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Robust Convex Optimization with Optimality, Duality and their Applications

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Robust Convex Optimization with Optimality, Duality and their Applications
(로바스트 볼록 최적화 문제에 대한 최적성, 쌍대성과 그 응용)

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요 약

볼록 추 제약식을 가지고 있는 로바스트 볼록 최적화 문제와 로바스트 분수 최적화 문제에 대해서 로바스트 최적화 접근 (최악의 경우의 접근) 방법을 이용하여 닫힌 추 제약조건 아래서 성립하는 근사해에 대한 근사 최적정리와 Wolfe형 쌍대문제의 근사 쌍대정리를 정립하였다.

그리고, 준 볼록 목적함수, 볼록함수 제약함수와 추 제약식으로 이루어진 로바스트 반무한 최적화 문제에 대해서 닫힌 추 제약조건 아래서 성립하는 대용 쌍대정리를 증명하였다. 나아가, 준 볼록 목적함수, 볼록함수 제약함수와 추 제약식으로 이루어진 로바스트 반무한 최적화 문제를 로바스트 분수 반무한 최적화 문제로 확장하여 닫힌 추 제약조건 아래서 성립하는 대용 쌍대정리를 증명하였다. 또한, 닫힌 추 제약조건 아래서 성립하는 로바스트 선형 반무한 최적화 문제의 대용 쌍대정리에 대한 결과를 얻었다.

마지막으로, Slater형 제약 조건 아래서 성립하는 SOS-오목 행렬 제약식으로 이루어진 볼록집합과 SOS-볼록 다항함수와 지지함수의 차로 이루어진 비 볼록집합의 포함관계와 다루기 쉬운 제곱의 합으로 표현되는 특성에 관한 정리를 정립하였다. 이러한 집합의 포함관계에 대한 특성을 이용하여 SOS-오목 행렬 제약식을 가지는 로바스트 최적화 문제, 이 문제의 제곱의 합으로 완화된 쌍대문제, 완화된 쌍대문제와 동치인 다루기 쉬운 반정부호 최적화 문제와 이 문제의 쌍대문제에 대해서 Slater형 제약 조건 아래서 성립하는 쌍대정리를 증명하였고 각 문제들의 최적해의 관계에 대해서 정립하였다.

Chapter 1
Introduction and Preliminaries

1.1 Motivation

Sometimes in engineering and economic problems, we do not exactly know input data. Robust convex optimization problems are to solve convex optimization problems with data uncertainty (incomplete data) by using the worst-case approach. Here, uncertainty means that input parameter of these problems are not known exactly at the time when solution has to be determined [8]. Generally, there are two main approaches to deal with constrained optimization with uncertainty: robust programming approach and stochastic programming approach; in robust programming one seeks for a solution which simultaneously satisfies all possible realizations of the constraints, and the stochastic programming approach works with the probabilistic distribution of uncertainty and the constraints are required to be satisfied up to prescribed level of probability [32]. So, sometimes it is convenient to use the robust approach for dealing with optimization problems with data uncertainty.

Many researchers [5, 43, 44, 54, 67] have investigated duality theory for linear or convex optimization problems under uncertainty with the worst-case approach (the robust approach).

The study of convex programs that are affected by data uncertainty [5, 7, 8, 9, 10, 44, 51] is becoming increasingly important in optimization. Recently, the duality theory for convex programs under uncertainty via the robust approach (the worst-case approach) have been studied in [5, 44, 45, 51]. It was shown that the value of the robust counterpart of primal problem is equal to the value of the optimistic counterpart of the dual primal ("primal worst equals dual best") [5, 44, 45].

In [4, 14, 18, 27, 28, 52, 55], many authors have treated fractional optimization problems in the absence of data uncertainty. Very recently, Jeyakumar and Li [42] have established a duality theory for fractional optimization problem in the face of data uncertainty via robust optimization.

The solution of the dual problem provides a lower bound to the solution of the primal problem. However, usually, the optimal value of the primal problem is different from the optimal value of the dual problem. The difference of the optimal values of the primal and dual problem is called the duality gap. There are a lot of dual problems, such as Lagrangian dual problem, Wolfe dual problem, Fenchel dual problem, the surrogate dual problem, etc.. Using of Lagrangian relaxation is effective to solve large-scale linear problem, as well as convex and nonconvex problems. On the other hand, surrogate dual problems are less known than Lagrangian dual problems. Nevertheless, surrogate dual problems have virtues, that is, surrogate duality gaps are equal to or less than Lagrangian duality gaps [26]. Surrogate dual problem is the primal problem with many constraints that is converted into a single constraint problem. Recently, many authors [25, 26, 57, 60, 61, 66, 67] have investigated surrogate duality for quasiconvex optimization problem. Surrogate duality is used not only in quasiconvex optimization problem but also in integer programming and the knapsack problem [19, 25, 26, 57, 60, 61]. In particular, Suzuki, Kuroiwa and Lee [67] proved a surrogate duality theorem for an optimization problem involving a quasiconvex objective function and finitely many convex constraint functions with data uncertainty and a surrogate duality theorem for a semidefinite optimization problem involving a quasiconvex objective function and a constraint set defined by a linear matrix inequality with data uncertainty.

On the other hand, duality theory for semi-infinite optimization problem have been extensively studied [20, 22, 23]. In particular, Goberna, Jeyakumar, G. Li and López [22] gave robust duality by establishing strong duality between the robust counterpart of an uncertain semi-infinite linear program and the optimistic counterpart of its uncertain Lagrangian dual.

The well-known Farkas' lemma provides a dual characterization of the containment of a polyhedral convex set in a closed half space. The generalizations of dual characterizations of set containments have been studied in [21, 24, 34, 37, 38, 40]. Such dual characterizations of containments have important and strong applications in optimization problems, for example, strong duality and optimality criteria.

Recently, many authors [1, 2, 30, 36, 47] have investigated SOS-convex polynomials and their applications. The class of SOS-convex polynomials includes separable convex polynomials and convex quadratic functions as their special cases. The important feature of the SOS-convexity, which distinguishes from the convexity of polynomials, is that one can numerically check whether a polynomial is SOS-convex or not by solving a related semidefinite optimization (feasibility) problem which can be solved efficiently via interior point methods [50]. In particular, the gap between SOS-convex polynomials and convex polynomials is completely characterized in [2]. Moreover, Lasserre [49] proved that under the Slater condition, SOS-convex optimization problems such as minimization of a SOS-convex polynomial subject to SOS-convex inequality constraints enjoys an exact SDP relaxation in the sense that the optimal value of the given SOS-convex optimization problem and its sum of squares relaxation problem are equal and the relaxation problem attains its optimal solution.

In particular, the exact semidefinite optimization problem relaxation or strong duality involving dual semidefinite programs is a highly desirable property because semidefinite optimization problem can be efficiently solved (e.g. using interior point methods) [3, 13, 30, 36]. Recently, Jeyakumar and Li [41] established exact SDP relaxations for classes of nonlinear semidefinite optimization problems with SOS-convex polynomials. Very recently, Jeyakuma et al. [46] established sums-of-squares polynomial representations characterizing robust solutions and exact SDP-relaxations of robust SOS-convex polynomial optimization problems under various commonly used uncertainty sets.

Optimization problems in the face of data uncertainty have been treated by the worst case approach or the stochastic approach. The worst case approach for optimization problems, which has emerged as a powerful deterministic approach for studying optimization problems with data uncertainty, associates an uncertain optimization problem with its robust counterpart.

Now, to explain the worst case approach for optimization problems, we consider the case of linear optimization problem:

$$\begin{aligned} \text{(LP)} \quad \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leqq b_i, \ i = 1, \ldots, m, \end{aligned}$$

where $a_i, c, x \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, $i = 1, \ldots, m$.

The linear optimization problem (LP) in the face of data uncertainty in the objective and constraint function can be captured by the problem

$$\begin{aligned} \text{(ULP)} \quad \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leqq b_i, \ i = 1, \ldots, m, \end{aligned}$$

where (a_i, b_i) is an uncertain parameter which belongs to the set $\mathcal{U}_i \subset \mathbb{R}^n \times \mathbb{R}$, $i = 1, \ldots, m$.

For the worst case of (ULP), the robust counterpart of (ULP) is given as follows [7];

$$\begin{aligned} \text{(RULP)} \quad \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leqq b_i, \ \forall (a_i, b_i) \in \mathcal{U}_i, \ i = 1, \ldots, m, \end{aligned}$$

or the same optimization problem

$$\text{(RULP)} \quad \min \left\{ t \mid c^T x \leqq t, \ a_i^T x \leqq b_i, \ \forall (a_i, b_i) \in \mathcal{U}_i, \ i = 1, \ldots, m \right\}.$$

In stochastic optimization, the uncertain parameters are assumed to be random variables. The stochastic programming approach works with the probabilistic distribution of uncertainty and the constraints are required to be satisfied up to the prescribed level of probability [32]. Now, we consider the stochastic model of (LP):

$$\text{(SLP)} \quad \min \quad c^T x$$
$$\text{s.t.} \quad \mathbf{Prob}(a_i^T x \leqq b_i) \geqq \eta, \; i = 1, \ldots, m,$$

where (a_i, b_i), $i = 1, \ldots m$, are random variables on some probability space and $\eta \in [0.1]$.

The robust optimization is associated with the choice of the uncertain set $\mathcal{U}$. There are various uncertain sets such as box (or interval), scenario data, ellipsoidal, polyhedral uncertain set, etc..

In this thesis, we consider optimization problems with data uncertainty which belongs to the interval uncertain sets.

1.2 Outline of the thesis

This thesis consists of three main parts. In the first part presented by Chapter 2, approximate solutions for a robust convex optimization problem in the face of data uncertainty are considered. Using the robust optimization approach (the worst-case approach), we establish an optimality theorem and duality theorems for approximate solutions for the robust convex optimization problem. Also, we extend the approximate optimality theorems and the approximate duality theorems for convex optimization problems to fractional optimization problems with data uncertainty. Moreover, we give an example illustrating the duality theorems.

In Chapter 3, a semi-infinite optimization problem involving a quasiconvex objective function and infinitely many convex constraint functions with data uncertainty are considered. A surrogate duality theorem for the semi-infinite optimization problem is given under a closed and convex cone constraint qualification. Moreover, we extend the surrogate duality theorem for the semi-infinite optimization problem to fractional semi-infinite optimization problem with data uncertainty. Also, we induce characterizations of the robust moment cone of Goberna et al. [22] by our results. Using a closed and convex cone constraint qualification, we present surrogate duality theorems for robust linear semi-infinite optimization problems. Moreover, we give an example illustrating the duality theorems.

In the last part given by Chapter 4, we consider the tractable containments of a convex semi-algebraic set, defined by a SOS-concave matrix polynomial constraint, in a non-convex semi-algebraic set, defined by difference between a SOS-convex and a support function. Moreover, using our set containment characterizations, we derive a zero duality gap result for robust SOS-convex polynomial optimization problem (RP), where the dual problem $(D)^{sos}$ can be represented by a sum of squares relaxation problem and other dual problem (SDP) and its dual problem (SDD) can be represented by a semidefinite program and which can be easily solved by interior-point methods. Also, we present the relations of the optimal solution of (RP) and the optimal solution of (SDD), and the optimal solution of $(D)^{sos}$ and (SDP). Finally, we illustrate our results through a simple numerical example.

1.3 Preliminaries

Let us first recall some notations and preliminary results which will be used throughout this thesis. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The nonnegative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_i \geqq 0\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set A in $\mathbb{R}^n$ is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$. Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$. Here, f is said to be proper if for all $x \in \mathbb{R}^n$, $f(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) \in \mathbb{R}$. We denote the domain of f by dom$f$, that is, $\text{dom}f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$. The epigraph of f, epi$f$, is defined as $\text{epi}f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqq r\}$, and f is said to be convex if for all $\mu \in [0, 1]$,

$$f((1-\mu)x + \mu y) \leqq (1-\mu)f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$, equivalently epi$f$ is convex. The function f is said to be concave whenever $-f$ is convex. Recall that f is said to be quasiconvex if for all x_1, $x_2 \in \mathbb{R}^n$ and $\lambda \in (0, 1)$, $f((1-\lambda)x_1 + \lambda x_2) \leqq \max\{f(x_1), f(x_2)\}$. We define level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as $L(f, \diamond, \beta) := \{x \in \mathbb{R}^n \mid f(x) \diamond \beta\}$ for any $\beta \in \mathbb{R}$. Then, f is quasiconvex if and only if for any $\beta \in \mathbb{R}$, $L(f, \leqq, \beta)$ is a convex set, or equivalently, for any $\beta \in \mathbb{R}$, $L(f, <, \beta)$ is a convex set. Any convex function is quasiconvex, but the converse is not true. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom}f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

More generally, for any $\epsilon \geqq 0$, the ϵ-subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x\rangle \leq f(y) - f(x) + \epsilon, \ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \mathrm{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We say f is a lower semicontinuous function if $\liminf_{y \to x} f(y) \geqq f(x)$ for all $x \in \mathbb{R}^n$. As usual, for any proper convex function g on $\mathbb{R}^n$, its conjugate function $g^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $g^*(x^*) = \sup\{\langle x^*, x\rangle - g(x) \mid x \in \mathbb{R}^n\}$ for any $x^* \in \mathbb{R}^n$. For a given set $A \subset \mathbb{R}^n$, we denote the closure, the convex hull, and the conical hull generated by A, by $\mathrm{cl}A$, $\mathrm{co}A$, and $\mathrm{cone}A$, respectively. The indicator function δ_A is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

We denote the relative interior of a convex set $S \subset \mathbb{R}^n$ by $\mathrm{ri}S$. Let C be a closed convex subset of $\mathbb{R}^n$ and let $x \in C$. Then the normal cone $N_C(x)$ to C at x is defined by

$$N_C(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leqq 0, \text{ for all } y \in C\},$$

and let $\epsilon \geqq 0$, then the ϵ-normal set $N_C^\epsilon(x)$ to C at x is defined by

$$N_C^\epsilon(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leqq \epsilon, \text{ for all } y \in C\}.$$

When C is a closed convex cone in $\mathbb{R}^n$, we denote $N_C(0)$ by C^* and call it the negative dual cone of C.

Chapter 2
Approximate Solutions for Robust Optimization Problems

2.1 Introduction

In this chapter, we consider approximate solutions for a robust convex optimization problem in the face of data uncertainty. Using the robust optimization approach (the worst-case approach), we establish optimality theorems and duality theorems for approximate solutions for the robust convex optimization problems. Moreover, we give an example illustrating the duality theorems.

A standard form of convex optimization problem [13, 64] with a geometric constraint set is given by

$$\begin{aligned} \text{(CP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leqq 0, \ i = 1, \cdots, m, \\ & x \in C, \end{aligned}$$

where $f, \ g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, m$, are convex functions and C is a closed convex cone of $\mathbb{R}^n$.

The convex optimization problem (CP) in the face of data uncertainty in the constraints can be captured by the problem

$$\begin{aligned} \text{(UCP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x, v_i) \leqq 0,\ i = 1, \cdots, m, \\ & x \in C, \end{aligned}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R},\ g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q,\ i = 1, \cdots, m$.

We study an approximate optimality theorem and approximate duality theorem for the uncertain convex optimization problem (UCP) by examining its robust counterpart [8]

$$\begin{aligned} \text{(RCP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \cdots, m, \\ & x \in C. \end{aligned}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R},\ g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is the uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q,\ i = 1, \cdots, m$. Clearly, $A := \{x \in C \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\}$ is the feasible set of the robust convex optimization problem (RCP).

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an approximate solution of (RCP) if for any $x \in A$,

$$f(x) \geqq f(\bar{x}) - \epsilon.$$

Recently, many authors have studied robust convex optimization problems [5, 7, 8, 9, 11, 12, 33, 44]. In particular, Jeyakumar and Li [44] has shown that when $C = \mathbb{R}^n$ and $\epsilon = 0$, the Lagrangian strong duality holds between s robust counterpart and an optimistic counterpart for robust convex optimization problem in the face of data uncertainty via robust optimization under a new robust characteristic cone constraint qualification (RCCCQ) that

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*$$

is convex and closed. Moreover, they gave numerical examples which present their duality theory insightfully.

In this chapter, we consider approximate solutions for a robust convex optimization problem with geometric constraint. We establish approximate optimality theorem for (RCP) under the following constraint qualification:

$$\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$$

is convex and closed. Moreover, we formulate a Wolfe type dual problem for the primal one and prove approximate weak duality and approximate strong duality between the primal problem and its Wolfe type dual problem, which hold under a weakened constraint qualification. We also give an example illustrating the duality theorems.

Proposition 2.1.1. [31] *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let δ_C be the indicator function with respect to a closed convex subset C of $\mathbb{R}^n$. Let*

$\epsilon \geqq 0$. *Then*

$$\partial_\epsilon (f+\delta_C)(\bar{x}) = \bigcup_{\substack{\epsilon_0 \geqq 0,\ \epsilon_1 \geqq 0 \\ \epsilon_0+\epsilon_1=\epsilon}} \{\partial_{\epsilon_0} f(\bar{x}) + \partial_{\epsilon_1}\delta_C(\bar{x})\}.$$

The following proposition, which describes the relationship between the epigraph of a conjugate function and the ϵ-subdifferential and plays a key role in deriving the main results, was recently given in [33].

Proposition 2.1.2. [33] *If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and if $a \in \text{dom} f$, then*

$$\text{epi} f^* = \bigcup_{\epsilon \geqq 0} \{(v, \langle v, a\rangle + \epsilon - f(a)) \mid v \in \partial_\epsilon f(a)\}.$$

Proposition 2.1.3. [35] *Let $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous convex functions. If $\text{dom}\, f \cap \text{dom}\, g \neq \emptyset$, then*

$$\text{epi}(f+g)^* = \text{cl}(\text{epi} f^* + \text{epi} g^*).$$

Moreover, if one of the functions f and g is continuous, then

$$\text{epi}(f+g)^* = \text{epi} f^* + \text{epi} g^*.$$

Proposition 2.1.4. [39, 53] *Let $g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ (where I is an arbitrary index set), be a proper lower semicontinuous convex function. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i\in I} g_i(x_0) < +\infty$. Then*

$$\text{epi}(\sup_{i\in I} g_i)^* = \text{cl}\,(\text{co} \bigcup_{i\in I} \text{epi}\, g_i^*).$$

Slightly modifying the proof of Proposition 3.2 in [44], we can obtain the following Proposition.

