beta}))$;

(4) $S(m(\tilde{\alpha}), n(\tilde{\alpha})) = S(m(\tilde{\alpha})^c, n(\tilde{\alpha})^c)$.

Corollary 2.1.19 Suppose that S is the similarity measure for GHFEs, then $S(m(\tilde{\alpha})^c, n(\tilde{\alpha})^c)$ is also the entropy of the GHFE $\tilde{\alpha}$.

Example 2.1.20 For a GHFE $\tilde{\alpha}$, we have some entropy formulas

$$\begin{aligned} &S_1(m(\tilde{\alpha}), n(\tilde{\alpha})) \\ &\quad = 1 - \frac{1}{4l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} \Big((\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^4 + 2|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| + (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2 \Big) \end{aligned} \tag{2.38}$$

$$\begin{aligned} &S_4(m(\tilde{\alpha}), n(\tilde{\alpha})) \\ &\quad = 1 - \frac{1}{4} \Big(\max_i \{ (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^4 + 2|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| + (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2 \} \Big) \end{aligned} \tag{2.39}$$

$$\begin{aligned} &S_7(m(\tilde{\alpha}), n(\tilde{\alpha})) \\ &\quad = 1 - \frac{1}{8} \Bigg(\frac{1}{l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} \Big((\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^4 + 2|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| + (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2 \Big) \\ &\qquad + \max_i \{ (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^4 + 2|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| + (\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)})^2 \} \Bigg) \end{aligned} \tag{2.40}$$

2.2 Cross-entropy measures for GHFEs

In this section, we shall present the axiomatic definition of cross-entropy measure for GHFE motivated by Bhandai and Pal [6], Shang and Jiang [49], Vlachos and Sergiadis [57], Hung and Yang [27], and Xu and Xia [76], from which we can get some entropy measures for GHFEs.

Definition 2.2.1 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, then the cross-entropy $C(\tilde{\alpha}, \tilde{\beta})$ of $\tilde{\alpha}$ and $\tilde{\beta}$ should satisfy the following conditions:

(1) $C(\tilde{\alpha}, \tilde{\beta}) \geq 0$;

(2) $C(\tilde{\alpha}, \tilde{\beta}) = 0$ if and only if $\mu_{\tilde{\alpha}}^{\sigma(i)} = \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$.

Based on Definition 2.2.1, we can construct two cross-entropy formulas of $\tilde{\alpha}$ and $\tilde{\beta}$ defined as

$$
\begin{aligned}
C_1(\tilde{\alpha},\tilde{\beta}) = \frac{1}{lT}\sum_{i=1}^{l}\Bigg(& \frac{(1+q\mu_{\tilde{\alpha}}^{\sigma(i)})\ln(1+q\mu_{\tilde{\alpha}}^{\sigma(i)})+(1+q\mu_{\tilde{\beta}}^{\sigma(i)})\ln(1+q\mu_{\tilde{\beta}}^{\sigma(i)})}{2} \\
& -\frac{2+q\mu_{\tilde{\alpha}}^{\sigma(i)}+q\mu_{\tilde{\beta}}^{\sigma(i)}}{2}\ln\frac{2+q\mu_{\tilde{\alpha}}^{\sigma(i)}+q\mu_{\tilde{\beta}}^{\sigma(i)}}{2} \\
& +\frac{(1+q\nu_{\tilde{\alpha}}^{\sigma(i)})\ln(1+q\nu_{\tilde{\alpha}}^{\sigma(i)})+(1+q\nu_{\tilde{\beta}}^{\sigma(i)})\ln(1+q\nu_{\tilde{\beta}}^{\sigma(i)})}{2} \\
& -\frac{2+q\nu_{\tilde{\alpha}}^{\sigma(i)}+q\nu_{\tilde{\beta}}^{\sigma(i)}}{2}\ln\frac{2+q\nu_{\tilde{\alpha}}^{\sigma(i)}+q\nu_{\tilde{\beta}}^{\sigma(i)}}{2}\Bigg),
\end{aligned}
\tag{2.41}
$$

where $T=(1+q)\ln(1+q)-(2+q)(\ln(2+q)-\ln 2)$ and $q>0$.

$$
\begin{aligned}
C_2(\tilde{\alpha},\tilde{\beta}) = \frac{1}{(1-2^{1-p})l}\sum_{i=1}^{l}\Bigg(& \frac{(\mu_{\tilde{\alpha}}^{\sigma(i)})^p+(\mu_{\tilde{\beta}}^{\sigma(i)})^p}{2}+\frac{(\nu_{\tilde{\alpha}}^{\sigma(i)})^p+(\nu_{\tilde{\beta}}^{\sigma(i)})^p}{2} \\
& -\left(\frac{\mu_{\tilde{\alpha}}^{\sigma(i)}+\mu_{\tilde{\beta}}^{\sigma(i)}}{2}\right)^p-\left(\frac{\nu_{\tilde{\alpha}}^{\sigma(i)}+\nu_{\tilde{\beta}}^{\sigma(i)}}{2}\right)^p\Bigg),\ p>1.
\end{aligned}
\tag{2.42}
$$

Remark 2.2.2 (1) Since $\frac{dT}{dq}=\ln\frac{2+2q}{1+q}>0$, then T is an increasing function about q, and thus $T>0$. In addition, let $f(x)=(1+qx)\ln(1+qx)$, $0\le x\le 1$, then $\frac{df(x)}{dx}=q\ln(1+qx)+q\ge 0$ and $\frac{d^2f(x)}{dx^2}=\frac{q^2}{1+qx}>0$. Then $f(x)$ is a concave upward function of x and $f(\frac{a+b}{2})=\frac{f(a)}{2}+\frac{f(b)}{2}$ if and only if $a=b$. Therefore, $C_1(\tilde{\alpha},\tilde{\beta})\ge 0$ and $C_1(\tilde{\alpha},\tilde{\beta})=0$ if and only if $\mu_{\tilde{\alpha}}^{\sigma(i)}=\mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}=\nu_{\tilde{\beta}}^{\sigma(i)}$, $i=1,2,\ldots,l$. According to Definition 2.2.1, $C_1(\tilde{\alpha},\tilde{\beta})$ is the cross-entropy of $\tilde{\alpha}$ and $\tilde{\beta}$.

(2) Let $g(x)=x^p$, $0\le x\le 1$ and $p>1$, since $\frac{dg(x)}{dx}=px^{p-1}$ and $\frac{d^2g(x)}{dx^2}=p(p-1)x^{p-2}>0$, $g(x)$ is a concave upward function of x, and then $C_2((\tilde{\alpha},\tilde{\beta}))\ge 0$ and $C_2(\tilde{\alpha},\tilde{\beta})=0$ if and only if $\mu_{\tilde{\alpha}}^{\sigma(i)}=\mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)}=\nu_{\tilde{\beta}}^{\sigma(i)}$, $i=1,2,\ldots,l$. From Definition 2.2.1, $C_2(\tilde{\alpha},\tilde{\beta})$ is the cross-entropy of $\tilde{\alpha}$ and $\tilde{\beta}$.

Theorem 2.2.3 *Let $\tilde{\alpha}$ be a GHFE, then $E(\tilde{\alpha}) = 1 - C_1(\tilde{\alpha}, \tilde{\alpha}^c)$ is the entropy of $\tilde{\alpha}$.*

Proof Let

$$\begin{aligned} E(\tilde{\alpha}) &= 1 - C_1(\tilde{\alpha}, \tilde{\alpha}^c) \\ &= 1 - \frac{2}{l_{\tilde{\alpha}}T}\sum_{i=1}^{l_{\tilde{\alpha}}}\left(\frac{(1+q\mu_{\tilde{\alpha}}^{\sigma(i)})\ln(1+q\mu_{\tilde{\alpha}}^{\sigma(i)}) + (1+q\nu_{\tilde{\alpha}}^{\sigma(i)})\ln(1+q\nu_{\tilde{\alpha}}^{\sigma(i)})}{2}\right. \\ &\qquad \left. - \frac{2+q\mu_{\tilde{\alpha}}^{\sigma(i)}+q\nu_{\tilde{\alpha}}^{\sigma(i)}}{2}\ln\frac{2+q\mu_{\tilde{\alpha}}^{\sigma(i)}+q\nu_{\tilde{\alpha}}^{\sigma(i)}}{2}\right), \ q > 0, \end{aligned} \tag{2.43}$$

where $T = (1+q)\ln(1+q) - (2+q)(\ln(2+q) - \ln 2)$.

(1) By the definition of $E(\tilde{\alpha})$, $E(\tilde{\alpha}) = 0 \Leftrightarrow C_1(\tilde{\alpha}, \tilde{\alpha}^c) = 1 \Leftrightarrow \tilde{\alpha} = (1,0)$ or $\tilde{\alpha} = (0,1)$;

(2) $E(\tilde{\alpha}) = 1 \Leftrightarrow C_1(\tilde{\alpha}, \tilde{\alpha}^c) = 0 \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(3) If $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l_{\tilde{\alpha}}$, then we have $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, which implies $|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| \geq |\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|$, $i = 1, 2, \ldots, l$. Let $0 \leq x, y \leq 1$, $t = |x - y|$, and

$$\begin{aligned} f(x,y) = &\frac{(1+qx)\ln(1+qx) + (1+qy)\ln(1+qy)}{2} \\ &- \frac{2+qx+qy}{2}\ln\frac{2+qx+qy}{2}, \ q > 0. \end{aligned} \tag{2.44}$$

If $x \geq y$, then $x = t + y$, and

$$\begin{aligned} f(t,y) = &\frac{(1+q(t+y))\ln(1+q(t+y)) + (1+qy)\ln(1+qy)}{2} \\ &- \frac{2+q(t+y)+qy}{2}\ln\frac{2+q(t+y)+qy}{2}, \ q > 0. \end{aligned} \tag{2.45}$$

and thus

$$\begin{aligned} \frac{\partial f(t,y)}{\partial t} &= \frac{q + q\ln(1+q(t+y))}{2} - \frac{q}{2} - \frac{q}{2}\ln\frac{2+q(t+y)+qy}{2} \\ &= \frac{q}{2}\ln\frac{2(1+q(t+y))}{2+q(t+y)+qy} \geq 0, \ q > 0. \end{aligned} \tag{2.46}$$

Therefore, $f(x,y)$ is a nondecreasing function of $|x-y|$, for $x \geq y$. With the same reason, we can also prove that it is true for $x \leq y$. Hence $E(\tilde{\alpha}) \leq E(\tilde{\beta})$.

(4) $E(\tilde{\alpha}) = E(\tilde{\alpha}^c)$.

Theorem 2.2.4 *Let $\tilde{\alpha}$ be a GHFE, then $E(\tilde{\alpha}) = 1 - C_2(\tilde{\alpha}, \tilde{\alpha}^c)$ is the entropy of $\tilde{\alpha}$.*

Proof Let

$$\begin{aligned} E(\tilde{\alpha}) &= 1 - C_2(\tilde{\alpha}, \tilde{\alpha}^c) \\ &= 1 - \frac{2}{(1-2^{1-p})l} \sum_{i=1}^{l} \left(\frac{(\mu_{\tilde{\alpha}}^{\sigma(i)})^p + (\nu_{\tilde{\alpha}}^{\sigma(i)})^p}{2} - \left(\frac{\mu_{\tilde{\alpha}}^{\sigma(i)} + \nu_{\tilde{\alpha}}^{\sigma(i)}}{2} \right)^p \right), \ p > 1. \end{aligned} \tag{2.47}$$

(1) By the definition of $E(\tilde{\alpha})$, $E(\tilde{\alpha}) = 0 \Leftrightarrow C_2(\tilde{\alpha}, \tilde{\alpha}^c) = 1 \Leftrightarrow \tilde{\alpha} = (1,0)$ or $\tilde{\alpha} = (0,1)$;

(2) $E(\tilde{\alpha}) = 1 \Leftrightarrow C_2(\tilde{\alpha}, \tilde{\alpha}^c) = 0 \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$;

(3) If $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$ for $\mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l_{\tilde{\alpha}}$, which means $|\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}| \geq |\mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|$, $i = 1, 2, \ldots, l$. Let $t = |x-y|$ and

$$g(x,y) = \frac{x^p + y^p}{2} - \left(\frac{x+y}{2} \right)^p, \ 0 \leq x, y \leq 1, \ p > 1. \tag{2.48}$$

If $x \geq y$, then $x = t + y$, and

$$g(t,y) = \frac{(t+y)^p + y^p}{2} - \left(y + \frac{t}{2} \right)^p, \ p > 1. \tag{2.49}$$

Since

$$\frac{\partial g(t,y)}{\partial t} = \frac{p}{2} \left((t+y)^{p-1} - \left(y + \frac{t}{2} \right)^{p-1} \right), \ p > 1, \tag{2.50}$$

$g(x,y)$ is a nondecreasing function of $|x-y|$, for $x \geq y$. With the same reason, we can also prove that it is true for $x \leq y$. Hence $E(\tilde{\alpha}) \leq E(\tilde{\beta})$.

(4) $E(\tilde{\alpha}) = E(\tilde{\alpha}^c)$.

2.3 Methods based on information measures for multiple attribute decision making with generalized hesitant fuzzy information

Suppose that there are m alternatives y_i $(i = 1, 2, \ldots, m)$ and n attributes x_j $(j = 1, 2, \ldots, n)$ with the attribute weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \in [0, 1]$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$. Suppose that a decision organization is authorized to provide all the possible degrees that the alternative y_i satisfies the attribute x_j, denoted by a GHFE $\tilde{\alpha}_{ij}$.

In following, we develop two approach to multiple attribute group decision making with generalized hesitant fuzzy information. First, we extend the entropy method to generalized hesitant fuzzy environment and obtain the final optimal alternative by comparing the cross-entropy measures with the ideal solutions.

Approach I

Step 1. The decision organization provides all possible evaluations the alternative y_i under the attribute x_j, denoted by the GHFE $\tilde{\alpha}_{ij}$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots, n)$.

Step 2. If the information about the weight w_j of the attribute x_j is unknown completely, then we establish an exact model of entropy weights for determining the attribute weights:

$$w_j = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} E(\tilde{\alpha}_{ij})}{n - \sum_{j=1}^{n}\left(\frac{1}{m}\sum_{i=1}^{m} E(\tilde{\alpha}_{ij})\right)}, \; j = 1, 2, \ldots, n, \tag{2.51}$$

where $E(\tilde{\alpha}_{ij})$ is the entropy of $\tilde{\alpha}_{ij}$ given by Eqs. (2.43) or (2.47).

Step 3. Let J_1 and J_2 be the sets of benefit attributes and cost attributes, respectively. Suppose that the generalized hesitant fuzzy positive-ideal solution is $\tilde{\alpha}^+ = (\tilde{\alpha}_1^+, \tilde{\alpha}_2^+, \ldots, \tilde{\alpha}_n^+)$ and the generalized hesitant fuzzy negative-ideal solution is $\tilde{\alpha}^- = (\tilde{\alpha}_1^-, \tilde{\alpha}_2^-, \ldots, \tilde{\alpha}_n^-)$, where $\tilde{\alpha}_j^+ = (1, 0)$, $\tilde{\alpha}_j^- = (0, 1)$, $j \in J_1$ and $\tilde{\alpha}_j^+ = (0, 1)$, $\tilde{\alpha}_j^- = (1, 0)$, $j \in J_2$. Then we calculate the cross-entropy between the alternative

y_i and positive-ideal solution and the negative-ideal solution:

$$C^{+}(y_i) = \sum_{j=1}^{n} w_j C(\tilde{\alpha}_{ij}, \tilde{\alpha}_j^{+}),\ i = 1, 2, \ldots, m, \tag{2.52}$$

$$C^{-}(y_i) = \sum_{j=1}^{n} w_j C(\tilde{\alpha}_{ij}, \tilde{\alpha}_j^{-}),\ i = 1, 2, \ldots, m. \tag{2.53}$$

Step 4. Calculate the closeness degree of the alternative y_i to the positive-ideal solution by using the following

$$C(y_i) = \frac{C^{+}(y_i)}{C^{+}(y_i) + C^{-}(y_i)},\quad i = 1, 2, \ldots, m. \tag{2.54}$$

Step 5. Rank the alternatives y_i $(i = 1, 2, \ldots, m)$ according the values of $C(y_i)$ $(i = 1, 2, \ldots, m)$ in ascending order, and the smaller the value of $C(y_i)$, the better the alternative y_i.

Next, if we utilize the maximizing deviation method to derive the weight vector of the attributes in Step 2 of Approach I, and use the TOPSIS method [58] to compare the alternatives in Steps 3 and 4 of Approach I, then we can obtain the following approach:

Approach II

Step 1. For this step, see Approach I.

Step 2. Utilize the maximizing deviation method to calculate the attribute weight w_j of the attribute x_j:

$$w_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})}{\sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})},\ j = 1, 2, \ldots, n, \tag{2.55}$$

where $d(\tilde{\alpha}_{ij}, \tilde{\alpha}_{kj})$ is distance between $\tilde{\alpha}_{ij}$ and $\tilde{\alpha}_{kj}$ such that for two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, the distance between $\tilde{\alpha}$ and $\tilde{\beta}$, denoted as $d(\tilde{\alpha}, \tilde{\beta})$, defined by

$$d(\tilde{\alpha}, \tilde{\beta}) = \frac{1}{2l} \sum_{i=1}^{l} \left(|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}| + |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}| \right). \tag{2.56}$$

Step 3. Calculate the distance between the alternative y_i and the positive-ideal solution $\tilde{\alpha}^+ = (\tilde{\alpha}_1^+, \tilde{\alpha}_2^+, \ldots, \tilde{\alpha}_n^+)$ and the negative-ideal solution $\tilde{\alpha}^- = (\tilde{\alpha}_1^-, \tilde{\alpha}_2^-, \ldots, \tilde{\alpha}_n^-)$:

$$d^+(y_i) = \sum_{j=1}^{n} \left(w_j d(\tilde{\alpha}_{ij}, \tilde{\alpha}_j^+) \right), \ i = 1, 2, \ldots, m, \tag{2.57}$$

$$d^-(y_i) = \sum_{j=1}^{n} \left(w_j d(\tilde{\alpha}_{ij}, \tilde{\alpha}_j^-) \right), \ i = 1, 2, \ldots, m. \tag{2.58}$$

Step 4. Calculate the closeness degree of the alternative y_i to the positive-ideal solution $\tilde{\alpha}^+$ by using the following

$$D(y_i) = \frac{d^-(y_i)}{d^-(y_i) + d^+(y_i)}, \quad i = 1, 2, \ldots, m. \tag{2.59}$$

Step 5. Rank the alternatives y_i $(i = 1, 2, \ldots, m)$ according the values of $D(y_i)$ $(i = 1, 2, \ldots, m)$ in descending order, and the larger the value of $D(y_i)$, the better the alternative y_i.

In the following, we use a multiple attribute decision making problem of determining what kind of air-conditioning systems should be installed in a library (adapted from [78] [88]) to illustrate the proposed approaches.

Example 2.3.1 A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers four feasible alternatives y_i $(i = 1, 2, 3, 4)$, which might be adapted to the physical structure of the library. The offered air-conditioning system must take a decision according to the following five attributes: (1) performance (x_1), (2) maintainability (x_2), (3) flexibility (x_3), (4) cost (x_4), (5) safety (x_5). Let $J = \{x_1, x_2, x_3, x_4, x_5\}$ be the set of five attributes, and assume that x_1, x_2, x_3 and x_5 are benefit attributes and x_4 is cost attribute. That is, $J_1 = \{x_1, x_2, x_3, x_5\}$ and $J_2 = \{x_4\}$.

To get the optimal alternative, the following steps are given if Approach I is used:

Step 1. The decision organization provides all possible evaluations of the alternative y_i, by a GHFE $\tilde{\alpha}_{ij}$, with respect to the attribute x_j, listed in Table 2.1 (i.e. generalized hesitant fuzzy decision matrix $D = (\tilde{\alpha}_{ij})_{4\times 5}$).

Table 2.1: Generalized hesitant fuzzy decision matrix

	x_1	x_2
y_1	$\{(0.3, 0.2), (0.3, 0.4)\}$	$\{(0.6, 0.2), (0.5, 0.2), (0.4, 0.3)\}$
y_2	$\{(0.7, 0.2), (0.5, 0.2)\}$	$\{(0.5, 0.1), (0.4, 0.2), (0.3, 0.1)\}$
y_3	$\{(0.6, 0.3), (0.5, 0.2)\}$	$\{(0.9, 0.05), (0.8, 0.1), (0.7, 0.1)\}$
y_4	$\{(0.5, 0.3), (0.5, 0.4)\}$	$\{(0.8, 0.1), (0.8, 0.3), (0.6, 0.3)\}$
	x_3	x_4
y_1	$\{(0.4, 0.5), (0.3, 0.4)\}$	$\{(0.4, 0.2), (0.3, 0.4), (0.2, 0.6), (0.2, 0.7)\}$
y_2	$\{(0.8, 0.1), (0.7, 0.2)\}$	$\{(0.8, 0.1), (0.7, 0.2), (0.6, 0.3), (0.5, 0.3)\}$
y_3	$\{(0.4, 0.3), (0.4, 0.4)\}$	$\{(0.8, 0.1), (0.7, 0.2), (0.6, 0.1), (0.4, 0, 1)\}$
y_4	$\{(0.7, 0.3), (0.5, 0.4)\}$	$\{(0.8, 0.1), (0.7, 0.3), (0.6, 0.3), (0.4, 0, 2)\}$
	x_5	
y_1	$\{(0.8, 0.1), (0.7, 0.2)\}$	
y_2	$\{(0.7, 0.2), (0.6, 0.3)\}$	
y_3	$\{(0.2, 0.5), (0.2, 0.6)\}$	
y_4	$\{(0.6, 0.3), (0.4, 0.5)\}$	

Step 2. Suppose that the information about the attribute weight w_j of the attribute x_j is unknown completely, then we utilize Eq. (2.43) (let $q = 2$) to calculate the entropy matrix (see Table 2.2)

and then by Eq. (2.51), we can obtain the attribute weight vector:

$$w = (0.0927, 0.2995, 0.1416, 0.2471, 0.2191)^T.$$

Step 3. Utilize Eqs. (2.41) (let $q = 2$), (2.52) and (2.53) to calculate the cross-entropy between the alternative y_i and the positive-ideal solution $\tilde{\alpha}^+$ or the

Table 2.2: Entropy matrix determined by the cross-entropy C_1

	x_1	x_2	x_3	x_4	x_5
y_1	0.9880	0.9052	0.9893	0.8788	0.6204
y_2	0.8219	0.9014	0.6204	0.7769	0.8273
y_3	0.9037	0.4578	0.9944	0.7100	0.8635
y_4	0.9737	0.7249	0.9180	0.8006	0.9495

negative-ideal solution $\tilde{\alpha}^-$:

$$C^+(y_1) = 0.2193, C^+(y_2) = 0.1086, C^+(y_3) = 0.1749, C^+(y_4) = 0.1352,$$
$$C^-(y_1) = 0.3720, C^-(y_2) = 0.5161, C^-(y_3) = 0.4785, C^-(y_4) = 0.4553.$$

Step 4. Utilize Eq. (2.54) to calculate the closeness degree of the alternative y_i to positive-ideal solution $\tilde{\alpha}^+$:

$$C(y_1) = 0.3709, C(y_2) = 0.1739, C(y_3) = 0.2677, C(y_4) = 0.2290.$$

Step 5. Rank the alternatives y_i $(i = 1, 2, 3, 4)$ according to the values of $C(y_i)$ $(i = 1, 2, 3, 4)$ in ascending order:

$$y_2 \succ y_4 \succ y_3 \succ y_1.$$

If we utilize Eq. (2.47) (let $p = 2$) in the above Approach I, then the following steps are given:

Step 1. See the above.

