Thesis for the Degree of Doctor of Philosophy

Dynamical Analysis in Complex Networks and Quantum Motions

by

Woonhak Baek

Department of Physics

The Graduate School

Pukyong National University

February 2016

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복잡계 네트워크와 양자 운동에서 동역학 분석에 관한 연구

Advisor: Prof. Kyungsik Kim

by
Woonhak Baek

A thesis submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in Department of Physics The Graduate School,
pukyong National University

February 2016

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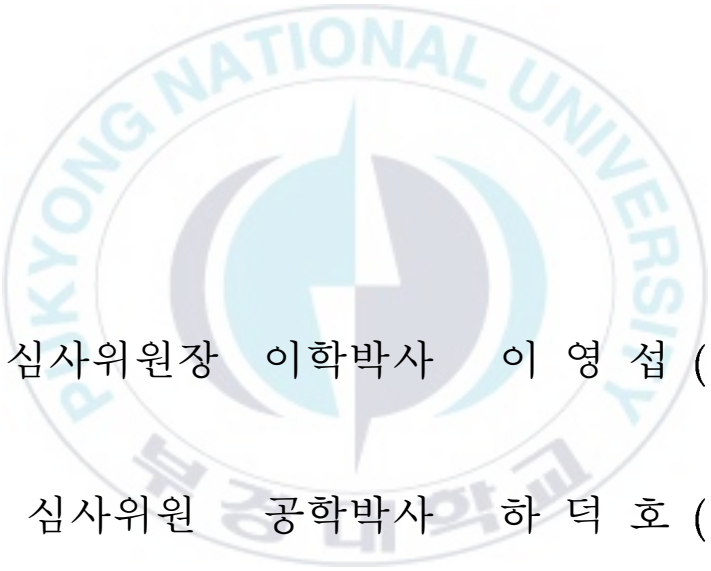

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심사위원 이학박사 김 경 식 (인)

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Woonhak Baek

Department of Physics, Graduate School,
Pukyong National University

Abstract

In this paper, we have studied seismic network. Abe and Suzuki investigate novel seismic network from a singular seismic time series by considering cell resolution and temporal causality. But their approach to construct network does not express the meaning of aftershock. Because, an aftershock is a smaller earthquake to occur after a previous large one. Therefore, we suggest new method to construct seismic network using relationship between aftershock and main earthquake. With the new method, we have examined some topological properties of the earthquake such as the mean degree, the characteristic path length, the clustering coefficient, the global efficiency, the hierarchy, and probability distribution. And the our results compare with Abe and Suzuki's method

We have simulated dynamical phase transitions in a Boolean network with initial random connections. The nature of the phase transition is found numerically and analytically in two connecting

probability density function. By using the noise intensity, we show that a critical value exists for the noise intensity. In addition, we find that the critical exponent of our simulation is similar to the theoretical result 1/2.

Lastly, we have studied the non-Markovian Caldeira - Leggett master equation for the Brownian motion of a free particle. The Fokker - Planck equation with the effective potential in the long time limit contains the Markovian Klein - Kramers equation with the diffusion energy. We mainly analyze the quantum Brownian motion with the harmonic oscillation in the one-dimensional quantum space. By using the Wigner function technique from the non-Markovian Caldeira-Leggett equation, we calculate the velocity distribution function with the diffusion energy and the correlational function. Since such three correlational functions are considered as the exponential, Gaussian, and complementary error functions. The quantum force can be analyzed from the velocity distribution function. Particularly, the quantum force is found to be proportional to the angular frequency $\omega^{1/2}$ in the quantum limit and the steady regime, while the classical force is proportional to the temperature $T^{1/2}$ in the classical limit.

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I Introduction

Most scientists have dreamed that they can uncover the important principles or clearly describe some phenomenon. But there are many scientific phenomena which we couldn't cognize exactly before. For example, economic activity, social interaction, World Wild Web, ecosystem etc, it seems that they behave irregular. And dynamics of above examples are so sensitive to the initial conditions that it is very difficult to predict. These some special properties are the feature of a complex system. Although, there is no absolute definition of what complex system means, some agreements among researchers are widespread. First, the system is basically composed of many components, which are interacting time-dependently. The behavior of individual agents is affected by information coming from the system or other agents. Therefore, it is unlikely that there is a typical agent. Second, the presence of feedback implies that the system is remembering its past and responding to it in a non-trivial way. Third, each agent adapts behavior toward improving performance. Fourth, if the entire agent population evolves, the system is driven by an ecology of agents who interact and adapt under the influence of feedback. Then, the system is typically far from equilibrium and hence exhibits extreme behaviors. They called this phenomena emergency. One of the methods to analysis complex system is the network approach which means that the ingredient of

complex system is regarded one vertex and the interaction is considered some edge to connect each vertexes.

Our purpose in this thesis is to study some complex phenomena. Firstly, we suggest new methode to construct seismic network. Research of complex systems has recently been applied to new methods and techniques of studying the intermittent nature of turbulence [1,2], various financial time series [3,4], wavelet transform approaches [5,6], growing and non-growing networks [7,8] and seismic phenomena [9], amongst others. In Particular, over the last two decades, the remarkable potential of complex networks to simulate and analyze the dynamical behavior of complex systems has gradually been an increasing trend in new fields of research in the social, natural, engineering, and medical sciences. In network theory, small-world and scale-free network models [10,11] have been studied widely in various applications of these scientific fields. These two network models have played a crucial role in understanding complex phenomena [12-14]. Of the many systems of current interest, the degree distribution for scale-free networks is interesting, because it follows a power law, and it decays faster than exponentially for random networks.

The network by new method compares with Abe and Suzuki's one. For comparison, we have constructed earthquake networks in two ways and examined some topological properties such as the mean degree, the characteristic path length, the clustering coefficient, the global efficiency, the hierarchy, and probability distribution.

Secondly, we have performed analysis of dynamical phase transitions in a Boolean neural network. Boolean neural networks have been described an genetic models for the dynamics of complex systems of interaction entities, such as social and economic networks, neural networks, and gene or protein interaction networks [36]. Kauffman [37] as a model for gene regulation was introduced and studied the simplest and most widely neural network models. Derrida and Pomeau [38] have performed calculations of random automata model by using a Boolean function. Their work has given annealed approximation and quantitative predictions for distances between iterated configurations. Boolean networks have been used to describe wide variety complex system. We have shown that there is a critical value of noise intensity on neural network with connection distribution c_{ij}. Also, we find that the critical exponent of our simulation is similar to the theoretical result 1/2.

Lastly, we have studied the quantum Brownian motion with the harmonic oscillation in the one-dimensional quantum space. The quantum Brownian motion has been one crucial subject among problems of statistical mechanics [61,62,63] for decades. The vast published works on quantum Brownian motion are treated with microscopic models in which the coupling of the Brownian particle is linear and nonlinear in the bosonic bath. The nonlinear case of a coupling system corresponds to a situation in which damping diffusion are spatially inhomogeneous, and such nonlinearity might have both classical and quantum consequences. We have investigated a non-Markovian Caldeira-Leggett master equation for the Brownian motion. We calculate the velocity distribution function with the diffusion energy and

the correlation functions, that is, the exponential, Gaussian, and complementary error functions are treated, and the statistical quantities such as the average energy, the velocity fluctuation, and the quantum and classical forces are calculated and analyzed from the velocity distribution function.

In chapter Ⅱ, we shall study theoretical background. For seismic network, we configure new method of seismic network, and introduce some network properties, For Boolean Neural network, we derive the probability density function, then show there exist phase transition. Also we shall show that the Markovian assumption for Brownian motion is not physical reality. So by using Non-Markove approach, we analysis quantum Brownian motion.

In chapter Ⅲ, we compare some network properties with Abe's method and our new method. And we simulate the Boolean neural network, so we find that the critical exponent is about 1/2 under existence of phase transition. For Brownian motion, we calculate some physical properties with three correlation function under classical and quantum limit.

In conclusion, we summarize the results of our simulation and calculation. Our plans of further study are mentioned as well.

II Theoretical Background

1 Seismic Network

Seismicity is a phenomenon of dynamical behavior in complex seismic time series [15-19], similar to a tsunami wave train. A shallow earthquake is well known to construct and analyze the distribution in the relevant area that leads to many aftershocks [20-23]. Two celebrated empirical law, that is, the Gutenberg-Richter law and the Omori law [17,18], have used to measure the number of aftershocks and analyzed the computational simulation of earthquakes by a theoretical formula. Abe and Suzuki [24] have analyzed the spatio-temporal properties of seismicity from the viewpoint of the Tsallis entropy under appropriate constraints. They have found the spatial distance and the time interval(between two successive earthquakes) by using q-exponetial distributions, which are characteristics of the nonextensive statistical mechanics [22]. In particular, the correlation function has been a main issue in theoretical and numerical investigations of aftershock phenomena. Several theoretical formulae have been used to carry out the computational simulation of earthquakes.

Futhermore, network theory is a topologically and dynamically useful tool for investigating and analyzing a seismic system, which can be simplified into processes for storing and transmitting energy

via the crust. Abe and Suzuki [9] have discussed a novel method, which uses the concept of complex networks, and small-world and scale-free networks for seismic complexity. Abe and Suzuki also introduced a complex-network approach [25] to the seismicity. and they showed that earthquake network behaves like a complex network.

1.1 New Method

It is first time suggested to construct complex network from seismic data by Abe and Suzuki [9] However, their proposal for constructing complex network using single time series does not express the meaning of aftershock. According to the wikipedea [26], doopedia [27] and The Korea Economic Daily [28] once a main shock of earthquake occurs , an aftershock is a smaller earthquake that it occurs after a previous large earthquake, in the same area . If an aftershock is larger than the main shock, the aftershock is role of the main shock and the original main shock is redesignated as a foreshock. Aftershocks are formed as the crust around the displaced fault plane adjusts to the effects of the main shock. Therefore, we suggest new method to construct complex network using property of aftershock.

An earthquake network is constructed by segmenting the whole region into three-dimensional cubic cells and making a link between consecutive events. Each cubic cell is regarded as node of a network, and the network

constructed in that manner is basically directed, but we transform it into an undirected one because we focus on the topology of the network.

The procedure is as follows:

A. (by Abe and Suzuki) [9]

(1) Segment the whole region into N-by-N-by-L cubic cells, each of which has the same size.

(2) Link two earthquakes occurring consecutively.

(3) If two consecutive events belong to the same cubic cell, their link is disregarded.

(4) If two directed links form between two cubic cells, the number of links is counted as one.

(5) By considering each cubic cell as a node, we regard the links made by all events belonging to the cubic cell with others in another cubic cell as links of a network.

B. (by Baek) [98]

(1) same as (1) of A.

(2) If the magnitude of second earthquake is smaller than the first one, link two earthquakes.

(3) If the magnitude of third earthquake is smaller than the second one, link first earthquake and third one. In this manner, smaller earthquakes as role of aftershock are linked main shock.

(4) Otherwise, that is, the magnitude of third earthquake is bigger than the second one, the third one is another new main shock.

(5) same as (3), (4) and (5) of A.