Proposition 2.1.5. *Let* $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, *be continuous functions such that for each* $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ *is a convex function and let* C *be a closed convex cone of* $\mathbb{R}^n$. *Suppose that each* $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$, *is compact and convex, and there exists* $x_0 \in C$ *such that*

$$g_i(x_0, v_i) < 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m. \tag{2.1}$$

Then $\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$ *is closed.*

Proof. Let $\{(z^k, s^k)\}$ be a sequence in the set $\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i))^*$ $+C^* \times \mathbb{R}_+$ such that (z^k, s^k) converges to (z, s). Then there exist $v_i^k \in \mathcal{V}_i$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, m$, $c^k \in C^*$ and $r^k \in \mathbb{R}_+$ such that $(z^k, s^k) \in \mathrm{epi}(\sum_{i=1}^m \lambda_i^k g_i(\cdot, v_i^k))^* + (c^k, r^k)$, that is, $(z^k - c^k, s^k - r^k) \in \mathrm{epi}(\sum_{i=1}^m \lambda_i^k g_i(\cdot, v_i^k))^*$. Since $\mathcal{V}_i$ is compact, we may assume that $v_i^k \to v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$. Let $\lambda^k := \sum_{i=1}^m \lambda_i^k$. We first show that $\{\lambda^k\}$ is bounded. Otherwise, we may assume that $\lambda^k \to +\infty$. Since $0 \leqq \frac{\lambda_i^k}{\lambda^k} \leqq 1$, $i = 1, \ldots, m$, we may assume that $\frac{\lambda_i^k}{\lambda^k} \to \delta_i \in \mathbb{R}_+$, $i = 1, \ldots, m$. Since $\lambda^k := \sum_{i=1}^m \lambda_i^k$, $\sum_{i=1}^m \delta_i = 1$. For each $x \in C$,

$$\begin{aligned}
(z^k)^T x - \sum_{i=1}^m \lambda_i^k g_i(x, v_i^k) &\leqq (z^k - c^k)^T x - \sum_{i=1}^m \lambda_i^k g_i(x, v_i^k) \\
&\leqq (\sum_{i=1}^m \lambda_i^k g_i(\cdot, v_i^k))^* (z^k - c^k) \\
&\leqq s^k - r^k \leqq s^k,
\end{aligned}$$

and so

$$\frac{(z^k)^T x}{\lambda^k} - \sum_{i=1}^{m} \frac{\lambda_i^k}{\lambda^k} g_i(x, v_i^k) \leqq \frac{s^k}{\lambda^k}.$$

Passing to the limit and noting that g_i is continuous, we see that, for each $x \in C$, $\sum_{i=1}^{m} \delta_i g_i(x, v_i) \geqq 0$. This contradicts (2.1) as $\sum_{i=1}^{m} \delta_i = 1$.

Now, as $\{\lambda^k\}$ is bounded, we may assume that $\lambda_i^k \to \lambda_i$. As for each $x \in C$,

$$(z^k)^T x - \sum_{i=1}^{m} \lambda_i^k g_i(x, v_i^k) \leqq s^k,$$

it follows by passing to the limit and noting that each g_i is continuous that for each $x \in C$,

$$z^T x - \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \leqq s.$$

Thus, for any $x \in \mathbb{R}^n$,

$$z^T x - \sum_{i=1}^{m} \lambda_i g_i(x, v_i) - \delta_C(x) \leqq s,$$

and hence $(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) + \delta_C)^*(z) \leqq s$. So, by Proposition 2.3, $(z, s) \in \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) + \delta_C)^* = \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$. □

Using Proposition 2.3 in [44], we can obtain the following proposition.

Proposition 2.1.6. *Let* $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, *be continuous functions and let* C *be a closed convex cone of* $\mathbb{R}^n$. *Suppose that each* $\mathcal{V}_i \subseteq \mathbb{R}^q$,

$i = 1, \ldots, m$, is convex, for all $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex function, and for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is concave on $\mathcal{V}_i$. Then

$$\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+,$$

is convex.

Proof. By Proposition 2.3 in [44], $\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^*$ is convex. Since $C^* \times \mathbb{R}_+$ is convex, $\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$ is convex. □

Slightly modifying Example 2.1 in [44], we can obtain the following example showing that the cone $\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$ may not be convex:

Example 2.1.1. Let $v_1 \in \mathcal{V}_1 := [0, 1]$. Let $g_1 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g_1(x, v_1) = v_1^2 |x_1| + \max\{x_2, 0\} - 2v_1.$$

Let $C = -\mathbb{R}^2_+$. Then for each $\lambda_1 \geqq 0$ and $v_1 \in \mathcal{V}_1$,

$$\begin{aligned} & (\lambda_1 g_1(\cdot, v_1))^*(a_1, a_2) \\ = & \sup_{(x_1, x_2) \in \mathbb{R}^2} \{a_1 x_1 + a_2 x_2 - \lambda_1 (v_1^2 |x_1| + \max\{x_2, 0\} - 2v_1)\} \\ = & \begin{cases} 2\lambda_1 v_1, & \text{if } -\lambda_1 v_1^2 \leqq a_1 \leqq \lambda_1 v_1^2 \text{ and } 0 \leqq a_2 \leqq \lambda_1, \\ +\infty, & \text{else.} \end{cases} \end{aligned}$$

So, $\bigcup_{\substack{v_1 \in \mathcal{V}_1 \\ \lambda_1 \geqq 0}} \text{epi}(\lambda g_1(\cdot, v_i))^* = \bigcup_{\substack{v_1 \in [0,1] \\ \lambda_1 \geqq 0}} [-\lambda_1 v_1^2, \lambda_1 v_1^2] \times [0, \lambda_1] \times [2\lambda_1 v_1, +\infty)$

$$= \bigcup_{s \geqq r \geqq 0} [-r, r] \times [0, s] \times [2\sqrt{rs}, +\infty).$$

Hence, we have

$$\bigcup_{\substack{v_1 \in \mathcal{V}_1 \\ \lambda_1 \geqq 0}} \text{epi}(\lambda_1 g_1(\cdot, v_1))^* + C^* \times \mathbb{R}_+ = \bigcup_{s \geqq r \geqq 0} [-r, +\infty) \times [0, +\infty) \times [2\sqrt{rs}, +\infty).$$

Let $a = (0,1,0)$ and $b = (1,1,2)$. Then, $a, b \in \bigcup_{\substack{v_1 \in [0,1] \\ \lambda_1 \geqq 0}} \text{epi}(\lambda_1 g_1(\cdot, v_1))^* + C^* \times \mathbb{R}_+$. On the other hand, $c := \frac{a+b}{2} = (0.5, 1, 1) \notin \bigcup_{\substack{v_1 \in [0,1] \\ \lambda_1 \geqq 0}} \text{epi}(\lambda_1 g_1(\cdot, v_1))^* + C^* \times \mathbb{R}_+$. Otherwise,

$$(0.5, 1, 1) \in \bigcup_{s \geqq r \geqq 0} [-r, +\infty) \times [0, +\infty) \times [2\sqrt{rs}, +\infty),$$

and so, there exist $s \geqq r \geqq 0$ such that $r \geqq 0.5$, $s \geqq 1$ and $2\sqrt{rs} \leqq 1$. Note that for any $r \geqq 0.5$ and $s \geqq 1$, we have $2\sqrt{rs} \geqq 2\sqrt{0.5} > 1$. This is a contradiction, and hence, the cone, $\bigcup_{\substack{v_1 \in \mathcal{V}_1 \\ \lambda_1 \geqq 0}} \text{epi}(\lambda_1 g_1(\cdot, v_1))^* + C^* \times \mathbb{R}_+$, is not convex.

Now we give an example illustrating Propositions 2.1.5 and 2.1.6.

Example 2.1.2. Let $v_1 \in \mathcal{V}_1 := [1, 2]$ and let $g_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g_1(x, v_1) = x^2 - 2v_1 x.$$

Let $C = \mathbb{R}_+$. Then we can easily find points which satisfy the Slater condition. Moreover, for each $\lambda_1 \geqq 0$ and $v_1 \in \mathcal{V}_1$,

$$g_1(\cdot, v_1)^*(a) = \frac{(a+2v_1)^2}{4},$$

and

$$\begin{aligned}\bigcup_{\substack{v_1 \in \mathcal{V}_1 \\ \lambda_1 \geqq 0}} \text{epi}(\lambda_1 g_1(\cdot, v_i))^* &= \bigcup_{\substack{v_1 \in [1,2] \\ \lambda_1 > 0}} \text{epi}(\lambda_1 g_1(\cdot, v_i))^* \cup (\{0\} \times [0, +\infty)) \\ &= \bigcup_{\substack{v_1 \in [1,2] \\ \lambda_1 > 0}} \lambda_1 \{(a, r) \mid r \geqq \frac{(a+2v_1)^2}{4}\} \cup (\{0\} \times [0, +\infty)) \\ &= \{(a, \alpha) \mid \max\{0, 2a\} \leqq \alpha\}.\end{aligned}$$

So, the cone, $\bigcup_{\substack{v_1 \in [1,2] \\ \lambda_1 \geqq 0}} \text{epi}(\lambda_1 g_1(\cdot, v_i))^* + C^* \times \mathbb{R}_+ = \{(a, \alpha) \mid \max\{0, 2a\} \leqq \alpha\}$, is closed and convex.

2.2 Approximate Optimality Theorem

Slightly extending Theorem 2.4 in [44] to a robust convex optimization problem with a geometric constraint, we can obtain the following lemma which is the robust version of Farkas' lemma for convex functions in [39]:

Lemma 2.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex function. Let C be a closed convex cone of $\mathbb{R}^n$. Let $\mathcal{V}_i \subseteq \mathbb{R}^q$,*

$i = 1, \ldots, m$, be compact and let $A := \{x \in C \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \neq \emptyset$. Then the following statements are equivalent:

(i) $\{x \in C \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \subseteq \{x \in \mathbb{R}^n \mid f(x) \geqq 0\}$;

(ii) $(0, 0) \in \text{epi} f^* + \text{cl co}\,(\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+)$.

Proof. Let $D := \{x \in \mathbb{R}^n \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\}$. Then $A = C \cap D$. We will prove that $\text{epi}\delta_A^* = \text{cl co}\,(\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+)$. For any $x \in \mathbb{R}^n$,

$$\delta_A(x) = \delta_C(x) + \delta_D(x) \text{ and } \delta_D(x) = \sup_{\substack{v_i \in \mathcal{V}_i \\ \lambda_i \geqq 0}} \sum_{i=1}^{m} \lambda_i g_i(x, v_i). \tag{2.2}$$

Thus we have $\text{epi}\delta_A^* = \text{epi}(\delta_D + \delta_C)^*$. So, by Proposition 2.1.3, $\text{epi}(\delta_D + \delta_C)^* = \text{cl}(\text{epi}\delta_D^* + \text{epi}\delta_C^*)$. So, from (2.2),

$$\text{cl}(\text{epi}\delta_D^* + \text{epi}\delta_C^*) = \text{cl}(\text{epi}(\sup_{\substack{v_i \in \mathcal{V}_i \\ \lambda_i \geqq 0}} \sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + \text{epi}\delta_C^*).$$

Hence, by Proposition 2.1.4,

$$\begin{aligned} \text{cl}(\text{epi}(\sup_{\substack{v_i \in \mathcal{V}_i \\ \lambda_i \geqq 0}} \sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + \text{epi}\delta_C^*) &= \text{cl}(\text{cl co} \bigcup_{\substack{v_i \in \mathcal{V}_i \\ \lambda_i \geqq 0}} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + \text{epi}\delta_C^*) \\ &= \text{cl co}(\bigcup_{\substack{v_i \in \mathcal{V}_i \\ \lambda_i \geqq 0}} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+). \end{aligned}$$

Thus, we see that

$$\mathrm{epi}\delta_A^* = \mathrm{cl}\,\mathrm{co}\Big(\bigcup_{\substack{v_i\in\mathcal{V}_i\\ \lambda_i\geqq 0}} \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i g_i(\cdot, v_i)\Big)^* + C^*\times\mathbb{R}_+\Big). \tag{2.3}$$

Now, we assume that (ii) holds. So, from (2.3), (ii) is equivalent to $(0,0)\in \mathrm{epi} f^* + \mathrm{epi}\delta_A^*$. From Proposition 2.1.3, equivalently, $(0,0)\in \mathrm{epi}(f+\delta_A)^*$. So, by the definition of epigraph, we see that $(f+\delta_A)^*(0)\leqq 0$. Also, from the definition of conjugate function, we see that $(f+\delta_A)(x)\geqq 0$, for any $x\in\mathbb{R}^n$. It means that $f(x)\geqq 0$, for any $x\in A$. Thus, we have (ii) $\Leftrightarrow$ (i). $\square$

Using Lemma 2.2.1, we can obtain the following theorem:

Theorem 2.2.1. *Let $\bar{x}\in A$ and let $g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$, $i=1,\ldots,m$, be continuous functions such that for each $v_i\in\mathbb{R}^q$, $g_i(\cdot,v_i)$ is convex on $\mathbb{R}^n$. Let $\mathcal{V}_i\subseteq\mathbb{R}^q$, $i=1,\ldots,m$, be compact. Suppose that $\bigcup_{v_i\in\mathcal{V}_i,\,\lambda_i\geqq 0}\mathrm{epi}(\sum_{i=1}^m\lambda_i g_i(\cdot,v_i))^* + C^*\times\mathbb{R}_+$ is closed and convex. Then the following statements are equivalent:*

(i) *$\bar{x}$ is an approximate solution of* (RCP)*;*

(ii) *if there exist $\bar{\lambda}_i\geqq 0$ and $\bar{v}_i\in\mathcal{V}_i$, $i=1,\ldots,m$, such that for any $x\in C$,*

$$f(x)+\sum_{i=1}^m\bar{\lambda}_i g_i(x,\bar{v}_i)\geqq f(\bar{x})-\epsilon.$$

Proof. [(i) ⇒ (ii)] Let $\bar{x}$ be an approximate solution of (RCP). Then $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. So, $A \subseteq \{x \in C \mid f(x) - f(\bar{x}) + \epsilon \geqq 0\}$. By Lemma 2.2.1 and the assumption,

$$(0, \epsilon - f(\bar{x})) \in \mathrm{epi} f^* + \bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+.$$

So, there exist $\bar{\lambda}_i \geqq 0$ and $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, such that

$$(0, \epsilon - f(\bar{x})) \in \mathrm{epi} f^* + \mathrm{epi}(\sum_{i=1}^{m} \bar{\lambda}_i g_i(\cdot, \bar{v}_i))^* + C^* \times \mathbb{R}_+.$$

Then there exist $u^* \in \mathbb{R}^n$, $\alpha \geqq 0$, $w_i^* \in \mathbb{R}^n$, $\beta_i \geqq 0$, $i = 1, \ldots, m$, $c^* \in C^*$ and $r \in \mathbb{R}_+$ such that

$$(0, \epsilon - f(\bar{x})) \in (u^*, f^*(u^*) + \alpha) + \sum_{i=1}^{m} \bar{\lambda}_i (w_i^*, g_i^*(w_i^*, \bar{v}_i) + \beta_i) + (c^*, r).$$

So, $0 = u^* + \sum_{i=1}^{m} \bar{\lambda}_i w_i^* + c^*$ and $\epsilon - f(\bar{x}) = f^*(u^*) + \alpha + \sum_{i=1}^{m} \bar{\lambda}_i (g_i^*(w_i^*, \bar{v}_i) + \beta_i) + r$. Hence, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
& -\langle \sum_{i=1}^{m} \bar{\lambda}_i w_i^*, x \rangle - \langle c^*, x \rangle - f(x) \\
= \ & \langle u^*, x \rangle - f(x) \\
\leqq \ & f^*(u^*) \\
= \ & \epsilon - f(\bar{x}) - \alpha - \sum_{i=1}^{m} \bar{\lambda}_i (g_i^*(w_i^*, \bar{v}_i) + \beta_i) - r.
\end{aligned}$$

Thus, for any $x \in C$,

$$\begin{aligned} f(\bar{x}) - \epsilon &\leqq \langle \sum_{i=1}^{m} \bar{\lambda}_i w_i^*, x \rangle + \langle c^*, x \rangle + f(x) - \alpha - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(w_i^*, \bar{v}_i) - \sum_{i=1}^{m} \bar{\lambda}_i \beta_i - r \\ &\leqq \langle \sum_{i=1}^{m} \bar{\lambda}_i w_i^*, x \rangle + f(x) - \sum_{i=1}^{m} \bar{\lambda}_i g_i^*(w_i^*, \bar{v}_i) \\ &\leqq \sum_{i=1}^{m} (\bar{\lambda}_i g_i(x, \bar{v}_i)) + f(x). \end{aligned}$$

[(ii) $\Rightarrow$ (i)] Suppose that there exist $\bar{\lambda}_i \geqq 0$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, such that for any $x \in C$, $f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) \geqq f(\bar{x}) - \epsilon$. Then we have for any $x \in A$,

$$f(x) \geqq f(x) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(x, \bar{v}_i) \geqq f(\bar{x}) - \epsilon.$$

Thus $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. Hence $\bar{x}$ is an approximate solution of (RCP). □

Using Lemma 2.2.1, we can obtain the following approximate optimality theorem for approximate solution of (RCP).

Theorem 2.2.2. (Approximate Optimality Theorem) *Let $\bar{x} \in A$ and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$. Let $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$, be compact. Suppose that $\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$ is closed and convex. Then the following statements are equivalent:*

(i) $\bar{x}$ *is an approximate solution of* (RCP)*;*

(ii) $(0, \epsilon - f(\bar{x})) \in \text{epi} f^* + \bigcup_{v_i \in \mathcal{V}_i,\ \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+;$

(iii) *there exist* $\bar{v}_i \in \mathcal{V}_i$, $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, *and* $\epsilon_i \geqq 0$, $i = 0, 1, \ldots, m+1$ *such that*

$$0 \in \partial_{\epsilon_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i g_i(\cdot, \bar{v}_i))(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x})$$

$$\textit{and} \quad \sum_{i=0}^{m+1} \epsilon_i - \epsilon = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i).$$

Proof. Let $\bar{x}$ be an approximate solution of (RCP). Then $f(x) \geqq f(\bar{x}) - \epsilon$, for any $x \in A$. Let $h(x) = f(x) - f(\bar{x}) + \epsilon$. Then

$$\begin{aligned} h^*(v) &= \sup\{\langle v, x\rangle - h(x) \mid x \in \mathbb{R}^n\} \\ &= \sup\{\langle v, x\rangle - f(x) + f(\bar{x}) - \epsilon \mid x \in \mathbb{R}^n\} \\ &= \sup\{\langle v, x\rangle - f(x) \mid x \in \mathbb{R}^n\} + f(\bar{x}) - \epsilon \\ &= f^*(v) + f(\bar{x}) - \epsilon. \end{aligned}$$

So, by Lemma 2.2.1, (i) $\Leftrightarrow$ (ii).