Step 2. Utilize utilize Eq. (2.47) (let $p = 2$) to calculate the entropy matrix (see Table 2.3)

and then by Eq. (2.51), we can get the attribute weight vector:

$$w = (0.0909, 0.2982, 0.1448, 0.2458, 0.2203)^T.$$

Step 3. Calculate the cross-entropy between the alternative y_i and the positive-ideal solution $\tilde{\alpha}^+$ or the negative-ideal solution $\tilde{\alpha}^-$ by Eqs. (2.42) (let

Table 2.3: Entropy matrix determined by the cross-entropy C_2

	x_1	x_2	x_3	x_4	x_5
y_1	0.9900	0.9133	0.9900	0.8850	0.6300
y_2	0.8300	0.9200	0.6300	0.7825	0.8300
y_3	0.9100	0.4758	0.9950	0.7300	0.8750
y_4	0.9750	0.7233	0.9150	0.8050	0.9500

$p = 2$), (2.52) and (2.53):

$$C^+(y_1) = 0.2268, C^+(y_2) = 0.1183, C^+(y_3) = 0.1817, C^+(y_4) = 0.1342,$$
$$C^-(y_1) = 0.3748, C^-(y_2) = 0.5137, C^-(y_3) = 0.4757, C^-(y_4) = 0.4534.$$

Step 4. Utilize Eq. (2.54) to calculate the closeness degree of the alternative y_i to positive-ideal solution $\tilde{\alpha}^+$:

$$C(y_1) = 0.3769, C(y_2) = 0.1872, C(y_3) = 0.2764, C(y_4) = 0.2283.$$

Step 5. Rank the alternatives y_i $(i = 1, 2, 3, 4)$ according to the values of $C(y_i)$ $(i = 1, 2, 3, 4)$ in ascending order:

$$y_2 \succ y_4 \succ y_3 \succ y_1.$$

If we use Approach II, then the following steps are given:

Step 1. For this step, see Approach I.

Step 2. Calculate the attribute weight w_j of the attribute x_j by Eqs. (2.55) and (2.56):

$$w = (0.1375, 0.1818, 0.2040, 0.1973, 0.2794)^T.$$

Step 3. Utilize Eqs. (2.57) and (2.58) to calculate the distance between the alternative y_i and the positive-ideal solution $\tilde{\alpha}^+$ or the negative-ideal solution $\tilde{\alpha}^-$:

$$d^+(y_1) = 0.4219, d^+(y_2) = 0.2893, d^+(y_3) = 0.4087, d^+(y_4) = 0.3653,$$
$$d^-(y_1) = 0.5781, d^-(y_2) = 0.7107, d^-(y_3) = 0.5913, d^-(y_4) = 0.6347.$$

Step 4. Calculate the closeness degree of the alternative y_i to negative-ideal solution $\tilde{\alpha}^+$ by Eq. (2.59):

$$D(y_1) = 0.5781, D(y_2) = 0.7107, D(y_3) = 0.5913, D(y_4) = 0.6347.$$

Step 5. Rank the alternatives y_i $(i = 1, 2, 3, 4)$ according to the values of $D(y_i)$ $(i = 1, 2, 3, 4)$ in descending order:

$$y_2 \succ y_4 \succ y_3 \succ y_1.$$

which is the same result as that in Approach I.

If we use the extension principle [47] of generalized hesitant fuzzy sets (i.e. we conduct Arithmetic Mean associated with the IFA operator [68]) to aggregate the generalized hesitant fuzzy information for each alternative, then by the score function [47] of GHFEs, we get the score $s(y_i)$ of each alternative y_i $(i = 1, 2, 3, 4)$:

$$s(y_1) = 0.9783, s(y_2) = 0.9962, s(y_3) = 0.9916, s(y_4) = 0.9937.$$

Ranking the alternatives y_i $(i = 1, 2, 3, 4)$ according to the values of $s(y_i)$ $(i = 1, 2, 3, 4)$ in ascending order, we obtain the same ranking order result: $y_2 \succ y_4 \succ y_3 \succ y_1$.

From the analysis presented above, when comparing three approaches, we know that the first approach focuses on the entropy and cross-entropy measures, the second one utilize the distance measures to apply the TOPSIS method, and both of them are suitable for dealing with the situations that the weight vector of the attributes are unknown; the last one is only suitable for the situations that the weights of the attributes are equal. The first two approaches are much simpler than the last one, because the aggregation operator in the last approach need a lot of computation.

2.4 Subsethood measures for GHFEs

The purpose of this section is to establish a unified formulation of subsethood, entropy, cardinality and similarity for genralized hesitant fuzzy elements. We present an axiomatic skeleton for subsethood measures in the generalized hesitant fuzzy setting, in order for subsethood to reduce an entropy measure. The notion of average possible cardinality is presented and its connection to least and biggest cardinalities is established. Moreover, the entropy-subsethood and entropy theorems in generalized hesitant fuzzy setting are stated and algebraically proved, which generalize the works of Kosko [29] for FSs and Liu and Xiong [36] for IFSs. Finally, we investigate the relationship between generalized hesitant fuzzy subsethood and generalized hesitant similarity measures.

According to Section 2.1, we define the inclusion between two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$ as follows:

$$\tilde{\alpha} \subset \tilde{\beta} \text{ if and only if } \mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}, \text{ for } i = 1, 2, \ldots, l. \tag{2.60}$$

In this section, we shall present the axiomatic definition of subsethood measure for GHFE motivated by Liu and Xiong [36], Vlachos and Sergiadis [57] and Park et al. [43], from which we can establish a connection between subsethood, entropy and similarity measures for GHFEs.

Definition 2.4.1 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, then the subsethood measure of $\tilde{\alpha}$ to $\tilde{\beta}$, denoted as $s(\tilde{\alpha}, \tilde{\beta})$, should satisfy the following conditions:

(1) $s(\tilde{\alpha}, \tilde{\beta}) = 1$ if and only if $\tilde{\alpha} \subset \tilde{\beta}$, i.e., $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, for $i = 1, 2, \ldots, l$;

(2) If $\tilde{\alpha}^c \subset \tilde{\alpha}$, then $s(\tilde{\alpha}, \tilde{\alpha}^c) = 0$ if and only if $\tilde{\alpha} = (1, 0)$;

(3) If $\tilde{\beta} \subset \tilde{\alpha}_1 \subset \tilde{\alpha}_2$, then $s(\tilde{\alpha}_1, \tilde{\beta}) \geq s(\tilde{\alpha}_2, \tilde{\beta})$, and if $\tilde{\beta}_1 \subset \tilde{\beta}_2$, then $s(\tilde{\alpha}, \tilde{\beta}_1) \leq s(\tilde{\alpha}, \tilde{\beta}_2)$.

Theorem 2.4.2 *Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, then*

$$s_1(\tilde{\alpha}, \tilde{\beta}) = 1 - \frac{\sum_{i=1}^{l} \left(\max\{0, \mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}\} + \max\{0, \nu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\} \right)}{\sum_{i=1}^{l} \left(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)} \right)} \tag{2.61}$$

is subsethood measure of $\tilde{\alpha}$ to $\tilde{\beta}$.

Proof (1) $s_1(\tilde{\alpha}, \tilde{\beta}) = 1 \Leftrightarrow \max\{0, \mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}\} = 0$ and $\max\{0, \nu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\} = 0$, $i = 1, 2, \ldots, l \Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$.

(2) Suppose that $\tilde{\alpha}^c \subset \tilde{\alpha}$, then $\mu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $i = 1, 2, \ldots, l$. From Eq. (2.61), we obtain

$$\begin{aligned} s_1(\tilde{\alpha}, \tilde{\alpha}^c) &= 1 - \frac{2\sum_{i=1}^{l}\left(\max\{0, \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\}\right)}{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\right)} \\ &= 1 - \frac{2\sum_{i=1}^{l}\left(\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\right)}{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\right)}. \end{aligned} \tag{2.62}$$

Thus, we have $s_1(\tilde{\alpha}, \tilde{\alpha}^c) = 0 \Leftrightarrow \tilde{\alpha} = (1, 0)$.

(3) Suppose that $\tilde{\beta} \subset \tilde{\alpha}_1 \subset \tilde{\alpha}_2$, then $\mu_{\tilde{\alpha}_2}^{\sigma(i)} \geq \mu_{\tilde{\alpha}_1}^{\sigma(i)} \geq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}_2}^{\sigma(i)} \leq \nu_{\tilde{\alpha}_1}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$ for $i = 1, 2, \ldots, l$. Since $\mu_{\tilde{\alpha}_2}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)} \geq \mu_{\tilde{\alpha}_1}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\alpha}_2}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\alpha}_1}^{\sigma(i)}$ for $i = 1, 2, \ldots, l$, we have

$$s_1(\tilde{\alpha}_1, \tilde{\beta}) = \frac{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}\right)}{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\alpha}_1}^{\sigma(i)} - \nu_{\tilde{\alpha}_1}^{\sigma(i)}\right)} \geq \frac{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\beta}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}\right)}{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\alpha}_2}^{\sigma(i)} - \nu_{\tilde{\alpha}_2}^{\sigma(i)}\right)} = s_1(\tilde{\alpha}_2, \tilde{\beta}). \tag{2.63}$$

Next, suppose that $\tilde{\beta}_1 \subset \tilde{\beta}_2$, then $\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}_1}^{\sigma(i)} \geq \mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}_2}^{\sigma(i)}$ and $\nu_{\tilde{\beta}_1}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}_2}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$. Due to the monotonicity of max operator, it follows that

$$\begin{aligned} s_1(\tilde{\alpha}, \tilde{\beta}_1) &= 1 - \frac{\sum_{i=1}^{l}\left(\max\{0, \mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}_1}^{\sigma(i)}\} + \max\{0, \nu_{\tilde{\beta}_1}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\}\right)}{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\right)} \\ &\leq 1 - \frac{\sum_{i=1}^{l}\left(\max\{0, \mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}_2}^{\sigma(i)}\} + \max\{0, \nu_{\tilde{\beta}_2}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\}\right)}{\sum_{i=1}^{l}\left(1 + \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)}\right)} \\ &= s_1(\tilde{\alpha}, \tilde{\beta}_2). \end{aligned} \tag{2.64}$$

Remark 2.4.3 Note that if $\tilde{\alpha} = (0, 1)$, Eq. (2.61) is undefined. However, since $(0, 1) \subset \tilde{\beta}$ for any GHFE $\tilde{\beta}$, by the definition, we have $s_1((0, 1), \tilde{\beta}) = 1$.

Theorem 2.4.4 *For two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$, we define*

$$s_2(\tilde{\alpha}, \tilde{\beta}) = \frac{1}{l}\sum_{i=1}^{l} \min\{1, g(\phi(\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)} + 1), \psi(\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)} + 1))\}, \qquad (2.65)$$

where $g : [0,2] \times [0,2] \to [0,2]$ is real function with the properties: 1) $x > y \Rightarrow g(x,z) < g(y,z)$, $g(z,x) > g(z,y)$ for $x,y,z \in [0,2]$; 2) $g(x,y) = 0 \Leftrightarrow x = 2, y = 0$; 3) $g(1,1) = 1$ and $\phi, \psi : [0,2] \to [0,2]$ are real functions with the following properties: 1) $x > y \Rightarrow \phi(x) > \phi(y)$, $\psi(x) > \psi(y)$ for $x, y \in [0,2]$; 2) $\phi(x) = 2 \Leftrightarrow x = 2$, $\psi(y) = 0 \Leftrightarrow y = 0$; 3) $\phi(1) = \psi(1) = 1$.

Then $s_2(\tilde{\alpha}, \tilde{\beta})$ is subsethood measure of $\tilde{\alpha}$ to $\tilde{\beta}$.

Proof (1) Suppose that $\tilde{\alpha} \subset \tilde{\beta}$, let $\alpha_i = \mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)} + 1$ and $\beta_i = \nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)} + 1$, $i = 1, 2, \ldots, l$. Since $\alpha_i \leq 1$ and $\beta_i \geq 1$, we have $\phi(\alpha_i) \leq 1$ and $\psi(\beta_i) \geq 1$ and then $g(\phi(\alpha_i), \psi(\beta_i)) \geq g(1, \psi(\beta_i)) \geq g(1,1) = 1$. Thus, we have $s_2(\tilde{\alpha}, \tilde{\beta}) = \frac{1}{l}\sum_{i=1}^{l} \min\{1, g(\phi(\alpha_i), \psi(\beta_i))\} = 1$. Suppose that $s_2(\tilde{\alpha}, \tilde{\beta}) = 1$, then $g(\phi(\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)} + 1), \psi(\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)} + 1)) \geq 1$, $i = 1, 2, \ldots, l$. Thus, we get $\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)} + 1 \leq 1$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)} + 1 \geq 1$, $i = 1, 2, \ldots, l$. In fact, suppose that there exists j such that $\mu_{\tilde{\alpha}}^{\sigma(j)} - \mu_{\tilde{\beta}}^{\sigma(j)} + 1 > 1$ or $\nu_{\tilde{\alpha}}^{\sigma(j)} - \nu_{\tilde{\beta}}^{\sigma(j)} + 1 < 1$. If $\alpha = \mu_{\tilde{\alpha}}^{\sigma(j)} - \mu_{\tilde{\beta}}^{\sigma(j)} + 1 > 1$, then $\phi(\alpha) > 1$ and thus $g(\phi(\alpha), \psi(\beta)) < g(1, \psi(\beta)) \leq g(1,1) = 1$, which is a contradiction. If $\beta = \nu_{\tilde{\alpha}}^{\sigma(j)} - \nu_{\tilde{\beta}}^{\sigma(j)} + 1 < 1$, then $\psi(\alpha) < 1$ and thus $g(\phi(\alpha), \psi(\beta)) < g(\phi(\alpha), 1)) \leq g(1,1) = 1$, a contradiction. So, we have $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, and thus $\tilde{\alpha} \subset \tilde{\beta}$.

(2) Suppose that $\tilde{\alpha}^c \subset \tilde{\alpha}$, then we have

$$\begin{aligned} s_2(\tilde{\alpha}, \tilde{\alpha}^c) = 0 &\Leftrightarrow \phi(\mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)} + 1) = 2, \psi(\nu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\alpha}}^{\sigma(i)} + 1) = 0, i = 1, 2, \ldots, l \\ &\Leftrightarrow \mu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\alpha}}^{\sigma(i)} + 1 = 2, \nu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\alpha}}^{\sigma(i)} + 1 = 0, i = 1, 2, \ldots, l \\ &\Leftrightarrow \tilde{\alpha} = (1, 0). \end{aligned}$$

(3) Suppose that $\tilde{\beta} \subset \tilde{\alpha}_1 \subset \tilde{\alpha}_2$, let $\alpha_{k,i} = \mu_{\tilde{\alpha}_k}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)} + 1$ and $\beta_{k,i} = \nu_{\tilde{\alpha}_k}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)} + 1$, $k = 1, 2$; $i = 1, 2, \ldots, l$. Then $\phi(\alpha_{1,i}) \leq \phi(\alpha_{2,i})$ and $\psi(\beta_{1,i}) \geq \psi(\beta_{2,i})$, $i = 1, 2, \ldots, l$, which implies $g(\phi(\alpha_{1,i}), \psi(\beta_{1,i})) \geq g(\phi(\alpha_{2,i}), \psi(\beta_{1,i})) \geq$

$g(\phi(\alpha_{2,i}), \psi(\beta_{2,i}))$, $i = 1, 2, \ldots, l$. Thus $s_2(\tilde{\alpha}_2, \tilde{\beta}) \leq s_2(\tilde{\alpha}_1, \tilde{\beta})$. With the same reason, we can prove that $s_2(\tilde{\alpha}, \tilde{\beta}_1) \leq s_2(\tilde{\alpha}, \tilde{\beta}_2)$ is also true for $\tilde{\beta}_1 \subset \tilde{\beta}_2$.

Now, to generalize the fuzzy entropy theorem in the setting of GHFEs, we start with the following definition.

Definition 2.4.5 Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, then
(1) $\tilde{\alpha}\tilde{\cup}\tilde{\beta} = \cup_{i=1}^{l}\{(\max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\beta}}^{\sigma(i)}\}, \min\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\beta}}^{\sigma(i)}\})\}$;
(2) $\tilde{\alpha}\tilde{\cap}\tilde{\beta} = \cup_{i=1}^{l}\{(\min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\beta}}^{\sigma(i)}\}, \max\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\beta}}^{\sigma(i)}\})\}$.

Theorem 2.4.6 *Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be three GHFEs , then*
(1) $\tilde{\alpha}\tilde{\cup}\tilde{\beta} = \tilde{\beta}\tilde{\cup}\tilde{\alpha}$;
(2) $\tilde{\alpha}\tilde{\cap}\tilde{\beta} = \tilde{\beta}\tilde{\cap}\tilde{\alpha}$;
(3) $\tilde{\alpha}^c\tilde{\cup}\tilde{\beta}^c = (\tilde{\alpha}\tilde{\cap}\tilde{\beta})^c$;
(4) $\tilde{\alpha}^c\tilde{\cap}\tilde{\beta}^c = (\tilde{\alpha}\tilde{\cup}\tilde{\beta})^c$;
(5) $\tilde{\alpha}\tilde{\cup}(\tilde{\beta}\tilde{\cap}\tilde{\gamma}) = (\tilde{\alpha}\tilde{\cup}\tilde{\beta})\tilde{\cap}(\tilde{\alpha}\tilde{\cup}\tilde{\gamma})$;
(6) $\tilde{\alpha}\tilde{\cap}(\tilde{\beta}\tilde{\cup}\tilde{\gamma}) = (\tilde{\alpha}\tilde{\cap}\tilde{\beta})\tilde{\cup}(\tilde{\alpha}\tilde{\cap}\tilde{\gamma})$.

Proof We prove only (5).
(5) By Definitions 2.1.4 and 2.4.5, we have

$$\begin{aligned}
\tilde{\alpha}\tilde{\cup}(\tilde{\beta}\tilde{\cap}\tilde{\gamma}) &= \cup_{i=1}^{l}\{(\max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \min\{\mu_{\tilde{\beta}}^{\sigma(i)}, \mu_{\tilde{\gamma}}^{\sigma(i)}\}, \min\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \max\{\nu_{\tilde{\beta}}^{\sigma(i)}, \nu_{\tilde{\gamma}}^{\sigma(i)}\})\} \\
&= \cup_{i=1}^{l}\Big\{\Big(\min\{\max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\beta}}^{\sigma(i)}\}, \max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\gamma}}^{\sigma(i)}\}\}, \\
&\qquad \max\{\min\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\beta}}^{\sigma(i)}\}, \min\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\gamma}}^{\sigma(i)}\}\}\Big)\Big\} \\
&= (\tilde{\alpha}\tilde{\cup}\tilde{\beta})\tilde{\cap}(\tilde{\alpha}\tilde{\cup}\tilde{\gamma}).
\end{aligned}$$

The relationships between subsethood measures and entropy have been studied by many authors under different environments, such as fuzzy sets, interval-valued fuzzy sets and interval-valued intuitionistic fuzzy sets. In the following, we investigate the relationships between generalized hesitant fuzzy subsethood measures and generalized hesitant fuzzy entropy:

Theorem 2.4.7 *Let $\tilde{\alpha}$ be a GHFE, then $E(\tilde{\alpha}) = s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c)$ is an entropy for $\tilde{\alpha}$.*

Proof (1) Suppose that $\tilde{\alpha} = (1,0)$ or $\tilde{\alpha} = (0,1)$, then $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c = (1,0)$ and $\tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c = (0,1)$. Since $\tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c = (\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c)^c$, we have $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c = (1,0) \supset (\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c)^c$ and thus by (2) of Definition 2.4.1, $E(\tilde{\alpha}) = s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) = 0$. Suppose that $E(\tilde{\alpha}) = s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) = 0$, that is $s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, (\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c)^c) = 0$. Then, since $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c \supset \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c$, by (2) of Definition 2.4.1, we obtain $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c = (1,0)$. Hence $\tilde{\alpha} = (1,0)$ or $\tilde{\alpha} = (0,1)$.

(2) Suppose that $\mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $i = 1, 2, \dots, l_{\tilde{\alpha}}$, then $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c = \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c$ and thus by (1) of Definition 2.4.1, $E(\tilde{\alpha}) = s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) = 1$. Suppose that $E(\tilde{\alpha}) = s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) = 1$, then from (1) of Definition 2.4.1, we deduce $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c = \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c$, which implies $\mu_{\tilde{\alpha}}^{\sigma(i)} = \nu_{\tilde{\alpha}}^{\sigma(i)}$ for $i = 1, 2, \dots, l_{\tilde{\alpha}}$.