For example, there are successive time series of seismic data (see Table. 1), on Fig. 1, we have shown two complex networks by the manner of A and B.

occurrence time	magnitude	latitude	longitude
1	3.2	3	3
2	2.0	1	5
3	1.8	3	6
4	1.0	5	3
5	0.9	1	2
6	2.5	4	6
7	2.0	3	6
8	1.2	4	7
9	4.0	6	2
10	2.0	5	3
11	1.5	6	4
12	1.1	7	1

Table. 1: An example of seismic data. Seismic data basically consists of the occurrence time, magnitude, and focus of each earthquake. Depth is omitted for simple comparison.

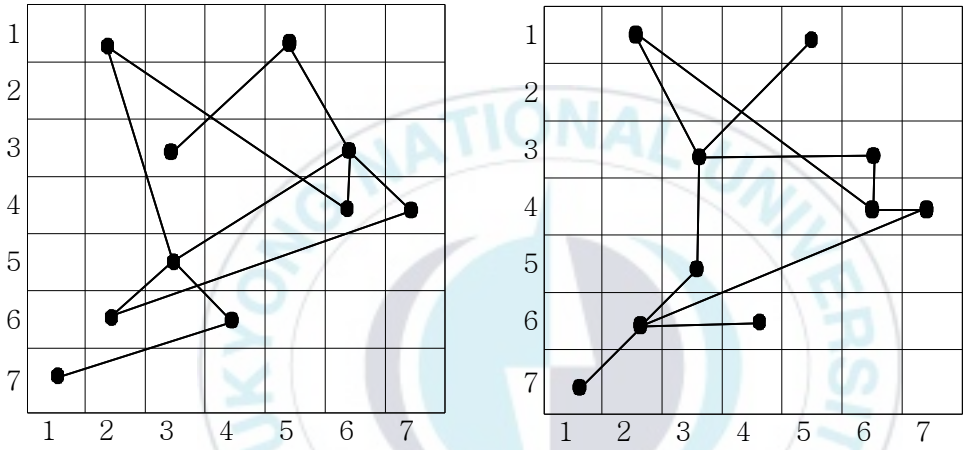

Fig. 1: Baek's method compares with Abe and Suzuki's method. The left one is constructed by Abe and Suzuki's method. Each node linked in order. On the other hand, the right network is constructed by Baek's method. It is applied the relationship between main shock and aftershock.

1.2 Topological Properties

There are some important ingredients of complex networks. They are different from ingredients of random networks. We have calculated some fundamental network metrics such as the mean degree, the characteristic path length, the clustering coefficient, the global efficiency, the hierarchy, and probability distribution.

First of all, the mean degree is defined as

$$< k > = \frac{1}{N}\sum_{i=1}^{N} k_i \qquad (1)$$

where k_i is the degree of ith node. Degree is the number of connected link. N is the total number of nodes. The random network is that it constructed by randomizing the seismic network under fixed links and nodes.

We introduce the characteristic path length L [30] given by

$$L = \frac{1}{N(N-1)} \sum_{i,j \in N} d_{ij} \qquad (2)$$

where d_{ij} is the number of links in the shortest path between ith and jth nodes. The characteristic path length is the average of the shortest paths each node and the other nodes in the network. Generally, most complex networks or real networks have a very short average path length.

To examine the cliquishness of seismic networks, we can calculate the mean clustering coefficient [30] as

$$C = \ < c > \ = \frac{1}{N}\sum_{i \in N} c_i \tag{3}$$

$$c_i = \frac{2e_i}{k_i(k_i - 1)} \tag{4}$$

where e_i denotes the number of links among k_i neighbors of a node i. This quantity shows tendency of two nodes of the subgraph being connected to each other. Where a subgraph is a set of directly linked nodes(that is, $d_{ij} = 1$). For complex network, the clustering coefficient is a large value compare to random network.

The global efficiency [31] is defined by

$$E_{glob} = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \tag{5}$$

This is just a harmonic mean of distance between two nodes. In network theory, the efficiency of a network is a property of how efficiently it exchanges information.

Fig. 7: A fully connected cluster of four nodes. This figure shows three steps to construct completely hierarchical. In this case, the hierarchy coefficient $\beta = 1$.

Generally, we can find that there is hierarchy frame in a complex network. In this case, the clustering coefficient of node i with degree k follows the scaling law.

$$C(k) \sim k^{-\beta} \tag{6}$$

where the scaling exponent β is a hierarchy coefficient [32]. If a network has completely hierarchical structure(see Fig. 7), the hierarchy coefficient $\beta = 1$.

Lastly, from our method for constructing network, a newly created node of the growing seismic network is linked with preferential attachment probability $\Pi(k_i)$.

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j} \tag{7}$$

A network generated with this rule characterized by power-law connectivity distribution. The probability distribution function of degrees is represented in terms of

$$P(k) \sim k^{-\gamma} \tag{8}$$

where γ is the degree exponent. The network with above mentioned feature of Eq. (7) and (8) is called the scale free network[11]. Theoretically, they call

the network ultrasmall where the degree exponent γ of the scale free network is in the range of 2 to 3 [33,34]. A scale free network governed the rule Eq. (8) is in contrast to the random graph, the degree distribution of which in Poissonian [11].

2 Neural Network

Boolean networks have been used to describe various models in complex systems such as neural networks with associative memory [39,40], spin glasses [41-44], dynamics of evolution [45,46], and cellular automata [47,48]. It is well known that a typical Boolean network consists of a set of binary elements which are connected among them to indicate a net, and he uses of common tools which have revealed a robust parallel between Boolean networks and dynamical systems. Historically, neural networks have approximated universal and nonlinear functions with arbitrary accuracy [49]. This approximate method is an important advance for neural networks, because of the huge number of possible nonlinear patterns for real world problems. Neural networks have been described to be effective in modeling and forecasting nonlinear time series with noise [50]. Until now, many scientists [51] have made the comparison between the neural network and the traditional method in time series modelling and forecasting performances.

Futhermore, over the last two decades, the remarkable potential of complex networks to simulate and analyze the dynamical behavior of complex

systems has gradually been an increasing trend in new fields of research in the social, natural, engineering, and medical sciences.

Until now, several papers have analyzed the non-equilibrium dynamics of deterministic Boolean neural networks [52,53] and suggested the existence of a variety of possible collective behaviors such as synchronized oscillations or chaos [54,55]. The influence of noise on the dynamics of Boolean networks has been analyzed in several published papers [56,57] as well. Scientists who have researched on neural networks have been interested in considering the changes in the dynamical properties of a deterministic system in the presence of noise. Following this motivation, we study random network models exhibiting self-organization and analyze its tolerance to the effect of noise. We mainly show from two connections (of probability density function) and one random network that the system undergoes a dynamical phase transition as its amount of randomness is increased.

2.1 Boolean Neural Network

Consider a neural network composed of N elements, each of which can only take the values $\sigma_i = +1$ or $\sigma_i = -1$. Every σ_i is randomly connected to any L elements, of the network, which define its set of linkages. The parameter L is the connectivity of the network, and each linkage is weighted by an independent random variable. The NL connections of a network and its corresponding weights remain fixed throughout the

evolution of the system. In our model, the input functions [58] at discrete time step t is represented in terms of

$$I(c_{i1},\ \cdots,\ c_{iL};\sigma_{i_1}(t),\ \cdots,\ \sigma_{i_L}(t))\ =\ Sign\left\{\sum_{j=1}^{L} c_{ij}\sigma_{i_j}(t)\right\} \tag{9}$$

Here $\sigma_i(t)$ is connected to any L elements having its set of linkages $\{\sigma_{i_j}(t)\}$ for $j\ =\ 1,\ \cdots,\ L$, and each linkage $\sigma_{i_j}(t)$ is weighted by an independent random variable c_{ij}. The input function takes the same value as the majority of the linkages, if it corresponds to the majority rule.

Using Eq. (9), we introduce a stochastic evolution rule for $\sigma_i(t+1)$ with a noise intensity γ such that

$$\sigma_i(t+1)\ =\ I(c_{i1},\ \cdots,\ c_{iL};\sigma_{i_1}(t),\ \cdots,\ \sigma_{i_L}(t))\ \text{with}\ 1-\gamma \tag{10}$$

$$\sigma_i(t+1)\ =\ -I(c_{i1},\ \cdots,\ c_{iL};\sigma_{i_1}(t),\ \cdots,\ \sigma_{i_L}(t))\ \text{with}\ \gamma \tag{11}$$

In above equation, we can select randomly a varying noise intensity γ between 0 and $1/2$. In the case with $\gamma=0$, a neural network system at time $t+1$ will converge to an ordered state in which all the $\sigma_i(t+1)$ are equal.

Next, we consider that the neural network system undergoes a dynamical phase transition from an ordered to a disordered state as the noise intensity γ is increased. In order to define the order parameter adequately described the degree of alignment of the elements of the network, we introduce a statistical quantity as

$$\sigma(t) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i(t) \tag{12}$$

where $|\sigma(t)| \rightarrow 1$ for an ordered system in which all elements take the same value, while $|\sigma(t)| \rightarrow 0$ for a disordered system.

For systems where the time-average of $|\sigma(t)|$ converges, an order parameter Φ is defined as

$$\Phi = \frac{1}{t-t_0}\sum_{t=t_0}^{t}|\sigma(t)| \tag{13}$$

where t_0 can take any arbitrary finite time without changing Φ. If the state is fully ordered, then $\Phi = 1$, and $\Phi = 0$ for a fully disordered state.

2.2 Dynamical Phase Transition

Huepe and Aldana [59] have shown that, under very general conditions, the neural network model in Eq. (10) and Eq. (11) undergoes a dynamical second order phase transitions. And, if the probability density of the connection weights is a non-symmetric, but otherwise arbitrary function, and if the linkages of the network are chosen randomly, then there exists a critical value λ_c of the noise in the vicinity of the phase transition.

In order to calculate the exact analytic expression of the order

parameter Φ that relates the noise intensity control parameter γ. Let us define $\phi(t)$ as the probability that at time t any arbitrary node σ_i acquires the value $+1$

$$\phi(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma_i(t)+1}{2} \tag{14}$$

If the state is fully ordered, then $\phi(t)=0$ or 1, and $\phi(t)=1/2$ for a fully disordered state.

The products $c_{ij}\sigma_{i_j}(t)$ can be considered a independent random variable, when the linkages of every node are assigned in a sufficiently random way. Therefore, if we denote by $P_{s(t)}(x)$ and $P_{c\sigma(t)}(x)$ the probability density function associated to $s_i = \sum_{j=1}^{L} c_{ij}\sigma_{i_j}(t)$ and to the product $c_{ij}\sigma_{i_j}$ respectively, then

$$P_{s(t)} = [P_{c\sigma(t)}(x)]^L \tag{15}$$

The probability $g(t)$ of having the input function $I=+1$ at time t can be computed as

$$g(t) = \int_0^{\infty} P_{s(t)}(x)dx \tag{16}$$

Using Eq. (10) the probability $\phi(t+1)$ of having $\sigma_i(t+1)=+1$ in terms of

$g(t)$ and γ

$$\phi(t+1) = g(t)[1-\gamma] + [1-g(t)]\gamma \tag{17}$$

Now, we should find the relation between $g(t)$ and $\phi(t)$. For this, Eq. (15) acquires the form

$$\tilde{P}_{s(t)}(\lambda) = \left[\tilde{P}_{c\sigma(t)}(\lambda)\right]^L \tag{18}$$

when $\tilde{P}(\lambda)$ is the Fourier transform of $P(x)$.