Now we will show that (ii) $\Leftrightarrow$ (iii). Let $A := \{x \in C \mid g_i(x, v_i) \leqq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\}$. Then $A \neq \emptyset$. Now we suppose that (ii) holds. Since $C^* \times \mathbb{R}_+ = \text{epi}\delta_C^*$, we have

$$(0, \epsilon - f(\bar{x})) \in \text{epi} f^* + \bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \text{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + \text{epi}\delta_C^*.$$

It means that there exist $\bar{v}_i \in \mathcal{V}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, such that

$$(0, \epsilon - f(\bar{x})) \in \mathrm{epi} f^* + \mathrm{epi}(\sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i))^* + \mathrm{epi}\delta_C^*.$$

So, by Proposition 2.1.1 and Proposition 2.1.2, equivalently, there exist $\bar{v}_i \in \mathcal{V}_i$, $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, and $\epsilon_i \geqq 0$, $i = 0, 1, \ldots, m+1$, such that $\sum_{i=1}^m \epsilon_i = \epsilon^*$,

$$\begin{aligned}(0, \epsilon - f(\bar{x})) \in & \bigcup_{\epsilon_0 \geqq 0} \{(\xi_0, \langle \xi_0, \bar{x} \rangle + \epsilon_0 - f(\bar{x})) \mid \xi_0 \in \partial_{\epsilon_0} f(\bar{x})\} \\ & + \bigcup_{\epsilon^* \geqq 0} \{(\xi^*, \langle \xi^*, \bar{x} \rangle + \epsilon^* - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)) \mid \xi^* \in \partial_{\epsilon^*} \sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})\} \\ & + \bigcup_{\epsilon_{m+1} \geqq 0} \{(\xi_{m+1}, \langle \xi_{m+1}, \bar{x} \rangle + \epsilon_{m+1} - \delta_C(\bar{x})) \mid \xi_{m+1} \in \partial_{\epsilon_{m+1}} \delta_C(\bar{x})\}.\end{aligned}$$

It means that there exist $\bar{v}_i \in \mathcal{V}_i$, $\bar{\lambda}_i \geqq 0$, $\bar{\xi}_0 \in \partial_{\epsilon_0} f(\bar{x})$, $\bar{\xi}^* \in \partial_{\epsilon^*} \sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}} \delta_C(\bar{x})$, $i = 1, \ldots, m$, and $\epsilon_i \geqq 0$, $i = 0, 1, \ldots, m+1$, such that $\sum_{i=1}^m \epsilon_i = \epsilon^*$,

$$\begin{aligned}(0, \epsilon - f(\bar{x})) &= (\bar{\xi}_0, \langle \bar{\xi}_0, \bar{x} \rangle + \epsilon_0 - f(\bar{x})) + (\bar{\xi}^*, \langle \bar{\xi}^*, \bar{x} \rangle + \epsilon^* - \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)) \\ & \quad + (\bar{\xi}_{m+1}, \langle \bar{\xi}_{m+1}, \bar{x} \rangle + \epsilon_{m+1} - \delta_C(\bar{x})).\end{aligned}$$

Thus, equivalently, there exist $\bar{v}_i \in \mathcal{V}_i$, $\bar{\lambda}_i \geqq 0$, $\bar{\xi}_0 \in \partial_{\epsilon_0} f(\bar{x})$, $\bar{\xi}_i \in \partial_{\epsilon_i} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)$, $\bar{\xi}_{m+1} \in N_C^{\epsilon_{m+1}}(\bar{x})$, $i = 1, \ldots, m$, and $\epsilon_i \geqq 0$, $i = 0, 1, \ldots, m+1$, such that $0 = \sum_{i=0}^{m+1} \bar{\xi}_i$ and $\sum_{i=0}^{m+1} \epsilon_i - \epsilon = \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i)$. □

2.3 Approximate Duality Theorem

As usual convex programs, the dual problem of (RCP) is sometimes more treatable than (RCP). So, we formulate a dual problem (RLD) for (RCP) as follows:

$$
\begin{aligned}
\text{(RLD)} \qquad \text{Maximize}_{(x,v,\lambda)} \quad & f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \\
\text{subject to} \quad & 0 \in \partial_{\epsilon_0} f(x) + \sum_{i=1}^{m} \partial_{\epsilon_i} \lambda_i g_i(x, v_i) + N_C^{\epsilon_{m+1}}(x), \\
& \lambda_i \geqq 0,\ v_i \in \mathcal{V}_i,\ i = 1, \ldots, m, \\
& \epsilon_i \geqq 0,\ i = 0, 1, \ldots, m+1,\ \sum_{i=0}^{m+1} \epsilon_i \leqq \epsilon.
\end{aligned}
$$

If $\epsilon = 0$ and $g_i(x, v_i) = g_i(x)$, $i = 1, \ldots, m$, then (RCP) becomes (CP), and (RLD) collapses to the Wolfe dual problem (D) for (CP) as follows:

$$
\begin{aligned}
\text{(D)} \qquad \text{Maximize}_{(x,\lambda)} \quad & f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \\
\text{subject to} \quad & \partial f(x) + \sum_{i=1}^{m} \partial \lambda_i g_i(x) + N_C(x) = 0, \\
& \lambda_i \geqq 0,\ i = 1, \ldots, m.
\end{aligned}
$$

Now, we prove an approximate weak duality theorem and an approximate strong duality theorem which hold between (RCP) and (RLD).

Theorem 2.3.1. (Approximate Weak Duality Theorem) *For any feasible solution x of* (RCP) *and any feasible solution (y, v, λ) of* (RLD),

$$f(x) \geqq f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \epsilon.$$

Proof. Let x and (y, v, λ) be feasible solutions of (RCP) and (RLD), respectively. Then there exist $\epsilon_0 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$, $\epsilon_{m+1} \geqq 0$, $\bar{\xi}_0 \in \partial_{\epsilon_0} f(y)$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\lambda_i g_i)(y, v_i)$, $i = 1, \ldots, m$, and $\bar{\xi}_{m+1} \in N_C^{\epsilon_{m+1}}(y)$ such that $\epsilon = \sum_{i=0}^{m+1} \epsilon_i$ and $\sum_{i=0}^{m+1} \bar{\xi}_i = 0$. Thus, we have

$$\begin{aligned}
& f(x) - \{f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)\} \\
\geqq\ & \langle \bar{\xi}_0, x - y \rangle - \epsilon_0 - \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
=\ & -\langle \sum_{i=1}^{m} \bar{\xi}_i, x - y \rangle - \langle \bar{\xi}_{m+1}, x - y \rangle - \epsilon_0 - \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
\geqq\ & -\sum_{i=1}^{m} \lambda_i (g_i(x, v_i) - g_i(y, v_i)) - \sum_{i=1}^{m} \epsilon_i - \epsilon_{m+1} - \epsilon_0 - \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
=\ & -\sum_{i=1}^{m} \lambda_i g_i(x, v_i) - \sum_{i=1}^{m} \epsilon_i - \epsilon_{m+1} - \epsilon_0 \\
\geqq\ & -\sum_{i=0}^{m+1} \epsilon_i \\
=\ & -\epsilon.
\end{aligned}$$

Hence $f(x) \geqq f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \epsilon$. □

Theorem 2.3.2. (Approximate Strong Duality Theorem) *Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$. Suppose that*

$$\bigcup_{v_i \in \mathcal{V}_i,\, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i))^* + C^* \times \mathbb{R}_+$$

is closed and convex. If $\bar{x}$ is an approximate solution of (RCP)*, then there exist $\bar{\lambda} \in \mathbb{R}^m_+$ and $\bar{v} \in \mathbb{R}^q$ such that $(\bar{x}, \bar{v}, \bar{\lambda})$ is a 2-approximate solution of* (RLD)*.*

Proof. Let $\bar{x} \in A$ be an approximate solution of (RCP). Then, by Theorem 2.2.2, there exist $\bar{\lambda}_i \geqq 0$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, and $\epsilon_i \geqq 0$, $i = 0, 1, \ldots, m+1$, such that

$$0 \in \partial_{\epsilon_0} f(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) + N_C^{\epsilon_{m+1}}(\bar{x}) \quad \text{and} \quad \sum_{i=0}^{m+1} \epsilon_i - \epsilon = \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i).$$

So, $(\bar{x}, \bar{v}, \bar{\lambda})$ is a feasible solution of (RLD). Hence, by Theorem 2.3.1, for any feasible (y, v, λ) of (RLD),

$$\begin{aligned} f(\bar{x}) \; + \; & \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) - \{f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)\} \\ \geqq \; & -\epsilon + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) \\ \geqq \; & -\epsilon + \sum_{i=0}^{m+1} \epsilon_i - \epsilon \\ \geqq \; & -2\epsilon. \end{aligned}$$

Thus $(\bar{x}, \bar{v}, \bar{\lambda})$ is a 2-approximate solution of (RLD). $\square$

Now, we give an example illustrating our approximate duality theorems:

Example 2.3.1. Consider the following convex optimization problem with uncertainty:

$$
\begin{aligned}
\text{(RCP)} \quad \min \quad & x_1 + x_2^2 \\
\text{s.t.} \quad & x_1^2 - 2v_1 x_1 \leqq 0, \ v_1 \in [-1, 1], \\
& (x_1, x_2) \in \mathbb{R}^2_+.
\end{aligned}
$$

Let $f(x_1, x_2) = x_1 + x_2^2$, $g_1((x_1, x_2), v_1) = x_1^2 - 2v_1 x_1$ and $0 < \epsilon \leqq \frac{1}{4}$. Then $A := \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geqq 0\}$ is the set of all robust feasible solutions of (RCP) and $\bar{A} := \{(0, x_2) \in \mathbb{R}^2 \mid x_1 = 0, \ 0 \leqq x_2 \leqq \sqrt{\epsilon}\}$ is the set of all robust approximate solutions of (RCP). Moreover, we can check that $\bigcup_{v_1 \in [-1,1], \ \lambda_1 \geqq 0} \operatorname{epi}(\lambda_1 g_1(\cdot, v_1))^* + C^* \times \mathbb{R}_+ = \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_+$. Let $F := \{((x_1, x_2), v_1, \lambda_1) \mid 0 \in \partial_{\epsilon_0} f((x_1, x_2)) + \partial_{\epsilon_1} \lambda_1 g((x_1, x_2), v_1) + N^{\epsilon_2}_{\mathbb{R}^2_+}(x), \ \lambda_1 \geqq 0, \ v_1 \in [-1, 1], \ \epsilon_i \geqq 0, \ i = 0, 1, 2, \ \epsilon_0 + \epsilon_1 + \epsilon_2 \leqq \epsilon\}$. Then $F := \widetilde{A} \cup \widetilde{B} \cup \widetilde{C} \cup \widetilde{D}$ is a set of all robust feasible solutions of (RLD), where

$$
\begin{aligned}
\widetilde{A} \ := \ & \{((0,0), v_1, \lambda_1) \mid 0 \in \partial_{\epsilon_0} f(0,0) + \partial_{\epsilon_1} \lambda_1 g((0,0), v_1) + N^{\epsilon_2}_{\mathbb{R}^2_+}(0,0), \\
& \lambda_1 \geqq 0, \ v_1 \in [-1, 1], \ \epsilon_i \geqq 0, \ i = 0, 1, 2, \ \epsilon_0 + \epsilon_1 + \epsilon_2 \leqq \epsilon\} \\
= \ & \{((0,0), v_1, \lambda_1) \mid 0 \in \{1\} \times [-2\sqrt{\epsilon_0}, 2\sqrt{\epsilon_0}] + [-2\lambda_1 v_1 - 2\sqrt{\lambda_1 \epsilon_1}, \\
& -2\lambda_1 v_1 + 2\sqrt{\lambda_1 \epsilon_1}] \times \{0\} - \mathbb{R}^2_+, \ \lambda_1 \geqq 0, \ v_1 \in [-1, 1], \ \epsilon_i \geqq 0, \\
& i = 0, 1, 2, \ \epsilon_0 + \epsilon_1 \leqq \epsilon\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{((0,0),v_1,\lambda_1) \mid 0 \leqq \lambda_1 \leqq 1/2v_1,\ 0 \leqq v_1 \leqq 1\} \cup \{((0,0),v_1,\lambda_1) \mid \\
&\quad 1/2v_1 \leqq \lambda_1 \leqq (v_1+\epsilon_1+\sqrt{\epsilon_1^2+2v_1\epsilon_1})/2v_1^2,\ 0 \leqq v_1 \leqq 1, \epsilon_0 \geqq 0, \\
&\quad \epsilon_2 \geqq 0,\ 0 \leqq \epsilon_1 \leqq \epsilon\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B} &:= \{(0,x_2),v_1,\lambda_1) \mid x_2 > 0,\ 0 \in \partial_{\epsilon_0} f(0,x_2) + \partial_{\epsilon_1}\lambda_1 g((0,x_2),v_1) \\
&\quad + N_{\mathbb{R}_+^2}^{\epsilon_2}(0,x_2),\ \lambda_1 \geqq 0,\ v_1 \in [-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\} \\
&= \{((0,x_2),v_1,\lambda_1) \mid x_2 > 0,\ 0 \in \{1\} \times [2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}] + \\
&\quad [-2\lambda_1 v_1 - 2\sqrt{\lambda_1\epsilon_1}, -2\lambda_1 v_1 + 2\sqrt{\lambda_1\epsilon_1}] \times \{0\} + (-\infty,0] \times [-\epsilon_2/x_2, 0], \\
&\quad \lambda_1 \geqq 0,\ v_1 \in [-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\} \\
&= \{((0,x_2),v_1,\lambda_1) \mid x_2 > 0,\ 0 \leqq 1 - 2\lambda_1 v_1 + 2\sqrt{\lambda_1\epsilon_1}, 2x_2 - 2\sqrt{\epsilon_0} \\
&\quad -\epsilon_2/x_2 \leqq 0,\ \lambda_1 \geqq 0, v_1 \in [-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\} \\
&= \{((0,x_2),v_1,\lambda_1) \mid 0 \leqq \lambda_1 \leqq 1/2v_1,\ 0 < x_2 \leqq (\sqrt{\epsilon_0} + \sqrt{\epsilon_0+2\epsilon_2})/2, \\
&\quad \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_2 \leqq \epsilon\} \cup \{((0,x_2),v_1,\lambda_1) \mid 1/2v_1 \leqq \lambda_1 \leqq \\
&\quad (v_1+\epsilon_1+\sqrt{\epsilon_1^2+2v_1\epsilon_1})/2v_1^2,\ 0 \leqq v_1 \leqq 1,\ 0 < x_2 \leqq (\sqrt{\epsilon_0} + \\
&\quad \sqrt{\epsilon_0+2\epsilon_2})/2,\ 0 \leqq v_1 \leqq 1,\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{C} &:= \{(x_1,0),v_1,\lambda_1) \mid x_1 > 0,\ 0 \in \partial_{\epsilon_0} f(x_1,0) + \partial_{\epsilon_1}\lambda_1 g((x_1,0),v_1) \\
&\quad + N_{\mathbb{R}_+^2}^{\epsilon_2}(x_1,0),\ \lambda_1 \geqq 0,\ v_1 \in [-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\} \\
&= \{((x_1,0),v_1,\lambda_1) \mid x_1 > 0,\ 0 \in \{1\} \times [-2\sqrt{\epsilon_0}, 2\sqrt{\epsilon_0}] + [-2\lambda_1 v_1 + 2\lambda_1 x_1 \\
&\quad -2\sqrt{\lambda_1\epsilon_1}, -2\lambda_1 v_1 + 2\lambda_1 x_1 + 2\sqrt{\lambda_1\epsilon_1}] \times \{0\} + [-\epsilon_2/x_1, 0] \times (-\infty,0], \\
&\quad \lambda_1 \geqq 0,\ v_1 \in [-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{((x_1,0),v_1,\lambda_1) \mid x_1>0,\ 0 \geqq 1-2\lambda_1 v_1+2\lambda_1 x_1-2\sqrt{\lambda_1\epsilon_1}-\epsilon_2/x_1,\\
&\quad 0 \leqq 1-2\lambda_1 v_1+2\lambda_1 x_1+2\sqrt{\lambda_1\epsilon_1},\ \lambda_1 \geqq 0, v_1\in[-1,1],\ \epsilon_i \geqq 0,\\
&\quad i=0,1,2,\ \epsilon_1+\epsilon_2 \leqq \epsilon\}\\
&= \{((x_1,0),v_1,\lambda_1) \mid x_1>0,\ x_1-2\lambda_1 v_1 x_1+\lambda_1 x_1^2 \leqq -\lambda_1 x_1^2+2\sqrt{\lambda_1\epsilon_1}x_1\\
&\quad +\epsilon_2,\ 0 \leqq 1-2\lambda_1 v_1+2\lambda_1 x_1+2\sqrt{\lambda_1\epsilon_1},\ \lambda_1 \geqq 0, v_1\in[-1,1],\ \epsilon_i \geqq 0,\\
&\quad i=0,1,2,\ \epsilon_1+\epsilon_2 \leqq \epsilon\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{D} &:= \{(x_1,x_2),v_1,\lambda_1) \mid x_1,x_2>0,\ 0\in \partial_{\epsilon_0} f(x_1,x_2)+\partial_{\epsilon_1}\lambda_1 g((x_1,x_2),v_1)+\\
&\quad N^{\epsilon_2}_{\mathbb{R}^2_+}(x_1,x_2),\ \lambda_1 \geqq 0,\ v_1\in[-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\}\\
&= \{((x_1,x_2),v_1,\lambda_1) \mid x_1>0,\ x_2>0,\ 0\in\{1\}\times[2x_2-2\sqrt{\epsilon_0},2x_2+2\sqrt{\epsilon_0}]\\
&\quad +[-2\lambda_1 v_1+2\lambda_1 x_1-2\sqrt{\lambda_1\epsilon_1},-2\lambda_1 v_1+2\lambda_1 x_1+2\sqrt{\lambda_1\epsilon_1}]\times\{0\}\\
&\quad +[-\epsilon_2^1/x_1,0]\times[-\epsilon_2^2/x_2,0],\ \lambda_1 \geqq 0,\ v_1\in[-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\\
&\quad \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\}\\
&= \{((x_1,x_2),v_1,\lambda_1) \mid x_1>0,\ x_2>0,\ 0 \geqq 1-2\lambda_1 v_1+2\lambda_1 x_1-2\sqrt{\lambda_1\epsilon_1}\\
&\quad -\epsilon_2^1/x_1,\ 0 \leqq 1-2\lambda_1 v_1+2\lambda_1 x_1+2\sqrt{\lambda_1\epsilon_1},\ 0 \geqq 2x_2-2\sqrt{\epsilon_0}-\epsilon_2^2/x_2,\\
&\quad \lambda_1 \geqq 0,\ v_1\in[-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_2^1+\epsilon_2^2=\epsilon_2,\ \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\}\\
&= \{((x_1,x_2),v_1,\lambda_1) \mid x_1>0,\ x_2>0,\ x_1-2\lambda_1 v_1 x_1+\lambda_1 x_1^2 \leqq -\lambda_1 x_1^2\\
&\quad +2\sqrt{\lambda_1\epsilon_1}x_1+\epsilon_2^1,\ 0 \leqq 1-2\lambda_1 v_1+2\lambda_1 x_1+2\sqrt{\lambda_1\epsilon_1},\ 0<x_2 \leqq (\sqrt{\epsilon_0}\\
&\quad +\sqrt{\epsilon_0+2\epsilon_2^2})/2,\ \lambda_1 \geqq 0,\ v_1\in[-1,1],\ \epsilon_i \geqq 0,\ i=0,1,2,\ \epsilon_2^1+\epsilon_2^2=\epsilon_2,\\
&\quad \epsilon_0+\epsilon_1+\epsilon_2 \leqq \epsilon\}.
\end{aligned}
$$