(3) Suppose that $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \dots, l$, then $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\alpha}}^{\sigma(i)}$. By (3) of Definition 2.4.1, $s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) \leq s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\beta}\tilde{\cap}\tilde{\beta}^c) \leq s(\tilde{\beta}\tilde{\cup}\tilde{\beta}^c, \tilde{\beta}\tilde{\cap}\tilde{\beta}^c)$ and thus $E(\tilde{\alpha}) \leq E(\tilde{\beta})$. With the same reason, when $\mu_{\tilde{\alpha}}^{\sigma(i)} \geq \mu_{\tilde{\beta}}^{\sigma(i)}$ and $\nu_{\tilde{\alpha}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)}$, for $\mu_{\tilde{\beta}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)}$, $i = 1, 2, \dots, l$, we can prove $E(\tilde{\alpha}) \leq E(\tilde{\beta})$.

(4) $E(\tilde{\alpha}^c) = s(\tilde{\alpha}^c\tilde{\cup}\tilde{\alpha}, \tilde{\alpha}^c\tilde{\cap}\tilde{\alpha}) = s(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) = E(\tilde{\alpha})$.

Remark 2.4.8 Theorem 2.4.7 describes an interesting relationship between the entropy and subsethood measure for GHFEs. It states that the entropy $E(\tilde{\alpha})$ expresses the degree to which the supset $\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c$ is a subset of its own subset $\tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c$. Evaluating for the proposed subsethood measures Eqs. (2.61) and (2.65), yields two new entropy measures for GHFEs given by

$$\begin{aligned} E_1(\tilde{\alpha}) &= s_1(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) \\ &= \frac{\sum_{i=1}^{l}\left(1 - \max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} + \min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\}\right)}{\sum_{i=1}^{l}\left(1 + \max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} - \min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\}\right)}, \end{aligned} \tag{2.66}$$

$$\begin{aligned} E_2(\tilde{\alpha}) &= s_2(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c, \tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) \\ &= \frac{1}{l}\sum_{i=1}^{l} \min\Big\{1, g\Big(\phi(\max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} - \min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} + 1), \\ &\qquad \psi(\min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} - \max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} + 1)\Big)\Big\}. \end{aligned} \tag{2.67}$$

Szmit and Kacprzyk [51] defined the concept of cardinality for IFSs. Vlachos and Sergiadis [57] provided an interpretation of cardinality under a geometrical framework and presented the concept of average possible cardinality for IFSs. We extend these concepts in the generalized hesitant fuzzy setting.

Definition 2.4.9 For a GHFE $\tilde{\alpha}$, the following two cardinalities are defined:

- the least cardinality or min-sigma-count, which is given by

$$\min \sum Count(\tilde{\alpha}) = \frac{1}{l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} \mu_{\tilde{\alpha}}^{\sigma(i)} \tag{2.68}$$

- the biggest cardinality or max-sigma-count defined as

$$\max \sum Count(\tilde{\alpha}) = \frac{1}{l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} \left(1 - \nu_{\tilde{\alpha}}^{\sigma(i)}\right). \tag{2.69}$$

The cardinality of the GHFE $\tilde{\alpha}$ is defined as the interval

$$card(\tilde{\alpha}) = \left[\min \sum Count(\tilde{\alpha}), \max \sum Count(\tilde{\alpha})\right]. \tag{2.70}$$

Definition 2.4.10 For a GHFE $\tilde{\alpha}$, the average possible cardinality $\mathcal{M}(\tilde{\alpha})$ is defined as

$$\mathcal{M}(\tilde{\alpha}) = d_h((0,1), \tilde{\alpha}) = \frac{1}{2l_{\tilde{\alpha}}} \sum_{i=1}^{l_{\tilde{\alpha}}} (\mu_{\tilde{\alpha}}^{\sigma(i)} + 1 - \nu_{\tilde{\alpha}}^{\sigma(i)}), \tag{2.71}$$

where $d_h(\tilde{\alpha}, \tilde{\beta})$ is the Hamming distance between $\tilde{\alpha}$ and $\tilde{\beta}$ given by $d_h(\tilde{\alpha}, \tilde{\beta}) = \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{\tilde{\alpha}}^{\sigma(i)} - \mu_{\tilde{\beta}}^{\sigma(i)}| + |\nu_{\tilde{\alpha}}^{\sigma(i)} - \nu_{\tilde{\beta}}^{\sigma(i)}|)$.

From Eqs. (2.71) and (2.70) it follows that $\mathcal{M}(\tilde{\alpha})$ is the midpoint of the interval $[\min \sum Count(\tilde{\alpha}), \max \sum Count(\tilde{\alpha})]$. It should be point out that Eq. (2.71) encompasses the notions of least, biggest, and average possible cardinalities.

We are going to generalize the fuzzy entropy theorem in the setting of GHFEs, by stating the following theorem.

Theorem 2.4.11 (Generalized hesitant fuzzy entropy theorem) *Let $\tilde{\alpha}$ be a GHFE and $\mathcal{M}$ be an average possible cardinality of GHFEs, then*

$$E(\tilde{\alpha}) = \frac{\mathcal{M}(\tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c)}{\mathcal{M}(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c)} \tag{2.72}$$

is an entropy for $\tilde{\alpha}$.

Proof For a GHFE $\tilde{\alpha}$ and its complement $\tilde{\alpha}^c$, it holds that

$$\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c = \cup_{i=1}^{l_{\tilde{\alpha}}}\{(\max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\}, \min\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\alpha}}^{\sigma(i)}\})\}, \tag{2.73}$$

$$\tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c = \cup_{i=1}^{l_{\tilde{\alpha}}}\{(\min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\}, \max\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\alpha}}^{\sigma(i)}\})\}. \tag{2.74}$$

From the definition of average possible cardinality, we obtain that

$$\mathcal{M}(\tilde{\alpha}\tilde{\cup}\tilde{\alpha}^c) = \frac{1}{2l_{\tilde{\alpha}}}\sum_{i=1}^{l_{\tilde{\alpha}}}\left(1 + \max\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} - \min\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\alpha}}^{\sigma(i)}\}\right), \tag{2.75}$$

$$\mathcal{M}(\tilde{\alpha}\tilde{\cap}\tilde{\alpha}^c) = \frac{1}{2l_{\tilde{\alpha}}}\sum_{i=1}^{l_{\tilde{\alpha}}}\left(1 + \min\{\mu_{\tilde{\alpha}}^{\sigma(i)}, \nu_{\tilde{\alpha}}^{\sigma(i)}\} - \max\{\nu_{\tilde{\alpha}}^{\sigma(i)}, \mu_{\tilde{\alpha}}^{\sigma(i)}\}\right). \tag{2.76}$$

Substituting Eqs. (2.75) and (2.76) into Eq. (2.66) yields Eq. (2.72). This completes the proof.

In the following, we investigate the relationships between generalized hesitant fuzzy subsethood measure and generalized hesitant similarity measure:

Theorem 2.4.12 *Let $\tilde{\alpha}$ and $\tilde{\beta}$ be two GHFEs, then $S(\tilde{\alpha}, \tilde{\beta}) = s(\tilde{\alpha}, \tilde{\beta}) \wedge s(\tilde{\beta}, \tilde{\alpha})$ is a similarity measure of $\tilde{\alpha}$ and $\tilde{\beta}$.*

Proof (1) If $\tilde{\alpha} = (0, 1)$ and $\tilde{\beta} = (1, 0)$, then $\tilde{\alpha} = \tilde{\beta}^c \subset \tilde{\beta}$. By (2) of Definition 2.4.1, we have $S(\tilde{\alpha}, \tilde{\beta}) = s(\tilde{\alpha}, \tilde{\beta}) \wedge s(\tilde{\beta}, \tilde{\alpha}) = 0 \wedge s(\tilde{\beta}, \tilde{\alpha}) = 0$. With the same reason, when $\tilde{\alpha} = (1, 0)$ and $\tilde{\beta} = (0, 1)$, we can prove $S(\tilde{\alpha}, \tilde{\beta}) = 0$.

(2) $S(\tilde{\alpha}, \tilde{\beta}) = 1 \Leftrightarrow s(\tilde{\alpha}, \tilde{\beta}) = s(\tilde{\beta}, \tilde{\alpha}) = 1 \Leftrightarrow \tilde{\alpha} = \tilde{\beta}$.

(3) If $\mu_{\tilde{\alpha}}^{\sigma(i)} \leq \mu_{\tilde{\beta}}^{\sigma(i)} \leq \mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \geq \nu_{\tilde{\beta}}^{\sigma(i)} \geq \nu_{\tilde{\gamma}}^{\sigma(i)}$ for $i = 1, 2, \ldots, l$, then $\tilde{\alpha} \subset \tilde{\beta} \subset \tilde{\gamma}$ and thus, by (1) and (3) of Definition 2.4.1, we have

$$\begin{aligned} S(\tilde{\alpha}, \tilde{\beta}) &= s(\tilde{\alpha}, \tilde{\beta}) \wedge s(\tilde{\beta}, \tilde{\alpha}) = s(\tilde{\beta}, \tilde{\alpha}) \geq s(\tilde{\gamma}, \tilde{\alpha}) = 1 \wedge s(\tilde{\gamma}, \tilde{\alpha}) \\ &= s(\tilde{\alpha}, \tilde{\gamma}) \wedge s(\tilde{\gamma}, \tilde{\alpha}) = S(\tilde{\alpha}, \tilde{\gamma}), \qquad (2.77) \\ S(\tilde{\alpha}, \tilde{\gamma}) &= s(\tilde{\alpha}, \tilde{\gamma}) \wedge s(\tilde{\gamma}, \tilde{\alpha}) = s(\tilde{\gamma}, \tilde{\alpha}) \leq s(\tilde{\gamma}, \tilde{\beta}) = 1 \wedge s(\tilde{\gamma}, \tilde{\beta}) \\ &= s(\tilde{\beta}, \tilde{\gamma}) \wedge s(\tilde{\gamma}, \tilde{\beta}) = S(\tilde{\beta}, \tilde{\gamma}). \qquad (2.78) \end{aligned}$$

With the same reason, when $\mu_{\tilde{\alpha}}^{\sigma(i)} \geq \mu_{\tilde{\beta}}^{\sigma(i)} \geq \mu_{\tilde{\gamma}}^{\sigma(i)}$, $\nu_{\tilde{\alpha}}^{\sigma(i)} \leq \nu_{\tilde{\beta}}^{\sigma(i)} \leq \nu_{\tilde{\gamma}}^{\sigma(i)}$, $i = 1, 2, \ldots, l$, we can prove $S(\tilde{\alpha}, \tilde{\beta}) \geq S(\tilde{\alpha}, \tilde{\gamma})$ and $S(\tilde{\alpha}, \tilde{\gamma}) \leq S(\tilde{\beta}, \tilde{\gamma})$.

(4) Obviously, $S(\tilde{\alpha}, \tilde{\beta}) = S(\tilde{\beta}, \tilde{\alpha})$.

2.5 Conclusions

In this chapter, the entropy, cross-entropy and similarity measures for GHFEs were proposed, and several theorems that the entropy, cross-entropy and similarity measures for GHFEs can be transformed by each other were proved. Besides, two approaches of multiple attribute decision making problems where attribute weights are unknown and the evaluation values of attributes for each alternative are given in the form of GHFEs were investigated. To get optimal weight vector of attributes, the first approach utilized the entropy method which focuses on the fuzziness of the provided information; while the second one utilized the maximizing deviation method which focuses on the deviations among the decision information. These two approaches utilized the weights of attributes to calculate closeness degrees of alternatives and to get their ranking. Furthermore, the illustrative example demonstrated the practicality and effectiveness of the developed approaches. The prominent feature of two approaches is that they can provide a flexible way to facilitate the decision process under generalized hesitant fuzzy environment and be more applicable than existing ones, because our approaches can avoid complex computations. Besides, in Section 2.4, we presented a unified framwork for subsethood, entropy, cardinality and similarity for GHFEs. An axiomatic skeleton for subsethood was introduced and new subsethood and entropy measures in the generalized hesitant fuzzy setting were proposed. The notion of average possible cardinality was presented. Moreover, generalized hesitant fuzzy version of the entropy and entropy-subsethood theorems were stated and proved, which generalized the works of Kosko [29] for FSs and Liu and Xiong [36] for IFSs. Finally, we investigated the relationship between generalized hesitant fuzzy subsethood and generalized hesitant similarity measures.

Chapter 3

Interval-valued generalized hesitant fuzzy sets and their application in decision making

In this chapter, we extend GHFSs to interval-valued generalized hesitant fuzzy sets (IVGHFSs). Some basic operations and them are defined, such as union, intersection and some arithmetic operations on their elements. And their properties and relationships with IVIFVs are discussed as well. Then we develop a comparison law to distinguish information of IVGHFEs. A corresponding extension principle is introduced for further application to multiple attribute decision making.

3.1 Interval-valued intuitionistic fuzzy sets

As a generalization of the notion of IFSs, Atanassov and Gargov [4] introduced the notion of interval-valued intuitionistic fuzzy sets in the spirit of interval-valued fuzzy sets.

Definition 3.1.1 [4] Let X be ordinary non-empty set. An interval-valued intuitionistic fuzzy set (IVIFS) A on X is defined as

$$A = \{(x, \tilde{\mu}_A(x), \tilde{\nu}_A(x))|x \in X\}, \tag{3.1}$$

where $\tilde{\mu}_A : X \to D[0,1]$, $\tilde{\nu}_A : X \to D[0,1]$ are two functions, where $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$, with the condition $\sup \tilde{\mu}_A(x) + \sup \tilde{\nu}_A(x) \leq 1$ for all $x \in X$.

The intervals $\tilde{\mu}_A(x)$ and $\tilde{\nu}_A(x)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element x to A. Then for each $x \in X$, $\tilde{\mu}_A(x)$ and $\tilde{\nu}_A(x)$ are closed intervals and their lower and upper end points are denoted by $\tilde{\mu}_{AL}(x)$, $\tilde{\mu}_{AU}(x)$, $\tilde{\nu}_{AL}(x)$ and $\tilde{\nu}_{AU}(x)$, respectively, and thus we can replace Eq. (3.1) with

$$A = \{\langle x, [\tilde{\mu}_{AL}(x), \tilde{\mu}_{AU}(x)], [\tilde{\nu}_{AL}(x), \tilde{\nu}_{AU}(x)]\rangle|x \in X\}, \tag{3.2}$$

where $0 \leq \tilde{\mu}_{AU}(x) + \tilde{\nu}_{AU}(x) \leq 1$ for all $x \in X$.

Bustince and Burillo [9] proposed a new operator, so that each point $x \in X$ we take a value p and a value r corresponding to that point. For each $x \in X$, we take $p_x, r_x \in [0,1]$ and we consider H_{p_x,r_x} : IVIFSs$(X) \to$ IFSs(X) given by

$$H_{p_x,r_x}(A) = \{\langle x, \tilde{\mu}_{AL}(x) + p_x W_{\tilde{\mu}A}(x), \tilde{\nu}_{AL}(x) + r_x W_{\tilde{\nu}A}(x)\rangle|x \in X\}, \tag{3.3}$$

where $W_{\tilde{\mu}A}(x) = \tilde{\mu}_{AU}(x) - \tilde{\mu}_{AL}(x)$ and $W_{\tilde{\nu}A}(x) = \tilde{\nu}_{AU}(x) - \tilde{\nu}_{AL}(x)$ is amplitudes of $\tilde{\mu}_A(x)$ and $\tilde{\nu}_A(x)$, respectively. Evidently, $H_{p_x,r_x}(A)$ is an IFS for all IVIFS A. The most important properties of this operator can be found in [8]. They [9] presented a theorem for the construction of IVIFSs from an IFS as follows.

Let A be an IFS A and let us consider mappings $X \to [0,1] \times [0,1]$, $x \to (\lambda_x, \rho_x)$, such that if $\pi_A(x) \neq 0$, λ_x and ρ_x satisfy $\lambda_x \leq \frac{\mu_A(x)}{\pi_A(x)}$ and $\rho_x \leq \frac{\nu_A(x)}{\pi_A(x)}$.

Theorem 3.1.2 [9] *Let* $\zeta_x, \eta_x \in [0,1]$ *such that* $0 \leq \zeta_x + \eta_x \leq 1$. *Let* Γ : IFSs$(X) \to$ IVIFSs(X) *be a function given by* $\Gamma(A) = \{\langle x, \tilde{\mu}_{\Gamma(A)}(x), \tilde{\nu}_{\Gamma(A)}(x)\rangle|x \in X\}$ *such that*

(1) $\tilde{\mu}_{\Gamma(A)L}(x) = a + b\mu_A(x) - \lambda_x \pi_A(x)$ *with fixed* $a, b \in R$ *for all IFSs* A*;*

(2) $W_{\tilde{\mu}\Gamma(A)}(x) = (\zeta_x + \lambda_x)\pi_A(x)$ *for all* $x \in X$*;*

(3) $\tilde{\nu}_{\Gamma(A)L}(x) = a' + b'\nu_A(x) - \rho_x \pi_A(x)$ *with fixed* $a', b' \in R$ *for all IFSs* A*;*

(4) $W_{\tilde{\nu}\Gamma(A)}(x) = (\eta_x + \rho_x)\pi_A(x)$ *for all* $x \in X$*;*

(5) *If* A *is a FS, then* $\Gamma(A) = A$*.*

Then we have

(a) $\tilde{\mu}_{\Gamma(A)L}(x) = \mu_A(x) - \lambda_x \pi_A(x)$, $\tilde{\mu}_{\Gamma(A)U}(x) = \mu_A(x) + \zeta_x \pi_A(x)$*;*

(b) $\tilde{\nu}_{\Gamma(A)L}(x) = \nu_A(x) - \rho_x \pi_A(x)$, $\tilde{\nu}_{\Gamma(A)U}(x) = \nu_A(x) + \eta_x \pi_A(x)$

and conversely.

By means of the H_{p_x, r_x} operators, they [9] studied the way of to recover the IFS A used in the construction of IVIFS $\Gamma(A)$ with the above theorem.

Theorem 3.1.3 [9] *Let* A *be an IFS and* $\Gamma(A)$ *be the IVIFS constructed in the previous theorem, such that* $0 < \zeta_x + \lambda_x \leq 1$ *and* $0 < \eta_x + \rho_x \leq 1$ *for all* $x \in X$*. Then*

$$H_{\frac{\lambda_x}{\zeta_x + \lambda_x}, \frac{\rho_x}{\eta_x + \rho_x}}(\Gamma(A)) = A. \tag{3.4}$$

For convenience, Xu [69] called the ordered pair $\tilde{\alpha}(x) = (\tilde{\mu}_{\tilde{\alpha}}(x), \tilde{\nu}_{\tilde{\alpha}}(x))$ an interval-valued intuitionistic fuzzy value (IVIFV), where $\tilde{\mu}_{\tilde{\alpha}}(x), \tilde{\nu}_{\tilde{\alpha}}(x) \subset [0, 1]$ and $\sup \tilde{\mu}_{\tilde{\alpha}}(x) + \sup \tilde{\nu}_{\tilde{\alpha}}(x) \leq 1$. Atanassov [3] and Atanassov and Gargov [4] introduced some basic operations on IVIFSs, which not only can ensure that the operational results are IVIFSs but also are useful in the calculus of variables under interval-valued intuitionistic fuzzy environment. Motivated by the operations in [4, 3], Xu [69] and Xu and Chen [72] defined some operational laws of IVIFVs, which are useful in the remainder of this thesis, as follows:

Definition 3.1.4 [69, 72] Let $\tilde{\alpha} = (\tilde{\mu}_{\tilde{\alpha}}, \tilde{\nu}_{\tilde{\alpha}})$, $\tilde{\beta} = (\tilde{\mu}_{\tilde{\beta}}, \tilde{\nu}_{\tilde{\beta}})$ be two IVIFVs and $\lambda > 0$, then

(1) $\tilde{\alpha} \cup \tilde{\beta} = ([\max(\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\beta}L}), \max(\tilde{\mu}_{\tilde{\alpha}U}, \tilde{\mu}_{\tilde{\beta}U})], [\min(\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\beta}L}), \min(\tilde{\nu}_{\tilde{\alpha}U}, \tilde{\nu}_{\tilde{\beta}U})])$;

(2) $\tilde{\alpha} \cap \tilde{\beta} = ([\min(\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\beta}L}), \min(\tilde{\mu}_{\tilde{\alpha}U}, \tilde{\mu}_{\tilde{\beta}U})], [\max(\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\beta}L}), \max(\tilde{\nu}_{\tilde{\alpha}U}, \tilde{\nu}_{\tilde{\beta}U})])$;

(3) $\tilde{\alpha}^c = ([\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\alpha}U}], [\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\alpha}U}])$;
(4) $\tilde{\alpha} \oplus \tilde{\beta} = ([\tilde{\mu}_{\tilde{\alpha}L} + \tilde{\mu}_{\tilde{\beta}L} - \tilde{\mu}_{\tilde{\alpha}L}\tilde{\mu}_{\tilde{\beta}L}, \tilde{\mu}_{\tilde{\alpha}U} + \tilde{\mu}_{\tilde{\beta}U} - \tilde{\mu}_{\tilde{\alpha}U}\tilde{\mu}_{\tilde{\beta}U}], [\tilde{\nu}_{\tilde{\alpha}L}\tilde{\nu}_{\tilde{\beta}L}, \tilde{\nu}_{\tilde{\alpha}U}\tilde{\nu}_{\tilde{\beta}U}])$;
(5) $\tilde{\alpha} \otimes \tilde{\beta} = ([\tilde{\mu}_{\tilde{\alpha}L}\tilde{\mu}_{\tilde{\beta}L}, \tilde{\mu}_{\tilde{\alpha}U}\tilde{\mu}_{\tilde{\beta}U}], [\tilde{\nu}_{\tilde{\alpha}L} + \tilde{\nu}_{\tilde{\beta}L} - \tilde{\nu}_{\tilde{\alpha}L}\tilde{\nu}_{\tilde{\beta}L}, \tilde{\nu}_{\tilde{\alpha}U} + \tilde{\nu}_{\tilde{\beta}U} - \tilde{\nu}_{\tilde{\alpha}U}\tilde{\nu}_{\tilde{\beta}U}])$;
(6) $\lambda\tilde{\alpha} = ([1 - (1 - \tilde{\mu}_{\tilde{\alpha}L})^\lambda, 1 - (1 - \tilde{\mu}_{\tilde{\alpha}U})^\lambda], [\tilde{\nu}_{\tilde{\alpha}L}^\lambda, \tilde{\nu}_{\tilde{\alpha}U}^\lambda])$;
(7) $\tilde{\alpha}^\lambda = ([\tilde{\mu}_{\tilde{\alpha}L}^\lambda, \tilde{\mu}_{\tilde{\alpha}U}^\lambda], [1 - (1 - \tilde{\nu}_{\tilde{\alpha}L})^\lambda, 1 - (1 - \tilde{\nu}_{\tilde{\alpha}U})^\lambda])$.