Since the connection weights c_{ij} are distributed according to the probability function $P_c(x)$, and the variables $\sigma_i(t)$ evaluate to $+1$ with probability $\phi(t)$, the probability density function of the products $c_{ij}\sigma_{i_j}$ is given by

$$P_{c\sigma(t)}(x) = \phi(t)P_c(x) + [1-\phi(t)]P_c(-x) \tag{19}$$

Inserting Eq. (19) into Eq. (18), we have

$$\tilde{P}_{s(t)}(\lambda) = \left[\tilde{P}_c^*(\lambda) + \left(\tilde{P}_c(\lambda)-\tilde{P}_c^*(\lambda)\right)\phi(t)\right]^L \tag{20}$$

where the P^* is the complex conjugation. Substituting $\phi(t)=[\sigma(t)+1]/2$, we get

$$\tilde{P}_{s(t)}(\lambda) = \left[\frac{\tilde{P}_c(\lambda) + \tilde{P}_c^*(\lambda)}{2} + \frac{\tilde{P}_c(\lambda) - \tilde{P}_c^*(\lambda)}{2}\sigma(t)\right]^L \tag{21}$$

According to denoting the real and imaginary parts of $\tilde{P}_c(\lambda)$ as

$$\frac{\tilde{P}_c(\lambda) + \tilde{P}_c^*(\lambda)}{2} = Re(\tilde{P}_c),\ \frac{\tilde{P}_c(\lambda) - \tilde{P}_c^*(\lambda)}{2} = iIm(\tilde{P}_c) \tag{22}$$

Now from Eq. (21) and Eq. (22) we have

$$\tilde{P}_{s(t)}(\lambda) = \sum_{m=0}^{L}\binom{L}{m}\left[Re(\tilde{P}_c)\right]^{L-m}\left[iIm(\tilde{P}_c)\right]^m \tag{23}$$

The inverse Fourier transform of Eq. (23) is

$$P_{s(t)}(x) = \sum_{m=0}^{L}\left\{\frac{i^m}{2\pi}\binom{L}{m}\int_{-\infty}^{\infty}\left[Re(\tilde{P}_c)\right]^{L-m}\left[Im(\tilde{P}_c)\right]^m e^{-i\lambda x}dx\right\}[\sigma(t)]^m \tag{24}$$

Using Eq. (16) and Eq. (24), we have

$$g(t) = \sum_{m=0}^{L} a_m[\sigma(t)]^m \tag{25}$$

where the a_m are constant coefficients that depend only on $\tilde{P}_c(\lambda)$ and are

given by

$$a_m = \frac{i^m}{2\pi}\binom{L}{m}\int_0^{\infty}\int_{-\infty}^{\infty}\left[Re(\widetilde{P}_c)\right]^{L-m}\left[Im(\widetilde{P}_c)\right]^m e^{-i\lambda x}d\lambda dx$$

$$= \frac{-i^{m+1}}{2\pi}\binom{L}{m}\int_{-\infty}^{\infty}\frac{1}{\lambda}\left[Re(\widetilde{P}_c)\right]^{L-m}\left[Im(\widetilde{P}_c)\right]^m d\lambda \qquad (26)$$

Note that $a_m = 0$ for all even values of $m \geq 2$ and $a_0 = 1/2$ [59].

We have shown that $g(t)$ is a polynomial of degree L in $\sigma(t)$. Thus Eq. (17) becomes

$$\sigma(t+1) = 2(1-2\gamma)(a_1\sigma(t) + a_3[\sigma(t)]^3 + \cdots + a_L[\sigma(t)]^L) \qquad (27)$$

In the limit $t \to \infty$, $\sigma(t)$ will asymptotically approach a fixed point σ

$$\sigma = 2(1-2\gamma)(a_1\sigma + a_3\sigma^3 + \cdots + a_L\sigma^L) \qquad (28)$$

Discarding the solution $\sigma = 0$ and solving Eq. (28) for γ, we have

$$\gamma = \frac{a_1 - 1/2 + a_3\sigma^3 + \cdots + a_L\sigma^{L-1}}{2(a_1 + a_3\sigma^2 + \cdots + a_L\sigma^{L-1}} \qquad (29)$$

and in the vicinity of the phase transition $\sigma \approx 0$. From Eq. (28) we have

$$\gamma_c = \frac{1}{2}\left(1 - \frac{1}{2a_1}\right) \tag{30}$$

By dividing the polynomials of Eq. (29) and neglecting the terms of order σ^4 and higher we obtain

$$\gamma - \gamma_c = \frac{a_3}{4a_1^2}\sigma^2 \tag{31}$$

From Eq. (26) $a_3 < 0$, therefore if $\gamma < \gamma_c$ Eq. (31) implies that real non-zero solutions for σ in Eq. (28) only exist. For $\gamma > \gamma_c$ the solutions of Eq. (31) are imaginary and therefore $\sigma = 0$ is the only acceptable solution of Eq. (28).

In the last analysis, the explicit behavior of the order parameter $\Phi = |\sigma|$ near the transition will be

$$\Phi = [C(\gamma - \gamma_c)]^{1/2} \qquad \text{for } \gamma < \gamma_c \tag{32}$$

and

$$\Phi = 0 \qquad \text{for } \gamma > \gamma_c \tag{33}$$

3 Brownian Motion

Recently, the type of inhomogeneity has been rigorously investigated in classical Brownian motion and other classical diffusive systems [64,65].

Applications have included Brownian motion in diffusion gradient [66,67], noisy electrical circuits [68], and thermophoresis [69]. The diffusion in inhomogeneous and disordered media has been one of the developing problems in the theory of random walks and Brownian motion [70,71,72]. There is a considerable interest in the studies of various forms of anomalous diffusion and nonergodicity nowadays.

The application of the Lindblad formalism can be really found in quantum optics in which the environment is represented by a quantized radiation field while the Brownian particle is considered to an atom or a molecule. In the experimental apparatus, the decoherence process that is the destruction of the superposition states by the presence of environment can be controlled [73,74]. Recently, we should not attribute fundamental significance to the Lindblad master equation, and the Lindblad theory is not appliable in the problems of condensed matter physics at low temperatures for which neither the Born approximation is valid or the Markov assumption holds [75]. The recent controversies have pointed out the inadequacy of the Lindblad approach to fathoming the true physical fields of open quantum systems [76,77].

Furthermore, the Markovian quantum master equation can be classified as two sorts of Caldeira-Leggett equations [78-82]. Firstly, the Markovian Calderea-Leggett equation [79] holds valid at high temperatures for any friction constant. Such Markovian Caldeira-Leggett equations may give rise to unphysical results, for they are not of the Lindblad form [83,84,85], although the high-temperature Caldeira-Leggett equation ha been employed for looking at the decoherence phenomenon [78,79]. It has been claimed that

Markovian Caldeira-Leggett equations cannot be considered as a bona fide description of quantum Brownian motion [75,85]. Secondly, the Markovian Caldeira-Leggett equation [80] holds for any temperature and very weak damping. The non-Markovian master equation has an applicability range for this class of non-Lindblad quantum master equations.

The purpose of this work is to study the quantum Brownian motion with the harmonic oscillation in the one-dimensional quantum space. We calculate the velocity distribution function with the diffusion energy and the correlation functions, that is, the exponential, Gaussian, and complementary error functions are treated, and the statistical quantities such as the average energy, the velocity fluctuation, and the quantum and classical forces are calculated and analyzed from the velocity distribution function. Our numerical result can be compared to the strength of the quantum thermal force measured the trapped ions[86].

3.1 Markovian Assumption

The Brownian motion had been mathematically studied by Einstein [87] and Langevin [88] on the basis of the concept of probability. In 1905, Einstein had built up a theoretical model in which the envioronment acts on the Brownian particle in a probabilistic fashion. Acoording to him, the random motion of a free Brownian particle could be described by the diffusion equation for the probability distribution function $F(x,t)$.

$$\frac{\partial F(x,t)}{\partial t} = D\frac{\partial^2 F(x,t)}{\partial x^2} \tag{34}$$

where D is called diffusion constant with dimensions of $[length^2 \times time^{-1}]$. In other to derive the diffusion equation Eq. (34), Einstein posited the motion of the Brownian particle as a translation transformation from the point $x_1 = x + \Delta x$ at $t_1 = t$ to the point $x_2 = x$ at $t_2 = t + \Delta t$. That movement evolves through the following integral equation

$$F(x, t+\Delta t) = \int_{-\infty}^{\infty} F(x+\Delta x, t)\phi(\Delta x)d(\Delta x) \tag{35}$$

where the $\phi(\Delta x)$ is a time-independent function with normalization condition. Markovianity property in the integral equation Eq. (35) is important assumption underpinning the theory of Brownian motion.

Einstein [87,89,90] derived the following expression for the diffusion constant

$$D = \frac{k_B T}{\beta m} \tag{36}$$

where k_B is the Boltzmann constant, and T, β are the temperature and the frictional constant. m is a mass of the Brownian particle. The solution to Eq. (35) reads

$$F(x,t) = \sqrt{\frac{m\beta}{4\pi k_B T t}}\, e^{\frac{-m\beta x^2}{4k_B T t}} \tag{37}$$

The position fluctuation of the free Brownian particle is then represented by the following Einstein's root mean square displacement

$$\boldsymbol{X}(t) = \sqrt{\frac{2k_B T}{m\beta}t} \tag{38}$$

the instantaneous velocity

$$\boldsymbol{V}(t) = \frac{d\boldsymbol{X}}{dt} = \sqrt{\frac{k_B T}{2m\beta t}} \tag{39}$$

blows up at short time $t \to 0$. In other words, the displacement fluctuation is not a differentiable quantity at $t=0$ hence the concept of instantaneous velocity of a non-inertial free particle cannot exist in the Einstein picture of Brownian motion.

On the other hand, Langevin [88] set out to address the problem of Brownian motion by focusing on the concept of random variable $X=X(t)$ whose time evolution is given by a stochastic differential equation. According to that approach, a free Brownian particle with position $X(t)$, mass m is described by the following Langevin equation

$$m\frac{d^2X(t)}{dt^2} = -\frac{dV(X)}{dX} - \beta m\frac{dX(t)}{dt} + b\Psi(t) \quad (40)$$

where the term $md^2X(t)/dt^2$ denotes an inertial force offsetting a conservative force, $F_c \equiv -dV(X)/dX$, derived from the Brownian particle's potential energy $V = V(X)$, and two kinds of environmental forces: a linearly velocity-dependent dissipative force, $F_d \equiv -\beta m dX/dt$, and a fluctuating force, $L(t) \equiv b\Psi(t)$, that is, Langevin's force.