We can check for any $(x_1, x_2) \in A$ and any $((y_1, y_2), v_1, \lambda_1) \in F$,

$$f(x_1, x_2) \geqq f((y_1, y_2)) + \lambda_1 g((y_1, y_2), v_1) - \epsilon, \tag{2.4}$$

that is, approximate weak duality holds. Indeed, let $x \in A$ and let $((y_1, y_2), v_1, \lambda_1) \in \widetilde{D}$ be any fixed. Then

$$\begin{aligned}
& f(y_1, y_2) + \lambda_1 g((y_1, y_2), v_1) - \epsilon \\
= \ & y_1 + y_2^2 + \lambda_1 y_1^2 - 2\lambda_1 v_1 x_1 - \epsilon \\
\leqq \ & -\lambda_1 y_1^2 + 2\sqrt{\lambda_1 \epsilon_1} y_1 + \epsilon_2^1 + y_2^2 - \epsilon \\
\leqq \ & \epsilon_1 + \epsilon_2^1 + (\epsilon_0 + \epsilon_2^2 + \sqrt{(\epsilon_0 + \epsilon_2^2)^2})/2 - \epsilon \\
= \ & \epsilon_1 + \epsilon_2^1 + \epsilon_0 + \epsilon_2^2 - \epsilon \\
\leqq \ & 0 \\
\leqq \ & f(x_1, x_2).
\end{aligned}$$

Let $(\bar{x_1}, \bar{x_2}) \in \bar{A}$ be an approximate solution of (RCP). Then $\bar{x}_1 = 0$, $0 \leqq \bar{x}_2 \leqq \sqrt{\epsilon}$. Let $\bar{\lambda}_1 = 1$, $\bar{v}_1 = \sqrt{\epsilon}$. Then we can easily check that $((\bar{x}_1, \bar{x}_2), \sqrt{\epsilon}, 1) \in F$. Moreover, for any $((y_1, y_2), v_1, \lambda_1) \in F$,

$$\begin{aligned}
& f(\bar{x}_1, \bar{x}_2) + \bar{\lambda}_1 g((\bar{x}_1, \bar{x}_2), \bar{v}_1) - \{f(y_1, y_2) + \lambda_1 g((y_1, y_2), v_1)\} \\
\geqq \ & -\epsilon + \bar{\lambda}_1 g((\bar{x}_1, \bar{x}_2), \bar{v}_1) \quad \text{(by (2.4))} \\
= \ & -\epsilon + \bar{\lambda}_1 (\bar{x}_1^2 - 2\bar{v}_1 \bar{x}_2) \\
= \ & -\epsilon + \bar{\lambda}_1 (\bar{x}_1 - \bar{v}_1)^2 - (\bar{v}_1)^2 \\
= \ & -\epsilon + (\bar{x}_1 - \sqrt{\epsilon})^2 - \epsilon \\
\geqq \ & -\epsilon + -\epsilon = -2\epsilon.
\end{aligned}$$

So, $((\bar{x}_1, \bar{x}_2), \sqrt{\epsilon}, 1)$ is an approximate solution of (RLD). Hence approximate strong duality theorem holds.

2.4 Robust Fractional Optimization Problems

The purpose of this section is to extend the approximate optimality theorems and approximate duality theorems from Section 2.2 and Section 2.3, respectively, to fractional optimization problems with data uncertainty.

Now, we consider approximate solutions for a fractional optimization problem in the face of data uncertainty. Using the robust optimization approach (the worst-case approach), we establish optimality theorems and duality theorems for approximate solutions for the robust fractional optimization problem. Moreover, we give an example illustrating our duality theorems.

Consider the following standard form of fractional optimization problem with a geometric constraint set:

$$
\begin{aligned}
\text{(FP)} \quad \min \quad & \frac{f(x)}{g(x)} \\
\text{s.t.} \quad & h_i(x) \leqq 0, \; i = 1, \cdots, m, \\
& x \in C,
\end{aligned}
$$

where $f, h_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \cdots, m$, are convex functions, C is a closed convex cone of $\mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a concave function such that for any $x \in C$, $f(x) \geqq 0$ and $g(x) > 0$.

The fractional optimization problem (FP) in the face of data uncertainty in the constraints can be captured by the problem:

$$\begin{aligned}
\text{(UFP)} \quad \min \quad & \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \\
\text{s.t.} \quad & h_i(x,w_i) \leqq 0,\ i=1,\cdots,m, \\
& x \in C,
\end{aligned}$$

where $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, $h_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$, $f(\cdot,u)$ and $h_i(\cdot,w_i)$ are convex, and $g:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$, $g(\cdot,v)$ is concave, and $u\in\mathbb{R}^p$, $v\in\mathbb{R}^p$ and $w_i\in\mathbb{R}^q$ are uncertain parameters which belongs to the convex and compact uncertainty sets $\mathcal{U}\subset\mathbb{R}^p$, $\mathcal{V}\subset\mathbb{R}^p$ and $\mathcal{W}_i\subset\mathbb{R}^q$, $i=1,\cdots,m$, respectively.

We study approximate optimality theorems and approximate duality theorems for the uncertain fractional optimization problem (UFP) by examining its robust counterpart [8]:

$$\begin{aligned}
\text{(RFP)} \quad \min \quad & \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \\
\text{s.t.} \quad & h_i(x,w_i) \leqq 0,\ \forall w_i\in\mathcal{W}_i,\ i=1,\cdots,m, \\
& x \in C.
\end{aligned}$$

Clearly, $A:=\{x\in C \mid h_i(x,w_i)\leqq 0,\ \forall w_i\in\mathcal{W}_i,\ i=1,\ldots,m\}$ is the feasible set of (RFP).

Let $\epsilon\geqq 0$. Then $\bar{x}$ is called an approximate solution of (RFP) if for any $x\in A$,

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon.$$

Using parametric approach, we transform the problem (RFP) into the robust non-fractional convex optimization problem $(\text{RNCP})_r$ with a parameter $r \in \mathbb{R}_+$:

$$
\begin{aligned}
(\text{RNCP})_r \quad & \min \quad \max_{u \in \mathcal{U}} f(x,u) - r \min_{v \in \mathcal{V}} g(x,v) \\
& \text{s.t.} \quad h_i(x, w_i) \leqq 0, \ \forall w_i \in \mathcal{W}_i, \ i = 1, \cdots, m, \\
& \qquad\ \ x \in C.
\end{aligned}
$$

Let $\epsilon \geqq 0$. Then $\bar{x}$ is called an approximate solution of $(\text{RNCP})_r$ if for any $x \in A$,

$$\max_{u \in \mathcal{U}} f(x,u) - r \min_{v \in \mathcal{V}} g(x,v) \geqq \max_{u \in \mathcal{U}} f(\bar{x},u) - r \min_{v \in \mathcal{V}} g(\bar{x},v) - \epsilon.$$

Proposition 2.4.1. [31] *Let $\epsilon \geqq 0$. Let $h_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, m$, be proper lower semicontinuous convex functions. If $\bigcup_{i=1}^{m} \text{ri}(\text{dom} h_i) \neq \emptyset$, then for all $x \in \bigcup_{i=1}^{m} \text{dom} h_i$,*

$$\partial_\epsilon (\sum_{i=1}^{m} h)(x) = \bigcup \{ \sum_{i=1}^{m} \partial_{\epsilon_i} h_i(x) \mid \epsilon_i \geqq 0, \ i = 1. \cdots, m, \ \sum_{i=1}^{m} \epsilon_i = \epsilon \}.$$

Now we give the following relation between approximate solution of (RFP) and $(\text{RNCP})_{\bar{r}}$.

Lemma 2.4.1. *Let $\bar{x} \in A$ and let $\epsilon \geqq 0$. If $\max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq 0$, then the following statements are equivalent:*

(i) $\bar{x}$ *is an approximate solution of* (RFP)*;*

(ii) $\bar{x}$ *is an* $\bar{\epsilon}$*-approximate solution of* (RNCP)$_{\bar{r}}$*, where* $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ *and* $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v)$.

Proof. [(i) $\Rightarrow$ (ii)] Let $\bar{x} \in A$ be an approximate solution of (RFP). Then for any $x \in A$, $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. Put $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v)$. Then we have for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \min_{v\in\mathcal{V}} \bar{r} g(x,v) \geqq 0$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, for any $x \in A$,

$$\begin{aligned}\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) &\geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v)\\ &= \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \bar{\epsilon}.\end{aligned}$$

Hence $\bar{x}$ is and $\bar{\epsilon}$-approximate solution of (RNCP)$_{\bar{r}}$.

[(ii) $\Rightarrow$ (i)] Let $\bar{x} \in A$ be an $\bar{\epsilon}$-approximate solution of (RNCP)$_{\bar{r}}$. Then for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \bar{\epsilon}$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq 0$. So, we have $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \bar{r}$. Since $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$,

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon.$$

Hence $\bar{x}$ is an approximate solution of (RFP). $\square$

Now, we give the following lemma which is the robust version of Farkas Lemma for non-fractional convex functions.

Lemma 2.4.2. *Let* $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ *and* $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ *be functions such that for any* $u \in \mathbb{R}^p$, $f(\cdot, u)$ *and for each* $w_i \in \mathbb{R}^q$, $h_i(\cdot, w_i)$, $i = 1, \ldots, m$, *are convex functions, and for any* $x \in \mathbb{R}^n$, $f(x, \cdot)$ *is a concave function. Let* $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ *be a function such that for any* $v \in \mathbb{R}^p$, $g(\cdot, v)$ *is a concave function, and for all* $x \in \mathbb{R}^n$, $g(x, \cdot)$ *is a convex function. Let* $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ *and* $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$, *be convex and compact sets. Let* $r \geqq 0$ *and let* C *be a closed convex cone of* $\mathbb{R}^n$. *Assume that* $A := \{x \in C \mid h_i(x, w_i) \leqq 0, \ \forall w_i \in \mathcal{W}_i, \ i = 1, \ldots, m\} \neq \emptyset$. *Then the following statements are equivalent:*

(i) $\{x \in C \mid h_i(x, w_i) \leqq 0, \ \forall w_i \in \mathcal{W}_i, \ i = 1, \ldots, m\} \subseteq \{x \in \mathbb{R}^n \mid \max_{u \in \mathcal{U}} f(x, u) - r \min_{v \in \mathcal{V}} g(x, v) \geqq 0\}$;

(ii) *there exist* $\bar{u} \in \mathcal{U}$ *and* $\bar{v} \in \mathcal{V}$ *such that*

$$A \subseteq \{x \in \mathbb{R}^n \mid f(x, \bar{u}) - r g(x, \bar{v}) \geqq 0\};$$

(iii)
$$(0,0) \in \bigcup_{u \in \mathcal{U}} \mathrm{epi}(f(\cdot, u))^* + \bigcup_{v \in \mathcal{V}} \mathrm{epi}(-r g(\cdot, v))^* + \mathrm{clco}\Big(\bigcup_{w_i \in \mathcal{W}_i, \lambda_i \geqq 0} \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i h_i(\cdot, w_i)\Big)^* + C^* \times \mathbb{R}_+\Big);$$

(iv)
$$(0,0) \in \mathrm{epi}(\max_{u \in \mathcal{U}} f(\cdot, u))^* + \mathrm{epi}(-r \min_{v \in \mathcal{V}} g(\cdot, v))^* + \mathrm{clco}\Big(\bigcup_{w_i \in \mathcal{W}_i, \lambda_i \geqq 0} \mathrm{epi}\Big(\sum_{i=1}^m \lambda_i h_i(\cdot, w_i)\Big)^* + C^* \times \mathbb{R}_+\Big).$$

Proof. Notice that $\mathrm{epi}\delta_A^* = \mathrm{cl}\,\mathrm{co}\,(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\mathrm{epi}(\sum_{i=1}^m \lambda_i h_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+)$ by the proof of Lemma 2.2.1.

[(i) $\Leftrightarrow$ (iv)] Now we assume that the statement (iv) holds. Then, by Proposition 2.1.3, the statement (iv) is equivalent to

$$(0,0) \in \mathrm{epi}(\max_{u\in\mathcal{U}} f(\cdot,u))^* + \mathrm{epi}(-r\min_{v\in\mathcal{V}} g(\cdot,v))^* + \mathrm{epi}\delta_A^*$$
$$= \mathrm{epi}(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A)^*.$$

Equivalently, by the definition of epigraph of $\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A)^*$,

$$(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A)^*(0) \leqq 0.$$

From the definition of conjugate function, for any $x\in\mathbb{R}^n$,

$$(\max_{u\in\mathcal{U}} f(\cdot,u) - r\min_{v\in\mathcal{V}} g(\cdot,v) + \delta_A)(x) \geqq 0.$$

It is equivalent to the statement that for any $x\in A$,

$$\max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq 0.$$

[(ii) $\Leftrightarrow$ (iii)] Now we assume that the statement (iii) holds. Then the statement (iii) is equivalent to

$$(0,0) \in \bigcup_{u\in\mathcal{U}} \mathrm{epi}(f(\cdot,u))^* + \bigcup_{v\in\mathcal{V}} \mathrm{epi}(-rg(\cdot,v))^* + \mathrm{epi}\delta_A^*.$$

It means that there exist $\bar{u}\in\mathcal{U}$ and $\bar{v}\in\mathcal{V}$ such that

$$(0,0) \in \mathrm{epi}(f(\cdot,\bar{u}) - rg(\cdot,\bar{v}) + \delta_A)^*.$$

It is equivalent to the statement that there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that

$$(f(\cdot, \bar{u}) - rg(\cdot, \bar{v}) + \delta_A)^*(0) \leqq 0,$$

From the definition of conjugate function, there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that for any $x \in \mathbb{R}^n$,

$$(f(\cdot, \bar{u}) - rg(\cdot, \bar{v}) + \delta_A)(x) \geqq 0.$$

It means that there exist $\bar{u} \in \mathcal{U}$ and $\bar{v} \in \mathcal{V}$ such that for any $x \in A$,

$$f(x, \bar{u}) - rg(x, \bar{v}) \geqq 0.$$

[(iii) $\Leftrightarrow$ (iv)] To get a desired result, it suffices to show that

$$\bigcup_{u \in \mathcal{U}} \operatorname{epi}(f(\cdot, u))^* = \operatorname{epi}(\max_{u \in \mathcal{U}} f(\cdot, u))^* \tag{2.5}$$

$$\bigcup_{v \in \mathcal{V}} \operatorname{epi}(-rg(\cdot, v))^* = \operatorname{epi}(-r \min_{v \in \mathcal{V}} g(\cdot, v))^*. \tag{2.6}$$

By Proposition 2.1.4, $\operatorname{epi}(\max_{u \in \mathcal{U}} f(\cdot, u))^* = \operatorname{cl}\operatorname{co} \bigcup_{u \in \mathcal{U}} \operatorname{epi}(f(\cdot, u))^*$. Let (z_1, α_1), $(z_2, \alpha_2) \in \bigcup_{u \in \mathcal{U}} \operatorname{epi}(f(\cdot, u))^*$ and let $\mu \in [0, 1]$. Then there exist $u_1, u_2 \in \mathcal{U}$ such that $(z_1, \alpha_1) \in \operatorname{epi}(f(\cdot, u_1))^*$ and $(z_2, \alpha_2) \in \operatorname{epi}(f(\cdot, u_2))^*$, that is, $(f(\cdot, u_1))^*(z_1) \leqq \alpha_1$ and $(f(\cdot, u_2))^*(z_2) \leqq \alpha_2$. Using the definition of conjugate function, we have for all $x \in \mathbb{R}^n$,

$$\langle z_1, x \rangle - f(x, u_1) \leqq \alpha_1 \quad \text{and} \quad \langle z_2, x \rangle - f(x, u_2) \leqq \alpha_2. \tag{2.7}$$

Since for all $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave, we have $f(x, \mu u_1 + (1-\mu)u_2) \geqq \mu f(x, u_1) + (1-\mu) f(x, u_2)$, i.e.,

$$-f(x, \mu u_1 + (1-\mu)u_2) \leqq -\mu f(x, u_1) - (1-\mu) f(x, u_2). \tag{2.8}$$

So, from (2.7) and (2.8), we have for all $x \in \mathbb{R}^n$,

$$\langle \mu z_1 + (1-\mu)z_2, x\rangle - f(x, \mu u_1 + (1-\mu)u_2) \leqq \mu\alpha_1 + (1-\mu)\alpha_2,$$

and so, $(f(\cdot, \mu u_1 + (1-\mu)u_2))^*(\mu z_1 + (1-\mu)z_2) \leqq \mu\alpha_1 + (1-\mu)\alpha_2$. Hence, we have

$$(\mu z_1 + (1-\mu)z_2, \mu\alpha_1 + (1-\mu)\alpha_2) \in \operatorname{epi}(f(\cdot, \mu u_1 + (1-\mu)u_2))^*.$$

So, $\bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot, u))^*$ is convex.

Now we will show that $\bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot, u))^*$ is closed. Let

$$(z_n, \alpha_n) \in \bigcup_{u\in\mathcal{U}} \operatorname{epi}(f(\cdot, u))^*$$

with $(z_n, \alpha_n) \to (z^*, \alpha^*)$ as $n \to \infty$. Then there exists $u_n \in \mathcal{U}$ such that $(f(\cdot, u_n))^*(z_n) \leqq \alpha_n$. Since $\mathcal{U}$ is compact, we may assume that $u_n \to u^* \in \mathcal{U}$ as $n \to \infty$. So, for each $x \in \mathbb{R}^n$,

$$\langle z_n, x\rangle - f(x, u_n) \leqq \alpha_n.$$

Since for all $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave, $f(x, \cdot)$ is continuous. Passing to the limit as $n \to \infty$, we get that, for each $x \in \mathbb{R}^n$, $\langle z^*, x\rangle - f(x, u^*) \leqq \alpha^*$. Hence, we have

$$(z^*, \alpha^*) \in \operatorname{epi}(f(\cdot, u^*))^*.$$

So, $\bigcup_{u\in\mathcal{U}} \text{epi}(f(\cdot,u))^*$ is closed. Thus, (2.5) holds.