By the above operations, Xu [69] and Xu and Chen [72], respectively, proposed the aggregation operators for IVIFVs as follows: For a collection of IVIFVs $\tilde{\alpha}_i = ([\tilde{\mu}_{\tilde{\alpha}_iL}, \tilde{\mu}_{\tilde{\alpha}_iU}], [\tilde{\nu}_{\tilde{\alpha}_iL}, \tilde{\nu}_{\tilde{\alpha}_iU}])$ $(i = 1, 2, \ldots, n)$, then

(1) the interval-valued intuitionistic fuzzy averaging (IIFA) operator [69]:

$$\begin{aligned}\text{IIFA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \oplus_{i=1}^n \left(\frac{1}{n}\tilde{\alpha}_i\right) \\ &= \left(\left[1 - \prod_{i=1}^n (1 - \tilde{\mu}_{\tilde{\alpha}_iL})^{\frac{1}{n}}, 1 - \prod_{i=1}^n (1 - \tilde{\mu}_{\tilde{\alpha}_iU})^{\frac{1}{n}}\right], \left[\prod_{i=1}^n (\tilde{\nu}_{\tilde{\alpha}_iL})^{\frac{1}{n}}, \prod_{i=1}^n (\tilde{\nu}_{\tilde{\alpha}_iU})^{\frac{1}{n}}\right]\right). \end{aligned} \tag{3.5}$$

(2) the interval-valued intuitionistic fuzzy geometric (IIFG) operator [72, 61]:

$$\begin{aligned}\text{IIFG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) &= \otimes_{i=1}^n \tilde{\alpha}_i^{\frac{1}{n}} \\ &= \left(\left[\prod_{i=1}^n (\tilde{\mu}_{\tilde{\alpha}_iL})^{\frac{1}{n}}, \prod_{i=1}^n (\tilde{\mu}_{\tilde{\alpha}_iU})^{\frac{1}{n}}\right], \left[1 - \prod_{i=1}^n (1 - \tilde{\nu}_{\tilde{\alpha}_iL})^{\frac{1}{n}}, 1 - \prod_{i=1}^n (1 - \tilde{\nu}_{\tilde{\alpha}_iU})^{\frac{1}{n}}\right]\right). \end{aligned} \tag{3.6}$$

3.2 Interval-valued generalized hesitant fuzzy sets

When considering the degree of an alternative satisfying a certain attribute, due to insufficiency in available information, we may have a doubt among several possible memberships with the form of both IFVs and IVIFVs. In order to handle this kind of assessment in decision making process, we extend the concept of HFSs by IVIFSs. Let us consider the following example.

Example 3.2.1 Four experts evaluate an alternative with respect to an attribute presented by interval-valued fuzzy value, IFV or IVIFV, which are $[0.6, 0.8]$,

$(0.6, 0.3)$, $([0.65, 0.7], [0.25, 0.3])$ and $(0.7, 0.2)$, respectively. By Definition 2.1.3, those evaluations form a GHFE such that

$$h = \{(0.6, 0.2), (0.6, 0.3), (0.7, 0.2)\} \cup \{([0.65, 0.7], [0.25, 0.3])\},$$

where $([0.65, 0.7], [0.25, 0.3])$ is an IVIFV.

Then we can conclude that a GHFS is a generalized extension of interval-valued fuzzy set, IFS and IVIFS by Definition 2.1.3. Without loss of generality, we consider each membership as an IVIFV. Then by Theorem 3.1.2 (let $\lambda = \frac{\mu}{\pi} \times 0.01$, $\rho = \frac{\nu}{\pi} \times 0.01$, $\zeta = \lambda$, $\eta = \rho$), the GHFE in Example 3.2.1 can be rewritten as

$$\begin{aligned} h = \{&([0.594, 0.606], [0.198, 0.202]), ([0.594, 0.606], [0.297, 0.303]), \\ &([0.693, 0.707], [0.198, 0.202]), ([0.65, 0.7], [0.25, 0.3])\}. \end{aligned}$$

However, Definition 2.1.3 emphasizes that possible memberships take the forms of both crisp and IFV as its elements. All existing literatures involved in HFSs and GHFSs focused in this case. As shown in Example 3.2.1, from the necessity of potential applications, we extend HFSs by using IVIFSs to modify Definition 2.1.2.

Definition 3.2.2 Given a set of N membership functions:

$$\begin{aligned} M = \{\tilde{\alpha}_i = (\tilde{\mu}_{\tilde{\alpha}_i}, \tilde{\nu}_{\tilde{\alpha}_i}) | \tilde{\mu}_{\tilde{\alpha}_i} = [\tilde{\mu}_{\tilde{\alpha}_i L}, \tilde{\mu}_{\tilde{\alpha}_i U}], \tilde{\nu}_i = [\tilde{\nu}_{\tilde{\alpha}_i L}, \tilde{\nu}_{\tilde{\alpha}_i U}] \subset [0, 1], \\ \tilde{\mu}_{\tilde{\alpha}_i U} + \tilde{\nu}_{\tilde{\alpha}_i U} \leq 1, i = 1, 2, \ldots, N\}, \end{aligned} \tag{3.7}$$

the interval-valued generalized hesitant fuzzy set (IVGHFS) associated with M, that is $\tilde{h}_M$, is defined as follows:

$$\begin{aligned} \tilde{h}_M(x) &= \cup_{\tilde{\alpha}_i \in M} \{(\tilde{\mu}_{\tilde{\alpha}_i}(x), \tilde{\nu}_{\tilde{\alpha}_i}(x))\} \\ &= \cup_{\tilde{\alpha}_i \in M} \{([\tilde{\mu}_{\tilde{\alpha}_i L}(x), \tilde{\mu}_{\tilde{\alpha}_i U}(x)], [\tilde{\nu}_{\tilde{\alpha}_i L}(x), \tilde{\nu}_{\tilde{\alpha}_i U}(x)])\}. \end{aligned} \tag{3.8}$$

Note that GHFSs, IVIFSs and IFSs are special cases of IVGHFSs. In fact, if $\tilde{\mu}_{\tilde{\alpha}_i L} = \tilde{\mu}_{\tilde{\alpha}_i U}$ and $\tilde{\nu}_{\tilde{\alpha}_i L} = \tilde{\nu}_{\tilde{\alpha}_i U}$ for all $i = 1, 2, \ldots, N$, then IVGHFSs reduce to

GHFSs. If $N = 1$, then IVGHFSs reduce to IVIFSs. If $N = 1$, $\tilde{\mu}_{\tilde{\alpha}_N L} = \tilde{\mu}_{\tilde{\alpha}_N U}$ and $\tilde{\nu}_{\tilde{\alpha}_N L} = \tilde{\nu}_{\tilde{\alpha}_N U}$, then IVGHFSs reduce to IFSs. Thus, IVGHFSs are not only the extension of GHFSs, but also the generalized representation of IFSs, IVIFSs and GHFSs.

For convenience, given a $x \in X$, γ is considered as a real number in $\tilde{h}(x)$, $\alpha = (\mu_\alpha, \nu_\alpha)$ represents an IFV as well as an interval in $\tilde{h}(x)$ and $\tilde{\alpha} = (\tilde{\mu}_{\tilde{\alpha}}, \tilde{\nu}_{\tilde{\alpha}})$ represents an IVIFV in $\tilde{h}(x)$. Similar to [47, 63], $\tilde{\alpha}_i$ in an IVGHFS $\tilde{h}$ is referred to as interval-valued generalized hesitant fuzzy element (IVGHFE). In the rest of this section, an IVGHFS $\tilde{h}$, represented by its membership function $\tilde{h}_M$, is denoted by Eqs. (3.7) and (3.8) as default. Now, we first extend basic operations defined by Qjan et al. [47] in the new setting.

3.2.1 Basic operations

For a given IVGHFE $\tilde{h}$ with its elements $\tilde{\alpha}_i = ([\tilde{\mu}_{\tilde{\alpha}_i L}, \tilde{\mu}_{\tilde{\alpha}_i U}], [\tilde{\nu}_{\tilde{\alpha}_i L}, \tilde{\nu}_{\tilde{\alpha}_i U}])$ $(i = 1, 2, \ldots, N)$, the upper and lower bounds of $\tilde{h}$ are denoted by

(1) upper bound: $\tilde{h}^+ = \max_{i=1,2,\ldots,N}\{1 - \tilde{\nu}_{\tilde{\alpha}_i}\}$
$= [\max_{i=1,2,\ldots,N}\{1 - \tilde{\nu}_{\tilde{\alpha}_i U}\}, \max_{i=1,2,\ldots,N}\{1 - \tilde{\nu}_{\tilde{\alpha}_i L}\}]$;

(2) lower bound: $\tilde{h}^- = \min_{i=1,2,\ldots,N}\{\tilde{\mu}_{\tilde{\alpha}_i}\}$
$= [\min_{i=1,2,\ldots,N}\{\tilde{\mu}_{\tilde{\alpha}_i L}\}, \min_{i=1,2,\ldots,N}\{\tilde{\mu}_{\tilde{\alpha}_i U}\}]$.

Obviously, the pair of $\tilde{h}^-$ and $1 - \tilde{h}^+$ define an IVIFS, which form the envelope of the IVGHFE. We present it in the following definition.

Definition 3.2.3 Given an IVGHFE $\tilde{h}$, we define an IVIFV $A_{env}(\tilde{h})$ as the envelope of $\tilde{h}$, where $A_{env}(\tilde{h})$ can be represented as $(\tilde{\mu}_A, \tilde{\nu}_A)$, with $\tilde{\mu}_A = \tilde{h}^-$ and $\tilde{\nu}_A = 1 - \tilde{h}^+$.

Then according to the notions of upper and lower bounds, the envelope of an IVGHFE $\tilde{h}$ can be rewritten as

$$A_{env}(\tilde{h}) = \left(\min_{i=1,2,\ldots,N}\{\tilde{\mu}_{\tilde{\alpha}_i}\}, \min_{i=1,2,\ldots,N}\{\tilde{\nu}_{\tilde{\alpha}_i}\}\right)$$

$$= \left(\left[\min_{i=1,2,\ldots,N} \{\tilde{\mu}_{\tilde{\alpha}_i L}\}, \min_{i=1,2,\ldots,N} \{\tilde{\mu}_{\tilde{\alpha}_i U}\} \right], \left[\min_{i=1,2,\ldots,N} \{\tilde{\nu}_{\tilde{\alpha}_i L}\}, \min_{i=1,2,\ldots,N} \{\tilde{\nu}_{\tilde{\alpha}_i U}\} \right] \right). \quad (3.9)$$

Similar to [47, 63], we can define the union, intersection and complement of IVGHFEs as follows.

Definition 3.2.4 Given three IVGHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$, then

(1) Complement: $\tilde{h}^c = \cup_{\tilde{\alpha}\in\tilde{h}}\{(\tilde{\nu}_{\tilde{\alpha}}, \tilde{\mu}_{\tilde{\alpha}})\} = \cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\alpha}U}], [\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\alpha}U}])\}$;

(2) Union: $\tilde{h}_1 \cup \tilde{h}_2 = \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{(\tilde{\mu}_{\tilde{\alpha}_1}, \tilde{\nu}_{\tilde{\alpha}_1}) \cup (\tilde{\mu}_{\tilde{\alpha}_2}, \tilde{\nu}_{\tilde{\alpha}_2})\}$
$= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\max\{\tilde{\mu}_{\tilde{\alpha}_1 L}, \tilde{\mu}_{\tilde{\alpha}_2 L}\}, \max\{\tilde{\mu}_{\tilde{\alpha}_1 U}, \tilde{\mu}_{\tilde{\alpha}_2 U}\}], [\min\{\tilde{\nu}_{\tilde{\alpha}_1 L}, \tilde{\nu}_{\tilde{\alpha}_2 L}\}, \min\{\tilde{\nu}_{\tilde{\alpha}_1 U}, \tilde{\nu}_{\tilde{\alpha}_2 U}\}])\}$;

(3) Intersection: $\tilde{h}_1 \cap \tilde{h}_2 = \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{(\tilde{\mu}_{\tilde{\alpha}_1}, \tilde{\nu}_{\tilde{\alpha}_1}) \cap (\tilde{\mu}_{\tilde{\alpha}_2}, \tilde{\nu}_{\tilde{\alpha}_2})\}$
$= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\min\{\tilde{\mu}_{\tilde{\alpha}_1 L}, \tilde{\mu}_{\tilde{\alpha}_2 L}\}, \min\{\tilde{\mu}_{\tilde{\alpha}_1 U}, \tilde{\mu}_{\tilde{\alpha}_2 U}\}], [\max\{\tilde{\nu}_{\tilde{\alpha}_1 L}, \tilde{\nu}_{\tilde{\alpha}_2 L}\}, \max\{\tilde{\nu}_{\tilde{\alpha}_1 U}, \tilde{\nu}_{\tilde{\alpha}_2 U}\}])\}$.

As can be seen in Definitions 2.1.3 and 3.1.1, we define some useful operations to deal with IVGHFEs when making decision in interval-valued generalized hesitant fuzzy information.

Definition 3.2.5 Given three IVGHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ and $\lambda > 0$, then

(1) $\tilde{h}^\lambda = \cup_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\alpha}^\lambda\} = \cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\mu}^\lambda_{\tilde{\alpha}L}, \tilde{\mu}^\lambda_{\tilde{\alpha}U}], [1-(1-\tilde{\nu}_{\tilde{\alpha}L})^\lambda, 1-(1-\tilde{\nu}_{\tilde{\alpha}U})^\lambda])\}$;

(2) $\lambda\tilde{h} = \cup_{\tilde{\alpha}\in\tilde{h}}\{\lambda\tilde{\alpha}\} = \cup_{\tilde{\alpha}\in\tilde{h}}\{([1-(1-\tilde{\mu}_{\tilde{\alpha}L})^\lambda, 1-(1-\tilde{\mu}_{\tilde{\alpha}U})^\lambda], [\tilde{\nu}^\lambda_{\tilde{\alpha}L}, \tilde{\nu}^\lambda_{\tilde{\alpha}U}])\}$;

(3) $\tilde{h}_1 \oplus \tilde{h}_2 = \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\alpha}_1 \oplus \tilde{\alpha}_2\} = \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\mu}_{\tilde{\alpha}_1 L} + \tilde{\mu}_{\tilde{\alpha}_2 L} - \tilde{\mu}_{\tilde{\alpha}_1 L}\tilde{\mu}_{\tilde{\alpha}_2 L}, \tilde{\mu}_{\tilde{\alpha}_1 U} + \tilde{\mu}_{\tilde{\alpha}_2 U} - \tilde{\mu}_{\tilde{\alpha}_1 U}\tilde{\mu}_{\tilde{\alpha}_2 U}], [\tilde{\nu}_{\tilde{\alpha}_1 L}\tilde{\nu}_{\tilde{\alpha}_2 L}, \tilde{\nu}_{\tilde{\alpha}_1 U}\tilde{\nu}_{\tilde{\alpha}_2 U}])\}$;

(4) $\tilde{h}_1 \otimes \tilde{h}_2 = \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\alpha}_1 \otimes \tilde{\alpha}_2\} = \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\mu}_{\tilde{\alpha}_1 L}\tilde{\mu}_{\tilde{\alpha}_2 L}, \tilde{\mu}_{\tilde{\alpha}_1 U}\tilde{\mu}_{\tilde{\alpha}_2 U}], [\tilde{\nu}_{\tilde{\alpha}_1 L} + \tilde{\nu}_{\tilde{\alpha}_2 L} - \tilde{\nu}_{\tilde{\alpha}_1 L}\tilde{\nu}_{\tilde{\alpha}_2 L}, \tilde{\nu}_{\tilde{\alpha}_1 U} + \tilde{\nu}_{\tilde{\alpha}_2 U} - \tilde{\nu}_{\tilde{\alpha}_1 U}\tilde{\nu}_{\tilde{\alpha}_2 U}])\}$.

3.2.2 Properties

In this subsection, we focus on some properties of operations defined hereinbefore. Some relationships among basic operations are introduced in the following theorem.

Theorem 3.2.6 *Given three IVGHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ and $\lambda > 0$, then*

(1) $\tilde{h}_1^c \cup \tilde{h}_2^c = (\tilde{h}_1 \cap \tilde{h}_2)^c$.

(2) $\tilde{h}_1^c \cap \tilde{h}_2^c = (\tilde{h}_1 \cup \tilde{h}_2)^c$.

(3) $(\tilde{h}^c)^\lambda = (\lambda \tilde{h})^c$.

(4) $\lambda \tilde{h}^c = (\tilde{h}^\lambda)^c$.

(5) $\tilde{h}_1^c \oplus \tilde{h}_2^c = (\tilde{h}_1 \otimes \tilde{h}_2)^c$.

(6) $\tilde{h}_1^c \otimes \tilde{h}_2^c = (\tilde{h}_1 \oplus \tilde{h}_2)^c$.

Proof From Definitions 3.1.4, 3.2.4 and 3.2.5, we have:

(1)

$$\begin{aligned}
\tilde{h}_1^c \cup \tilde{h}_2^c &= \cup_{\tilde{\alpha}_1 \in \tilde{h}_1, \tilde{\alpha}_2 \in \tilde{h}_2} \{\tilde{\alpha}_1^c \cup \tilde{\alpha}_2^c\} \\
&= \cup_{\tilde{\alpha}_1 \in \tilde{h}_1, \tilde{\alpha}_2 \in \tilde{h}_2} \{([\max\{\tilde{\nu}_{\tilde{\alpha}_1 L}, \tilde{\nu}_{\tilde{\alpha}_2 L}\}, \max\{\tilde{\nu}_{\tilde{\alpha}_1 U}, \tilde{\nu}_{\tilde{\alpha}_2 U}\}], \\
&\qquad [\min\{\tilde{\mu}_{\tilde{\alpha}_1 L}, \tilde{\mu}_{\tilde{\alpha}_2 L}\}, \min\{\tilde{\mu}_{\tilde{\alpha}_1 U}, \tilde{\mu}_{\tilde{\alpha}_2 U}\}])\} \\
&= \cup_{\tilde{\alpha}_1 \in \tilde{h}_1, \tilde{\alpha}_2 \in \tilde{h}_2} \{(\tilde{\alpha}_1 \cap \tilde{\alpha}_2)^c\} = (\tilde{h}_1 \cap \tilde{h}_2)^c.
\end{aligned}$$

(2)

$$\begin{aligned}
\tilde{h}_1^c \cap \tilde{h}_2^c &= \cup_{\tilde{\alpha}_1 \in \tilde{h}_1, \tilde{\alpha}_2 \in \tilde{h}_2} \{\tilde{\alpha}_1^c \cap \tilde{\alpha}_2^c\} \\
&= \cup_{\tilde{\alpha}_1 \in \tilde{h}_1, \tilde{\alpha}_2 \in \tilde{h}_2} \{([\min\{\tilde{\nu}_{\tilde{\alpha}_1 L}, \tilde{\nu}_{\tilde{\alpha}_2 L}\}, \min\{\tilde{\nu}_{\tilde{\alpha}_1 U}, \tilde{\nu}_{\tilde{\alpha}_2 U}\}], \\
&\qquad [\max\{\tilde{\mu}_{\tilde{\alpha}_1 L}, \tilde{\mu}_{\tilde{\alpha}_2 L}\}, \max\{\tilde{\mu}_{\tilde{\alpha}_1 U}, \tilde{\mu}_{\tilde{\alpha}_2 U}\}])\} \\
&= \cup_{\tilde{\alpha}_1 \in \tilde{h}_1, \tilde{\alpha}_2 \in \tilde{h}_2} \{(\tilde{\alpha}_1 \cup \tilde{\alpha}_2)^c\} = (\tilde{h}_1 \cup \tilde{h}_2)^c.
\end{aligned}$$

(3)

$$\begin{aligned}
(\tilde{h}^c)^\lambda &= \cup_{\tilde{\alpha} \in \tilde{h}} \{(\tilde{\alpha}^c)^\lambda\} \\
&= \cup_{\tilde{\alpha} \in \tilde{h}} \{([\tilde{\nu}_{\tilde{\alpha}L}^\lambda, \tilde{\nu}_{\tilde{\alpha}U}^\lambda], [1 - (1 - \tilde{\mu}_{\tilde{\alpha}L})^\lambda, 1 - (1 - \tilde{\mu}_{\tilde{\alpha}U})^\lambda])\} \\
&= \cup_{\tilde{\alpha} \in \tilde{h}} \{(\lambda \tilde{\alpha})^c\} = (\lambda \tilde{h})^c.
\end{aligned}$$

(4)

$$\begin{aligned}
\lambda \tilde{h}^c &= \cup_{\tilde{\alpha} \in \tilde{h}} \{\lambda \tilde{\alpha}^c\} \\
&= \cup_{\tilde{\alpha} \in \tilde{h}} \{([1 - (1 - \tilde{\nu}_{\tilde{\alpha}L})^\lambda, 1 - (1 - \tilde{\nu}_{\tilde{\alpha}U})^\lambda], [\tilde{\mu}_{\tilde{\alpha}L}^\lambda, \tilde{\mu}_{\tilde{\alpha}U}^\lambda])\} \\
&= \cup_{\tilde{\alpha} \in \tilde{h}} \{(\tilde{\alpha}^\lambda)^c\} = (\tilde{h}^\lambda)^c.
\end{aligned}$$

(5)

$$\begin{aligned}\tilde{h}_1^c \oplus \tilde{h}_2^c &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\alpha}_1^c \oplus \tilde{\alpha}_2^c\}\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\nu}_{\tilde{\alpha}_1L}+\tilde{\nu}_{\tilde{\alpha}_2L}-\tilde{\nu}_{\tilde{\alpha}_1L}\tilde{\nu}_{\tilde{\alpha}_2L},\tilde{\nu}_{\tilde{\alpha}_1U}+\tilde{\nu}_{\tilde{\alpha}_2U}-\tilde{\nu}_{\tilde{\alpha}_1U}\tilde{\nu}_{\tilde{\alpha}_2U}],\\ &\qquad [\tilde{\mu}_{\tilde{\alpha}_1L}\tilde{\mu}_{\tilde{\alpha}_2L},\tilde{\mu}_{\tilde{\alpha}_1U}\tilde{\mu}_{\tilde{\alpha}_2U}])\}\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{(\tilde{\alpha}_1\otimes\tilde{\alpha}_2)^c\} = (\tilde{h}_1\otimes\tilde{h}_2)^c.\end{aligned}$$

(6)

$$\begin{aligned}\tilde{h}_1^c \otimes \tilde{h}_2^c &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\alpha}_1^c \oplus \tilde{\alpha}_2^c\}\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\mu}_{\tilde{\alpha}_1L}\tilde{\mu}_{\tilde{\alpha}_2L},\tilde{\nu}_{\tilde{\alpha}_1U}\tilde{\nu}_{\tilde{\alpha}_2U}],\\ &\qquad [\tilde{\mu}_{\tilde{\alpha}_1L}+\tilde{\mu}_{\tilde{\alpha}_2L}-\tilde{\mu}_{\tilde{\alpha}_1L}\tilde{\mu}_{\tilde{\alpha}_2L},\tilde{\mu}_{\tilde{\alpha}_1U}+\tilde{\mu}_{\tilde{\alpha}_2U}-\tilde{\mu}_{\tilde{\alpha}_1U}\tilde{\mu}_{\tilde{\alpha}_2U}])\}\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{(\tilde{\alpha}_1\oplus\tilde{\alpha}_2)^c\} = (\tilde{h}_1\oplus\tilde{h}_2)^c.\end{aligned}$$

Moreover, the relations of these operational laws are given as:

Theorem 3.2.7 *Given three IVGHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ and $\lambda, \lambda_1, \lambda_2 > 0$, then*

(1) $\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1$.
(2) $\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1$.
(3) $\lambda(\tilde{h}_1 \oplus \tilde{h}_2) = \lambda\tilde{h}_1 \oplus \lambda\tilde{h}_2$.
(4) $(\tilde{h}_1 \otimes \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \otimes \tilde{h}_2^\lambda$.
(5) $\lambda_1\tilde{h} \oplus \lambda_2\tilde{h} = (\lambda_1+\lambda_2)\tilde{h}$.
(6) $\tilde{h}^{\lambda_1} \otimes \tilde{h}^{\lambda_2} = \tilde{h}^{\lambda_1+\lambda_2}$.