The Langevin equation Eq. (40), for the inertial free particle, $V = 0$, may be written in terms of the stochastic momentum $P(t) = mdX(t)/dt$ as

$$\frac{dP(t)}{dt} = -\beta P(t) + b\Psi(t) \quad (41)$$

whose formal solution is the following

$$P(t) = P(0)e^{-\beta t} + b\int_0^t e^{-\beta(t-s)}\Psi(s)ds \quad (42)$$

The mean square momentum then reads

$$< P^2(t) > = < P^2(0) > e^{-2\beta t} + b^2 e^{-2\beta t}\int_0^t\int_0^t e^{\beta(s+s')} < \Psi(s)\Psi(s') > dsds' \quad (43)$$

Substitution Markovian property

$$< \Psi(s)\Psi(s') > \ = \ \delta(s-s') \qquad (44)$$

into Eq. (44), gives the following

$$< P^2(t) > \ = \ < P^2(0) > e^{-2\beta t} \ + \ \frac{b^2}{2\beta}(1 \ - \ e^{-2\beta t}) \qquad (45)$$

And assuming that the energy equipartition is valid in the steady regime, $< P^2(\infty) > \ = mk_BT$, then we obtain the fluctuation-dissipation

$$b \ = \ \sqrt{2\beta m k_B T} \qquad (46)$$

For initial condition, $< P^2(0) > = < P(0) > = 0$, and $< \Psi(t) > = 0$, the root mean square momentum $\boldsymbol{P}(t) \equiv \sqrt{< P^2(t) > - < P(t) >^2}$ is obtained

$$\boldsymbol{P}(t) \ = \ \sqrt{mk_BT(1 \ - \ e^{-2\beta t})} \qquad (47)$$

Eq. (47) is nondifferentiable at $t \rightarrow 0$, so implying that the stochastic differntial equation Eq. (41), or the Langevin equation Eq. (40), is devoid of any mathematical significance because the force $dP(t)/dt$, or the acceleration $d^2X(t)/dt^2$, cannot exist.

3.2 Non-Markovian Brownian Motion

The stochastic differential equations I phase space Eq. (40) give rise to the following Kolmogorov equation in phase space

$$\frac{\partial F(x,p,t)}{\partial t} = \boldsymbol{K}F(x,p,t) \tag{48}$$

for the marginal probability distribution function

$$F(x,p,t) = \int_{-\infty}^{\infty} F_{XP\Psi}(x,p,\psi,t)d\psi \tag{49}$$

where $F_{XP\Psi}(x,p,\psi,t)$ expressed in terms of the possible values $x=\{x_i(t)\}$, $p=\{p_i(t)\}$, and $\psi=\{\psi_i(t)\}$, with $i \geq 1$, distributed about the sharp values x', p', and ψ' of $X(t)$, $P(t)$ and $\Psi(t)$, respectively.

The Kolmogorov equation Eq. (48) reduces to the Fokker-Planck equation[91],

$$\frac{\partial F}{\partial t} = -\frac{P}{m}\frac{\partial F}{\partial x} + \frac{\partial}{\partial p}\left[\frac{\partial V_{eff}(x,t)}{\partial x} + \beta p\right] + D_p(t)\frac{\partial^2 F}{\partial p^2} \tag{50}$$

where the potential energy $V(x)$ present in the effective potential $V_{eff}(x,t)$,

$$V_{eff}(x,t) = V(x) - xb<\Psi(t)> \tag{51}$$

and $b<\Psi(t)>$ being the average of the Langevin random force $L(t)=b\Psi(t)$

$$b<\Psi(t)> \;=\; b\int_{-\infty}^{\infty}\psi F_{\Psi}(\psi,t)d\psi \tag{52}$$

The time dependent diffusion coefficient $D_p(t)$ is given by

$$D_p(t) \;=\; \beta m\,\mathcal{E}(t) \tag{53}$$

where $\mathcal{E}(t)$ is the time-dependent diffusion energy

$$\mathcal{E}(t) \;=\; \mathcal{E}\,C(t) \tag{54}$$

The correlation function $C(t)$ is defined as

$$C(t) \;=\; \lim_{\epsilon\to o}\frac{1}{\epsilon}\int_{t}^{t+\epsilon}\int_{t}^{t+\epsilon}<\Psi(t')\Psi(t'')>dt'dt'' \tag{55}$$

and $\mathcal{E}$ is the time-independent diffusion energy

$$\mathcal{E} \;=\; \frac{b^2}{2\beta m} \tag{56}$$

For long times $t\to\infty$, if the correlation function Eq. (54) displays the following steady behavior

$$\lim_{t\to\infty} C(t) = 1 \tag{57}$$

then the fluctuations is Markovian whereby the Brownian particle's diffusion energy Eq. (54) becomes stationary

$$\lim_{t\to\infty} \varepsilon(t) = \varepsilon(\infty) = \varepsilon \tag{58}$$

Accordingly, the particle-environment interaction is said to be non-Markovian in the non-steady, range $0 < t < \infty$. The non-Markovian character is also manifest if $C(t)$ does feature the following asymptotic behavior

$$\lim_{t\to\infty} C(t) \approx C'(t) \tag{59}$$

In that case the stochastic process remains ever non-Markovian.

If a non-Markovian correlation function $C(t)$ is given by

$$C(t) = 1 - e^{-\frac{t}{t_c}} \tag{60}$$

where the correlation time t_c explicitly represents the non-Markovianity parameter of the Brownian motion. At short times $t \to 0$, Eq. (60) approaches $C(t) \sim t/t_c$ so implying that $t_c > 0$, since the Markovian limit $t_c \to 0$ would lead to the unphysical result, $D_p(t) \to \infty$. On the other hand, Eq. (60) in the

overcorrelated case $t_c \rightarrow \infty$ predicts no diffusion phenomenon for all time t, $C(t) = 0$. Accordingly, the correlation time t_c is to be held within the range $0 < t_c < \infty$.

Let us consider an environment in thermodynamic equation at temperature T and characterized by the thermal energy $k_B T$ then the Brownian particle diffusion energy ε is identified with the reservoir's thermal energy $k_B T$,

$$\varepsilon \equiv k_B T \tag{61}$$

For convenience we could define the random variable $\Psi(t)$ as $\Psi(t) = \Phi(t) - <\Phi(t)>$, then we obtain the following statistical property [92,93]

$$<\Psi(t)> = 0 \tag{62}$$

then the non-Markovian Fokker-Planck equation (50) reads

$$\frac{\partial F}{\partial t} = -\frac{P}{m}\frac{\partial F}{\partial x} + \frac{\partial}{\partial p}\left[\frac{\partial V(x)}{\partial x} + \beta p\right] + \beta m k_B T C(t)\frac{\partial^2 F}{\partial p^2} \tag{63}$$

Eq. (63) is the non-Markovian Klein-Kramers equation[91].

3.3 Non-Markovian Quantum Brownian Motion

We now wish to quantize the non-Markovian Klein-Kramers equation (63) for a Brownian particle immersed in a generic non-Gaussian environment by means of the dynamical quantization process [65,94]. First, we introduce the following Fourier transform

$$X(x,\eta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x,p,t)e^{ip\eta}dp \tag{64}$$

where the exponential $e^{ip\eta}$ is dimensionless term. Inserting Eq. (64), (54), and $\beta = 2\gamma$ into Eq. (50), we obtain the classical equatoion of motion in space (x,η)

$$\frac{\partial X}{\partial t} = -i\eta\frac{\partial V_{eff}(x,t)}{\partial x}X + \frac{i}{m}\frac{\partial^2 X}{\partial x\partial\eta} - 2\gamma\eta\frac{\partial X}{\partial\eta} - 2\gamma m\,\mathcal{E}(\infty)C(t)\eta^2 X \tag{65}$$

The stochastic dynamics Eq. (50) is said to be quantized by introducing into the equation of motion Eq. (65) the quantization conditions through the change of variables $(x,\eta)\rightarrow(x_1,x_2)$ given by

$$x_1 = x + \frac{\eta\hbar}{2} \tag{66a}$$

and

$$x_2 = x - \frac{\eta\hbar}{2} \tag{66b}$$

where the transformation parameter $\hbar$ is Planck's constant. The geometric meaning of the quantization condition Eq. (66) has to do with the existence of a minimal distance between the points x_1 and x_2, i.e., $|x_2 - x_1| = |\eta\hbar|$, by virtue of the quantum nature of space.

Making use of the relations

$$\frac{\partial}{\partial \eta} = \frac{\partial x_1}{\partial \eta}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial \eta}\frac{\partial}{\partial x_2} = \frac{\hbar}{2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)$$

$$\frac{\partial}{\partial x} = \frac{\partial x_1}{\partial x}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x}\frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

and the Gaussian approximation, i.e., $|x_1 - x_2|^3 \ll 1$, in the quantum context

$$V_{eff}^{(\hbar)}(x_1,t) - V_{eff}^{(\hbar)}(x_2,t) \sim \eta\hbar\frac{\partial V_{eff}^{(\hbar)}(x,t)}{\partial x} \tag{67}$$

we obtain the non-Markovian quantum master equation

$$i\hbar\frac{\partial \rho}{\partial t} = \left[V_{eff}^{(\hbar)}(x_1,t) - V_{eff}^{(\hbar)}(x_2,t)\right]\rho - \frac{\hbar^2}{2m}\left(\frac{\partial^2 \rho}{\partial x_1^2} - \frac{\partial^2 \rho}{\partial x_2^2}\right)$$
$$-i\hbar\gamma(x_1 - x_2)\left(\frac{\partial \rho}{\partial x_1} - \frac{\partial \rho}{\partial x_2}\right) - \frac{iD_\hbar(t)}{\hbar}(x_1 - x_2)^2\rho \tag{68}$$

which describes the quantum Brownian motion of a particle moving in the effective potentials

$$V_{eff}^{(\hbar)}(x_i,t) = V(x_i) - x_i\sqrt{4\gamma m\,\varepsilon_{\hbar}(\infty)}<\Psi(t)> \tag{69}$$

Eq. (68) complies with quantum fluctuation-dissipation relationship given by the quantum diffusion coefficient

$$D_{\hbar}(t) = 2\gamma m\,\varepsilon_{\hbar}(t) \tag{70}$$

Let us assume that the environment can be devised as a heat bath comprising of a set of N quantum harmonic oscillators having the same oscillation frequency ω in thermal equilibrium at temperature T. The quantum diffusion energy $\varepsilon_{\hbar}$ is to be identified with the medium's internal energy U per particle

$$\varepsilon_{\hbar}(\infty) = \frac{U}{N} \tag{71}$$

The internal energy U of this system is given by $U=N\overline{E}$, where $\overline{E}=(\omega\hbar/2)\coth(\omega\hbar/2k_BT)$ is the mean energy of such oscillators[95]. Accordingly, the quantum diffusion energy of a Brownian particle immersed in such oscillators heat bath reads

$$\varepsilon_{\hbar}(\infty) = \frac{\omega\hbar}{2} coth\left(\frac{\omega\hbar}{2k_B T}\right) \tag{72}$$

The non-Markovian master equation (68), with Eq. (72), and $C(t)=1$, becomes the Markovian Caldeira-Leggett equation

$$i\hbar\frac{\partial\rho}{\partial t} = [V(x_1,t) - V(x_2,t)]\rho - \frac{\hbar^2}{2m}\left(\frac{\partial^2\rho}{\partial x_1^2} - \frac{\partial^2\rho}{\partial x_2^2}\right)$$
$$-i\hbar\gamma(x_1 - x_2)\left(\frac{\partial\rho}{\partial x_1} - \frac{\partial\rho}{\partial x_2}\right) - i\gamma m\omega\coth\left(\frac{\omega\hbar}{2k_B T}\right)(x_1 - x_2)^2\rho \tag{73}$$

Because non-Marlivian quantum master equation (68) contains the Markovian Caldeira-Leggett equation (73) as special case, we can call it the non-Markovian Caldeira-Leggett equation.