Moreover, since for all $x\in\mathbb{R}^n$, $g(x,\cdot)$ is convex and $r \geqq 0$, for all $x\in\mathbb{R}^n$, $-rg(x,\cdot)$ is concave. So, similarly, we can prove that (2.6) holds. □

Remark 2.4.1. *Using the convex-concave minimax theorem* (*Corollary 37.3.2 in* [64]), *we can prove that the statement* (i) *in Lemma 2.4.2 is equivalent to the statement* (ii) *in Lemma 2.4.2.*

Remark 2.4.2. *From proving Lemma 2.4.2 that the statement* (i) *is equivalent to the statement* (iv), *we see that we can prove the equivalent relation without the assumptions that for all $x\in\mathbb{R}^n$, $f(x,\cdot)$ and $g(x,\cdot)$ are concave and convex, respectively.*

From Lemmas 2.4.1 and 2.4.2, we can get the following theorem:

Theorem 2.4.1. *Let $f:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ and $h_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$ be functions such that for any $u\in\mathbb{R}^p$, $f(\cdot,u)$ and for each $w_i\in\mathbb{R}^q$, $h_i(\cdot,w_i)$, $i=1,\ldots,m$, are convex functions, and for any $x\in\mathbb{R}^n$, $f(x,\cdot)$ is concave function. Let $g:\mathbb{R}^n\times\mathbb{R}^p\to\mathbb{R}$ be a function such that for any $v\in\mathbb{R}^p$, $g(\cdot,v)$ is a concave function, and for all $x\in\mathbb{R}^n$, $g(x,\cdot)$ is a convex function. Let $\mathcal{U}\subset\mathbb{R}^p$, $\mathcal{V}\subset\mathbb{R}^p$ and $\mathcal{W}_i\subset\mathbb{R}^q$, $i=1,\ldots,m$, be convex and compact, and let $A:=\{x\in C \mid h_i(x,w_i)\leqq 0,\ \forall w_i\in\mathcal{W}_i,\ i=1,\ldots,m\}\neq\emptyset$. Let $\bar{r}=\max\limits_{(u,v)\in\mathcal{U}\times\mathcal{V}}\frac{f(\bar{x},u)}{g(\bar{x},v)}-\epsilon$.*

Suppose that $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0}\text{epi}(\sum\limits_{i=1}^{m}\lambda_i h_i(\cdot,w_i))^*+C^*\times\mathbb{R}_+$ is closed and convex. Then the following statements are equivalent:

(i) $\bar{x} \in A$ *is an approximate solution of* (RFP)*;*

(ii) *there exist* $\bar{u} \in \mathcal{U}$*,* $\bar{v} \in \mathcal{V}$*,* $\bar{w}_i \in \mathcal{W}_i$ *and* $\bar{\lambda}_i \geqq 0$*,* $i = 1, \ldots, m$*, such that for any* $x \in C$*,*

$$f(x, \bar{u}) - \bar{r} g(x, \bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(x, \bar{w}_i) \geqq 0.$$

Proof. [(i) $\Rightarrow$ (ii)] Let $\bar{x}$ be an approximate solution of (RFP). Then, by Lemma 2.4.1, equivalently, $\bar{x}$ is an $\bar{\epsilon}$-approximate solution of $(\text{RNCP})_{\bar{r}}$, where $\bar{r} = \max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v)$, that is, for any $x \in A$, $\max_{u \in \mathcal{U}} f(x, u) - \bar{r} \min_{v \in \mathcal{V}} g(x, v) \geqq \max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) - \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v)$. Since $\max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) - \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v) = 0$, we have $A \subseteq \{x \in C \mid \max_{u \in \mathcal{U}} f(x, u) - \bar{r} \min_{v \in \mathcal{V}} g(x, v) \geqq 0\}$. Then, by Lemma 2.4.2, we have

$$\begin{aligned}(0,0) \in \bigcup_{u \in \mathcal{U}} \operatorname{epi}(f(\cdot, u))^* &+ \bigcup_{v \in \mathcal{V}} \operatorname{epi}(-\bar{r} g(\cdot, v))^* \\ &+ \operatorname{cl} \operatorname{co} \Big(\bigcup_{w_i \in \mathcal{W}_i, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot, w_i))^* + C^* \times \mathbb{R}_+ \Big).\end{aligned}$$

Moreover, by assumption,

$$\begin{aligned}(0,0) \in \bigcup_{u \in \mathcal{U}} \operatorname{epi}(f(\cdot, u))^* &+ \bigcup_{v \in \mathcal{V}} \operatorname{epi}(-\bar{r} g(\cdot, v))^* \\ &+ \bigcup_{w_i \in \mathcal{W}_i, \lambda_i \geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot, w_i))^* + C^* \times \mathbb{R}_+.\end{aligned}$$

So, there exist $\bar{u} \in \mathcal{U}$, $\bar{v} \in \mathcal{V}$, $\bar{w}_i \in \mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, such that

$$(0,0) \in \operatorname{epi}(f(\cdot,\bar{u}))^* + \operatorname{epi}(-\bar{r}g(\cdot,\bar{v}))^* + \operatorname{epi}(\sum_{i=1}^{m} \bar{\lambda}_i h_i(\cdot,\bar{w}_i))^* + C^* \times \mathbb{R}_+.$$

Then there exist $s \in \mathbb{R}^n$, $\eta \geqq 0$, $t \in \mathbb{R}^n$, $\mu \geqq 0$, $z_i \in \mathbb{R}^n$, $\rho_i \geqq 0$, $i = 1, \ldots, m$, $c^* \in C^*$ and $\gamma \in \mathbb{R}_+$ such that

$$\begin{aligned}(0,0) = (s, (f(\cdot,\bar{u}))^*(s) + \eta) + (t, (-\bar{r}g(\cdot,\bar{v}))^*(t) + \mu)& \\ + \sum_{i=1}^{m}(z_i, (\bar{\lambda}_i h_i(\cdot,\bar{w}_i))^*(z_i) + \rho_i) + (c^*, \gamma).&\end{aligned}$$

So, $0 = s + t + \sum_{i=1}^{m} z_i + c^*$ and $0 = (f(\cdot,\bar{u}))^*(s) + \eta + (-\bar{r}g(\cdot,\bar{v}))^*(t) + \mu + \sum_{i=1}^{m}((\bar{\lambda}_i h_i(\cdot,\bar{w}_i))^*(z_i) + \rho_i) + \gamma$. Hence, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} & -\langle \sum_{i=1}^{m} z_i, x\rangle - \langle c^*, x\rangle - f(x,\bar{u}) - (-\bar{r}g(x,\bar{v})) \\ &= \langle s, x\rangle + \langle t, x\rangle - f(x,\bar{u}) - (-\bar{r}g(x,\bar{v})) \\ &\leqq (f(\cdot,\bar{u}))^*(s) + (-\bar{r}g(\cdot,v))^*(t) \\ &= -\eta - \mu - \sum_{i=1}^{m}((\bar{\lambda}_i h_i(\cdot,\bar{w}_i))^*(z_i) + \rho_i) - \gamma. \end{aligned} \tag{2.9}$$

Since $\eta \geqq 0$, $\mu \geqq 0$, $\rho_i \geqq 0$, $i = 1, \ldots, m$, and $c^* \in C^*$, from (2.9), for any

$x \in C$,

$$
\begin{aligned}
0 &\leqq \langle \sum_{i=1}^{m} z_i, x\rangle + \langle c^*, x\rangle + f(x,\bar{u}) + (-\bar{r}g(x,\bar{v})) - \eta - \mu \\
&\quad -\sum_{i=1}^{m}(\bar{\lambda}_i h_i(\cdot,\bar{w}_1))^*(z_i) - \sum_{i=1}^{m}\bar{\lambda}_i\rho_i - \gamma \\
&\leqq \langle \sum_{i=1}^{m} z_i, x\rangle + f(x,\bar{u}) - \bar{r}g(x,\bar{v}) - \sum_{i=1}^{m}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))^*(z_i) \\
&\leqq f(x,\bar{u}) - \bar{r}g(x,\bar{v}) + \sum_{i=1}^{m}(\bar{\lambda}_i h_i(x,\bar{w}_i)).
\end{aligned}
$$

[(ii) $\Rightarrow$ (i)] Suppose that there exist $\bar{u} \in \mathcal{U}$, $\bar{v} \in \mathcal{V}$, $\bar{w}_i \in \mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, such that for any $x \in C$,

$$
f(x,\bar{u}) - \bar{r}g(x,\bar{v}) + \sum_{i=1}^{m}\bar{\lambda}_i h_i(x,\bar{w}_i) \geqq 0. \tag{2.10}
$$

Since $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$, we have $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = 0$. So, from (2.10), we have for any $x \in A$,

$$
\begin{aligned}
\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) &\geqq \max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) + \sum_{i=1}^{m}\bar{\lambda}_i h_i(x,\bar{w}_i) \\
&\geqq f(x,\bar{u}) - \bar{r}g(x,\bar{v})) + \sum_{i=1}^{m}\bar{\lambda}_i h_i(x,\bar{w}_i) \\
&\geqq 0 \\
&= \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v).
\end{aligned}
$$

Hence, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v)$. It means that $\bar{x}$ is an $\bar{\epsilon}$-approximate solution of $(\mathrm{RNCP})_{\bar{r}}$. Thus, by Lemma 2.4.1, $\bar{x}$ is an approximate solution of (RFP). $\square$

Using Remark 2.4.2 and Lemma 2.4.1 and Lemma 2.4.2, we can obtain the following characterization of approximate solution for (RFP).

Theorem 2.4.2. (Approximate Optimality Theorem) *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be functions such that for any $u \in \mathbb{R}^p$, $f(\cdot,u)$ and for each $w_i \in \mathbb{R}^q$, $h_i(\cdot,w_i)$, $i = 1,\ldots,m$, are convex functions. Let $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be a function such that for any $v \in \mathbb{R}^p$, $g(\cdot,v)$ is a concave function. Let $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1,\ldots,m$, be convex and compact and let $A := \{x \in C \mid h_i(x,w_i) \leqq 0,\ \forall w_i \in \mathcal{W}_i,\ i = 1,\ldots,m\} \neq \emptyset$. Let $\bar{x} \in A$ and let $\epsilon \geqq 0$. Let $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$. If $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} < \epsilon$, then $\bar{x}$ is an approximate solution of* (RFP). *If $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} \geqq \epsilon$ and $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^* \times \mathbb{R}_+$ is closed and convex, then the following statements are equivalent:*

(i) *$\bar{x}$ is an approximate solution of* (RFP)*;*

(ii) *There exist $\bar{w}_i \in \mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1,\ldots,m$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$ and $\epsilon_i \geqq 0$, $i = 1,\ldots,m+1$ such that*

$$0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})$$

$$+N_C^{\epsilon_{m+1}}(\bar{x}), \tag{2.11}$$

$$\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) \quad \text{and} \tag{2.12}$$

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v) = \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x},\bar{w}_i). \tag{2.13}$$

Proof. [(i) $\Rightarrow$ (ii)] We assume that $\bar{x}$ is an approximate solution of (RFP). Then, by Lemma 2.4.1, $\bar{x}$ is an $\bar{\epsilon}$-approximate solution of $(\text{RNCP})_{\bar{r}}$, where $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x},v)$, that is, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v)$. Since $\max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) - \epsilon\min_{v\in\mathcal{V}} g(\bar{x},v) = 0$, we have $A \subseteq \{x \in C \mid \max_{u\in\mathcal{U}} f(x,u) - \bar{r}\min_{v\in\mathcal{V}} g(x,v) \geqq 0\}$. By Lemma 2.4.2,

$$(0,0) \in \operatorname{epi}(\max_{u\in\mathcal{U}} f(\cdot,u))^* + \operatorname{epi}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))^* + \operatorname{cl}\operatorname{co}\Big(\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+\Big).$$

By assumption,

$$(0,0) \in \operatorname{epi}(\max_{u\in\mathcal{U}} f(\cdot,u))^* + \operatorname{epi}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))^* + \bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \operatorname{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+.$$

Notice that $C^* \times \mathbb{R}_+ = \text{epi}\delta_C^*$. So, there exist $\bar{w}_i \in \mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, m$, such that

$$(0,0) \in \text{epi}(\max_{u\in\mathcal{U}} f(\cdot,u))^* + \text{epi}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))^* + \text{epi}(\sum_{i=1}^m \bar{\lambda}_i h_i(\cdot,\bar{w}_i))^* + \text{epi}\delta_C^*.$$

By Proposition 2.1.2, we obtain that

$$\begin{aligned}(0,0) \in & \bigcup_{\epsilon_0^1 \geqq 0} \{(\xi_0^1, \langle \xi_0^1, \bar{x}\rangle + \epsilon_0^1 - \max_{u\in\mathcal{U}} f(\bar{x},u)) \mid \xi_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x})\} \\ & + \bigcup_{\epsilon_0^2 \geqq 0} \{(\xi_0^2, \langle \xi_0^2, \bar{x}\rangle + \epsilon_0^2 + \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v)) \mid \xi_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x})\} \\ & + \bigcup_{\epsilon^* \geqq 0} \{(\xi^*, \langle \xi^*, \bar{x}\rangle + \epsilon^* - \sum_{i=1}^m \bar{\lambda}_i h_i(\bar{x},\bar{w}_i)) \mid \xi^* \in \partial_{\epsilon^*}(\sum_{i=1}^m \bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})\} \\ & + \bigcup_{\epsilon_{m+1} \geqq 0} \{(\xi_{m+1}, \langle \xi_{m+1}, \bar{x}\rangle + \epsilon_{m+1} - \delta_C(\bar{x})) \mid \xi_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})\}.\end{aligned}$$

So, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x})$, $\bar{\xi}^* \in \partial_{\epsilon^*}(\sum_{i=1}^m \bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon^* \geqq 0$ and $\epsilon_{m+1} \geqq 0$ such that $0 = \bar{\xi}_0^1 + \bar{\xi}_0^2 + \bar{\xi}^* + \bar{\xi}_{m+1}$ and $\epsilon_0^1 + \epsilon_0^2 + \epsilon^* + \epsilon_{m+1} = \max_{u\in\mathcal{U}} f(\bar{x},u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x},v) + \sum_{i=1}^m \bar{\lambda}_i h_i(\bar{x},\bar{w}_i)$. Hence, by Proposition 2.4.1, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x})$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$, and $\epsilon_{m+1} \geqq 0$ such that

$$0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot,v))(\bar{x}) + \sum_{i=1}^m \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x})$$

and

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i = \max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i). \qquad (2.14)$$

Hence, (2.11) holds. Moreover, since $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$,

$$\max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) - \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v) = 0. \qquad (2.15)$$

So, (2.12) holds, and so, from (2.14) and (2.15), we have

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v) = \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i).$$

Thus, (2.13) holds.

[(ii) ⇒ (i)] Taking into account the converse of the process for proving (i) ⇒ (ii), we can easily check that the statement (ii) ⇒ (i) holds. □

If for all $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave, and for all $x \in \mathbb{R}$, $g(x, \cdot)$ is convex, then using Lemma 2.4.1 and Lemma 2.4.2, we can obtain the following characterization of approximate solution for (RFP).

Theorem 2.4.3. (Approximate Optimality Theorem) *Let $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $h_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be functions such that for any $u \in \mathbb{R}^p$, $f(\cdot, u)$ and for each $w_i \in \mathbb{R}^q$, $h_i(\cdot, w_i)$ are convex functions, and for all $x \in \mathbb{R}^n$, $f(x, \cdot)$ is concave function. Let $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be a function such that for any $v \in \mathbb{R}^p$, $g(\cdot, v)$ is concave, and for all $x \in \mathbb{R}^n$, $g(x, \cdot)$ is convex. Let $\mathcal{U} \subset \mathbb{R}^p$, $\mathcal{V} \subset \mathbb{R}^p$ and $\mathcal{W}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$, be convex and* *compact and let* $A := \{x \in C \mid h_i(x, w_i) \leqq 0,\ \forall w_i \in \mathcal{W}_i,\ i = 1, \dots, m\} \neq \emptyset$*. Let* $\bar{x} \in A$ *and let* $\epsilon \geqq 0$*. Let* $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$*. If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} < \epsilon$*, then* $\bar{x}$ *is an approximate solution of* (RFP)*. If* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} \geqq \epsilon$ *and* $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}(\sum_{i=1}^{m} \lambda_i h_i(\cdot, w_i))^* + C^* \times \mathbb{R}_+$ *is closed and convex, then the following statements are equivalent:*

(i) $\bar{x}$ *is an approximate solution of* (RFP)*;*

(ii) *There exist* $\bar{u} \in \mathcal{U}$*,* $\bar{v} \in \mathcal{V}$*,* $\bar{w}_i \in \mathcal{W}_i$*,* $\bar{\lambda}_i \geqq 0$*,* $i = 1, \dots, m$*,* $\epsilon_0^1 \geqq 0$*,* $\epsilon_0^2 \geqq 0$ *and* $\epsilon_i \geqq 0$*,* $i = 1, \dots, m+1$*, such that*

$$0 \in \partial_{\epsilon_0^1}(f(\cdot, \bar{u}))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}g(\cdot, \bar{v}))(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot, \bar{w}_i))(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x}), \tag{2.16}$$

$$\max_{u\in\mathcal{U}} f(\bar{x}, u) - \min_{v\in\mathcal{V}} \bar{r}g(\bar{x}, v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v) \text{ and} \tag{2.17}$$

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v) \leqq \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i). \tag{2.18}$$

Proof. [(i) $\Rightarrow$ (ii)] Let $\bar{x}$ be an approximate solution of (RFP). Then, by Lemma 2.4.1, $\bar{x}$ is an $\bar{\epsilon}$-approximate solution of $(\text{RNCP})_{\bar{r}}$, where $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$ and $\bar{\epsilon} = \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v)$, that is, for any $x \in A$, $\max_{u\in\mathcal{U}} f(x, u) - \bar{r} \min_{v\in\mathcal{V}} g(x, v) \geqq \max_{u\in\mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v\in\mathcal{V}} g(\bar{x}, v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v)$. Since $\max_{u\in\mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v\in\mathcal{V}} g(\bar{x}, v) - \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v) = 0$, we have $A \subseteq \{x \in C \mid \max_{u\in\mathcal{U}} f(x, u) - \bar{r} \min_{v\in\mathcal{V}} g(x, v) \geqq 0\}$. By

Lemma 2.4.2 and the assumption, $(0,0) \in \bigcup\limits_{u\in\mathcal{U}} \text{epi}(f(\cdot,u))^* + \bigcup\limits_{v\in\mathcal{V}} \text{epi}(-\bar{r}g(\cdot,v))^* +$ $\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \text{epi}(\sum\limits_{i=1}^{m} \lambda_i h_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+$. Since $C^*\times\mathbb{R}_+ = \text{epi}\delta_C^*$, there exist $\bar{u}\in\mathcal{U}$, $\bar{v}\in\mathcal{V}$, $\bar{w}_i\in\mathcal{W}_i$ and $\bar{\lambda}_i \geqq 0$, $i=1,\ldots,m$, such that

$$(0,0) \in \text{epi}(f(\cdot,\bar{u}))^* + \text{epi}(-\bar{r}g(\cdot,\bar{v}))^* + \text{epi}(\sum_{i=1}^{m} \bar{\lambda}_i h_i(\cdot,\bar{w}_i))^* + \text{epi}\delta_C^*.$$