Proof (1) and (2) are straightforward.

(3)

$$\begin{aligned}&\lambda\tilde{h}_1 \oplus \lambda\tilde{h}_2\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2\}\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([1-(1-\tilde{\mu}_{\tilde{\alpha}_1L})^\lambda(1-\tilde{\mu}_{\tilde{\alpha}_2L})^\lambda, 1-(1-\tilde{\mu}_{\tilde{\alpha}_1U})^\lambda(1-\tilde{\mu}_{\tilde{\alpha}_2U})^\lambda],\\ &\qquad [\tilde{\nu}_{\tilde{\alpha}_1L}^\lambda\tilde{\nu}_{\tilde{\alpha}_2L}^\lambda,\tilde{\nu}_{\tilde{\alpha}_1U}^\lambda\tilde{\nu}_{\tilde{\alpha}_2U}^\lambda])\}\\ &= \cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([1-(1-\tilde{\mu}_{\tilde{\alpha}_1L}-\tilde{\mu}_{\tilde{\alpha}_2L}+\tilde{\mu}_{\tilde{\alpha}_1L}\tilde{\mu}_{\tilde{\alpha}_2L})^\lambda,\end{aligned}$$

$$1-(1-\tilde{\mu}_{\tilde{\alpha}_1U}-\tilde{\mu}_{\tilde{\alpha}_2U}+\tilde{\mu}_{\tilde{\alpha}_1U}\tilde{\mu}_{\tilde{\alpha}_2U})^{\lambda}],[\tilde{\nu}_{\tilde{\alpha}_1L}^{\lambda}\tilde{\nu}_{\tilde{\alpha}_2L}^{\lambda},\tilde{\nu}_{\tilde{\alpha}_1U}^{\lambda}\tilde{\nu}_{\tilde{\alpha}_2U}^{\lambda}])\}$$
$$=\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\lambda(\tilde{\alpha}_1\oplus\tilde{\alpha}_2)\}$$
$$=\lambda(\tilde{h}_1\oplus\tilde{h}_2).$$

(4)

$$\tilde{h}_1^{\lambda}\otimes\tilde{h}_2^{\lambda}$$
$$=\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\alpha}_1^{\lambda}\otimes\tilde{\alpha}_2^{\lambda}\}$$
$$=\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\mu}_{\tilde{\alpha}_1L}^{\lambda}\tilde{\mu}_{\tilde{\alpha}_2L}^{\lambda},\tilde{\mu}_{\tilde{\alpha}_1U}^{\lambda}\tilde{\mu}_{\tilde{\alpha}_2U}^{\lambda}],$$
$$[1-(1-\tilde{\nu}_{\tilde{\alpha}_1L})^{\lambda}(1-\tilde{\nu}_{\tilde{\alpha}_2L})^{\lambda},1-(1-\tilde{\nu}_{\tilde{\alpha}_1U})^{\lambda}(1-\tilde{\nu}_{\tilde{\alpha}_2U})^{\lambda}])\}$$
$$=\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\mu}_{\tilde{\alpha}_1L}^{\lambda}\tilde{\mu}_{\tilde{\alpha}_2L}^{\lambda},\tilde{\mu}_{\tilde{\alpha}_1U}^{\lambda}\tilde{\mu}_{\tilde{\alpha}_2U}^{\lambda}],[1-(1-\tilde{\nu}_{\tilde{\alpha}_1L}-\tilde{\nu}_{\tilde{\alpha}_2L}+\tilde{\nu}_{\tilde{\alpha}_1L}\tilde{\nu}_{\tilde{\alpha}_2L})^{\lambda},$$
$$1-(1-\tilde{\nu}_{\tilde{\alpha}_1U}-\tilde{\nu}_{\tilde{\alpha}_2U}+\tilde{\nu}_{\tilde{\alpha}_1U}\tilde{\nu}_{\tilde{\alpha}_2U})^{\lambda}])\}$$
$$=\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{(\tilde{\alpha}_1\otimes\tilde{\alpha}_2)^{\lambda}\}$$
$$=(\tilde{h}_1\otimes\tilde{h}_2)^{\lambda}.$$

(5)

$$\lambda_1\tilde{h}\oplus\lambda_2\tilde{h}$$
$$=\cup_{\tilde{\alpha}\in\tilde{h}}\{\lambda_1\tilde{\alpha}\oplus\lambda_2\tilde{\alpha}\}$$
$$=\cup_{\tilde{\alpha}\in\tilde{h}}\{([1-(1-\tilde{\mu}_{\tilde{\alpha}L})^{\lambda_1}(1-\tilde{\mu}_{\tilde{\alpha}L})^{\lambda_2},1-(1-\tilde{\mu}_{\tilde{\alpha}U})^{\lambda_1}(1-\tilde{\mu}_{\tilde{\alpha}U})^{\lambda_2}],$$
$$[\tilde{\nu}_{\tilde{\alpha}L}^{\lambda_1}\tilde{\nu}_{\tilde{\alpha}L}^{\lambda_2},\tilde{\nu}_{\tilde{\alpha}U}^{\lambda_1}\tilde{\nu}_{\tilde{\alpha}U}^{\lambda_2}])\}$$
$$=\cup_{\tilde{\alpha}\in\tilde{h}}\{([1-(1-\tilde{\mu}_{\tilde{\alpha}L})^{\lambda_1+\lambda_2},1-(1-\tilde{\mu}_{\tilde{\alpha}U})^{\lambda_1+\lambda_2}],[\tilde{\nu}_{\tilde{\alpha}L}^{\lambda_1+\lambda_2},\tilde{\nu}_{\tilde{\alpha}U}^{\lambda_1+\lambda_2}])\}$$
$$=\cup_{\tilde{\alpha}\in\tilde{h}}\{(\lambda_1+\lambda_2)\tilde{\alpha}\}$$
$$=(\lambda_1+\lambda_2)\tilde{h}.$$

(6)

$$\tilde{h}^{\lambda_1}\otimes\tilde{h}^{\lambda_2}$$
$$=\cup_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\alpha}^{\lambda_1}\otimes\tilde{\alpha}^{\lambda_2}\}$$
$$=\cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\mu}_{\tilde{\alpha}L}^{\lambda_1}\tilde{\mu}_{\tilde{\alpha}L}^{\lambda_2},\tilde{\mu}_{\tilde{\alpha}U}^{\lambda_1}\tilde{\mu}_{\tilde{\alpha}U}^{\lambda_2}],[1-(1-\tilde{\nu}_{\tilde{\alpha}L})^{\lambda_1}(1-\tilde{\nu}_{\tilde{\alpha}L})^{\lambda_2},$$

$$\begin{aligned}
&1-(1-\tilde{\nu}_{\tilde{\alpha}U})^{\lambda_1}(1-\tilde{\nu}_{\tilde{\alpha}U})^{\lambda_2}])\} \\
&= \cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\mu}_{\tilde{\alpha}L}^{\lambda_1+\lambda_2}, \tilde{\mu}_{\tilde{\alpha}U}^{\lambda_1+\lambda_2}], [1-(1-\tilde{\nu}_{\tilde{\alpha}L})^{\lambda_1+\lambda_2}, 1-(1-\tilde{\nu}_{\tilde{\alpha}U})^{\lambda_1+\lambda_2}])\} \\
&= \cup_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\alpha}^{\lambda_1+\lambda_2}\} \\
&= \tilde{h}^{\lambda_1+\lambda_2}.
\end{aligned}$$

Then, we give the further study of the relationship between IVGHFEs and IVIFVs, that is, the notion of envelope brings some relationships between IVGHFEs and IVIFVs.

Theorem 3.2.8 *Given three IVGHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$ and $\lambda > 0$, then*

(1) $A_{env}(\tilde{h}^c) = (A_{env}(\tilde{h}))^c$.

(2) $A_{env}(\tilde{h}_1 \cup \tilde{h}_2) = A_{env}(\tilde{h}_1) \cup A_{env}(\tilde{h}_2)$.

(3) $A_{env}(\tilde{h}_1 \cap \tilde{h}_2) = A_{env}(\tilde{h}_1) \cap A_{env}(\tilde{h}_2)$.

(4) $A_{env}(\tilde{h}^\lambda) = (A_{env}(\tilde{h}))^\lambda$.

(5) $A_{env}(\lambda\tilde{h}) = \lambda(A_{env}(\tilde{h}))$.

(6) $A_{env}(\tilde{h}_1 \oplus \tilde{h}_2) = A_{env}(\tilde{h}_1) \oplus A_{env}(\tilde{h}_2)$.

(7) $A_{env}(\tilde{h}_1 \otimes \tilde{h}_2) = A_{env}(\tilde{h}_1) \otimes A_{env}(\tilde{h}_2)$.

Proof We prove only (1), (2), (4) and (6).

(1) From Definitions 3.2.3 and 3.2.4, we have

$$\begin{aligned}
A_{env}(\tilde{h}^c) &= A_{env}\left(\cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\alpha}U}], [\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\alpha}U}])^c\}\right) \\
&= A_{env}\left(\cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\alpha}U}], [\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\alpha}U}])\}\right) \\
&= \left(\left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}L}\}, \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}U}\}\right], \left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}L}\}, \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}U}\}\right]\right) \\
&= \left(\left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}L}\}, \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}U}\}\right], \left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}L}\}, \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}U}\}\right]\right)^c \\
&= (A_{env}(\tilde{h}))^c.
\end{aligned}$$

(2) From Definitions 3.2.3 and 3.2.4, we have

$A_{env}(\tilde{h}_1 \cup \tilde{h}_1)$

$$
\begin{aligned}
&= A_{env}\left(\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\max\{\tilde{\mu}_{\tilde{\alpha}_1L},\tilde{\mu}_{\tilde{\alpha}_2L}\},\max\{\tilde{\mu}_{\tilde{\alpha}_1U},\tilde{\mu}_{\tilde{\alpha}_2U}\}],\right.\\
&\qquad\qquad\qquad\qquad \left.[\min\{\tilde{\nu}_{\tilde{\alpha}_1L},\tilde{\nu}_{\tilde{\alpha}_2L}\},\min\{\tilde{\nu}_{\tilde{\alpha}_1U},\tilde{\nu}_{\tilde{\alpha}_2U}\}])\}\right)\\
&= \left(\left[\min_{\tilde{\alpha}_1\in\tilde{h},\tilde{\alpha}_2\in\tilde{h}_2}\{\max\{\tilde{\mu}_{\tilde{\alpha}_1L},\tilde{\mu}_{\tilde{\alpha}_2L}\}\},\min_{\tilde{\alpha}_1\in\tilde{h},\tilde{\alpha}_2\in\tilde{h}_2}\{\max\{\tilde{\mu}_{\tilde{\alpha}_1U},\tilde{\mu}_{\tilde{\alpha}_2U}\}\}\right],\right.\\
&\qquad \left.\left[\min_{\tilde{\alpha}_1\in\tilde{h},\tilde{\alpha}_2\in\tilde{h}_2}\{\min\{\tilde{\nu}_{\tilde{\alpha}_1L},\tilde{\nu}_{\tilde{\alpha}_2L}\}\},\min_{\tilde{\alpha}_1\in\tilde{h},\tilde{\alpha}_2\in\tilde{h}_2}\{\min\{\tilde{\nu}_{\tilde{\alpha}_1U},\tilde{\nu}_{\tilde{\alpha}_2U}\}\}\right]\right)\\
&= \left(\left[\max\left\{\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1L}\},\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2L}\}\right\},\max\left\{\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1L}\},\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2L}\}\right\}\right],\right.\\
&\qquad \left.\left[\min\left\{\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1L}\},\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_2L}\}\right\},\min\left\{\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1L}\},\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_2L}\}\right\}\right]\right)\\
&= \left(\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1L}\},\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1U}\}\right],\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1L}\},\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1U}\}\right]\right)\\
&\quad \cup\left(\left[\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2L}\},\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2U}\}\right],\left[\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_2L}\},\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_2U}\}\right]\right)\\
&= A_{env}(\tilde{h}_1)\cup A_{env}(\tilde{h}_2).
\end{aligned}
$$

(4) From Definitions 3.1.4, 3.2.3 and 3.2.5, we have

$$
\begin{aligned}
&A_{env}(\tilde{h}^{\lambda})\\
&= A_{env}\left(\cup_{\tilde{\alpha}\in\tilde{h}}\{([\tilde{\mu}^{\lambda}_{\tilde{\alpha}L},\tilde{\mu}^{\lambda}_{\tilde{\alpha}U}],[1-(1-\tilde{\nu}_{\tilde{\alpha}L})^{\lambda},1-(1-\tilde{\nu}_{\tilde{\alpha}U})^{\lambda}])\}\right)\\
&= \left(\left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}^{\lambda}_{\tilde{\alpha}L}\},\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}^{\lambda}_{\tilde{\alpha}U}\}\right],\left[\min_{\tilde{\alpha}\in\tilde{h}}\{1-(1-\tilde{\nu}_{\tilde{\alpha}L})^{\lambda}\},\min_{\tilde{\alpha}\in\tilde{h}}\{1-(1-\tilde{\nu}_{\tilde{\alpha}U})^{\lambda}\}\right]\right)\\
&= \left(\left[\left(\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}L}\}\right)^{\lambda},\left(\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}U}\}\right)^{\lambda}\right],\right.\\
&\qquad \left.\left[1-\left(1-\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}L}\}\right)^{\lambda},1-\left(1-\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}U}\}\right)^{\lambda}\right]\right)\\
&= \left(\left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}L}\},\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}U}\}\right],\left[\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}L}\},\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}U}\}\right]\right)^{\lambda}\\
&= (A_{env}(\tilde{h}))^{\lambda}.
\end{aligned}
$$

(6) From Definitions 3.1.4, 3.2.3 and 3.2.5, we have

$A_{env}(\tilde{h}_1\oplus\tilde{h}_2)$

$$
\begin{aligned}
&= A_{env}\big(\cup_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{([\tilde{\mu}_{\tilde{\alpha}_1 L}+\tilde{\mu}_{\tilde{\alpha}_2 L}-\tilde{\mu}_{\tilde{\alpha}_1 L}\tilde{\mu}_{\tilde{\alpha}_2 L}, \tilde{\mu}_{\tilde{\alpha}_1 U}+\tilde{\mu}_{\tilde{\alpha}_2 U}-\tilde{\mu}_{\tilde{\alpha}_1 U}\tilde{\mu}_{\tilde{\alpha}_2 U}],\\
&\qquad\qquad\qquad\qquad [\tilde{\nu}_{\tilde{\alpha}_1 L}\tilde{\nu}_{\tilde{\alpha}_2 L}, \tilde{\nu}_{\tilde{\alpha}_1 U}\tilde{\nu}_{\tilde{\alpha}_2 U}])\}\big)\\
&= \left(\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_1 L}+\tilde{\mu}_{\tilde{\alpha}_2 L}-\tilde{\mu}_{\tilde{\alpha}_1 L}\tilde{\mu}_{\tilde{\alpha}_2 L}\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_1 U}+\tilde{\mu}_{\tilde{\alpha}_2 U}-\tilde{\mu}_{\tilde{\alpha}_1 U}\tilde{\mu}_{\tilde{\alpha}_2 U}\}\right],\right.\\
&\quad \left.\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_1 L}\tilde{\nu}_{\tilde{\alpha}_2 L}\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_1 U}\tilde{\nu}_{\tilde{\alpha}_2 U}\}\right]\right)\\
&= \left(\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{1-(1-\tilde{\mu}_{\tilde{\alpha}_1 L})(1-\tilde{\mu}_{\tilde{\alpha}_2 L})\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{1-(1-\tilde{\mu}_{\tilde{\alpha}_1 U})(1-\tilde{\mu}_{\tilde{\alpha}_2 U})\}\right],\right.\\
&\quad \left.\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_1 L}\tilde{\nu}_{\tilde{\alpha}_2 L}\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1,\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_1 U}\tilde{\nu}_{\tilde{\alpha}_2 U}\}\right]\right)\\
&= \left(\left[1-\left(1-\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1 L}\}\right)\left(1-\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2 L}\}\right), 1-\left(1-\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1 U}\}\right)\right.\right.\\
&\quad \left.\left.\times\left(1-\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2 U}\}\right)\right], \left[\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1 L}\}\cdot\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_2 L}\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1 U}\}\cdot\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\nu}_{\tilde{\alpha}_2 U}\}\right]\right)\\
&= \left(\left[\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1 L}\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\mu}_{\tilde{\alpha}_1 U}\}\right], \left[\min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1 L}\}, \min_{\tilde{\alpha}_1\in\tilde{h}_1}\{\tilde{\nu}_{\tilde{\alpha}_1 U}\}\right]\right)\\
&\quad \oplus\left(\left[\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2 L}\}, \min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2 U}\}\right], \left[\min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2 L}\}, \min_{\tilde{\alpha}_2\in\tilde{h}_2}\{\tilde{\mu}_{\tilde{\alpha}_2 U}\}\right]\right)\\
&= A_{env}(\tilde{h}_1)\oplus A_{env}(\tilde{h}_2).
\end{aligned}
$$

3.2.3 Comparison of IVGHFEs

Qjan et al. [47] defined the score function of a GHFE h, i.e. $s(h)=\frac{1}{l(h)}\sum_{\alpha\in h}E(\alpha)$, where $l(h)$ is the number of elements in h and $E(\alpha)$ is the expect value of element α in h given by $E(\alpha)=\frac{1}{2}(\mu_\alpha+1-\nu_\alpha)$. However, the definition is unavailable if some elements in h take the form of IVIFV. To deal with this situation, we extend the definition to IVGHFEs as follows.

Definition 3.2.9 Given an IVIFV $\tilde{\alpha}=([\tilde{\mu}_{\tilde{\alpha}L},\tilde{\mu}_{\tilde{\alpha}U}],[\tilde{\nu}_{\tilde{\alpha}L},\tilde{\nu}_{\tilde{\alpha}U}])$, the expect value of $\tilde{\alpha}$ is defined by

$$E(\tilde{\alpha})=\frac{1}{4}(\tilde{\mu}_{\tilde{\alpha}L}+1-\tilde{\nu}_{\tilde{\alpha}L}+\tilde{\mu}_{\tilde{\alpha}U}+1-\tilde{\nu}_{\tilde{\alpha}U}), \tag{3.10}$$

where $E(\tilde{\alpha})\in[0,1]$. The larger the value of $E(\tilde{\alpha})$, the higher the IVIFV $\tilde{\alpha}$. Especially, if $E(\tilde{\alpha})=1$, then $\tilde{\alpha}=([1,1],[0,0])$, which is the largest IVIFV; if $E(\tilde{\alpha})=0$, then $\tilde{\alpha}=([0,0],[1,1])$, which is the smallest IVIFV.

Note that $E(\tilde{\alpha}) = \frac{1}{2}(\tilde{\mu}_{\tilde{\alpha}L} + 1 - \tilde{\nu}_{\tilde{\alpha}L})$ if and only if $\tilde{\mu}_{\tilde{\alpha}L} = \tilde{\mu}_{\tilde{\alpha}U}$ and $\tilde{\nu}_{\tilde{\alpha}L} = \tilde{\nu}_{\tilde{\alpha}U}$, in other words, $\tilde{\alpha}$ reduces to an IFV; $E(\tilde{\alpha}) = \tilde{\mu}_{\tilde{\alpha}L}$ if and only if $\tilde{\mu}_{\tilde{\alpha}L} = \tilde{\mu}_{\tilde{\alpha}U}$, $\tilde{\nu}_{\tilde{\alpha}L} = \tilde{\nu}_{\tilde{\alpha}U}$ and $\tilde{\mu}_{\tilde{\alpha}L} + \tilde{\nu}_{\tilde{\alpha}L} = 1$, in other words, $\tilde{\alpha}$ reduces to a fuzzy set. Further, $E(\tilde{\alpha}) = \frac{S(\tilde{\alpha})}{2} + 0.5$, where $S(\tilde{\alpha})$ is the score function [69] of $\tilde{\alpha}$.