If there are no external potential, i.e., $V(x_1)=V(x_2)=0$, the non-Markovian quantum master equation (68) reads

$$i\hbar\frac{\partial\rho}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\rho}{\partial x_1^2} - \frac{\partial^2\rho}{\partial x_2^2}\right) - i\hbar\gamma(x_1 - x_2)\left(\frac{\partial\rho}{\partial x_1} - \frac{\partial\rho}{\partial x_2}\right)$$
$$-\frac{2i\gamma m\,\varepsilon_{\hbar}}{\hbar}C(t)(x_1 - x_2)^2\rho \tag{74}$$

In the Wigner representation of quantum phenomena [96] Eq. (74) changes into the following dynamics in quantum phase space

$$\frac{\partial W}{\partial t} = -\frac{p}{m}\frac{\partial W}{\partial x} + 2\gamma\frac{\partial}{\partial p}pW + 2\gamma m\varepsilon_{\hbar}C(t)\frac{\partial^2 W}{\partial p^2} \tag{75}$$

where $W \equiv W(x,p,t)$ is the Wigener function [96] as the Fourier transform

$$W(x,p,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\rho(x_1,x_2,t)e^{-i\frac{p}{\hbar}(x_1-x_2)}d(x_1-x_2) \tag{76}$$

This is a sort of quantum Fokker-Planck equation. We perform the integral transformation $W(p,t) = \int_{-\infty}^{\infty} W(x,p,t)dx$ on Eq. (75) and obtain the quantum Rayleigh equation in momentum space [80,97]

$$\frac{\partial W(p,t)}{\partial t} = -\frac{p}{m}\frac{\partial W(p,t)}{\partial x} + 2\gamma\frac{\partial}{\partial p}pW(p,t) + 2\gamma m\varepsilon_{\hbar}C(t)\frac{\partial^2 W(p,t)}{\partial p^2} \tag{77}$$

To solve Eq. (77), the quantum Brownian particle starts with the environment as the initial velocity $u_0 = 0$, at the initial position $x_0 = 0$. The time-dependent solution to Eq. (77) is calculated as

$$W(u,t) = \frac{1}{\sqrt{4\pi A(t)}}exp\left[-\frac{u^2}{4A(t)}\right] \tag{78}$$

where

$$A(t) = \frac{\hbar\omega}{4m} coth\left(\frac{\hbar\omega}{2k_B T}\right) C(t)\left(1 - e^{-t/t_r}\right) \tag{79}$$

and $t_r = 1/2\beta$ denotes a relaxation time. From the probability distribution $W(u,t)$, we can calculate the average energy, the momentum fluctuation, and the quantum force in the three cases of the correlation function.

The average energy is given by

$$\begin{aligned} E &= \frac{1}{2} m < u^2 > \\ &= \frac{\hbar\omega}{4} coth\left(\frac{\hbar\omega}{2k_B T}\right) C(t)\left(1 - e^{-t/t_r}\right) \end{aligned} \tag{80}$$

and the velocity fluctuation by

$$\begin{aligned} \Delta u(t) &= \left[< u^2 > - < u >^2\right]^{1/2} \\ &= \left[\frac{\hbar\omega}{2m} coth\left(\frac{\hbar\omega}{2k_B T}\right) C(t)\left(1 - e^{-t/t_r}\right)\right]^{1/2} \end{aligned} \tag{81}$$

We summarize the result of the average energy E and the velocity fluctuation $\Delta u(t)$ in Table 3 for the correlation function in the three forms.

The following quantum force is also calculated as

$$F_q = m\frac{d}{dt}\Delta u(t)$$

$$= \frac{1}{2}\left[\frac{m\hbar\omega}{2}coth\left(\frac{\hbar\omega}{2k_BT}\right)\right]^{1/2}\frac{\left(1-e^{-t/t_r}\right)dC(t)/dt+t_r^{-1}e^{-t/t_r}C(t)}{\left[C(t)\left(1-e^{-t/t_r}\right)\right]^{1/2}} \quad (82)$$

Furthermore, in the steady regime $t\rightarrow\infty$, we find that the velocity fluctuation becomes simply

$$\Delta u(\infty) = \left[\frac{\hbar\omega}{2m}coth\left(\frac{\hbar\omega}{2k_BT}\right)C(\infty)\right]^{1/2} \quad (83)$$

from Eq. (81).

In the quantum limit $k_BT \ll \hbar\omega$ as $\coth\left(\frac{\hbar\omega}{2k_BT}\right) \sim 1$, the force F_q^l is given by

$$F_q^l = \frac{1}{2}\left[\frac{m\hbar\omega}{2}\right]^{1/2}\frac{\left(1-e^{-t/t_r}\right)dC(t)/dt+t_r^{-1}e^{-t/t_r}C(t)}{\left[C(t)\left(1-e^{-t/t_r}\right)\right]^{1/2}} \quad (84)$$

Moreover, it follows from Eq. (82) that in the classical limit $k_BT \gg \hbar\omega$, as $\coth\left(\frac{\hbar\omega}{2k_BT}\right) \sim \frac{2k_BT}{\hbar\omega}$, the force F_c^l is given by

$$F_c^l = \frac{1}{2}\left[\frac{mk_BT}{2}\right]^{1/2}\frac{\left(1-e^{-t/t_r}\right)dC(t)/dt+t_r^{-1}e^{-t/t_r}C(t)}{\left[C(t)\left(1-e^{-t/t_r}\right)\right]^{1/2}} \quad (85)$$

Consequently, we can calculate and analyze the average energy, velocity

fluctuation, and the force from the three correlation function as the exponential, Gaussian, and complementary error functions.

Ⅲ Numerical results and Calculation

1 Comparison between Abe's and New Method

We have constructed and analyzed the seismic data collected from Southern California. The data sources are USCS [29] that the time intervals are between 20 May 2001 and 19 May 2010. And the seismic data were measured over the region with latitudes of $32^\circ N \sim 37^\circ N$ and longitudes of $115^\circ W \sim 120^\circ W$ on California. The maximal magnitude is 7.2, and maximal depth is $797km$. The total numbers of events are 147193. Two different seismic networks are configured by using the prior've written method.

We have formed 11 networks with various cubic cell scale from $(1^\circ/10)\times(1^\circ/10)\times(10km)$ to $(1^\circ/20)\times(1^\circ/20)\times(20km)$. In the case of $(1^\circ/10)\times(1^\circ/10)\times(10km)$, the cell have the sizes which are given as about $10km\times10km\times10km$, and the number of cells are 50-by-50-by-80. Due to the difference of method to construct network, the number of nodes and links are different too. On Fig. 2 we compare the links versus the nodes.

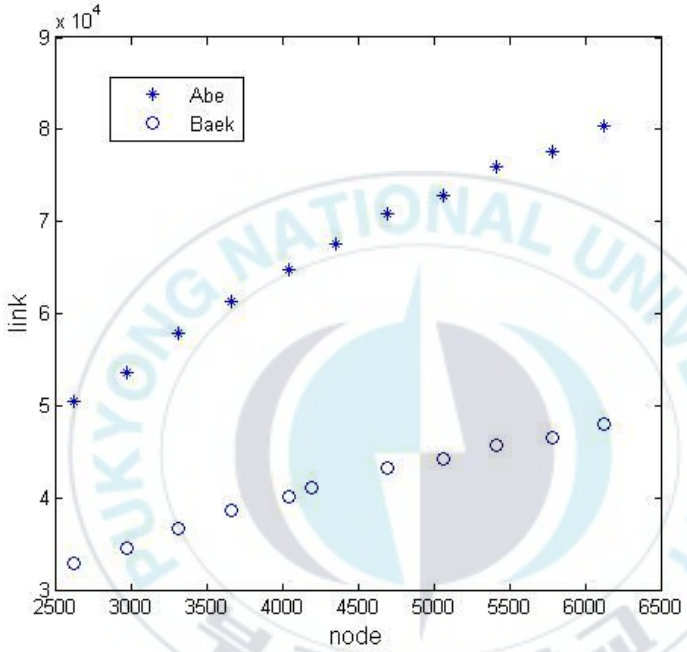

Fig. 2: Links versus nodes. Due to the difference of method to construct network, Abe and Suzuki's network have more links.

On Fig. 3, the mean degree of seismic network compared with random network. It showed that the nodes of the seismic network are more associated than in case of the random network. Fig. 4 shows our results that the average shortest path length of the seismic network is smaller than nracom network. The circle and the star denote the Baek's network and the Abe and Suzuki's network, respectively. Fig. 5 shows the clustering coefficient as a function of cell width. And On Fig. 6, the global efficiency of two seismic networks compare to two random networks.

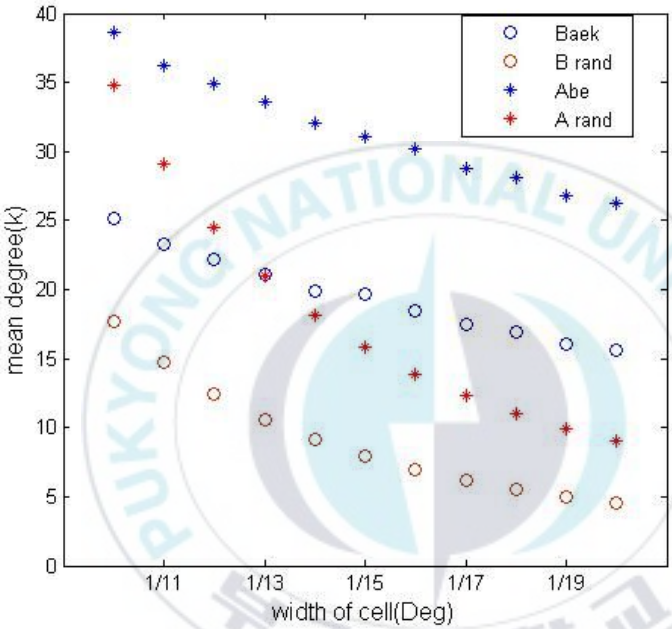

Fig. 3: Mean degree versus the cell width, where the blue circle(by Baek) and star(by Abe and Suzuki) denote the earthquake network, otherwise, the red one denotes random network.