By Proposition 2.1.2, we obtain that

$$\begin{aligned}(0,0) \in & \bigcup_{\epsilon_0^1\geqq 0} \{(\xi_0^1, \langle \xi_0^1,\bar{x}\rangle + \epsilon_0^1 - f(\bar{x},\bar{u})) \mid \xi_0^1 \in \partial_{\epsilon_0^1}(f(\cdot,\bar{u}))(\bar{x})\} \\ & + \bigcup_{\epsilon_0^2\geqq 0} \{(\xi_0^2, \langle \xi_0^2,\bar{x}\rangle + \epsilon_0^2 + \bar{r}g(\bar{x},\bar{v})) \mid \xi_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}g(\cdot,\bar{v}))(\bar{x})\} \\ & + \bigcup_{\epsilon^*\geqq 0} \{(\xi^*, \langle \xi^*,\bar{x}\rangle + \epsilon^* - \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x},\bar{w}_i)) \mid \xi^* \in \partial_{\epsilon^*}(\sum_{i=1}^{m} \bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})\} \\ & + \bigcup_{\epsilon_{m+1}\geqq 0} \{(\xi_{m+1}, \langle \xi_{m+1},\bar{x}\rangle + \epsilon_{m+1} - \delta_C(\bar{x})) \mid \xi_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})\}.\end{aligned}$$

So, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(f(\cdot,\bar{u}))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}g(\cdot,\bar{v}))(\bar{x})$, $\bar{\xi}^* \in \partial_{\epsilon^*}(\sum\limits_{i=1}^{m} \bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon^* \geqq 0$ and $\epsilon_{m+1} \geqq 0$ such that $0 = \bar{\xi}_0^1 + \bar{\xi}_0^2 + \bar{\xi}^* + \bar{\xi}_{m+1}$ and $\epsilon_0^1 + \epsilon_0^2 + \epsilon^* + \epsilon_{m+1} = f(\bar{x},\bar{u}) - \bar{r}g(\bar{x},\bar{v}) + \sum\limits_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x},\bar{w}_i)$. By Proposition 2.4.1, there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(f(\cdot,\bar{u}))(\bar{x})$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-\bar{r}g(\cdot,\bar{v}))(\bar{x})$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot,\bar{w}_i))(\bar{x})$, $\bar{\xi}_{m+1} \in \partial_{\epsilon_{m+1}}\delta_C(\bar{x})$, $\epsilon_0^1 \geqq 0$,

$\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$, and $\epsilon_{m+1} \geqq 0$ such that

$$0 \in \partial_{\epsilon_0^1}(f(\cdot, \bar{u}))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}g(\cdot, \bar{v}))(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot, \bar{w}_i))(\bar{x}) + N_C^{\epsilon_{m+1}}(\bar{x})$$

$$\text{and} \quad \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i = f(\bar{x}, \bar{u}) - \bar{r}g(\bar{x}, \bar{v}) + \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i). \tag{2.19}$$

Hence, (2.16) holds. Since $\bar{r} = \max\limits_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon$, we have $\max\limits_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min\limits_{v \in \mathcal{V}} g(\bar{x}, v) = \epsilon \min\limits_{v \in \mathcal{V}} g(\bar{x}, v)$. So, we have

$$f(\bar{x}, \bar{u}) - \bar{r}g(\bar{x}, \bar{v}) \leqq \max_{u \in \mathcal{U}} f(\bar{x}, u) - \bar{r} \min_{v \in \mathcal{V}} g(\bar{x}, v) = \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v). \tag{2.20}$$

Hence, (2.17) holds. Also, from (2.19) and (2.20), we have

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v \in \mathcal{V}} g(\bar{x}, v) \leqq \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i).$$

Consequently, (2.16) and (2.18) hold.

[(ii) $\Rightarrow$ (i)] Taking into account the converse of the process for proving (i) $\Rightarrow$ (ii), we can easily check that the statement (ii) $\Rightarrow$ (i) holds. $\square$

Remark 2.4.3. *Assume that $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and $g : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ are functions such that for all $x \in \mathbb{R}^n$, $f(x, \cdot)$ and $g(x, \cdot)$ are concave and convex, respectively. Then, we know that Theorem 2.4.2 is equivalent to Theorem 2.4.3 from Lemma 2.4.2, immediately.*

Now. following the approach in [27], we formulate a dual problem (RFD) for (RFP) as follows:

$$
\begin{aligned}
\text{(RFD)}\quad & \max \quad r \\
& \text{s.t.} \quad 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(x) + \partial_{\epsilon_0^2}(-r\min_{v\in\mathcal{V}} g(\cdot,v))(x) \\
& \qquad + \sum_{i=1}^{m} \partial_{\epsilon_i}(\lambda_i h_i(\cdot,w_i))(x) + N_C^{\epsilon_{m+1}}(x), \\
& \qquad \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq \epsilon \min_{v\in\mathcal{V}} g(x,v), \\
& \qquad \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(x,v) \leqq \sum_{i=1}^{m}\lambda_i h_i(x,w_i), \\
& \qquad r \geqq 0,\ w_i \in \mathcal{W}_i,\ \lambda_i \geqq 0,\ i=1,\ldots,m, \\
& \qquad \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_i \geqq 0,\ i=1,\ldots,m+1.
\end{aligned}
$$

Clearly, $F := \{(x,w,\lambda,r) \mid 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(x) + \partial_{\epsilon_0^2}(-r\min_{v\in\mathcal{V}} g(\cdot,v))(x) + \sum_{i=1}^{m}\partial_{\epsilon_i}(\lambda_i h_i(\cdot,w_i))\ (x) + N_{\mathbb{R}_+}^{\epsilon_2}(x),\ \max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) \geqq \epsilon g(x,v),\ \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1}\epsilon_i - \epsilon\min_{v\in\mathcal{V}} g(x,v) \leqq \sum_{i=1}^{m}\lambda_i h_i(x,w_i),\ r \geqq 0,\ w_i \in \mathcal{W}_i,\ \lambda_i \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_i \geqq 0,\ i=1,\ldots,m,\ \epsilon_{m+1} \geqq 0\}$ is the feasible set of (RFD).

Let $\epsilon \geqq 0$. Then $(\bar{x},\bar{w},\bar{\lambda},\bar{r})$ is called an approximate solution of (RFD) if for any $(y,w,\lambda,r) \in F$, $\bar{r} \geqq r - \epsilon$.

When $\epsilon = 0$, $\max_{u\in\mathcal{U}} f(x,u) = f(x)$, $\min_{v\in\mathcal{V}} g(x,v) = g(x)$ and $h_i(x,w_i) = h_i(x)$, $i = 1,\ldots,m$, (RFP) becomes (FP), and (RFD) collapses to the Mondwier type dual problem (FD) for (FP) as follows [56]:

$$\begin{aligned}
\text{(FD)} \quad & \max \quad r \\
& \text{s.t.} \quad 0 \in \partial f(x) + \partial(-rg)(x) + \sum_{i=1}^{m} \partial \lambda_i h_i(x) + N_C(x), \\
& \qquad\quad f(x) - rg(x) \geqq 0, \ \lambda_i h_i(x) \geqq 0, \\
& \qquad\quad r \geqq 0, \ \lambda_i \geqq 0, \ i = 1, \ldots, m.
\end{aligned}$$

Now, we prove an approximate weak duality theorem and an approximate strong duality theorem which hold between (RFP) and (RFD).

Theorem 2.4.4. (Approximate Weak Duality Theorem) *For any feasible solution x of* (RFP) *and any feasible solution (y, w, λ, r) of* (RFD),

$$\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq r - \epsilon.$$

Proof. Let x and (y, w, λ, r) be feasible for (RFP) and (RFD), respectively. Then there exist $\bar{\xi}_0^1 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot, u))(y)$, $\bar{\xi}_0^2 \in \partial_{\epsilon_0^2}(-r \min_{v\in\mathcal{V}} g(\cdot, v))(y)$, $\bar{\xi}_i \in \partial_{\epsilon_i}(\lambda_i h_i(\cdot, w_i)(y)$, $\bar{\xi}_{m+1} \in N_C^{\epsilon_{m+1}}(y)$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$ and $\epsilon_{m+1} \geqq 0$ such that

$$\bar{\xi}_0^1 + \bar{\xi}_0^2 + \sum_{i=1}^{m+1} \bar{\xi}_i = 0, \quad \max_{u\in\mathcal{U}} f(y,u) - r \min_{v\in\mathcal{V}} g(y,v) \geqq \epsilon \min_{v\in\mathcal{V}} g(y,v)$$

$$\text{and} \quad \epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(y,v) \leqq \sum_{i=1}^{m} \lambda_i h_i(y, w_i).$$

Thus, we have

$$
\begin{aligned}
&\max_{u\in\mathcal{U}} f(x,u) - r\min_{v\in\mathcal{V}} g(x,v) + \epsilon\min_{v\in\mathcal{V}} g(x,v)\\
\geqq&\max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) + \langle \bar{\xi}_0^1+\bar{\xi}_0^2, x-y\rangle - \epsilon_0^1 - \epsilon_0^2 + \epsilon\min_{v\in\mathcal{V}} g(x,v)\\
=&\max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) - \langle \sum_{i=1}^{m+1}\bar{\xi}_i, x-y\rangle - \epsilon_0^1 - \epsilon_0^2 + \epsilon\min_{v\in\mathcal{V}} g(x,v)\\
\geqq&\max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) + \sum_{i=1}^{m}\lambda_i h_i(y,w_i) - \sum_{i=1}^{m}\lambda_i h_i(x,w_i) - \epsilon_0^1 - \epsilon_0^2\\
&- \sum_{i=1}^{m+1}\epsilon_i + \epsilon\min_{v\in\mathcal{V}} g(x,v)\\
\geqq&\max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) + \sum_{i=1}^{m}\lambda_i h_i(y,w_i) - \epsilon_0^1 - \epsilon_0^2 - \sum_{i=1}^{m+1}\epsilon_i\\
\geqq&\max_{u\in\mathcal{U}} f(y,u) - r\min_{v\in\mathcal{V}} g(y,v) - \epsilon\min_{v\in\mathcal{V}} g(y,v)\\
\geqq&0
\end{aligned}
$$

Hence, we have $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq r-\epsilon$. □

Theorem 2.4.5. (Approximate Strong Duality Theorem) *Suppose that*

$$\bigcup_{w_i\in\mathcal{W}_i,\lambda_i\geqq 0} \mathrm{epi}(\sum_{i=1}^{m}\lambda_i g_i(\cdot,w_i))^* + C^*\times\mathbb{R}_+$$

is closed. If $\bar{x}$ *is an approximate solution of* (RFP) *and* $\max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)} - \epsilon \geqq 0$, *then there exist* $\bar{w} \in \mathbb{R}^q$, $\bar{\lambda} \in \mathbb{R}^m_+$ *and* $\bar{r} \in \mathbb{R}_+$ *such that* $(\bar{x}, \bar{w}, \bar{\lambda}, \bar{r})$ *is a* 2-*approximate solution of* (RFD).

Proof. Let $\bar{x} \in A$ be an approximate solution of (RFP). Let $\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x},u)}{g(\bar{x},v)}$.

Then, by Theorem 2.4.2, there exist $\bar{w}_i \in \mathcal{W}_i$, $\bar{\lambda}_i \geqq 0$, $\epsilon_0^1 \geqq 0$, $\epsilon_0^2 \geqq 0$, $\epsilon_i \geqq 0$, $i = 1, \ldots, m$ and ϵ_{m+1} such that

$$0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot, u))(\bar{x}) + \partial_{\epsilon_0^2}(-\bar{r}\min_{v\in\mathcal{V}} g(\cdot, v))(\bar{x}) + \sum_{i=1}^{m} \partial_{\epsilon_i}(\bar{\lambda}_i h_i(\cdot, \bar{w}_i))(\bar{x})$$

$$+ N_C^{\epsilon_{m+1}}(\bar{x}),$$

$$\max_{u\in\mathcal{U}} f(\bar{x}, u) - \bar{r}\min_{v\in\mathcal{V}} g(\bar{x}, v) = \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v) \quad \text{and}$$

$$\epsilon_0^1 + \epsilon_0^2 + \sum_{i=1}^{m+1} \epsilon_i - \epsilon \min_{v\in\mathcal{V}} g(\bar{x}, v) = \sum_{i=1}^{m} \bar{\lambda}_i h_i(\bar{x}, \bar{w}_i).$$

So, $(\bar{x}, \bar{w}, \bar{\lambda}, \bar{r})$ is a feasible solution of (RFD). For any feasible (y, u, v, w, λ, v) of (RFD), it follows from Theorem 2.4.4 (Approximate Weak Duality Theorem) that

$$\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x}, u)}{g(\bar{x}, v)} - \epsilon \geqq r - \epsilon - \epsilon = r - 2\epsilon.$$

Thus $(\bar{x}, \bar{w}, \bar{\lambda}, \bar{r})$ is a 2-approximate solution of (RFD). □

Remark 2.4.4. *Using the optimality conditions of Theorem 2.4.2, robust fractional dual problem* (RFD) *for a robust fractional problem* (RFP) *in the convex constraint functions with uncertainty is formulated. However, when we formulated the dual problem using optimality condition in Theorem 2.4.3,* *we could not know whether approximate weak duality theorem is established, or not. It is our open question.*

Now we give an example illustrating our duality theorems.

Example 2.4.1. Consider the following fractional optimization problem with uncertainty:

$$\text{(RFP)} \quad \min \quad \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{ux+1}{vx+2}$$

$$\text{s.t.} \quad 2w_1x - 3 \leqq 0,\ w_1 \in [1,2],\ x \in \mathbb{R}_+,$$

where $\mathcal{U} = [1,2]$ and $\mathcal{V} = [1,2]$.

Now we transform the problem (RFP) into the robust non-fractional convex optimization problem (RNCP)_r with a parameter $r \in \mathbb{R}_+$:

$$\text{(RNCP)}_r \quad \min \quad \max_{u\in[1,2]} (ux+1) - r \min_{v\in[1,2]} (vx+2)$$

$$\text{s.t.} \quad 2w_1x - 3 \leqq 0,\ w_1 \in [1,2],\ x \in \mathbb{R}_+.$$

Let $f(x,u) = ux+1$, $g(x,v) = vx+2$, $h_1(x,w_1) = -2w_1x-3$ and $\epsilon \in [0, \frac{9}{22}]$. Then $A := \{x \in \mathbb{R} \mid 0 \leqq x \leqq \frac{3}{4}\}$ is the set of all robust feasible solutions of (RFP) and $\bar{A} := \{x \in \mathbb{R} \mid 0 \leqq x \leqq \frac{4\epsilon}{3-2\epsilon}\}$ is the set of all approximate solutions of (RFP). Let $F := \{(y, w_1, \lambda_1, r) \mid 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot,u))(y) + \partial_{\epsilon_0^2}(-r \min_{v\in\mathcal{V}} g(\cdot,v))(y) + \partial_{\epsilon_1}(\lambda_1 h_1(\cdot,w_1))(y) + N_{\mathbb{R}_+}^{\epsilon_2}(x),\ \max_{u\in\mathcal{U}} f(y,u) - r \min_{v\in\mathcal{V}} g(y,v) \geqq \epsilon \min_{v\in\mathcal{V}} g(y,v),\ \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - \epsilon \min_{v\in\mathcal{V}} g(y,v) \leqq \lambda_1 h_1(y,w_1),\ r \geqq 0,\ w_1 \in$

$[1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\}$. Then we formulate a dual problem (RFD) for (RFP) as follows:

$$
\begin{array}{llc}
\text{(RFD)} & \max & r \\
 & \text{s.t.} & (y, w_1, \lambda_1, r) \in F.
\end{array}
$$

Then F is the set of all robust feasible solutions of (RFD). Now we calculate the set $F = \widetilde{A} \cup \widetilde{B}$, where

$$
\begin{aligned}
\widetilde{A} \ :=\ & \{(0, w_1, \lambda_1, r) \mid 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot, u))(0) + \partial_{\epsilon_0^2}(-r \min_{v\in\mathcal{V}} g(\cdot, v))(0) + \\
& \partial_{\epsilon_1}(\lambda_1 h_1(\cdot, w_1))(0) + N_{\mathbb{R}_+}^{\epsilon_2}(0),\ \max_{u\in\mathcal{U}} f(0, u) - r \min_{v\in\mathcal{V}} g(0, v) \geqq \\
& \epsilon \min_{v\in\mathcal{V}} g(0, v),\ \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - \epsilon \min_{v\in\mathcal{V}} g(0, v) \leqq \lambda_1 h_1(0, w_1),\ r \geqq 0, \\
& u \in [1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\} \\
=\ & \{(0, w_1, \lambda_1, r) \mid 0 \in \{2\} + \{-r\} + \{2\lambda_1 w_1\} + (-\infty, 0],\ 1 - 2r \geqq 2\epsilon, \\
& \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - 2\epsilon \leqq -3\lambda_1,\ r \geqq 0,\ w_1 \in [1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0, \\
& \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\} \\
=\ & \{(0, w_1, \lambda_1, r) \mid r \leqq 2 + 2\lambda_1 w_1,\ r \leqq \frac{1 - 2\epsilon}{2},\ \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - 2\epsilon \leqq \\
& -3\lambda_1,\ r \geqq 0,\ w_1 \in [1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\}, \\
\widetilde{B} \ :=\ & \{(y, w_1, \lambda_1, r) \mid\ 0 \in \partial_{\epsilon_0^1}(\max_{u\in\mathcal{U}} f(\cdot, u))(y) + \partial_{\epsilon_0^2}(-r \min_{v\in\mathcal{V}} g(\cdot, v))(y) + \\
& \partial_{\epsilon_1}(\lambda_1 h_1(\cdot, w_1))(y) + N_{\mathbb{R}_+}^{\epsilon_2}(y),\ \max_{u\in\mathcal{U}} f(y, u) - r \min_{v\in\mathcal{V}} g(y, v) \geqq \\
& \epsilon \min_{v\in\mathcal{V}} g(y, v),\ \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - \epsilon \min_{v\in\mathcal{V}} g(y, v) \leqq \lambda_1 h_1(y, w_1),\ y > 0, \\
& r \geqq 0,\ w_1 \in [1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\}
\end{aligned}
$$

$$\begin{aligned}
&= \{(y, w_1, \lambda_1, r) \mid 0 \in \{2 - r + 2\lambda_1 w_1\} + [-\frac{\epsilon_2}{y}, 0],\ 2y + 1 - r(y+2) \geqq \\
&\quad \epsilon(y+2),\ y > 0,\ \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - \epsilon(y+2) \leqq \lambda_1(2w_1 y - 3),\ r \geqq 0, \\
&\quad w_1 \in [1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\} \\
&= \{(y, w_1, \lambda_1, r) \mid 0 \in [2 - r + 2\lambda_1 w_1 - \frac{\epsilon_2}{y}, 2 - r + 2\lambda_1 w_1],\ 2y + 1 - \\
&\quad r(y+2) \geqq \epsilon(y+2),\ \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 - \epsilon(y+2) \leqq \lambda_1(2w_1 y - 3), \\
&\quad y > 0,\ r \geqq 0,\ w_1 \in [1,2],\ \lambda_1 \geqq 0,\ \epsilon_0^1 \geqq 0,\ \epsilon_0^2 \geqq 0,\ \epsilon_1 \geqq 0,\ \epsilon_2 \geqq 0\}.
\end{aligned}$$