Definition 3.2.10 Given an IVGHFE $\tilde{h}$, the score function of $\tilde{h}$, denoted by $s(\tilde{h})$, is defined by

$$s(\tilde{h}) = \frac{1}{l(\tilde{h})} \sum_{\tilde{\alpha} \in \tilde{h}} E(\tilde{\alpha}), \tag{3.11}$$

where $l(\tilde{h})$ the number of elements in $\tilde{h}$ and $\tilde{\alpha}$ is an element in $\tilde{h}$ taken the form of IVIFV.

Definition 3.2.11 Given an IVGHFE $\tilde{h}$, the consistency function of $\tilde{h}$, denoted by $c(\tilde{h})$, is defined by

$$c(\tilde{h}) = \frac{1}{2}\left(\min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}L}\} + \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\mu}_{\tilde{\alpha}U}\} + \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}L}\} + \min_{\tilde{\alpha}\in\tilde{h}}\{\tilde{\nu}_{\tilde{\alpha}U}\}\right), \tag{3.12}$$

where $\tilde{\alpha} = ([\tilde{\mu}_{\tilde{\alpha}L}, \tilde{\mu}_{\tilde{\alpha}U}], [\tilde{\nu}_{\tilde{\alpha}L}, \tilde{\nu}_{\tilde{\alpha}U}])$ is an element in $\tilde{h}$ taken the form of IVIFV.

The score function represents the average of expect values of all elements in $\tilde{h}$, while the consistency function focus on the degree of consistency of all elements in $\tilde{h}$. For example, the committee of some experts represent the characteristic of an alternative by an IVGHFE with respect to an attribute, then the score function quantizes the average opinion of experts, but the consistency function reflects how they agree with each other. Based on these two definitions, we introduce the following method for comparing any two IVGHFEs.

Definition 3.2.12 Given two IVGHFEs $\tilde{h}_1$ and $\tilde{h}_2$, then

(1) if $s(\tilde{h}_1) < s(\tilde{h}_2)$, then $\tilde{h}_1$ is small than $\tilde{h}_2$, denoted by $\tilde{h}_1 < \tilde{h}_2$;

(2) if $s(\tilde{h}_1) = s(\tilde{h}_2)$, then

(a) if $c(\tilde{h}_1) < c(\tilde{h}_2)$, then $\tilde{h}_1$ is small than $\tilde{h}_2$, denoted by $\tilde{h}_1 < \tilde{h}_2$;

(b) if $c(\tilde{h}_1) = c(\tilde{h}_2)$, then $\tilde{h}_1$ and $\tilde{h}_2$ represent the same information, denoted by $\tilde{h}_1 = \tilde{h}_2$.

Example 3.2.13 Let $\tilde{h}_1 = \{(0.5, 0.4)\}$, $\tilde{h}_2 = \{([0.4, 0.5], [0.3, 0.4])\}$ and $\tilde{h}_3 = \{([0.3, 0.35], [0.4, 0.55]), (0.4, 0.55)\}$ be three IVGHFEs. To compare three IVGFEs, we firstly construct IVIFVs from IFVs in IVGHFEs $\tilde{h}_1$ and $\tilde{h}_3$. From Theorem 3.1.2 (let $\lambda = \frac{\mu}{\pi} \times 0.005$, $\rho = \frac{\nu}{\pi} \times 0.005$, $\zeta = \lambda$, $\eta = \rho$), we have $\tilde{h}_1 = \{[0.4975, 0.5025], [0.398, 0.402]\}$ and $\tilde{h}_3 = \{([0.3, 0.35], [0.4, 0.55]), ([0.398, 0.402], [0.5473, 0.5528])\}$, then we calculate the score of $\tilde{h}_i$ $(i = 1, 2, 3)$:

$$
\begin{aligned}
s(\tilde{h}_1) &= \frac{1}{4}(0.4975 + 1 - 0.398 + 0.5025 + 1 - 0.402) = 0.55, \\
s(\tilde{h}_2) &= \frac{1}{4}(0.4 + 1 - 0.3 + 0.5 + 1 - 0.4) = 0.55, \\
s(\tilde{h}_3) &= \frac{1}{2}\bigg(\frac{1}{4}(0.3 + 1 - 0.4 + 0.35 + 1 - 0.55) \\
&\qquad + \frac{1}{4}(0.398 + 1 - 0.5473 + 0.402 + 1 - 0.5528)\bigg) = 0.425.
\end{aligned}
$$

and thus, $s(\tilde{h}_1) = s(\tilde{h}_2) > s(\tilde{h}_3)$, we get $\tilde{h}_1 > \tilde{h}_3$ and $\tilde{h}_2 > \tilde{h}_3$. On the other hand, we calculate the consistency degrees of $\tilde{h}_1$ and $\tilde{h}_2$:

$$
\begin{aligned}
c(\tilde{h}_1) &= \frac{1}{2}(0.4975 + 0.5025 + 0.398 + 0.402) = 0.9, \\
c(\tilde{h}_2) &= \frac{1}{2}(0.4 + 0.5 + 0.3 + 0.4) = 0.8,
\end{aligned}
$$

then since $c(\tilde{h}_1) > c(\tilde{h}_2)$, we have $\tilde{h}_1 > \tilde{h}_2$. Therefore, $\tilde{h}_1 > \tilde{h}_2 > \tilde{h}_3$.

3.2.4 Extension principle

As discussed in Subsection 3.2.3, IVGHFSs permit us to represent the situation in which a group, or even several group, of experts have to make decisions on a set of alternatives. Whereas we need to develop a function or mechanism to aggregate evaluations taking the form of IVGHFSs, so as to obtain the overall satisfaction degree to select the most relevant one. Motivated by [54, 47], we present an extension principle to export operations on IVIFSs to IVGHFSs as follows.

Definition 3.2.14 Let Θ be a function $\Theta : (D[0,1]\times D[0,1])^N \to D[0,1]\times D[0,1]$ and $\tilde{H} = \{\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_N\}$ be a set of IVGHFSs on the reference set X. Then the extension of Θ on $\tilde{H}$ is defined, for each $x \in X$, by:

$$\Theta_{\tilde{H}}(x) = \cup_{\tilde{\alpha}\in\tilde{h}_1(x)\times\tilde{h}_2(x)\times\cdots\times\tilde{h}_N(x)}\{\Theta(\tilde{\alpha})\}. \tag{3.13}$$

To deal with aggregation of IVGHFEs according to Definition 3.2.14, we can employ some aggregation operators on IVIFVs such as the IIFA operator (Eq. (3.5)) and the IIFG operator (Eq. (3.6)) etc. Let's clarify it in the following example.

Example 3.2.15 Let $\tilde{h}_1 = \{(0.4, 0.5)\}$, $\tilde{h}_2 = \{([0.3, 0.4], [0.4, 0.45])\}$ and $\tilde{h}_3 = \{([0.3, 0.35], [0.45, 0.55]),\ (0.35, 0.45)\}$ be three IVGHFEs, then, from Theorem 3.1.2 (let $\lambda = \frac{\mu}{\pi}\times 0.008$, $\rho = \frac{\nu}{\pi}\times 0.008$, $\zeta = \lambda$, $\eta = \rho$), we construct IVIFVs from IFVs in IVGHFEs $\tilde{h}_1$ and $\tilde{h}_3$ as follows: $\tilde{h}_1 = \{([0.3968, 0.4032], [0.496, 0.504])\}$ and $\tilde{h}_3 = \{([0.3, 0.35], [0.45, 0.55]),\ ([0.3472, 0.3528], [0.4464, 0.4536])\}$. Then the Arithmetic Mean (AM) of them is conducted, associated with the IIFA operator (Eq. (3.5)), as follows:

$$\begin{aligned}
\mathrm{AM}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3) &= \cup_{\tilde{\alpha}\in\tilde{h}_1\times\tilde{h}_2\times\tilde{h}_3}\{\mathrm{IIFA}(\tilde{\alpha})\}\\
&= \{\mathrm{IIFA}(([0.3968, 0.4032], [0.496, 0.504]), ([0.3, 0.4], [0.4, 0.45]), ([0.3, 0.35], [0.45, 0.55]))\}\\
&\quad \cup\{\mathrm{IIFA}(([0.3968, 0.4032], [0.496, 0.504]), ([0.3, 0.4], [0.4, 0.45]), ([0.3472, 0.3528], [0.4464,\\
&\qquad 0.4536]))\}\\
&= \{([0.3339, 0.3849], [0.4469, 0.4997]), ([0.3492, 0.3858], [0.4457, 0.4686])\}.
\end{aligned}$$

The Geometric Mean (GM) of them is also conducted, associated with the IIFG operator (Eq. (3.6)), as follows:

$$\begin{aligned}
\mathrm{GM}(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3) &= \cup_{\tilde{\alpha}\in\tilde{h}_1\times\tilde{h}_2\times\tilde{h}_3}\{\mathrm{IIFG}(\tilde{\alpha})\}\\
&= \{\mathrm{IIFG}(([0.3968, 0.4032], [0.496, 0.504]), ([0.3, 0.4], [0.4, 0.45]), ([0.3, 0.35], [0.45, 0.55]))\}\\
&\quad \cup\{\mathrm{IIFG}(([0.3968, 0.4032], [0.496, 0.504]), ([0.3, 0.4], [0.4, 0.45]), ([0.3472, 0.3528], [0.4464,\\
&\qquad 0.4536]))\}\\
&= \{([0.3293, 0.3836], [0.4501, 0.503]), ([0.3457, 0.3846], [0.4489, 0.4698])\}.
\end{aligned}$$

Furthermore, based on interval-valued intuitionistic fuzzy aggregation operators such as in [69, 72, 71, 70, 78, 44], we can also develop other versions of aggregation operators for aggregating IVGHFEs. For example, if IVGHFEs have different relative weights, the weighting aggregation operator should be considered; if the ordering of IVGHFEs is provided, the ordered weighting aggregation operator should be dealt with.

3.2.5 Distance measure of IVGHFEs

Because that distance and similarity measures can be applied to many areas such as pattern recognition, cluster analysis, approximate reasoning and decision making, they have attracted a lot of attention. A lot of distance measures have been developed for FSs, IFSs, IVIFSs and HFSs as mentioned in introduction, but there is little research on IVGHFEs. Thus, it is very necessary to develop some distance measure under interval-valued generalized hesitant fuzzy environment. We first present this issue by proposing the axioms for distance measure.

Definition 3.2.16 Let $\tilde{h}_1$ and $\tilde{h}_2$ be two IVGHFEs, then the distance measure between $\tilde{h}_1$ and $\tilde{h}_2$ is defined as $d(\tilde{h}_1, \tilde{h}_2)$, which satisfies the following properties:

(D1) $0 \leq d(\tilde{h}_1, \tilde{h}_2) \leq 1$;

(D2) $d(\tilde{h}_1, \tilde{h}_2) = 0$ if and only if $\tilde{h}_1 = \tilde{h}_2$;

(D3) $d(\tilde{h}_1, \tilde{h}_2) = d(\tilde{h}_2, \tilde{h}_1)$.

In most cases of two IVGHFEs $\tilde{h}_1$ and $\tilde{h}_2$, the numbers of elements of $\tilde{h}_1$ and $\tilde{h}_2$ may be different, i.e. $l(\tilde{h}_1) \neq l(\tilde{h}_2)$, and for convenience, let $l_{\tilde{h}} = \max\{l(\tilde{h}_1), l(\tilde{h}_2)\}$ To operate correctly, we should extend the shorter ones, until both of them have the same length when we compare them. To extend the shorter one, the best way is to add the same values several times in it. In fact, we can extend the shorter one by adding any values in it. The selection of this value mainly depends on the decision makers' risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value. For example, let $\tilde{h}_1 = \{([0.4, 0.5], [0.3, 0.4]), ([0.5, 0.6], [0.2.0.3]), ([0.5, 0.6], [0.1, 0.2])\}$,

$\tilde{h}_2 = \{([0.5, 0.6], [0.2, 0.3]), ([0.3, 0.4], [0.4, 0.5])\}$, then we get $l(\tilde{h}_1) > l(\tilde{h}_2)$. To operate correctly, we should extend $\tilde{h}_2$ until it has the same length of $\tilde{h}_1$. Using the score function [69] of IVIFVs, the optimist may extend $\tilde{h}_2$ as $\tilde{h}_2 = \{([0.5, 0.6], [0.2, 0.3]), ([0.5, 0.6], [0.2, 0.3]), ([0.3, 0.4], [0.4, 0.5])\}$, and the pessimist may extend it as $\tilde{h}_2 = \{([0.5, 0.6], [0.2, 0.3]), ([0.3, 0.4], [0.4, 0.5]), ([0.3, 0.4], [0.4, 0.5])\}$ (If some elements of IVGHFE are in the form of IFVs, then we use Theorem 3.1.2 to construct to IVIFVs from IFVs). Although the results may be different if we extend the shorter one by adding different values, it is reasonable because the decision makers' risk preferences can directly influence the final decision. In this chapter, we assume that the decision makers are all pessimistic (other situation can be studied similarly).

Based on the Hamming distance and the Euclidean distance, we define the interval-valued generalized hesitant normalized Hamming distance:

$$d_{ivghnh}(\tilde{h}_1, \tilde{h}_2) = \frac{1}{4l_{\tilde{h}}} \sum_{j=1}^{l_{\tilde{h}}} \left(\left| \tilde{\mu}_{\tilde{\alpha}_1^{\sigma(j)}L} - \tilde{\mu}_{\tilde{\alpha}_2^{\sigma(j)}L} \right| + \left| \tilde{\mu}_{\tilde{\alpha}_1^{\sigma(j)}U} - \tilde{\mu}_{\tilde{\alpha}_2^{\sigma(j)}U} \right| + \left| \tilde{\nu}_{\tilde{\alpha}_1^{\sigma(j)}L} - \tilde{\nu}_{\tilde{\alpha}_2^{\sigma(j)}L} \right| + \left| \tilde{\nu}_{\tilde{\alpha}_1^{\sigma(j)}U} - \tilde{\nu}_{\tilde{\alpha}_2^{\sigma(j)}U} \right| \right) \tag{3.14}$$

and the interval-valued generalized hesitant normalized Euclidean distance:

$$d_{ivghne}(\tilde{h}_1, \tilde{h}_2) = \left[\frac{1}{4l_{\tilde{h}}} \sum_{j=1}^{l_{\tilde{h}}} \left(\left| \tilde{\mu}_{\tilde{\alpha}_1^{\sigma(j)}L} - \tilde{\mu}_{\tilde{\alpha}_2^{\sigma(j)}L} \right|^2 + \left| \tilde{\mu}_{\tilde{\alpha}_1^{\sigma(j)}U} - \tilde{\mu}_{\tilde{\alpha}_2^{\sigma(j)}U} \right|^2 + \left| \tilde{\nu}_{\tilde{\alpha}_1^{\sigma(j)}L} - \tilde{\nu}_{\tilde{\alpha}_2^{\sigma(j)}L} \right|^2 + \left| \tilde{\nu}_{\tilde{\alpha}_1^{\sigma(j)}U} - \tilde{\nu}_{\tilde{\alpha}_2^{\sigma(j)}U} \right|^2 \right) \right]^{\frac{1}{2}}, \tag{3.15}$$

where $\tilde{\alpha}_i^{\sigma(j)} = \left(\left[\tilde{\mu}_{\tilde{\alpha}_i^{\sigma(j)}L}, \tilde{\mu}_{\tilde{\alpha}_i^{\sigma(j)}U} \right], \left[\tilde{\nu}_{\tilde{\alpha}_i^{\sigma(j)}L}, \tilde{\nu}_{\tilde{\alpha}_i^{\sigma(j)}U} \right] \right), i = 1, 2$ are the jth largest values in $\tilde{h}_1$ and $\tilde{h}_2$, respectively.

Example 3.2.17 Let $\tilde{h}_1 = \{([0.7, 0.8], [0.3, 0.45]), ([0.35, .39], [0.4, 0.45]), ([0.5, 0.57], [0.6, 0.65]), ([0.1, 0.3], [0.2, 0.8]), ([0.4, 0.5], [0.7, 0.9])\}$ and $\tilde{h}_2 = \{([0.7, 0.9], [0.3, 0.7]), ([0.5, 0.65], [0.2, 0.4]), ([0.2, 0.3], [0.1, 0.2]), ([0.4, 0.7], [0.5, 0.6]), ([0.35,$

0.45], [0.7, 0.85])} be two IVGHFEs. By definition of the interval-valued generalized hesitant normalized Hamming distance (Eq.(3.14)), we calculate the $d_{ivghnh}(\tilde{h}_1, \tilde{h}_2)$:

$$d_{ivghnh}(\tilde{h}_1, \tilde{h}_2) = 0.194$$

and we also calculate the $d_{ivghne}(\tilde{h}_1, \tilde{h}_2)$ by definition of the interval-valued generalized hesitant normalized Euclidean distance (Eq.(3.15)):

$$d_{ivghne}(\tilde{h}_1, \tilde{h}_2) = 0.2458.$$

3.3 Decision making based on interval-valued generalized hesitant fuzzy information

In some practical problems, for example, the presidential election or the blind peer review of thesis, anonymity is required in order to protect the decision makers' privacy or avoid influencing each other. In this section, we develop two approaches for solving multiple attribute decision making with anonymity under interval-valued generalized hesitant fuzzy information.

3.3.1 Two approaches to multiple attribute decision making

Suppose that there are m alternatives O_i $(i = 1, 2, \ldots, m)$ and n attributes x_j $(j = 1, 2, \ldots, n)$. An none specific weighting operator has been developed, we suppose that the weights of attributes are indifferent. If the decision makers provide several values such as IVIFVs or IFVs for the alternatives O_i under the attribute x_j with anonymity, these values can be considered as IVGHFE $\tilde{h}_{ij}$. In the case where two decision makers provide the same value, then the value emerges only once in $\tilde{h}_{ij}$. Suppose that the decision matrix $\tilde{H} = (\tilde{h}_{ij})_{m\times n}$ is the interval-valued generalized hesitant fuzzy decision matrix, where $\tilde{h}_{ij}$ $(i = 1, 2, \ldots, m;$ $j = 1, 2, \ldots n)$ are in the form of IVGHFEs.

Then, we utilize the extension principle, i.e. the AM or GM in Example 3.2.15, to develop an approach to multiple attribute decision making problems with interval-valued generalized hesitant fuzzy information, which can be described as following:

Approach III

Step 1. Construct the decision matrix $\tilde{H} = (\tilde{h}_{ij})_{m\times n}$, where all the arguments $\tilde{h}_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots n)$ are IVGHFEs, given by decision makers, for alternative O_i with respect to the attribute x_j.

Step 2. Utilize Theorem 3.1.2 to construct IVIFVs from IFVs in IVGHFEs $\tilde{h}_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots n)$.

Step 3. Aggregate all IVGHFEs $\tilde{h}_{ij}$ $(j = 1, 2, \ldots, n)$ of alternatives O_i $(i = 1, 2, \ldots, m)$ with respect to attributes x_j $(j = 1, 2, \ldots, n)$ into the overall values $\tilde{h}_i$ by using the AM, associated with IIFA operator on IVIFVs, in Example 3.2.15:

$$\begin{aligned}
\tilde{h}_i &= \mathrm{AM}(\tilde{h}_{i1}, \tilde{h}_{i2}, \ldots, \tilde{h}_{in}) \\
&= \cup_{\tilde{\alpha}_{i1}\in\tilde{h}_{i1}, \tilde{\alpha}_{i2}\in\tilde{h}_{i2}, \ldots, \tilde{\alpha}_{in}\in\tilde{h}_{in}} \{\mathrm{IIFA}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2}, \ldots, \tilde{\alpha}_{in})\} \\
&= \cup_{\tilde{\alpha}_{i1}\in\tilde{h}_{i1}, \tilde{\alpha}_{i2}\in\tilde{h}_{i2}, \ldots, \tilde{\alpha}_{in}\in\tilde{h}_{in}} \left\{ \left(\left[1 - \prod_{j=1}^{n} (1 - \tilde{\mu}_{\tilde{\alpha}_{ij}L})^{\frac{1}{n}}, 1 - \prod_{j=1}^{n} (1 - \tilde{\mu}_{\tilde{\alpha}_{ij}U})^{\frac{1}{n}} \right], \right. \right. \\
&\qquad \left. \left. \left[\prod_{j=1}^{n} \tilde{\nu}_{\tilde{\alpha}_{ij}L}^{\frac{1}{n}}, \prod_{j=1}^{n} \tilde{\nu}_{\tilde{\alpha}_{ij}U}^{\frac{1}{n}} \right] \right) \right\},
\end{aligned} \tag{3.16}$$

or the GM, associated with IIFG operator on IVIFSs, in Example 3.2.15:

$$\begin{aligned}
\tilde{h}_i &= \mathrm{GM}(\tilde{h}_{i1}, \tilde{h}_{i2}, \ldots, \tilde{h}_{in}) \\
&= \cup_{\tilde{\alpha}_{i1}\in\tilde{h}_{i1}, \tilde{\alpha}_{i2}\in\tilde{h}_{i2}, \ldots, \tilde{\alpha}_{in}\in\tilde{h}_{in}} \{\mathrm{IIFG}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2}, \ldots, \tilde{\alpha}_{in})\} \\
&= \cup_{\tilde{\alpha}_{i1}\in\tilde{h}_{i1}, \tilde{\alpha}_{i2}\in\tilde{h}_{i2}, \ldots, \tilde{\alpha}_{in}\in\tilde{h}_{in}} \left\{ \left(\left[\prod_{j=1}^{n} \tilde{\mu}_{\tilde{\alpha}_{i1}L}^{\frac{1}{n}}, \prod_{j=1}^{n} \tilde{\mu}_{\tilde{\alpha}_{i1}U}^{\frac{1}{n}} \right], \right. \right. \\
&\qquad \left. \left. \left[1 - \prod_{j=1}^{n} (1 - \tilde{\nu}_{\tilde{\alpha}_{i1}L})^{\frac{1}{n}}, 1 - \prod_{j=1}^{n} (1 - \tilde{\nu}_{\tilde{\alpha}_{i1}U})^{\frac{1}{n}} \right] \right) \right\},
\end{aligned} \tag{3.17}$$

where $\tilde{\alpha}_{ij} = ([\tilde{\mu}_{\tilde{\alpha}_{ij}L}, \tilde{\mu}_{\tilde{\alpha}_{ij}U}], [\tilde{\nu}_{\tilde{\alpha}_{ij}L}, \tilde{\nu}_{\tilde{\alpha}_{ij}U}])$ is an element in $\tilde{h}_{ij}$ taken the form of IVIFV.