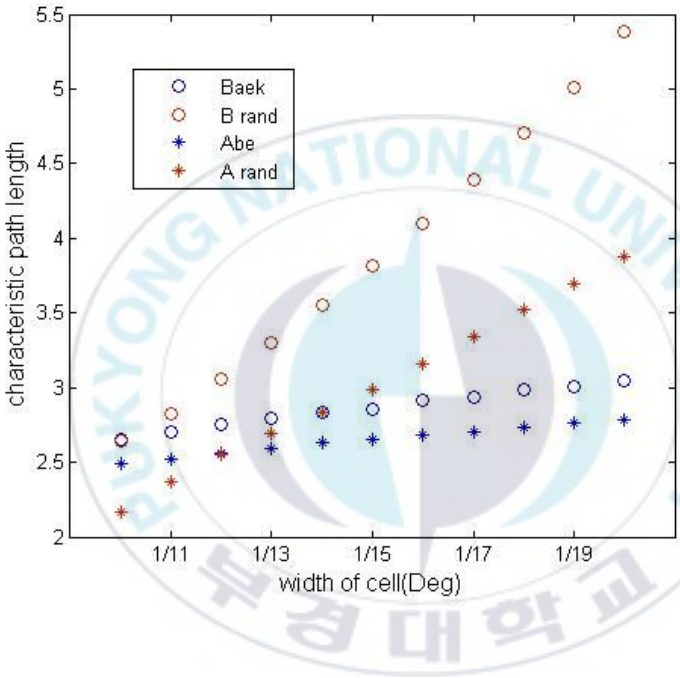

Fig. 4: Characteristic path length. The results show that the average shortest path length of the seismic network is smaller than random network.

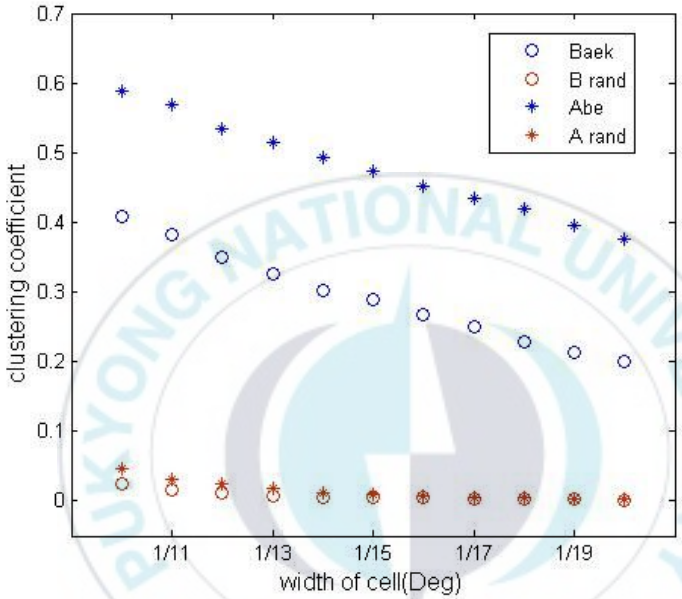

Fig. 5: Clustering coefficient as a function of the cell width. Both the Abe and Baek′ network, the clustering coefficient is larger than random network.

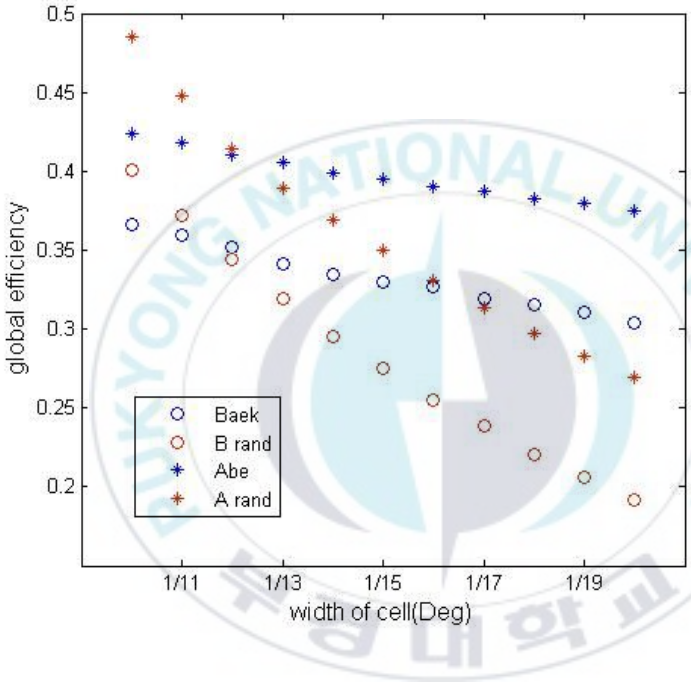

Fig. 6: Global efficiency as a function of the cell width. Both the Abe and Baek′ network, the global efficiency is larger than that of random network.

Above mentioned Clustering coefficient and Global efficiency is large value in the case of Abe and Suzuki's network compare to Baek's network. It is due to that there are more links in Abe and Suzuki's network than Baek's network. Each case of Abe and Baek, the number of links in various scale network are given in Table. 2.

size	node	link(Abe)	link(Baek)
1°/10	2616	50465	32863
1°/11	2963	53604	34501
1°/12	3309	57832	36735
1°/13	3660	61386	38557
1°/14	4041	64690	40156
1°/15	4350	67578	41154
1°/16	4691	70811	43245
1°/17	5063	72824	44186
1°/18	5408	75902	45764
1°/19	5783	77580	46499
1°/20	6127	80398	47955

Table. 2: Links and nodes with various scale cubic cell. The smaller size of cell, nodes and links increase both case of Abe and Baek.

Fig. 8 shows the log-log plot of clustering coefficient as a function of degree k. (a), (b) are the case of the cell size $(1^\circ/15)\times(1^\circ/15)\times(10km)$, and (c), (d) are the case of the cell size $(1^\circ/20)\times(1^\circ/20)\times(10km)$. (a), (c) and (b), (d) are constructed by Abe's and Baek's method respectively. In this simulation, hierarchy coefficient β are about 0.81, 0.85, 0.74, and 0.80 for (a), (b), (c), and (d) respectively.

On Fig. 9, we have shown the log-log plots of the degree distribution. (a), (b) are the case of the cell size $(1^\circ/15)\times(1^\circ/15)\times(10km)$, and (c), (d) are the case of the cell size $(1^\circ/20)\times(1^\circ/20)\times(10km)$. (a), (c) and (b), (d) are constructed by Abe's and Baek's method respectively. In this simulation, degree exponent γ are about 1.37, 1.53, 1.36, and 1.60 for (a), (b), (c), and (d) respectively.

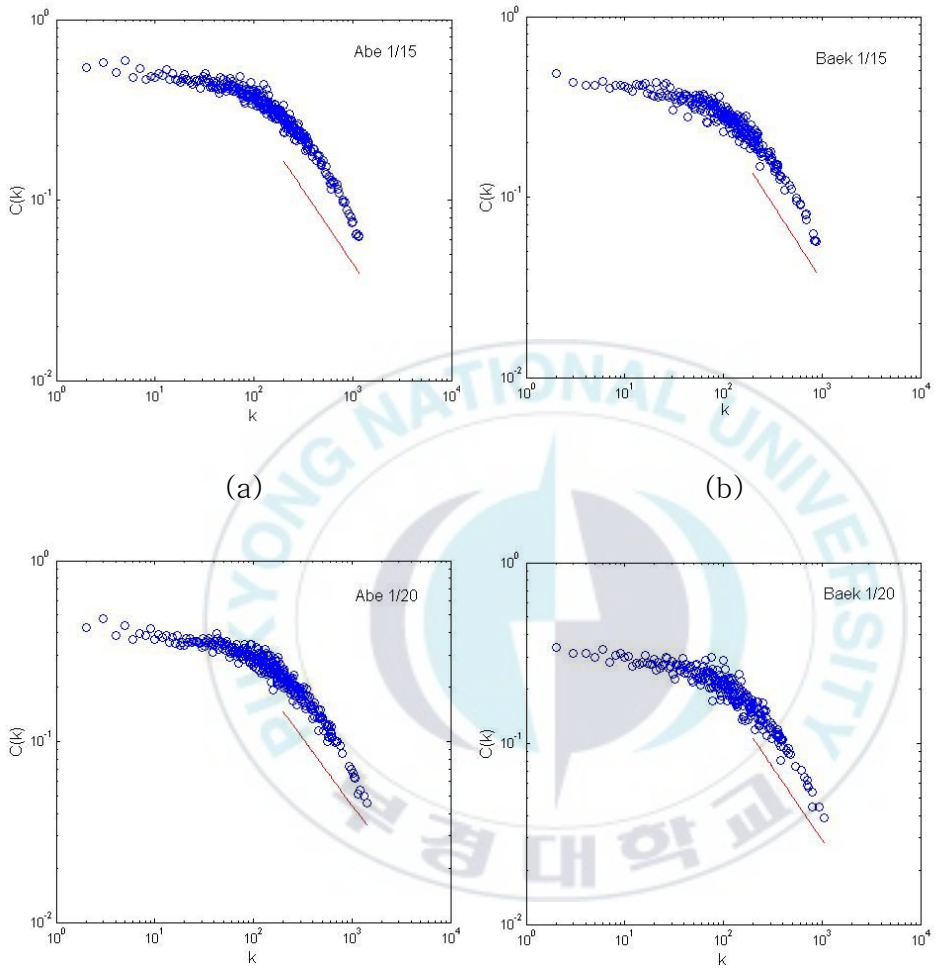

Fig. 8: The log-log plots of the clustering coefficient with degree k of the networks with the cell size (a), (b) $(1^\circ/15)\times(1^\circ/15)\times(10km)$, and (c), (d) $(1^\circ/20)\times(1^\circ/20)\times(10km)$. (a), (c) and (b), (d) are constructed by Abe's and Baek's method respectively.

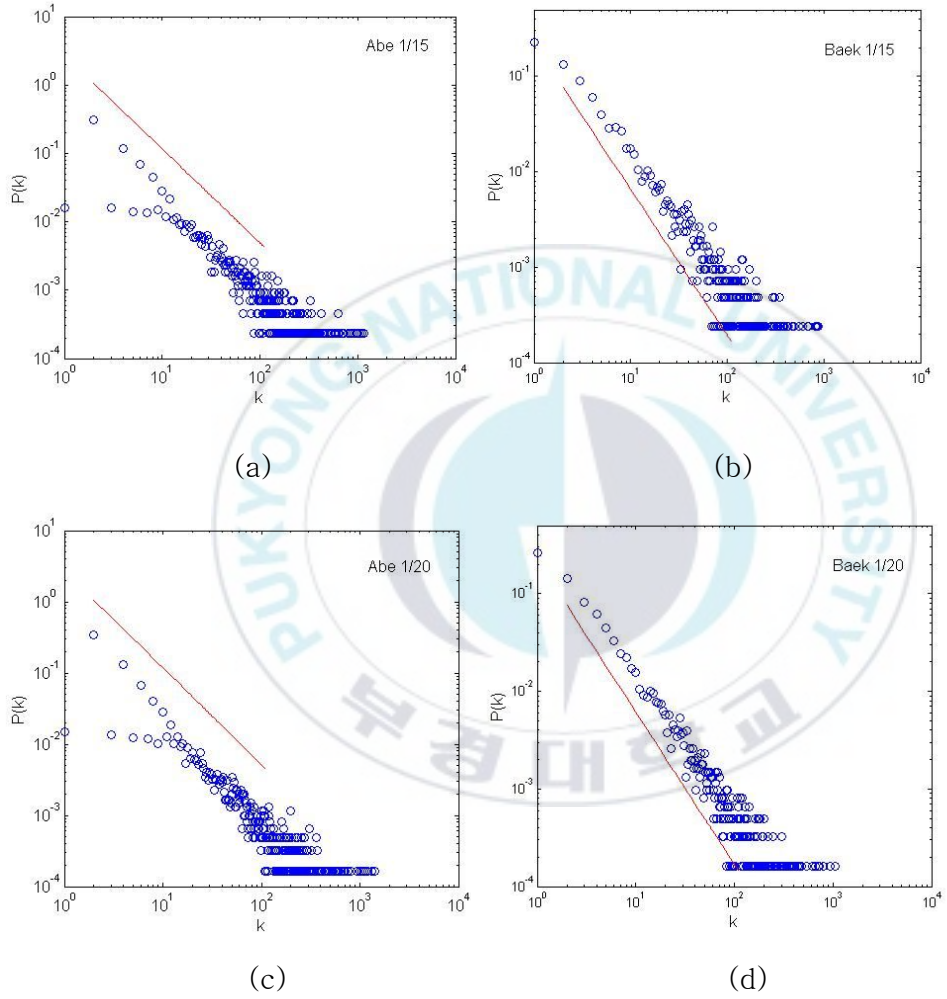

Fig. 9: The log-log plots of the probability distribution function with degree k of the networks with the cell size (a), (b) $(1^\circ/15)\times(1^\circ/15)\times(10km)$, and (c), (d) $(1^\circ/20)\times(1^\circ/20)\times(10km)$. (a), (c) and (b), (d) are constructed by Abe's and Baek's method respectively.