We can check for any $x \in A$ and any $(y, w_1, \lambda_1, r) \in F$,

$$\max_{(u,v) \in \mathcal{U} \times \mathcal{V}} \frac{f(x,u)}{g(x,v)} \geqq r - \epsilon,$$

that is, Theorem 2.4.4 (Approximate Weak Duality Theorem) holds. Indeed, let $x \in A$ and $(y, w_1, \lambda_1, r) \in \widetilde{A}$ be any fixed. Then

$$\begin{aligned}
&\max_{u \in [1,2]} f(x,u) - r \min_{v \in [1,2]} g(x,v) + \epsilon \min_{v \in [1,2]} g(x,v) \\
&= 2x + 1 - r(x+2) + \epsilon(x+2) \\
&= (2 - r)x + 1 - 2r + \epsilon(x+2) \\
&\geqq -2\lambda_1 w_1 x + 2\epsilon + \epsilon(x+2) \\
&\geqq -3\lambda_1 + 2\epsilon + \epsilon(x+2) \\
&\geqq -3\lambda_1 + \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 + 3\lambda_1 + \epsilon(x+2) \\
&\geqq 0.
\end{aligned}$$

Moreover, let $x \in A$ and $(y, u, , v, w_1, \lambda_1, r) \in \widetilde{B}$ be any fixed.

$$\begin{aligned}
& \max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
= \;& 2x + 1 - r(x+2) + \epsilon(x+2) \\
= \;& 2y + 1 - r(y+2) + (2-r)(x-y) + \epsilon(x+2).
\end{aligned}$$

If $x - y \geqq 0$, then

$$\begin{aligned}
& \max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
= \;& 2y + 1 - r(y+2) + (2-r)(x-y) + \epsilon(x+2) \\
\geqq \;& 2y + 1 - r(y+2) - 2\lambda_1 w_1 (x-y) + \epsilon(x+2) \\
\geqq \;& \epsilon(y+2) + 2\lambda_1 w_1 y - 3\lambda_1 + \epsilon(x+2) \\
\geqq \;& \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 + 3\lambda_1 - 3\lambda_1 + \epsilon(x+2) \\
\geqq \;& 0.
\end{aligned}$$

If $x - y < 0$, then

$$\begin{aligned}
& \max_{u\in[1,2]} f(x,u) - r \min_{v\in[1,2]} g(x,v) + \epsilon \min_{v\in[1,2]} g(x,v) \\
=\;& 2y + 1 - r(y+2) + (2-r)(x-y) + \epsilon(x+2) \\
\geqq\;& 2y + 1 - r(y+2) + (-2\lambda_1 w_1 + \tfrac{\epsilon_2}{y})(x-y) + \epsilon(x+2) \\
\geqq\;& \epsilon(y+2) + 2\lambda_1 w_1 y - \epsilon_2 - 3\lambda_1 + \tfrac{\epsilon_2}{y} x + \epsilon(x+2) \\
\geqq\;& \epsilon_0^1 + \epsilon_0^2 + \epsilon_1 + \epsilon_2 + 3\lambda_1 - \epsilon_2 - 3\lambda_1 + \tfrac{\epsilon_2}{y} x + \epsilon(x+2) \\
\geqq\;& 0.
\end{aligned}$$

Let $\epsilon = \frac{1}{3}$. Then $\bar{x} \in \bar{A} := \{x \in \mathbb{R} \mid 0 \leqq x \leqq \frac{4}{7}\}$ is the set of all approximate solutions of (RFP) and $\frac{1}{6} \leqq \bar{r} \leqq \frac{1}{2}$.

If $\bar{x} = 0$, then $\bar{r} = \frac{1}{6}$. When $\epsilon = \frac{1}{3}$, we can calculate the set $\tilde{A}$ as follows:

$$\tilde{A} := \{(0, w_1, \lambda_1, r) \mid 0 \leqq r \leqq \frac{1}{6},\ 0 \leqq \lambda_1 \leqq \frac{2}{9},\ w_1 \in [1, 2]\}.$$

Let $\bar{w}_1 = 2$, $\bar{\lambda}_1 = \frac{1}{9}$. Then, $(0, 2, \frac{5}{8}, \frac{1}{9}) \in \tilde{A}$. So, we have

$$\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x}, u)}{g(\bar{x}, v)} - \epsilon = \frac{1}{6} = \frac{5}{6} - 2\epsilon \geqq r - 2\epsilon.$$

Hence, $(0, 2, \frac{1}{9}, \frac{1}{6})$ is a 2-approximate solution of (RFD). If $0 < \bar{x} \leqq \frac{4}{7}$, then, $\frac{1}{6} < \bar{r} \leqq \frac{1}{2}$. When $\epsilon = \frac{1}{3}$, we can calculate the set $\tilde{B}$ as follows:

$$\widetilde{B} := \{(y, w_1, \lambda_1, r) \mid y > 0,\ 2 + 2\lambda_1 w_1 - \frac{\epsilon_2}{y} \leqq r \leqq \frac{5y + 1}{3(y + 2)},\ \epsilon_2 - \frac{1}{3}(y + 2) \leqq \lambda_1(2w_1 y - 3),\ r \geqq 0,\ u \in [1, 2],\ v \in [1, 2],\ w_1 \in [1, 2],\ \epsilon_2 \geqq 0\}.$$

Let $\bar{w}_1 = 2$, $\bar{\lambda}_1 = 0$ and $\epsilon_2 = \frac{\bar{x}+2}{3}$. Then, $(\bar{x}, 2, 0, \frac{5\bar{x}+1}{3(\bar{x}+2)}) \in \tilde{B}$. So, we have

$$\bar{r} = \max_{(u,v)\in\mathcal{U}\times\mathcal{V}} \frac{f(\bar{x}, u)}{g(\bar{x}, v)} - \epsilon = \frac{5\bar{x} + 1}{3(\bar{x} + 2)} \geqq r \geqq r - 2\epsilon.$$

Hence, $(\bar{x}, 2, 0, \frac{5\bar{x}+1}{3(\bar{x}+2)})$ is a 2-approximate solution of (RFD) So, Theorem 2.4.5 (Approximate Strong Duality Theorem) holds.

Chapter 3
Surrogate Duality for Robust Semi-infinite Optimization Problems

3.1 Introduction

In this chapter, a semi-infinite optimization problem involving a quasi-convex objective function and infinitely many convex constraint functions with data uncertainty are considered. A surrogate duality theorem for the semi-infinite optimization problem is given under a closed and convex cone constraint qualification. Moreover, we extend the surrogate duality theorem for the semi-infinite optimization problem to fractional semi-infinite optimization problem with data uncertainty. Also, we induce characterizations of the robust moment cone of Goberna et al. [22] by our results. Using a closed and convex cone constraint qualification, we present surrogate duality theorems for robust linear semi-infinite optimization problems.

Consider the following semi-infinite optimization problem in the absence of data uncertainty

$$\begin{aligned} \text{(SIP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_t(x) \leqq 0, \ \forall t \in T, \\ & x \in C, \end{aligned}$$

where $f, g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are functions, T is an index set with coordinately possible infinite and C is a closed convex cone of $\mathbb{R}^n$.

The semi-infinite optimization problem (SIP) in the face of data uncertainty in the constraints can be captured by the problem

$$\begin{aligned} \text{(USIP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_t(x, v_t) \leqq 0, \ \forall t \in T, \\ & x \in C, \end{aligned}$$

where $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_t(\cdot, v_t)$ is convex for all $t \in T$ and $u_t \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{U}_t \subset \mathbb{R}^q$, $t \in T$. The robust counterpart of (USIP):

$$\begin{aligned} \text{(RSIP)} \quad \min \quad & f(x) \\ \text{s.t.} \quad & g_t(x, v_t) \leqq 0, \ \forall v_t \in \mathcal{V}_t, \ \forall t \in T, \\ & x \in C. \end{aligned}$$

The robust feasible set F of (RSIP) is defined by

$$F := \{x \in C \mid g_t(x, v_t) \leqq 0, \ \forall v_t \in \mathcal{V}_t, \ t \in T\}.$$

The uncertainty set-valued mapping $\mathcal{V} : T \rightrightarrows \mathbb{R}^q$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$. We represent an element of an uncertainty set $\mathcal{V}_t$ by $v_t \in \mathcal{V}_t$ and $v \in \mathcal{V}$ means that v is a *selection* of $\mathcal{V}$, i.e., $v : T \to \mathbb{R}^q$ and $v_t \in \mathcal{V}_t$ for all $t \in T$ (v is denoted by $(v_t)_{t \in T}$). $\mathbb{R}_+^{(T)}$ denotes the set of mapping $\lambda : T \to \mathbb{R}_+$ (also denoted by $(\lambda_t)_{t \in T}$ such that $\lambda_t = 0$ except for finitely many indices).

The surrogate dual of (USIP) is given by

$$\text{(USD)} \quad \max_{\lambda \in \mathbb{R}^{(T)}_+} \inf\{f(x) \mid \sum_{t \in T} \lambda_t g_t(x, v_t) \leqq 0,\ x \in C\}.$$

x is optimistic feasible solution of (USD) if and only if for every $t \in T$, $\sum_{t \in T} g_t(x, v_t) \leqq 0$ for some $v_t \in \mathcal{V}_t$ and $\lambda_t \geq 0$ [4]. The optimistic counterpart of the uncertain surrogate dual (USD) over the set of optimistic feasible solutions is as follows:

$$\text{(OSD)} \quad \max_{\mathcal{V} \times \mathbb{R}^{(T)}_+} \inf\{f(x) \mid \sum_{t \in T} \lambda_t g_t(x, v_t) \leqq 0,\ x \in C\}.$$

By Lemma 2.2.1, we can obtain the following lemma which is the robust version of Farkas Lemma for convex functions:

Lemma 3.1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, be continuous functions such that for each $v_t \in \mathbb{R}^q$, $g_t(\cdot, v_t)$ is a convex function. Let C be a closed convex cone of $\mathbb{R}^n$. Let $\mathcal{V}_t \subseteq \mathbb{R}^q$, $t \in T$, and let $F := \{x \in C \mid g_t(x, v_t) \leqq 0,\ \forall v_t \in \mathcal{V}_t,\ t \in T\} \neq \emptyset$. Then the following statements are equivalent:*

(i) $\{x \in C \mid g_t(x, v_t) \leqq 0,\ \forall v_t \in \mathcal{V}_t,\ t \in T\} \subseteq \{x \in \mathbb{R}^n \mid f(x) \geqq 0\}$;

(ii) $(0, 0) \in \operatorname{epi} f^* + \operatorname{cl} \operatorname{co} \Big(\bigcup_{(v,\lambda) \in \mathcal{V} \times \mathbb{R}^{(T)}_+} \operatorname{epi}(\sum_{t \in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+ \Big)$.

Proof. We can easily prove this lemma in a similar way to the proof of Lemma 2.2.1. □

3.2 Surrogate Duality Theorem

In this section, we investigate a surrogate duality theorem for a semi-infinite optimization problem with a quasiconvex objective function and convex constraint functions with data uncertainty, that is, the value of the robust counterpart (RSIP) is equal to the value of the optimistic counterpart (OSD) ("primal worst equals dual best") in the sense that

$$\inf\{f(x) \mid x \in F\} = \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{f(x) \mid \sum_{t\in T}\lambda_t g_t(x,v_t) \leqq 0,\ x\in C\}.$$

under the robust characteristic cone constraint qualification that

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \text{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^* + C^* + \mathbb{R}_+$$

is closed and convex.

Now, we establish the surrogate duality theorem for the semi-infinite optimization problem with data uncertainty:

Theorem 3.2.1. *Let $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ be an upper semicontinuous quasiconvex function with $\text{dom} f\cap F\neq\emptyset$, and let $g_t:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$ be continuous functions such that for each $t\in T$ and $v_t\in\mathcal{V}_t$, $g_t(\cdot,v_t)$ is a convex function. Assume that the robust characteristic cone,*

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \text{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^* + C^* + \mathbb{R}_+$$

is closed and convex. Then

$$\inf\{f(x) \mid x \in F\} = \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{f(x) \mid \sum_{t\in T}\lambda_t g_t(x,v_t) \leqq 0,\ x\in C\}.$$

Proof. Suppose that the assumption holds. Let $m = \inf_{x \in F} f(x)$. If $m = -\infty$, then the conclusion holds trivially. So, assume that m is finite. If $L(f, <, m)$ is empty, then putting $\lambda = 0$ and taking any $v \in \mathcal{V}$, the equality holds. Suppose that $L(f, <, m)$ is not empty. Then $L(f, <, m) \cap F = \emptyset$, $L(f, <, m)$ is a nonempty open convex set, and F is closed and convex. So, by separation theorem, there exist a nonzero $x^* \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, such that for all $x \in F$ and $y \in L(f, <, m)$,

$$\langle x^*, x \rangle \leqq \alpha < \langle x^*, y \rangle . \tag{3.1}$$

Since $\langle x^*, x \rangle \leqq \alpha$ for any $x \in F$, $(x^*, \alpha) \in \mathrm{epi}\delta_F^*$. Let $A := \{x \in \mathbb{R}^n : g_t(x, v_t) \leqq 0, \ \forall t \in T, \ \forall v_t \in \mathcal{V}_t\}$. Then $F = A \cap C$. So, for any $x \in \mathbb{R}^n$,

$$\delta_F(x) = \delta_A(x) + \delta_C(x) \text{ and } \delta_A(x) = \sup_{\substack{v \in \mathcal{V} \\ \lambda \in \mathbb{R}_+^{(T)}}} \sum_{t \in T} \lambda_t g_t(\cdot, v_t).$$

By Lemma 2.1.3 and Lemma 2.1.4,

$$\begin{aligned} \mathrm{epi}\delta_F^* = \mathrm{epi}(\delta_A + \delta_C)^* &= \mathrm{epi}\delta_A^* + \mathrm{epi}\delta_C^* \\ &= \mathrm{epi}(\sup_{\substack{v \in \mathcal{V} \\ \lambda \in \mathbb{R}_+^{(T)}}} \sum_{t \in T} \lambda_t g_t(\cdot, v_t))^* + \mathrm{epi}\delta_C^* \\ &= \mathrm{cl}\,\mathrm{co}(\bigcup_{(v,\lambda) \in \mathcal{V} \times \mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t \in T} \lambda_t g_t(\cdot, v_t))^*) + C^* \times \mathbb{R}_+. \end{aligned}$$

So, by assumption,

$$(x^*, \alpha) \in \mathrm{epi}\delta_F^* = \bigcup_{(v,\lambda) \in \mathcal{V} \times \mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t \in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+.$$

Hence, there exist $\bar{\lambda} \in \mathbb{R}_+^{(T)}$, $\bar{v} \in \mathcal{V}$, $c^* \in C^*$ and $r \in \mathbb{R}_+$ such that

$$(x^*, \alpha) \in \text{epi}(\sum_{t \in T} \bar{\lambda}_t g_t(\cdot, \bar{v}_t))^* + (c^*, r).$$

So, $\sum_{t\in T} \bar{\lambda}_t g_t(\cdot, \bar{v}_t)^*(x^* - c^*) \leqq \alpha - r$, and hence $\langle x^* - c^*, x\rangle - \sum_{t\in T} \bar{\lambda}_t g_t(x, \bar{v}_t) \leqq \alpha - r$ for any C, that is, $\langle x^*, x\rangle - \sum_{t\in T} \bar{\lambda}_t g_t(x, \bar{v}_t) \leqq \alpha - r + \langle c^*, x\rangle \leqq \alpha$ for any C. Hence, for any $x \in F_{(\bar{v}, \bar{\lambda})}$, $\langle x^*, x\rangle \leqq \alpha$. Thus, from (3.1), for any $x \in F_{(\bar{v}, \bar{\lambda})}$, $x \notin L(f, <, m)$. So, for any $x \in F_{(\bar{v}, \bar{\lambda})}$, $f(x) \geqq m$, that is, $\inf\{f(x) \mid \sum_{t\in T} \bar{\lambda}_t g_t(x, \bar{v}_t) \leqq 0,\ x \in C\} \geqq m$. Since $\inf\{f(x) \mid \sum_{t\in T} \bar{\lambda}_t g_t(x, v_t) \leqq 0,\ x \in C\} \geqq \inf\{f(x) \mid x \in F\}$, we have

$$\inf\{f(x) \mid \sum_{t\in T} \bar{\lambda}_t g_t(x, \bar{v}_t) \leqq 0,\ x \in C\} = m.$$

□

Now we give an example illustrating Theorem 3.2.1.

Example 3.2.1. Consider the following semi-infinite optimization problem with uncertainty:

$$\begin{aligned} (\text{RSIP})_1 \quad & \min \quad x_1^3 + x_2^5 \\ & \text{s.t.} \quad x_1^2 - 2v_t x_1 \leqq 0,\ \forall v_t \in [t-2, t+2],\ \forall t \in [0,1], \\ & \qquad\ (x_1, x_2) \in \mathbb{R}_+^2. \end{aligned}$$

Let $f(x_1, x_2) = x_1^3 + x_2^2$ and $g_t((x_1, x_2), v_t) = x_1^2 - 2v_t x_1$. Then f is quasiconvex function on $\mathbb{R}^2$ and $F := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leqq x_1 \leqq 1,\ x_2 \geqq 0\}$ is the set of all robust feasible solutions of $(\text{RSIP})_1$. Also, we see that, for each $v_t \in \mathcal{V}_t = [t-2, t+2]$, $t \in [0,1]$,

$$g_t(\cdot, v_t)^*(a_1, a_2) = \begin{cases} \frac{(a_1+2v_t)^2}{4}, & a_2 = 0, \\ +\infty, & a_2 \neq 0. \end{cases}$$

So,

$$\begin{aligned}
& \bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \text{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* \\
= & \bigcup_{v_t\in\mathcal{V}_t,\lambda_t>0} \text{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* \cup \{0\} \times \{0\} \times \mathbb{R}_+ \\
= & \bigcup_{v_t\in\mathcal{V}_t,\lambda_t>0} \sum_{t\in T} \lambda_t \text{epi}(g_t(\cdot, v_t))^* \cup \{0\} \times \{0\} \times \mathbb{R}_+ \\
= & \bigcup_{v_t\in\mathcal{V}_t,\lambda_t>0} \sum_{t\in T} \lambda_t \{(a_1, 0, r_t) \mid r \geqq \tfrac{(a_1+2v_t)^2}{4}\} \cup \{0\} \times \{0\} \times \mathbb{R}_+ \\
= & \mathbb{R} \times \{0\} \times \mathbb{R}_+.
\end{aligned}$$

Hence, the cone, $\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \text{epi}(\lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+ = \mathbb{R} \times (-\mathbb{R}_+) \times \mathbb{R}_+$, is closed and convex. Moreover, let $\bar{\lambda}_0 = \bar{\lambda}_1 = 1$ and $\bar{\lambda}_t = 0$ for all $t \in (0,1)$ and $v_t \in [t-2, t+2]$, then

$$\begin{aligned}
0 &= \inf\{f(x_1, x_2) \mid (x_1, x_2) \in F\} \\
&= \inf\{f(x_1, x_2) \mid \sum_{t\in T} \lambda_t g_t(x_1, x_2, v_t) \leqq 0,\ (x_1, x_2) \in \mathbb{R}_+\}.
\end{aligned}$$

Thus, Theorem 3.2.1 holds.