Step 4. Compute the score values $s(\tilde{h}_i)$ $(i = 1, 2, \ldots, m)$ of $\tilde{h}_i$ $(i = 1, 2, \ldots, m)$ by Definition 3.2.10:

$$\begin{aligned} s(\tilde{h}_i) &= \frac{1}{l(\tilde{h}_i)} \sum_{\tilde{\alpha}_i \in \tilde{h}_i} E(\tilde{\alpha}_i) \\ &= \frac{1}{4l(\tilde{h}_i)} \sum_{\tilde{\alpha}_i \in \tilde{h}_i} (\tilde{\mu}_{\tilde{\alpha}_i L} + 1 - \tilde{\nu}_{\tilde{\alpha}_i L} + \tilde{\mu}_{\tilde{\alpha}_i U} + 1 - \tilde{\nu}_{\tilde{\alpha}_i U}), \end{aligned} \tag{3.18}$$

where $l(\tilde{h}_i)$ is the number of elements in $\tilde{h}_i$ and $\tilde{\alpha}_i = ([\tilde{\mu}_{\tilde{\alpha}_i L}, \tilde{\mu}_{\tilde{\alpha}_i U}], [\tilde{\nu}_{\tilde{\alpha}_i L}, \tilde{\nu}_{\tilde{\alpha}_i U}])$ is an element in $\tilde{h}_i$ taken the form of IVIFV. If there is no difference between two score values $s(\tilde{h}_i)$ and $s(\tilde{h}_j)$, then we need to calculate the consistency degrees $c(\tilde{h}_i)$ and $c(\tilde{h}_j)$ of the alternatives O_i and O_j $(i, j = 1, 2, \ldots, m, i \neq j)$, respectively, by Definition 3.2.11:

$$c(\tilde{h}_i) = \frac{1}{2} \left(\min_{\tilde{\alpha}_i \in \tilde{h}_i} \{\tilde{\mu}_{\tilde{\alpha}_i L}\} + \min_{\tilde{\alpha}_i \in \tilde{h}_i} \{\tilde{\mu}_{\tilde{\alpha}_i U}\} + \min_{\tilde{\alpha}_i \in \tilde{h}_i} \{\tilde{\nu}_{\tilde{\alpha}_i L}\} + \min_{\tilde{\alpha}_i \in \tilde{h}_i} \{\tilde{\nu}_{\tilde{\alpha}_i U}\} \right). \tag{3.19}$$

Step 5. Rank the alternatives O_i $(i = 1, 2, \ldots, m)$ according to Definition 3.2.12 and then select the most desirable alternative(s).

Step 6. End.

In the situations where the information about attribute weights is completely known, that is, the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ of the attributes x_j $(i = 1, 2, \ldots, n)$ can be completely determined in advance, then we can construct the weighted interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}^* = (\tilde{h}^*_{ij})_{m \times n}$, where $\tilde{h}^*_{ij} = w_j \tilde{h}_{ij} = \cup_{\tilde{\alpha}_{ij} \in \tilde{h}_{ij}} \{w_j \tilde{\alpha}_{ij}\} = \cup_{\tilde{\alpha}_{ij} \in \tilde{h}_{ij}} \{([1 - (1 - \tilde{\mu}_{\tilde{\alpha}_{ij} L})^{w_j}, 1 - (1 - \tilde{\mu}_{\tilde{\alpha}_{ij} U})^{w_j}], [\tilde{\nu}^{w_j}_{\tilde{\alpha}_{ij} L}, \tilde{\nu}^{w_j}_{\tilde{\alpha}_{ij} U}])\}$ is the weighted IVGHFE, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, and w_j is weight of the attribute x_j such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

The positive ideal solution and negative ideal solution can be denoted as $\tilde{h}^+ = \{([1, 1], [0, 0])\}$ and $\tilde{h}^- = \{([0, 0], [1, 1])\}$, respectively, within the interval-valued generalized hesitant fuzzy environment. The separation between alternatives can be usually measured by Hamming distance or Euclidean distance. The separation degrees, $S^+(O_i)$ and $S^-(O_i)$, of each alternative O_i $(i = 1, 2, \ldots, m)$ to the PIS $\tilde{h}^+$ and NIS $\tilde{h}^-$, respectively, are derived from Eqs. (3.14) and (3.15):

- Separation degree based on the interval-valued generalized hesitant normalized Hamming distance d_{ivghnh}:

$$\begin{aligned} S^+_{d_{ivghnh}}(O_i) &= \frac{1}{n}\sum_{j=1}^{n} d_{ivghnh}(\tilde{h}^*_{ij}, \tilde{h}^+) \\ &= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{4l(\tilde{h}^*_{ij})}\sum_{k=1}^{l(\tilde{h}^*_{ij})}\Big((1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}L})^{w_j} + (1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}U})^{w_j} \right. \\ &\qquad \left. + \tilde{\nu}^{w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}L} + \tilde{\nu}^{w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}U}\Big)\right], \end{aligned} \tag{3.20}$$

$$\begin{aligned} S^-_{d_{ivghnh}}(O_i) &= \frac{1}{n}\sum_{j=1}^{n} d_{ivghnh}(\tilde{h}^*_{ij}, \tilde{h}^-) \\ &= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{4l(\tilde{h}^*_{ij})}\sum_{k=1}^{l(\tilde{h}^*_{ij})}\Big(1-(1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}L})^{w_j} + 1-(1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}U})^{w_j} \right. \\ &\qquad \left. + 1-\tilde{\nu}^{w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}L} + 1-\tilde{\nu}^{w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}U}\Big)\right]. \end{aligned} \tag{3.21}$$

- Separation degree based on the interval-valued generalized hesitant normalized Euclidean distance d_{ivghne}:

$$\begin{aligned} S^+_{d_{ivghne}}(O_i) &= \frac{1}{n}\sum_{j=1}^{n} d_{ivghne}(\tilde{h}^*_{ij}, \tilde{h}^+) \\ &= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{4l(\tilde{h}^*_{ij})}\sum_{k=1}^{l(\tilde{h}^*_{ij})}\Big((1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}L})^{2w_j} + (1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}U})^{2w_j} \right. \\ &\qquad \left. + \tilde{\nu}^{2w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}L} + \tilde{\nu}^{2w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}U}\Big)\right]^{\frac{1}{2}}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} S^-_{d_{ivghne}}(O_i) &= \frac{1}{n}\sum_{j=1}^{n} d_{ivghne}(\tilde{h}^*_{ij}, \tilde{h}^-) \\ &= \frac{1}{n}\sum_{j=1}^{n}\left[\frac{1}{4l(\tilde{h}^*_{ij})}\sum_{k=1}^{l(\tilde{h}^*_{ij})}\Big((1-(1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}L})^{w_j})^2 + (1-(1-\tilde{\mu}_{\tilde{\alpha}^{\sigma(k)}_{ij}U})^{w_j})^2\right. \end{aligned}$$

$$+(1-\tilde{\nu}^{w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}L})^2+(1-\tilde{\nu}^{w_j}_{\tilde{\alpha}^{\sigma(k)}_{ij}U})^2\Big)\Big]^{\frac{1}{2}}. \tag{3.23}$$

Then the closeness coefficient $C^+(O_i)$ of an alternative O_i with respect to PIS $\tilde{h}^+$ is defined as the following:

$$C^+(O_i)=\frac{S^-(O_i)}{S^+(O_i)+S^-(O_i)},\quad i=1,2,\ldots,m. \tag{3.24}$$

The bigger the closeness coefficient $C^+(O_i)$, the better the alternative O_i will be, as the alternative O_i is closer to the PIS $\tilde{h}^+$. Therefore, the alternatives O_i $(i=1,2,\ldots,m)$ can be ranked according to the closeness coefficients so that the best alternative can be selected.

Approach IV

Step 1. For this step, see Approach III.

Step 2. For this step, see Approach III.

Step 3. Calculate the weighted interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}^*=(\tilde{h}^*_{ij})_{m\times n}$, where $\tilde{h}^*_{ij}=\cup_{\tilde{\alpha}_{ij}\in\tilde{h}_{ij}}\{([1-(1-\tilde{\mu}_{\tilde{\alpha}_{ij}L})^{w_j},1-(1-\tilde{\mu}_{\tilde{\alpha}_{ij}U})^{w_j}],[\tilde{\nu}^{w_j}_{\tilde{\alpha}_{ij}L},\tilde{\nu}^{w_j}_{\tilde{\alpha}_{ij}U}])\}$ is the weighted IVGHFE, $i=1,2,\ldots,m$; $j=1,2,\ldots,n$, and w_j is weight of the attribute x_j such that $w_j>0$ and $\sum_{j=1}^n w_j=1$.

Step 4. Utilize Eqs. (3.20)-(3.23) to calculate the separation degrees $S^+(O_i)$ and $S^-(O_i)$ of each alternative O_i $(i=1,2,\ldots,m)$ from PIS $\tilde{h}^+=\{([1,1],[0,0])\}$ and NIS $\tilde{h}^-=\{([0,0],[1,1])\}$, respectively.

Step 5. Utilize Eq. (3.24) to calculate the closeness coefficient $C^+(O_i)$ of each alternative O_i $(i=1,2,\ldots,m)$ to the PIS $\tilde{h}^+$.

Step 6. Rank the alternatives O_i $(i=1,2,\ldots,m)$ according to the closeness coefficient to the PIS $\tilde{h}^+$ and then select the most desirable one(s).

Step 7. End.

3.3.2 Illustrative Examples

Example 3.3.1 Let us suppose there is an investment company, which wants to invest a sum of money in the best option (adapted from [25]). There is a panel with four possible alternatives to invest the money: (1) O_1 is a car company; (2) O_2 is a food company; (3) O_3 is a computer company; (4) O_4 is an arms company. The investment company must take a decision according to the following three attributes: (1) x_1 is the market share analysis; (2) x_2 is the market growth analysis; (3) x_3 is the benefit analysis. In order to avoid influence each other, the decision makers are required to evaluate the four possible alternatives O_i ($i = 1, 2, 3, 4$), by using the IFVs or IVIFVs, under the above three attributes in anonymity, then the interval-valued generalized hesitant fuzzy decision matrix $\tilde{H} = (\tilde{h}_{ij})_{4\times 3}$ is constructed as shown in Table 3.1, where $\tilde{h}_{ij}$ ($i = 1, 2, 3, 4; j = 1, 2, 3$) are in the form of IVGHFEs.

Table 3.1: Interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}$

	x_1
O_1	$\{([0.5, 0.6], [0.2, 0.3]), (0.3, 0.6), (0.7, 0.2)\}$
O_2	$\{([0.3, 0.5], [0.4, 0.5]), ([0.6, 0.7], [0.1, 0.2]), (0.5, 0.4)\}$
O_3	$\{([0.6, 0.7], [0.2, 0.3]), (0.5, 0.4), (0.6, 0.3)\}$
O_4	$\{([0.5, 0.7], [0.1, 0.2]), ([0.5, 0.6], [0.2, 0.3]), (0.5, 0.4)\}$
	x_2
O_1	$\{([0.3, 0.4], [0.4, 0.6]), ([0, 4, 0.5], [0.3, 0.4]), (0.6, 0.3)\}$
O_2	$\{([0.1, 0.3], [0.2, 0.4]), (0.4, 0.5), (0.7, 0.2)\}$
O_3	$\{([0.3, 0.4], [0.4, 0.5]), ([0.5, 0.6], [0.1, 0.3]), (0.6, 0.3)\}$
O_4	$\{([0.2, 0.4], [0.5, 0.6]), (0.4, 0.5), (0.3, 0.6)\}$
	x_3
O_1	$\{([0.4, 0.5], [0.3, 0.5]), ([0.6, 0.7], [0.1, 0.2]), (0.5, 0.3)\}$
O_2	$\{([0.7, 0.8], [0.1, 0.2]), ([0.1, 0.2], [0.7, 0.8]), (0.3, 0.4)\}$
O_3	$\{([0.5, 0.8], [0.1, 0.2]), (0.5, 0.2), (0.8, 0.1)\}$
O_4	$\{([0.4, 0.6], [0.2, 0.3]), (0.6, 0.3), (0.7, 0.2)\}$

Then, we utilize the Approach III to get the most desirable alternative(s), which involves the following steps:

Step 1. Utilize Theorem 3.1.2 (here we take $\lambda = \frac{\mu}{\pi} \times 0.008$, $\rho = \frac{\nu}{\pi} \times 0.008$, $\zeta = \lambda$, $\eta = \rho$) to construct IVIFVs from IFVs in IVGHFEs $\tilde{h}_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3)$ and then the constructed interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}_0$ is shown in Table 3.2.

Table 3.2: Constructed interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}_0$

	x_1
O_1	$\{([0.5, 0.6], [0.2, 0.3]), ([0.2976, 0.3024], [0.5952, 0.6048]), ([0.6944, 0.7056], [0.1984, 0.2016])\}$
O_2	$\{([0.3, 0.5], [0.4, 0.5]), ([0.6, 0.7], [0.1, 0.2]), ([0.496, 0.504], [0.3968, 0.4032])\}$
O_3	$\{([0.6, 0.7], [0.2, 0.3]), ([0.496, 0.504], [0.3968, 0.4032]), ([0.5952, 0.6048], [0.2976, 0.3024])\}$
O_4	$\{([0.5, 0.7], [0.1, 0.2]), ([0.5, 0.6], [0.2, 0.3]), ([0.496, 0.504], [0.3968, 0.4032])\}$
	x_2
O_1	$\{([0.3, 0.4], [0.4, 0.6]), ([0, 4, 0.5], [0.3, 0.4]), ([0.5952, 0.6048], [0.2976, 0.3024])\}$
O_2	$\{([0.1, 0.3], [0.2, 0.4]), ([0.3968, 0.4032], [0.496, 0.504]), ([0.6944, 0.7056], [0.1984, 0.2016])\}$
O_3	$\{([0.3, 0.4], [0.4, 0.5]), ([0.5, 0.6], [0.1, 0.3]), ([0.5952, 0.6048], [0.2976, 0.3024])\}$
O_4	$\{([0.2, 0.4], [0.5, 0.6]), ([0.3968, 0.4032], [0.496, 0.504]), ([0.2976, 0.3024], [0.5952, 0.6048])\}$
	x_3
O_1	$\{([0.4, 0.5], [0.3, 0.5]), ([0.6, 0.7], [0.1, 0.2]), ([0.496, 0.504], [0.2976, 0.3024])\}$
O_2	$\{([0.7, 0.8], [0.1, 0.2]), ([0.1, 0.2], [0.7, 0.8]), ([0.2976, 0.3024], [0.3968, 0.4032])\}$
O_3	$\{([0.5, 0.8], [0.1, 0.2]), ([0.496, 0.504], [0.1984, 0.2016]), ([0.7936, 0.8064], [0.0992, 0.1008])\}$
O_4	$\{([0.4, 0.6], [0.2, 0.3]), ([0.5952, 0.6048], [0.2976, 0.3024]), ([0.6944, 0.7056], [0.1984, 0.2016])\}$

Step 2. Aggregate all IVGHFEs $\tilde{h}_{ij}$ of alternatives O_i $(i = 1, 2, 3, 4)$ with respect to attributes x_j $(j = 1, 2, 3)$ into the overall values $\tilde{h}_i$ by Eq. (3.16):

$$
\begin{aligned}
\tilde{h}_1 = \Big\{ & ([0.4056, 0.5068], [0.2884, 0.4481]), ([0.4808, 0.5840], [0.2000, 0.3302]), \\
& ([0.4392, 0.5018], [0.2877, 0.3790]), ([0.4354, 0.5358], [0.2621, 0.3915]), \\
& ([0.5068, 0.6085], [0.1817, 0.2884]), ([0.4673, 0.5371], [0.2614, 0.3311]), \\
& ([0.5048, 0.5708], [0.2614, 0.3566]), ([0.5674, 0.6380], [0.1812, 0.2628]), \\
& ([0.5328, 0.5720], [0.2607, 0.3016]), ([0.3343, 0.4063], [0.4149, 0.5661]), \\
& ([0.4185, 0.4992], [0.2877, 0.4171]), ([0.3719, 0.4079], [0.4138, 0.4788]), \\
& ([0.3676, 0.4413], [0.3770, 0.4946]), ([0.4476, 0.5288], [0.2614, 0.3644]),
\end{aligned}
$$

$$([0.4033, 0.4428], [0.3760, 0.4182]), ([0.4454, 0.4834], [0.3760, 0.4505]),$$
$$([0.5155, 0.5643], [0.2607, 0.3320]), ([0.4767, 0.4848], [0.3750, 0.3810]),$$
$$([0.4956, 0.5547], [0.2877, 0.3925]), ([0.5593, 0.6244], [0.1995, 0.2892]),$$
$$([0.5241, 0.5559], [0.2869, 0.3320]), ([0.5208, 0.5809], [0.2614, 0.3429]),$$
$$([0.5814, 0.6465], [0.1812, 0.2527]), ([0.5479, 0.5820], [0.2607, 0.2900]),$$
$$([0.5797, 0.6125], [0.2607, 0.3124]), ([0.6329, 0.6732], [0.1807, 0.2302]),$$
$$([0.6035, 0.6136], [0.2600, 0.2642])\Big\},$$

$$\tilde{h}_2 = \Big\{([0.4261, 0.5879], [0.2000, 0.3420]), ([0.1723, 0.3458], [0.3826, 0.5429]),$$
$$([0.2380, 0.3750], [0.3166, 0.4320]), ([0.4978, 0.6092], [0.2707, 0.3694]),$$
$$([0.2757, 0.3797], [0.5179, 0.5864]), ([0.3331, 0.4073], [0.4286, 0.4666]),$$
$$([0.5996, 0.6912], [0.1995, 0.2722]), ([0.4226, 0.5098], [0.3816, 0.4320]),$$
$$([0.4684, 0.5317], [0.3158, 0.3438]), ([0.5238, 0.6524], [0.1260, 0.2520]),$$
$$([0.3132, 0.4482], [0.2410, 0.4000]), ([0.3676, 0.4728], [0.1995, 0.3183]),$$
$$([0.5832, 0.6704], [0.1705, 0.2722]), ([0.3989, 0.4768], [0.3262, 0.4320]),$$
$$([0.4466, 0.5001], [0.2700, 0.3438]), ([0.6678, 0.7396], [0.1257, 0.2005]),$$
$$([0.5208, 0.5866], [0.2404, 0.3183]), ([0.5588, 0.6050], [0.1989, 0.2533]),$$
$$([0.4856, 0.5890], [0.1995, 0.3183]), ([0.2582, 0.3475], [0.3816, 0.5053]),$$
$$([0.3170, 0.3767], [0.3158, 0.4021]), ([0.5499, 0.6103], [0.2700, 0.3438]),$$
$$([0.3508, 0.3813], [0.5165, 0.5458]), ([0.4023, 0.4089], [0.4274, 0.4343]),$$
$$([0.6412, 0.6920], [0.1989, 0.2533]), ([0.4825, 0.5112], [0.3805, 0.4021]),$$
$$([0.5235, 0.5330], [0.3149, 0.3200])\Big\},$$

$$\tilde{h}_3 = \Big\{([0.4808, 0.6698], [0.2000, 0.3107]), ([0.4794, 0.5531], [0.2513, 0.3115]),$$
$$([0.6134, 0.6734], [0.1995, 0.2473]), ([0.5358, 0.7116], [0.1260, 0.2621]),$$
$$([0.5346, 0.6096], [0.1583, 0.2628]), ([0.6544, 0.7147], [0.1257, 0.2086]),$$

$$
\begin{aligned}
&([0.5674, 0.7127], [0.1812, 0.2628]), ([0.5662, 0.6111], [0.2277, 0.2635]),\\
&([0.6779, 0.7158], [0.1807, 0.2091]), ([0.4392, 0.6096], [0.2513, 0.3644]),\\
&([0.4377, 0.4715], [0.3158, 0.3654]), ([0.5824, 0.6138], [0.2506, 0.2900]),\\
&([0.4673, 0.6326], [0.2283, 0.3183]), ([0.4658, 0.5027], [0.2869, 0.3192]),\\
&([0.6033, 0.6365], [0.2277, 0.2533]), ([0.5328, 0.6603], [0.2277, 0.2900]),\\
&([0.5315, 0.5402], [0.2861, 0.2908]), ([0.6521, 0.6640], [0.2271, 0.2308]),\\
&([0.4787, 0.6380], [0.2283, 0.3311]), ([0.4773, 0.5101], [0.2869, 0.3320]),\\
&([0.6118, 0.6419], [0.2277, 0.2635]), ([0.5048, 0.6594], [0.2075, 0.2892]),\\
&([0.5035, 0.5389], [0.2607, 0.2900]), ([0.6313, 0.6631], [0.2069, 0.2302]),\\
&([0.5657, 0.6851], [0.2069, 0.2635]), ([0.5645, 0.5737], [0.2600, 0.2642]),\\
&([0.6766, 0.6885], [0.2063, 0.2097])\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{h}_4 = \Big\{&([0.3786, 0.5840], [0.2154, 0.3302]), ([0.4550, 0.5857], [0.2460, 0.3311]),\\
&([0.5037, 0.6244], [0.2149, 0.2892]), ([0.4344, 0.5847], [0.2149, 0.3115]),\\
&([0.5039, 0.5864], [0.2453, 0.3124]), ([0.5483, 0.6251], [0.2143, 0.2729]),\\
&([0.4049, 0.5625], [0.2283, 0.3311]), ([0.4781, 0.5643], [0.2607, 0.3320]),\\
&([0.5248, 0.6050], [0.2277, 0.2900]), ([0.4056, 0.5421], [0.2520, 0.3780]),\\
&([0.4787, 0.5440], [0.2877, 0.3790]), ([0.5253, 0.5866], [0.2513, 0.3311]),\\
&([0.4354, 0.5691], [0.2289, 0.3302]), ([0.5048, 0.5708], [0.2614, 0.3311]),\\
&([0.5491, 0.6110], [0.2283, 0.2892]), ([0.5048, 0.6016], [0.2283, 0.3008]),\\
&([0.5657, 0.6032], [0.2607, 0.3016]), ([0.6045, 0.6403], [0.2277, 0.2635]),\\
&([0.4040, 0.5081], [0.3166, 0.4171]), ([0.4773, 0.5101], [0.3615, 0.4182]),\\
&([0.5241, 0.5559], [0.3158, 0.3654]), ([0.4339, 0.5371], [0.2877, 0.3644]),\\
&([0.5035, 0.5389], [0.3284, 0.3654]), ([0.5479, 0.5820], [0.2869, 0.3192]),\\
&([0.5035, 0.5720], [0.2869, 0.3320]), ([0.5645, 0.5737], [0.3276, 0.3328]),\\
&([0.6035, 0.6136], [0.2861, 0.2908])\Big\}.
\end{aligned}
$$

Step 3. Compute the score values $s(\tilde{h}_i)$ $(i = 1, 2, 3, 4)$ of overall values $\tilde{h}_i$ $(i = 1, 2, 3, 4)$ by Eq. (3.18), the score values of alternatives O_i $(i = 1, 2, 3, 4)$ are obtained as follows:

$$s(\tilde{h}_1) = 0.5993, \ s(\tilde{h}_2) = 0.5726, \ s(\tilde{h}_3) = 0.6682, \ s(\tilde{h}_4) = 0.6199.$$

Step 4. Rank the alternatives O_i $(i = 1, 2, 3, 4)$ in accordance with the score values $s(\tilde{h}_i)$ $(i = 1, 2, 3, 4)$ of the overall interval-valued generalized hesitant fuzzy preference values: $O_3 \succ O_4 \succ O_1 \succ O_2$, where the symbol "$\succ$" means "superior to", and thus the most desirable investment alternative is O_3.