We have suggested new method to construct seismic network. The network by new method compares with Abe and Suzuki's one. For comparison, we have constructed earthquake networks in two ways and examined some topological properties such as the mean degree, the characteristic path length, the clustering coefficient, the global efficiency, the hierarchy, and probability distribution. From the results of our simulation, we conclude that seismic network is scale free network, and has hierarchical structure.

We could not be sure what is the right way to construct seismic network. Our new method is just another proposal. However, in that our new method expresses the meaning of aftershock, it seems that Baek's method is more advanced way of constructing seismic network. In order to verify the our suggestion, a further work is strongly needed. It is our expectation that the formalism of our analysis can be extended to both discrimination and the characterization of various earthquakes in other nations, as has similarly been done for complex systems

2 Critical Values of Neural Network

In this work, the N elements of our neural network can be extended to a larger number, but we only restrict ourselves to the cases for which the computer simulations are carried out for $N = 5\times10^3$ elements. From random initial conditions, we simulate numerically the evolution of lthe model

for a neural network with $N = 5\times10^3$ elements and three kinds of connectivities $L = 7, 9, 11$. Next, the order parameter Φ is obtained by integration $|\sigma(t)|$ from $t_0 = 1\times10^3$ until $t = 1\times10^4$.

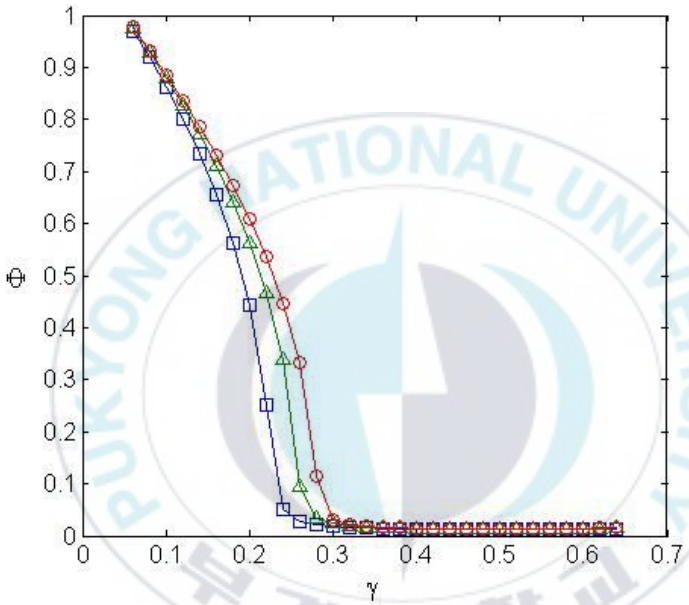

Fig. 10: Values of order parameter Φ for connection weights c_{ij} following that the probability density function is equal to 1 if $0 \leq x \leq 1$ and 0 otherwise. The numeric results only differ at $\gamma_c = 0.240$, 0.264 and 0.285. Here squares, triangles, and circles denote values of Φ for $L = 7$, 9 and 11, respectively.

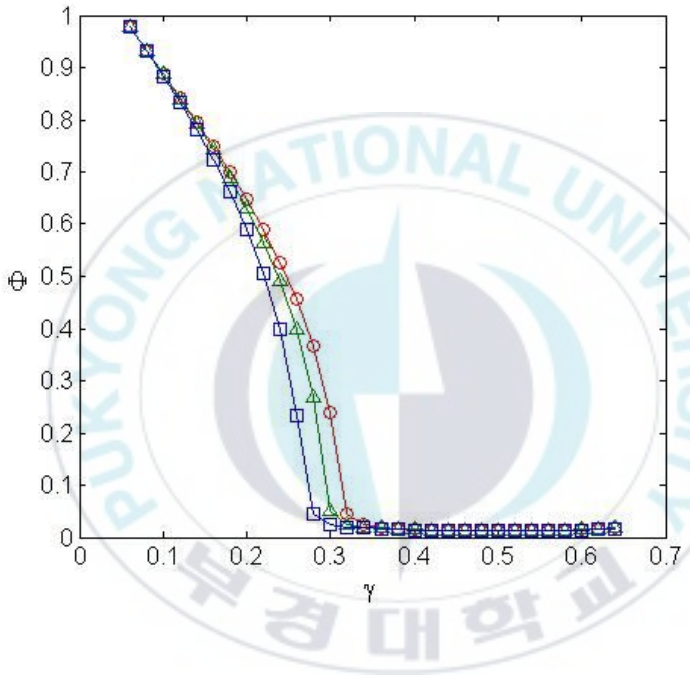

Fig. 11: Plot of the order parameter Φ as a function of the noise intensity γ for a neural network system in which $c_{ij} = 1$ for all weights. The phase transitions occur at γ_c, Where squares, triangles, and circles denote values of Φ for $L = 7$, 9 and 11, respectively.

In Fig. 10, we calculate the results for the case of a network with fixed connection weights c_{ij} following that the probability density function $P_c(x)$ is equal to 1 if $0 \leq x \leq 1$ and 0 otherwise. The phase transitions occur only at $\gamma_c = 0.240$, 0.264 and 0.285 for $L = 7$, 9 and 11 respectively. The numerical results presented on Fig. 3.2 show the bifurcation diagram of Φ as a function of the noise intensity γ for the case with $c_{ij} = 1$, in which all connection weights are equal. As the input function then becomes the majority rule, the system undergoes apparently a phase transition with $\gamma_c = 0.283$, 0.299 and 0.325 for $L = 7$, 9 and 11. For $\gamma < \gamma_c$, all elements in the system will tend to align either to $+1$ or -1

Next, we also study that there is a second order phase transition with $\Phi \sim (\gamma_c - \gamma)^{\beta}$ for $\gamma < \gamma_c$ where β is the critical exponent. On Fig. 12 and Fig. 13, the numerical results of Φ as a function of $\gamma_c - \gamma$ are presented. In The case of $c_{ij} = 1$, the results of our simulation for the critical exponent are 0.5345, 0.4959 and 0.5417 for $L = 7$, 9 and 11. and in the case of uniform distribution, the critical exponent are 0.5169, 0.4894 and 0.4866 for $L = 7$, 9 and 11.

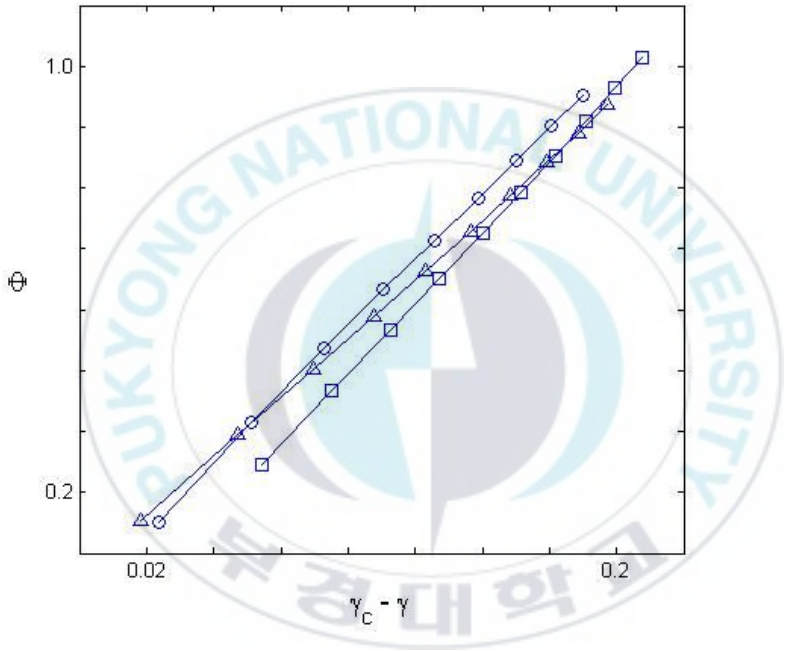

Fig. 12: Log plot of the order parameter Φ as a function of the distance to the critical noise value $\gamma_c - \gamma$ with equal connection weights $c_{ij} = 1$. The slope of the line is the critical exponent β for $L = 7$(circle), 9(triangle) and 11(square).

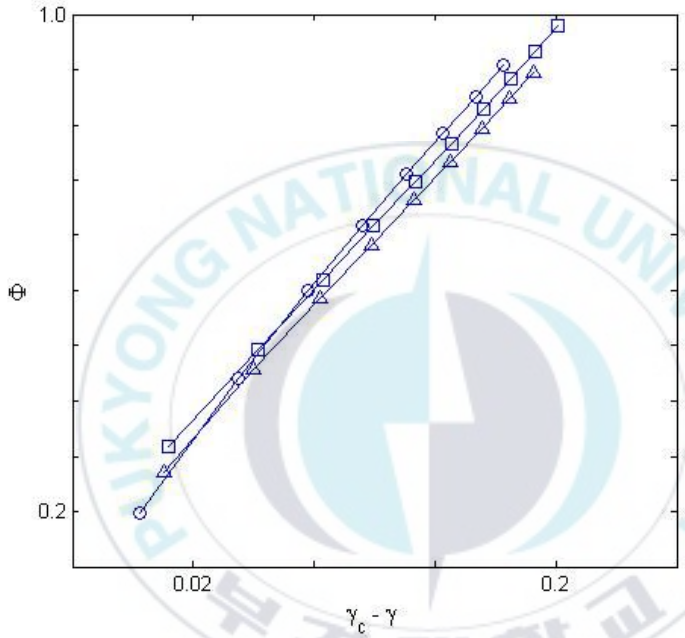

Fig. 13: Log plot of the order parameter Φ as a function of the distance to the critical noise value $\gamma_c - \gamma$ with uniformly distributed connection weights c_{ij}. The slope of the line is the critical exponent β for $L = 7$(circle), 9 (triangle) and 11(square).