Similarly, if Eq. (3.17) (i.e. the GM associated with IIFG operator) is utilized in Step 2, then the overall performance value $\tilde{h}_i$ corresponding to the alternative O_i $(i = 1, 2, 3, 4)$ can be calculated:

$$\begin{aligned}
\tilde{h}_1 = \Big\{ & ([0.3915, 0.4932], [0.3048, 0.4808]), ([0.4481, 0.5518], [0.2440, 0.3927]), \\
& ([0.4206, 0.4946], [0.3040, 0.4198]), ([0.4309, 0.5313], [0.2681, 0.4056]), \\
& ([0.4932, 0.5944], [0.2042, 0.3048]), ([0.4629, 0.5327], [0.2673, 0.3358]), \\
& ([0.4919, 0.5661], [0.2673, 0.3750]), ([0.5631, 0.6333], [0.2033, 0.2690]), \\
& ([0.5285, 0.5676], [0.2665, 0.3016]), ([0.3293, 0.3925], [0.4460, 0.5708]), \\
& ([0.3770, 0.4391], [0.3976, 0.4981]), ([0.3538, 0.3936], [0.4454, 0.5205]), \\
& ([0.3625, 0.4228], [0.4168, 0.5087]), ([0.4149, 0.4730], [0.3658, 0.4254]), \\
& ([0.3894, 0.4240], [0.4161, 0.4511]), ([0.4138, 0.4505], [0.4161, 0.4834]), \\
& ([0.4737, 0.5040], [0.3651, 0.3958]), ([0.4446, 0.4517], [0.4155, 0.4228]), \\
& ([0.4368, 0.5206], [0.3043, 0.4575]), ([0.5000, 0.5824], [0.2435, 0.3655]), \\
& ([0.4692, 0.5220], [0.3035, 0.3938]), ([0.4807, 0.5608], [0.2677, 0.3790]), \\
& ([0.5503, 0.6274], [0.2037, 0.2736]), ([0.5165, 0.5623], [0.2668, 0.3061]), \\
& ([0.5488, 0.5976], [0.2668, 0.3470]), ([0.6283, 0.6685], [0.2027, 0.2362]), \\
& ([0.5896, 0.5991], [0.2660, 0.2703]) \Big\},
\end{aligned}$$

$$\tilde{h}_2 = \Big\{([0.2759, 0.4932], [0.2440, 0.3786]), ([0.1442, 0.3107], [0.4759, 0.6085]),$$
$$([0.2075, 0.3566], [0.3384, 0.4364]), ([0.4368, 0.5443], [0.3520, 0.4168]),$$
$$([0.2283, 0.3429], [0.5507, 0.6326]), ([0.3284, 0.3936], [0.4329, 0.4710]),$$
$$([0.5264, 0.6560], [0.2435, 0.3165]), ([0.2752, 0.4132], [0.4755, 0.5694]),$$
$$([0.3958, 0.4743], [0.3380, 0.3801]), ([0.3476, 0.5518], [0.1347, 0.2732]),$$
$$([0.1817, 0.3476], [0.4000, 0.5421]), ([0.2614, 0.3990], [0.2427, 0.3408]),$$
$$([0.5503, 0.6089], [0.2582, 0.3178]), ([0.2877, 0.3836], [0.4856, 0.5703]),$$
$$([0.4138, 0.4403], [0.3508, 0.3813]), ([0.6632, 0.7338], [0.1341, 0.2005]),$$
$$([0.3467, 0.4623], [0.3996, 0.4964]), ([0.4987, 0.5306], [0.2422, 0.2749]),$$
$$([0.3262, 0.4946], [0.2427, 0.3408]), ([0.1705, 0.3115], [0.4749, 0.5847]),$$
$$([0.2453, 0.3576], [0.3373, 0.4021]), ([0.5165, 0.5458], [0.3508, 0.3813]),$$
$$([0.2700, 0.3438], [0.5499, 0.6103]), ([0.3884, 0.3946], [0.4319, 0.4389]),$$
$$([0.6224, 0.6577], [0.2422, 0.2749]), ([0.3254, 0.4143], [0.4746, 0.5432]),$$
$$([0.4680, 0.4755], [0.3368, 0.3424])\Big\},$$

$$\tilde{h}_3 = \Big\{([0.4481, 0.6073], [0.2440, 0.3458]), ([0.4469, 0.5206], [0.2727, 0.3462]),$$
$$([0.5227, 0.6089], [0.2438, 0.3198]), ([0.5313, 0.6952], [0.1347, 0.2681]),$$
$$([0.5299, 0.5960], [0.1674, 0.2686]), ([0.6198, 0.6971], [0.1344, 0.2391]),$$
$$([0.5631, 0.6971], [0.2033, 0.2690]), ([0.5616, 0.5976], [0.2334, 0.2695]),$$
$$([0.6569, 0.6989], [0.2030, 0.2399]), ([0.4206, 0.5443], [0.3120, 0.4241]),$$
$$([0.4195, 0.4666], [0.3380, 0.4245]), ([0.4906, 0.5458], [0.3117, 0.4012]),$$
$$([0.4629, 0.5864], [0.2757, 0.3408]), ([0.4617, 0.5027], [0.3031, 0.3412]),$$
$$([0.5400, 0.5879], [0.2755, 0.3146]), ([0.5285, 0.6248], [0.2748, 0.3068]),$$
$$([0.5271, 0.5356], [0.3023, 0.3073]), ([0.6165, 0.6264], [0.2746, 0.2793]),$$
$$([0.4469, 0.5784], [0.2761, 0.3934]), ([0.4457, 0.4959], [0.3035, 0.3938]),$$
$$([0.5213, 0.5800], [0.2759, 0.3693]), ([0.4919, 0.6231], [0.2380, 0.3056]),$$

$$([0.4906, 0.5342], [0.2668, 0.3061]), ([0.5738, 0.6248], [0.2377, 0.2780]),$$
$$([0.5616, 0.6639], [0.2371, 0.2698]), ([0.5601, 0.5691], [0.2660, 0.2703]),$$
$$([0.6551, 0.6657], [0.2369, 0.2408])\Big\},$$

$$\begin{aligned}
\tilde{h}_4 = \Big\{&([0.3420, 0.5518], [0.2886, 0.3927]), ([0.3904, 0.5533], [0.3188, 0.3934]),\\
&([0.4110, 0.5824], [0.2881, 0.3655]), ([0.4297, 0.5533], [0.2867, 0.3475]),\\
&([0.4906, 0.5547], [0.3170, 0.3483]), ([0.5165, 0.5840], [0.2863, 0.3183]),\\
&([0.3904, 0.5027], [0.3370, 0.3951]), ([0.4457, 0.5040], [0.3651, 0.3958]),\\
&([0.4692, 0.5306], [0.3365, 0.3680]), ([0.3915, 0.5241], [0.2732, 0.4191]),\\
&([0.4469, 0.5255], [0.3040, 0.4198]), ([0.4705, 0.5533], [0.2727, 0.3931]),\\
&([0.4309, 0.5646], [0.2348, 0.3351]), ([0.4919, 0.5661], [0.2673, 0.3358]),\\
&([0.5179, 0.5960], [0.2343, 0.3053]), ([0.4919, 0.6016], [0.2340, 0.3008]),\\
&([0.5616, 0.6032], [0.2665, 0.3016]), ([0.5912, 0.6350], [0.2334, 0.2695]),\\
&([0.3904, 0.4946], [0.3384, 0.4492]), ([0.4457, 0.4959], [0.3665, 0.4498]),\\
&([0.4692, 0.5220], [0.3380, 0.4245]), ([0.4297, 0.5327], [0.3036, 0.3695]),\\
&([0.4906, 0.5342], [0.3331, 0.3702]), ([0.5165, 0.5623], [0.3031, 0.3412]),\\
&([0.4906, 0.5676], [0.3028, 0.3370]), ([0.5601, 0.5691], [0.3324, 0.3378]),\\
&([0.5896, 0.5991], [0.3023, 0.3073])\Big\}.
\end{aligned}$$

Then, the score value $s(\tilde{h}_i)$ of overall values $\tilde{h}_i$ $(i = 1, 2, 3, 4)$ can be obtained:

$$s(\tilde{h}_1) = 0.5716,\ s(\tilde{h}_2) = 0.5100,\ s(\tilde{h}_3) = 0.6370,\ s(\tilde{h}_4) = 0.5905.$$

Then $s(\tilde{h}_3) > s(\tilde{h}_4) > s(\tilde{h}_1) > s(\tilde{h}_2)$ and so the final ranking is $O_3 \succ O_4 \succ O_1 \succ O_2$. Thus the best alternative is also O_3.

Example 3.3.2 Let us consider a factory which intends to select a new site for new buildings. Four alternatives O_i $(i = 1, 2, 3, 4)$ are available, and the decision makers consider three attribute to decide which site to choose: (1) x_1 is the price analysis; (2) x_2 is the location analysis; (3) x_3 is the environment analysis. The weight vector of the attributes x_j $(j = 1, 2, 3)$ is $w = (0.5, 0.3, 0.2)^T$. In order to avoid influence each other, the decision makers are required to evaluate the four possible alternatives O_i $(i = 1, 2, 3, 4)$, by using the IFVs or IVIFVs, under the above three attributes in anonymity, then the interval-valued generalized hesitant fuzzy decision matrix $\tilde{H} = (\tilde{h}_{ij})_{4\times 3}$ is constructed as shown in Table 3.3, where $\tilde{h}_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3)$ are in the form of IVGHFEs.

Table 3.3: Interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}$

	x_1
O_1	$\{([0.4, 0.5], [0.3, 0.4]), ([0.3, 0.4], [0.5, 0.6]), (0.6, 0.3)\}$
O_2	$\{([0.4, 0.5], [0.3, 0.4]), ([0.6, 0.7], [0.1, 0.2]), (0.7, 0.2)\}$
O_3	$\{([0.6, 0.7], [0.1, 0.2]), (0.5, 0.4), (0.6, 0.3)\}$
O_4	$\{([0.5, 0.6], [0.2, 0.3]), ([0.5, 0.6], [0.3, 0.4]), (0.5, 0.4)\}$
	x_2
O_1	$\{([0.5, 0.6], [0.3, 0.4]), ([0, 6, 0.7], [0.1, 0.2]), (0.7, 0.2)\}$
O_2	$\{([0.3, 0.4], [0.2, 0.3]), (0.5, 0.4), (0.8, 0.1)\}$
O_3	$\{([0.4, 0.5], [0.3, 0.4]), ([0.5, 0.6], [0.1, 0.2]), (0.6, 0.3)\}$
O_4	$\{([0.3, 0.4], [0.4, 0.5]), (0.5, 0.4), (0.3, 0.6)\}$
	x_3
O_1	$\{([0.5, 0.6], [0.1, 0.2]), ([0.7, 0.8], [0.1, 0.2]), (0.5, 0.4)\}$
O_2	$\{([0.7, 0.8], [0.1, 0.2]), ([0.5, 0.6], [0.3, 0.4]), (0.2, 0.7)\}$
O_3	$\{([0.5, 0.8], [0.1, 0.2]), (0.5, 0.4), (0.8, 0.1)\}$
O_4	$\{([0.4, 0.5], [0.3, 0.4]), (0.6, 0.3), (0.8, 0.1)\}$

Then, we utilize the Approach IV to get the most desirable alternative(s), which involves the following steps:

Step 1. Utilize Theorem 3.1.2 (let $\lambda = \frac{\mu}{\pi} \times 0.008$, $\rho = \frac{\nu}{\pi} \times 0.008$, $\zeta = \lambda$, $\eta = \rho$) to construct IVIFVs from IFVs in IVGHFEs $\tilde{h}_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3)$ and then transform the interval-valued generalized hesitant fuzzy decision matrix

$\tilde{H}$ into the constructed interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}_0$ (see Table 3.4).

Table 3.4: Constructed interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}_0$

	x_1
O_1	$\{([0.4,0.5],[0.3,0.4]),([0.3,0.4],[0.5,0.6]),([0.5952,0.6048],[0.2976,0.3024])\}$
O_2	$\{([0.4,0.5],[0.3,0.4]),([0.6,0.7],[0.1,0.2]),([0.6944,0.7056],[0.1984,0.2016])\}$
O_3	$\{([0.6,0.7],[0.1,0.2]),([0.496,0.504],[0.3968,0.4032]),([0.5952,0.6048],[0.2976,0.3024])\}$
O_4	$\{([0.5,0.6],[0.2,0.3]),([0.5,0.6],[0.3,0.4]),([0.496,0.504],[0.3968,0.4032])\}$
	x_2
O_1	$\{([0.5,0.6],[0.3,0.4]),([0,6,0.7],[0.1,0.2]),([0.6944,0.7056],[0.1984,0.2016])\}$
O_2	$\{([0.3,0.4],[0.2,0.3]),([0.496,0.504],[0.3968,0.4032]),([0.7936,0.8064],[0.0992,0.1008])\}$
O_3	$\{([0.4,0.5],[0.3,0.4]),([0.5,0.6],[0.1,0.2]),([0.5952,0.6048],[0.2976,0.3024])\}$
O_4	$\{([0.3,0.4],[0.4,0.5]),([0.496,0.504],[0.3968,0.4032]),([0.2976,0.3024],[0.5952,0.6048])\}$
	x_3
O_1	$\{([0.5,0.6],[0.1,0.2]),([0.7,0.8],[0.1,0.2]),([0.496,0.504],[0.3968,0.4032])\}$
O_2	$\{([0.7,0.8],[0.1,0.2]),([0.5,0.6],[0.3,0.4]),([0.1984,0.2016],[0.6944,0.7056])\}$
O_3	$\{([0.5,0.8],[0.1,0.2]),([0.496,0.504],[0.3968,0.4032]),([0.7936,0.8064],[0.0992,0.1008])\}$
O_4	$\{([0.4,0.5],[0.3,0.4]),([0.5952,0.6048],[0.2976,0.3024]),([0.7936,0.8064],[0.0992,0.1008])\}$

Step 2. Utilize Definition 3.2.5 (let $w = (0.5, 0.3, 0.2)^T$ be the weight vector of attributes x_j $(j = 1, 2, 3)$) to obtain the weighted IVGHFEs $\tilde{h}^*_{ij} = w_j\tilde{h}_{ij}$ of IVGHFEs $\tilde{h}_{ij}$ $(i = 1, 2, 3, 4; j = 1, 2, 3)$ in $\tilde{H}_0$ and then construct the weighted constructed interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}^* = (\tilde{h}^*_{ij})_{4\times3}$ (see Table 3.5).

Step 3. Utilize Eqs. (3.20) and (3.21) to calculate the separation degrees $S^+(O_i)$ and $S^-(O_i)$ of each alternative O_i $(i = 1, 2, 3, 4)$ from PIS $\tilde{h}^+ = \{([1, 1], [0, 0])\}$ and NIS $\tilde{h}^- = \{([0, 0], [1, 1])\}$, respectively:

$$S^+(O_1) = 0.7133,\ S^+(O_2) = 0.6947,\ S^+(O_3) = 0.6884,\ S^+(O_4) = 0.7461,$$
$$S^-(O_1) = 0.2867,\ S^-(O_2) = 0.3053,\ S^-(O_3) = 0.3116,\ S^-(O_4) = 0.2539.$$

Step 4. Calculate the closeness coefficient $C^+(O_i)$ of each alternative O_i $(i = 1, 2, 3, 4)$ to the PIS $\tilde{h}^+$ by Eq. (3.24):

$$C^+(O_1) = 0.2867,\ C^+(O_2) = 0.3053,\ C^+(O_3) = 0.3116,\ C^+(O_4) = 0.2539.$$

Table 3.5: Weighted constructed interval-valued generalized hesitant fuzzy decision matrix $\tilde{H}^*$

	x_1
O_1	$\{([0.2254, 0.2929], [0.5477, 0.6325]), ([0.1633, 0.2254], [0.7071, 0.7746]), ([0.3638, 0.3714], [0.5455, 0.5499])\}$
O_2	$\{([0.2254, 0.2929], [0.5477, 0.6325]), ([0.3675, 0.4523], [0.3162, 0.4472]), ([0.4472, 0.4574], [0.4454, 0.4490])\}$
O_3	$\{([0.3675, 0.4523], [0.3162, 0.4472]), ([0.2901, 0.2957], [0.6299, 0.6350]), ([0.3638, 0.3714], [0.5455, 0.5499])\}$
O_4	$\{([0.2929, 0.3675], [0.4472, 0.5477]), ([0.2929, 0.3675], [0.5477, 0.6325]), ([0.2901, 0.2957], [0.6299, 0.6350])\}$
	x_2
O_1	$\{([0.1877, 0.2403], [0.6968, 0.7597]), ([0, 2403, 0.3032], [0.5012, 0.6170]), ([0.2993, 0.3071], [0.6155, 0.6185])\}$
O_2	$\{([0.1015, 0.1421], [0.6170, 0.6968]), ([0.1858, 0.1897], [0.7578, 0.7615]), ([0.3771, 0.3890], [0.5000, 0.5024])\}$
O_3	$\{([0.1421, 0.1877], [0.6968, 0.7597]), ([0.1877, 0.2403], [0.5012, 0.6170]), ([0.2376, 0.2431], [0.6952, 0.6985])\}$
O_4	$\{([0.1015, 0.1421], [0.7597, 0.8123]), ([0.1858, 0.1897], [0.7578, 0.7615]), ([0.1006, 0.1024], [0.8559, 0.8600])\}$
	x_3
O_1	$\{([0.1294, 0.1674], [0.6310, 0.7248]), ([0.2140, 0.2752], [0.6310, 0.7248]), ([0.1281, 0.1308], [0.8312, 0.8339])\}$
O_2	$\{([0.2140, 0.2752], [0.6310, 0.7248]), ([0.1294, 0.1674], [0.7860, 0.8326]), ([0.0433, 0.0440], [0.9297, 0.9326])\}$
O_3	$\{([0.1294, 0.2752], [0.6310, 0.7248]), ([0.1281, 0.1308], [0.8312, 0.8339]), ([0.2706, 0.2799], [0.6299, 0.6320])\}$
O_4	$\{([0.0971, 0.1294], [0.7860, 0.8326]), ([0.1655, 0.1695], [0.7847, 0.7873]), ([0.2706, 0.2799], [0.6299, 0.6320])\}$

Step 5. Rank the alternatives O_i $(i = 1, 2, 3, 4)$ in accordance with the closeness coefficient $C^+(O_i)$ to the PIS $\tilde{h}^+$: $O_3 \succ O_2 \succ O_1 \succ O_4$, and thus the most desirable alternative is O_3.

Similarly, if Eqs. (3.22) and (3.23) are utilized in Step 3, then the separation degrees $S^+(O_i)$ and $S^-(O_i)$ of each alternative O_i $(i = 1, 2, 3, 4)$ from PIS $\tilde{h}^+$ and NIS $\tilde{h}^-$, respectively, can be calculated:

$$S^+(O_1) = 0.7187,\ S^+(O_2) = 0.7049,\ S^+(O_3) = 0.6954,\ S^+(O_4) = 0.7496,$$
$$S^-(O_1) = 0.3002,\ S^-(O_2) = 0.3285,\ S^-(O_3) = 0.3267,\ S^-(O_4) = 0.2641,$$

and the closeness coefficient $C^+(O_i)$ of each alternative O_i $(i = 1, 2, 3, 4)$ to the PIS $\tilde{h}^+$ can be obtained:

$$C^+(O_1) = 0.2946,\ C^+(O_2) = 0.3179,\ C^+(O_3) = 0.3197,\ C^+(O_4) = 0.2605.$$

Then $C^+(O_3) > C^+(O_2) > C^+(O_1) > C^+(O_4)$ and so the final ranking is $O_3 \succ O_2 \succ O_1 \succ O_4$. Thus the best alternative is also O_3.

3.4 Conclusions

In this chapter, we both generalized HFS [54, 56] and extended GHFS [47] using IVIFSs in group decision making framework. Firstly, the basic concept of IVGHFS was proposed. The IVGHFS is fit for the situation when decision makers have a hesitation among several interval-valued memberships with uncertainties. Then, we discussed the relationships between IVGHFSs and other types of FSs such as GHFSs, IVIFSs and IFSs. The envelop and basic operations of IVGHFEs were defined and then some relationships and operational laws among those operations were also discussed. In order to apply these IVHGFEs to group decision making, we proposed the extension principle which enables us to employ aggregation operators of IVIFSs to aggregate IVGHFEs. Finally, the effectiveness and applicability of the proposed approaches to solve multiple attribute decision making problem was illustrated with two practical examples. The examples showed the less aggregation time and the flexibility of expressing decision makers' opinion of IVGHFEs.

As future work, we consider the study of proper aggregation operators in interval-valued generalized hesitant fuzzy setting, and apply these operators in many actual fields such as decision making, pattern recognition, medical diagnosis and clustering analysis.

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