We have simulated and analyzed dynamical phase transitions in a Boolean neural network initial random connections. Due to the presence of noise and to the randomness with the initial assignment of the linkages, the statistical properties in the dynamics of the neural network do not change if the connection weights or the linkages are either time-independent or it they are randomly re-assigned at every time step. And we have shown the critical exponent is same as theoretical result $1/2$. Through Boolean networks, it means that the annealed and quenched dynamics are equivalent for our model presented in this work.

Until present, there exist a number of reasons to use the neural network for time series simulation and analysis. If an function has relation between the inputs and outputs for any forecasting model, then it is very important to identify accurately this function. All these features have made neural networks useful for time series modelling and forecasting[60] in real world problems. In the future, we will extend our model to scale-free networks of other scientific fields.

3 Forces Under Quantum and Classical Limit

In Table 3, we summarize the numerical result of the quantum force average energy E and the velocity fluctuation $\Delta u(t)$ in three correlation functions. The quantum force F_q^l and classical force F_c^l are summarized in Table 4.

In the quantum limit and the steady regime, $\Delta u(\infty) \sim \omega^{1/2}$ for the three correlation functions, but the third case of the complementary error function is satisfied only for $t \sim t_c^2$ from Eq. (83).

From Eq. (84), we find that the force is proportional to the angular frequency $\omega^{1/2}$ in the quantum limit. In the quantum limit $k_B T \ll \hbar\omega$ and the short time limit, we find that tha force F_q^l is proportional to $\omega^{1/2}$ for $t_c t_r \sim 1$, independent of t, in the correlation function having the exponential. In the correlation function having the Gaussian function, the force is proportional to $\omega^{1/2}$ for $t \sim t_c t_r$, while that in the complementary error function is the same for $t \sim (t_c t_r)^{-2}$.

E	$\Delta u(t)$
$C(t) = 1-e^{-t/t_c}$:	
$\frac{\hbar\omega}{4}coth\left(\frac{\hbar\omega}{2k_BT}\right)(1-e^{-t/t_c})(1-e^{-t/t_r})$	$\left[\frac{\hbar\omega}{2m}coth\left(\frac{\hbar\omega}{2k_BT}\right)(1-e^{-t/t_c})(1-e^{-t/t_r})\right]^{1/2}$
$C(t) = 1-e^{-t^2/t_c}$:	
$\frac{\hbar\omega}{4}coth\left(\frac{\hbar\omega}{2k_BT}\right)(1-e^{-t^2/t_c})(1-e^{-t/t_r})$	$\left[\frac{\hbar\omega}{2m}coth\left(\frac{\hbar\omega}{2k_BT}\right)(1-e^{-t^2/t_c})(1-e^{-t/t_r})\right]^{1/2}$
$C(t) = erfc(t^{1/2}/t_c)$:	
$\frac{\hbar\omega}{4}coth\left(\frac{\hbar\omega}{2k_BT}\right)[1-(\pi^{1/2}t^{1/2}/t_c)^{-1}](1-e^{-t/t_r})$	$\left[\frac{\hbar\omega}{2m}coth\left(\frac{\hbar\omega}{2k_BT}\right)[1-(\pi^{1/2}t^{1/2}/t_c)^{-1}](1-e^{-t/t_r})\right]^{1/2}$

Table. 3: Calculations of the average energy E and the velocity fluctuation $\Delta u(t)$ in the three correlation functions.

In the classical limit $k_BT \gg \hbar\omega$, the quantum force F_q^l is reduced to the classical thermal force F_c^l from Eq. (85). In the classical and the short time limits, the force F_c^l is proportional to $T^{1/2}$ for in the correlation function having the exponential function. In the correlation function having the Gaussian function, the force is proportional to $T^{1/2}$ for $t \sim t_c t_r$, independent of t, while that in the complementary error function is the same for $t \sim (t_c t_r)^{-2}$.

F_q^l	F_c^l
$C(t) = 1-e^{-t/t_c}$:	
$\frac{1}{2}\left[\frac{m\hbar\omega}{2}\right]^{1/2}\frac{t_c^{-1}e^{-t/t_c}(1-e^{-t/t_r})+t_r^{-1}e^{-t/t_r}(1-e^{-t/t_c})}{[(1-e^{-t/t_r})(1-e^{-t/t_c})]^{1/2}}$	$\left[\frac{mk_BT}{t_ct_r}\right]^{1/2}$
$C(t) = 1-e^{-t^2/t_c}$:	
$\left[\frac{m\hbar\omega}{2}\right]^{1/2}\frac{t\cdot t_c^{-1}e^{-t^2/t_c}(1-e^{-t/t_r})+(2t_r)^{-1}e^{-t/t_r}(1-e^{-t^2/t_c})}{[(1-e^{-t/t_r})(1-e^{-t/t_c})]^{1/2}}$	$\frac{3}{2}\left[\frac{mk_BT}{t_ct_r}\right]^{1/2}t^{1/2}$
$C(t) = erfc(t^{1/2}/t_c)$:	
$\left[\frac{m\hbar\omega}{2}\right]^{1/2}\frac{(4\pi^{1/2}t^{3/2}t_c)^{-1}(1-e^{-t/t_r})+t(2t_r)^{-1}e^{-t/t_r}[1-(\pi tt_c)^{-1/2}]}{[(1-e^{-t/t_r})(1-e^{-t/t_r})]^{1/2}}$	$\left[\frac{mk_BT}{2\pi^{1/2}t_ct_r}\right]^{1/2}t^{1/2}t^{-1/4}$

Table. 4: Calculations of the classical force F_c^l (in the short time limit) and the quantum force F_q^l in the three correlation functions.

We have investigated a non-Markovian Caldeira-Leggett master equation for the Brownian motion. We have analyzed the quantum Browian motion of a free particle with the harmonic oscillation in the one-dimensional quantum space. After we treat the three correlation functions as the exponential, Gaussian, and complementary error functions, we analyze the average energy, velocity fluctuation, and the quantum and classical forces from the velocity distribution function. Particularly, in the steady regime and the classical limit, we find that the average energy from Eq. (14) is approximately $k_B T/2$ in all three correlation function. The force is proportional to $T^{1/2}$ in the classical limit and the short time limit. We also find that the force is proportional to the angular frequency $\omega^{1/2}$ in the quantum limit.

We hope that further detailed analytical and numerical investigations of the quantum Brownian motion and other stochastic motions will be presented elsewhere. In the future, out estimated result should be demonstrated to be in an agreement with other experiments, and our result will be also compared to other findings.

Ⅳ Conclusion

Through these works, we have confirmed that the seismicity and neural network have the feature of complex network. And we have analyzed dynamics in Brownian motion that the force in quantum limit is different from classical limit.

Firstly, We have suggested new method constructing seismic network. The network by the new method compares with Abe and Suzuki's one. For comparison, we have constructed earthquake networks in two ways and examined some topological properties. From the results of our simulation, we conclude that seismic network is scale free network, and has hierarchical structure.

We still could not be sure what is the right way to construct seismic network. Our new method is just only another proposal. But it seems that Baek's method is more advanced way of constructing seismic network in that our new method expresses the meaning of aftershock. In order to verify the our suggestion, a further work is strongly needed. We sincerely expect that the formalism of our analysis can be extended to both discrimination and the characterization of various earthquakes in other nations, as similarly been done for complex systems

Secondly, we have performed the simulation and analysis of dynamical phase transitions in a Boolean neural network initial random connections. Due to the presence of noise and the randomness with the initial assignment of the linkages, the statistical properties in the dynamics of the neural network

do not change if the connection weights or the linkages are either time-independent or randomly re-assigned at every time step. And we have shown the critical exponent is the same as theoretical result $1/2$. Through Boolean networks, it means that the annealed and quenched dynamics are equivalent for our model presented in this work.

Lastly, we have investigated a non-Markovian Caldeira-Leggett master equation for the Brownian motion. We have analyzed the quantum Browian motion of a free particle with the harmonic oscillation in the one-dimensional quantum space. After we treat the three correlation functions as the exponential, Gaussian, and complementary error functions, we analyze the average energy, velocity fluctuation, and the quantum and classical forces from the velocity distribution function. Particularly, in the steady regime and the classical limit, we find that the average energy from Eq. (14) is approximately $k_B T/2$ in all three correlation function. The force is proportional to $T^{1/2}$ in the classical limit and the short time limit. We also find that the force is proportional to the angular frequency $\omega^{1/2}$ in the quantum limit.

In the future, we will study dynamics of seismic network that is reconstructed in succession by Omori's law [18]. Also we are performing more simulations with various situation, that is, neural network with scale free condition, input function under governed some different rule, and so on.

Summary (In Korean)

아베와 스즈키는 단일 시계열의 지진 데이터를 이용해서 지진 네트워크를 구성하는 방법을 제시했다. 그들은 지진 발생지역을 cell로 구분해서 발생 순서대로 연결해 네트워크를 만들었는데, 그들의 방법으로는 여진의 특성을 반영할 수 없었다. 일반적으로 여진은 메인 쇼크가 생긴 이 후 그 보다 작은 진도를 가지는 지진으로 정의 된다. 우리는 그러한 여진의 성질을 반영한 새로운 네트워크 구성 방법을 제안했다.

남부 캘리포니아의 2001년부터 2010년까지 10년간 발생한 지진 데이터를 이용해서 아베와 우리의 새로운 방법으로 구성한 네트워크에서 여러 가지 위상기하적인 특성들, 즉 mean degree, characteristic path length, clustering coefficient, global efficiency, hierarchy, 그리고 probability distribution등을 조사하였다. 그 결과로부터 두 가지 방식 모두에서 지진 네트워크는 계층구조를 이루고 있고 scale-free 네트워크의 성질을 보여준다는 것을 확인했다. 하지만 우리의 새로운 방식에서 각 위상기하적인 특성들은 아베의 방식에서의 것과 조금씩 다른 결과가 나옴을 알 수 있었다.

또 우리는 랜덤하게 연결된 Boolean 네트워크의 동역학적 상변이를 조사했다. 두 가지 연결확률, 즉 uniform connection probability c_{ij}와 $c_{ij}=1$을 가지는 경우에 대해 상변이의 특성이 수치적, 해석적으로 계산하였다. 우리의 시뮬레이션 결과 noise intensity에 따라 critical value가 존재 했으며 그 때 critical exponent의 값이 근사적으로 이론적 결과인 1/2과 같다는 것을 확인하였다.

마지막으로 non-Markovian 브라운 운동에 대해 연구하였다. 1차원 양자 공간에서 harmonic oscillation을 가지는 양자 브라운 운동에 대해

Wigner function 테크닉을 이용해서 확산 에너지를 가지는 속도 분포를 계산하였다. 이 때, 각각의 correlation function은 exponential, Gaussian, 그리고 complementary error functions인 경우를 적용시켰다. 속도분포함수로부터 quantum force를 구했으며 classical force가 온도 $T^{1/2}$에 비례하는데 비해 quantum force의 경우에는 각주파수 $\omega^{1/2}$에 비례한다는 것을 알 수 있었다.

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