We now show that if for each $x \in \mathbb{R}^n$, $g_t(x, \cdot)$ is concave on $\mathcal{V}_t$ and $\mathcal{V}_t \subseteq \mathbb{R}^q$, $t \in T$, is convex, then the robust characteristic cone is a convex cone.

Proposition 3.2.1. (cf. [*44*]) *Let $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $t \in T$, be continuous functions and let C be a closed convex cone. Suppose that each $\mathcal{V}_t \subseteq \mathbb{R}^q$, $t \in T$, is convex, for all $v_t \in \mathbb{R}^q$, $g_t(\cdot, v_t)$ is a convex function, and for each $x \in \mathbb{R}^n$, $g_t(x, \cdot)$ is concave on $\mathcal{V}_t$. Then, the cone,*

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+,$$

is convex.

Proof. Let $\Lambda := \bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+$. Let $(y_1, \alpha_1) \in \Lambda$, $(y_2, \alpha_2) \in \Lambda$, and $\mu \in [0, 1]$. We will show that $(\mu y_1 + (1-\mu)y_2, \mu\alpha_1 + (1-\mu)\alpha_2) \in \Lambda$. Since Λ is a cone, $\mu(y_1, \alpha_1) \in \Lambda$ and $(1-\mu)(y_2, \alpha_2) \in \Lambda$. So, there exist $v_t^1 \in \mathcal{V}_t$ for all $t \in T$, $(\lambda_t^1)_{t\in T} \in \mathbb{R}_+^{(T)}$, $c_1^* \in C^*$ and $r_1 \in \mathbb{R}_+$ such that

$$\mu(y_1, \alpha_1) \in \operatorname{epi}(\sum_{t\in T} \lambda_t^1 g_t(\cdot, v_t^1))^* + (c_1^*, r_1).$$

Similarly, there exist $v_t^2 \in \mathcal{V}_t$ for all $t \in T$, $(\lambda_t^2)_{t\in T} \in \mathbb{R}_+^{(T)}$, $c_2^* \in C^*$ and $r_2 \in \mathbb{R}_+$ such that

$$(1-\mu)(y_2, \alpha_2) \in \operatorname{epi}(\sum_{t\in T} \lambda_t^2 g_t(\cdot, v_t^2))^* + (c_2^*, r_2).$$

Hence, we have

$$
\begin{aligned}
& (\sum_{t\in T}\lambda_t^1 g_t(\cdot, v_t^1))^*(\mu y_1 - c_1^*) + (\sum_{t\in T}\lambda_t^2 g_t(\cdot, v_t^2))^*((1-\mu)y_2 - c_2^*) \\
\leqq \;& \mu\alpha_1 + (1-\mu)\alpha_2 - r_1 - r_2.
\end{aligned}
$$

Let for each $t \in T$, $\lambda_t = \lambda_t^1 + \lambda_t^2$ and

$$
v_t := \begin{cases} v_t^1, & \text{if } \lambda_t = 0, \\ \frac{\lambda_t^1}{\lambda_t}v_t^1 + \frac{\lambda_t^2}{\lambda_t}v_t^2, & \text{if } \lambda_t > 0. \end{cases}
$$

If $\lambda_t = 0$, then $\lambda_t^1 = \lambda_t^2 = 0$, and hence $\lambda_t^1 g_t(x, v_t^1) + \lambda_t^2 g_t(x, v_t^2) = \lambda_t g_t(x, v_t)$. Now we assume that $\lambda_t > 0$. Since $g_t(x, \cdot)$ is concave, we have

$$
\begin{aligned}
\lambda_t^1 g_t(x, v_t^1) + \lambda_t^2 g_t(x, v_t^2) &= \lambda_t(\frac{\lambda_t^1}{\lambda_t} g_t(x, v_t^1) + \frac{\lambda_t^2}{\lambda_t} g_t(x, v_t^2)) \\
&\leqq \lambda_t g_t(x, \frac{\lambda_t^1}{\lambda_t} v_t^1 + \frac{\lambda_t^2}{\lambda_t} v_t^2) \\
&= \lambda_t g_t(x, v_t).
\end{aligned}
$$

Thus for any $t \in T$, $\lambda_t^1 g_t(x, v_t^1) + \lambda_t^2 g_t(x, v_t^2) \leqq \lambda_t g_t(x, v_t)$. Moreover, we have

$$
\begin{aligned}
& \mu\alpha_1 + (1-\mu)\alpha_2 - r_1 - r_2 \\
\geqq \;& (\sum_{t\in T}\lambda_t^1 g_t(\cdot, v_t^1))^*(\mu y_1 - c_1^*) + (\sum_{t\in T}\lambda_t^2 g_t(\cdot, v_t^2))^*((1-\mu)y_2 - c_2^*) \\
= \;& \sup_{x\in\mathbb{R}^n}\{\langle \mu y_1 - c_1^*, x\rangle - \sum_{t\in T}\lambda_t^1 g_t(x, v_t^1)\} + \sup_{x\in\mathbb{R}^n}\{\langle (1-\mu)y_2 - c_2^*, x\rangle \\
& - \sum_{t\in T}\lambda_t^2 g_t(x, v_t^2)\}
\end{aligned}
$$

$$\geqq \sup_{x\in\mathbb{R}^n}\{\langle \mu y_1 + (1-\mu)y_2 - c_1^* - c_2^*, x\rangle - \sum_{t\in T}(\lambda_t^1 g_t(x, v_t^1) + \lambda_t^2 g_t(x, v_t^2))\}$$

$$\geqq \sup_{x\in\mathbb{R}^n}\{\langle \mu y_1 + (1-\mu)y_2 - c_1^* - c_2^*, x\rangle - \sum_{t\in T}\lambda_t g_t(x, v_t)\}$$

$$= (\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^*(\mu y_1 + (1-\mu)y_2 - c_1^* - c_2^*).$$

So, $(\mu y_1+(1-\mu)y_2-c_1^*-c_2^*, \mu\alpha_1+(1-\mu)\alpha_2-r_1-r_2) \in \mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^*$, and hence $(\mu y_1 + (1-\mu)y_2, \mu\alpha_1 + (1-\mu)\alpha_2) \in \mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^* + (c_1^* + c_2^*, r_1 + r_2)$. Since $(c_1^* + c_2^*, r_1 + r_2) \in C^* \times \mathbb{R}_+$, we see that $(\mu y_1 + (1-\mu)y_2, \mu\alpha_1 + (1-\mu)\alpha_2) \in \Lambda$. □

Let T be a compact metric space and $\mathcal{V} : T \rightrightarrows \mathbb{R}^q$ be a set-valued mapping. Let $g : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be a given function and let C be a closed convex cone of $\mathbb{R}^n$. Now, we will assume that the following conditions hold:

(C1) $g_{t^k}(\cdot, v_{t^k}) \to g_{t^k}(\cdot, v_t)$, when $x \in \mathbb{R}^n$, $t^k \to t \in T$ and $v_{t^k} \in \mathcal{V}_{t^k} \to v_t \in \mathcal{V}_t$ as $k \to \infty$.

(C2) (Slater condition) There exists $x_0 \in C$ such that

$$g_t(x_0, v_t) < 0,\ \forall v_t \in \mathcal{V}_t,\ t \in T.$$

Now we prove that the robust characteristic cone is closed under the conditons (C1) and (C2).

Proposition 3.2.2. *Let T be a compact metric space and let $\mathcal{V}$ be compact, convex and uniformly upper semicontinuous on T. Let $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$,*

$t \in T$, be continuous functions such that for all $v_t \in \mathbb{R}^q$, $g_t(\cdot, v_t)$ is a convex function and let C be a closed convex cone of $\mathbb{R}^n$. Suppose that the condition (C1) and the condition (C2) hold. Then

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+$$

is closed.

Proof. First we notice that

$$\begin{aligned}&\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+\\ =& \bigcup_{v\in\mathcal{V}} \mathrm{co\,cone}(\{\mathrm{epi}(g_t(\cdot, v_t))^* : t \in T\} \cup (0,1)) + C^* \times \mathbb{R}_+.\end{aligned}$$

Let $\Lambda := \bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+$ and let $(w^k, \alpha^k) \in \Lambda$ with $(w^k, \alpha^k) \to (x^*, \alpha) \in \mathbb{R}^n \times \mathbb{R}$. Now, we will show that $(x^*, \alpha) \in \Lambda$. Since $(w^k, \alpha^k) \in \Lambda$, for each $k \in \mathbb{N}$, there exist $v_t^k \in \mathcal{V}_t$, $t \in T$, $c^k \in C^*$ and $r^k \in \mathbb{R}_+$ such that $(w^k, \alpha^k) \in \mathrm{co\,cone}(\{\mathrm{epi}(g_t(\cdot, v_t^k))^* : t \in T\} \cup (0,1)) + C^* \times \mathbb{R}_+$. So, from Carathéodory theorem, for each $k \in \mathbb{N}$, there exist $v_{t_i^k} \in \mathcal{V}_{t_i^k}$, $t_i^k \in T$, $\lambda_i^k \geqq 0$, $i = 1, \ldots, n+1$, and $\lambda_0^k \geqq 0$ such that $(w^k, \alpha^k) \in \sum_{i=1}^{n+1} \lambda_i^k \mathrm{epi}(g_{t_i^k}(\cdot, v_{t_i^k}))^* + \lambda_0^k(0,1)$. Since T is compact, we may assume that $t_i^k \to t_i \in T$ as $t \to \infty$, $i = 1, \ldots, n+1$.

Fix $i = 1, \ldots, n+1$ and let $\epsilon > 0$ be any fixed. Since $\mathcal{V}$ is uniformly upper semicontinuous, there exist $\eta > 0$ such that $\mathcal{V}_t \subset \mathcal{V}_{t_i} + \epsilon\mathbb{B}$, for any $t \in T$ with $d(t, t_i) \leqq \eta$, where $\mathbb{B}$ is a unit ball in $\mathbb{R}^q$. Since $t_i^k \to t_i$ as $k \to \infty$, there exists

$k_i \in \mathbb{N}$ such that $k \geqq k_i$, $d(t_i^k, t_i) \leqq \eta$. So, for all $k \geqq k_i$, $\mathcal{V}_{t_i^k} \subset \mathcal{V}_{t_i} + \epsilon\mathbb{B}$. Since $v_{t_i^k} \in \mathcal{V}_{t_i^k}$, there exists $w_{t_i} \in \mathcal{V}_{t_i}$ such that $v_{t_i^k} \in w_{t_i} + \epsilon\mathbb{B}$, i.e., $\|v_{t_i^k} - w_{t_i}\| < \epsilon$. So, $\inf_{z_{t_i} \in \mathcal{V}_{t_i}} \|v_{t_i^k} - z_{t_i}\| < \epsilon$. It follows that there exists $k_i \in \mathbb{N}$ such that for all $k \geqq k_i$, $d(v_{t_i^k}, \mathcal{V}_{t_i}) \leqq \epsilon$. So, $d(v_{t_i^k}, \mathcal{V}_{t_i}) = 0$ as $k \to \infty$, i.e., $v_{t_i^k} \in \mathcal{V}_{t_i}$. Hence, there exists $z^*_{t_i^k} \in \mathcal{V}_{t_i}$, $k = 1, 2, \ldots$, such that $d(v_{t_i^k}, \mathcal{V}_{t_i}) = \|v_{t_i^k} - z^*_{t_i^k}\| \to 0$ as $k \to \infty$. Since $\mathcal{V}_{t_i}$ is compact, we may assume that there exists $v_{t_i} \in \mathcal{V}_{t_i}$ such that $z^*_{t_i^k} \to v_{t_i}$ as $k \to \infty$. Hence, we have

$$\begin{aligned}\lim_{k\to\infty} \|v_{t_i^k} - v_{t_i}\| &= \lim_{k\to\infty} \|(v_{t_i^k} - z^*_{t_i^k}) + (z^*_{t_i^k} - v_{t_i})\| \\ &\leqq \lim_{k\to\infty} \|v_{t_i^k} - z^*_{t_i^k}\| + \lim_{k\to\infty} \|z^*_{t_i^k} - v_{t_i}\| = 0.\end{aligned}$$

So, $v_{t_i^k} \to v_{t_i}$ as $k \to \infty$.

Now, we show that $l^k := \sum_{i=1}^{n+1} \lambda_i^k + \lambda_0^k$ is bounded. Otherwise, we may assume that $l^k \to +\infty$. By passing to subsequences, we may assume that $\frac{\lambda_i^k}{l^k} \to \delta_i \in \mathbb{R}_+$, $i = 1, \ldots, n+1$, $\frac{\lambda_0^k}{l^k} \to \delta_0 \in \mathbb{R}_+$ with $\sum_{i=1}^{n+1} \delta_i + \delta_0 = 1$. Then, for each $x \in \mathbb{R}^n$,

$$\begin{aligned}(w^k)^T x - \sum_{i=1}^{n+1} \lambda_i^k g_{t_i^k}(x, v_{t_i^k}) &\leqq (w^k - c^k)^T x - \sum_{i=1}^{n+1} \lambda_i^k g_{t_i^k}(x, v_{t_i^k}) \\ &\leqq (\sum_{i=1}^{n+1} \lambda_i^k g_{t_i^k}(\cdot, v_{t_i^k}))^*(w^k - c^k) \\ &\leqq \alpha^k - r^k - \lambda_0^k \\ &\leqq \alpha^k - \lambda_0^k.\end{aligned}$$

Dividing both sides of the last inequality by l^k and passing to the limit, we get that, for each $x \in C$, $\sum_{i=1}^{n+1} \delta_i g_{t_i}(x, v_{t_i}) \geqq \delta_0$. If $\delta_i = 0$, for all $i = 1, \ldots, n+1$, then we see that $0 = \sum_{i=1}^{n+1} \delta_i g_{t_i}(x, v_{t_i}) \geqq 1$. This is a contradiction. Also, if $\delta_i \neq 0$, for some i, then $\sum_{i=1}^{n+1} \delta_i g_{t_i}(x, v_{t_i}) \geqq 0$. This contradicts (C2) as $0 < \sum_{i=1}^{n+1} \delta_i \leqq 1$.

Now, as l^k is bounded, we may assume that $\lambda_i^k \to \lambda_i$ and $\lambda_0^k \to \lambda_0$. As, for each $x \in C$,

$$(w^k)^T x - \sum_{i=1}^{n+1} \lambda_i^k g_{t_i}^k(x, v_{t_i}^k) \leqq \alpha^k - \lambda_0^k,$$

it follows, by passing to the limit and noting that g_t is continuous, that, for each $x \in C$,

$$(x^*)^T x - \sum_{i=1}^{n+1} \lambda_i g_{t_i}(x, v_{t_i}) \leqq \alpha - \lambda_0.$$

Hence, for any $x \in \mathbb{R}^n$,

$$(x^*)^T x - \sum_{i=1}^{n+1} \lambda_i g_{t_i}(x, v_{t_i}) - \delta_C \leqq \alpha - \lambda_0,$$

and so $(\sum_{i=1}^{n+1} \lambda_i g_{t_i}(\cdot, v_{t_i}) + \delta_C)^*(x^*) \leqq \alpha - \lambda_0$. By Lemma 2.1.3, it follows that

$$(x^*, \alpha - \lambda_0) \in \operatorname{epi}(\sum_{i=1}^{n+1} \lambda_i g_{t_i}(\cdot, v_{t_i}) + \delta_C)^* = \operatorname{epi}(\sum_{i=1}^{n+1} \lambda_i g_{t_i}(\cdot, v_{t_i}))^* + C^* \times \mathbb{R}_+.$$

Hence, we have

$$(x^*, \alpha) \in \mathrm{epi}(\sum_{i=1}^{n+1} \lambda_i g_{t_i}(\cdot, v_{t_i}))^* + \lambda_0(0,1) + C^* \times \mathbb{R}_+$$

$$\subseteq \mathrm{epi}(\sum_{t \in T} \lambda_t g_t(\cdot, v_t))^* + \{0\} \times \mathbb{R}_+ + C^* \times \mathbb{R}_+$$

$$\subseteq \bigcup_{(v,\lambda) \in \mathcal{V} \times \mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t \in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+.$$

Thus, the cone $\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \mathrm{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+$ is closed. $\square$

We give an example illustrating Proposition 3.2.1 and Proposition 3.2.2.

Example 3.2.2. Consider the following semi-infinite optimization problem with uncertainty:

$$\begin{aligned} (\mathrm{RSIP})_2 \quad \min \quad & x_1^3 + x_2^5 \\ \text{s.t.} \quad & t x_1^2 - 2 v_t x_1 \leqq 0, \ \forall v_t \in [t, t+1], \ \forall t \in [1,2], \\ & (x_1, x_2) \in \mathbb{R}_+^2. \end{aligned}$$

Let $f(x_1, x_2) = x_1^3 + x_2^5$ and $g_t((x_1, x_2), v_t) = t x_1^2 - 2 v_t x_1$. Then f is quasi-convex function on $\mathbb{R}^2$, for each $x \in \mathbb{R}^2$, $g_t(x, \cdot)$ is concave on $\mathcal{V}_t$ and for all $v_t \in \mathbb{R}$, $g_t(\cdot, v_t)$ is convex on $\mathbb{R}^2$. $F := \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leqq x_1 \leqq 2, \ x_2 \geqq 0\}$ is the set of all robust feasible solutions of $(\mathrm{RSIP})_2$. Clearly, the Slater condition holds for $(\mathrm{RSIP})_2$. Also, we see that, for each $v_t \in \mathcal{V}_t = [t, t+1]$, $t \in [1,2]$,

$$g_t(\cdot, v_t)^*(a_1, a_2) = \begin{cases} \frac{(a_1 + 2v_t)^2}{4t}, & a_2 = 0, \\ +\infty, & a_2 \neq 0. \end{cases}$$

So, we have

$$
\begin{aligned}
&\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^* \\
=& \bigcup_{v_t\in\mathcal{V}_t,\lambda_t>0} \sum_{t\in T}\lambda_t \operatorname{epi}(g_t(\cdot,v_t))^* \cup \{0\}\times\{0\}\times\mathbb{R}_+ \\
=& \bigcup_{v_t\in[t,t+1],\lambda_t>0} \sum_{t\in[1,2]} \lambda_t\{(a_1,0,r_t) \mid r_t \geqq \frac{(a_1+2v_t)^2}{4t}\} \cup \{0\}\times\{0\}\times\mathbb{R}_+ \\
=& \{(a_1,0,\alpha) \mid \max\{0,2a\} \leqq \alpha\}.
\end{aligned}
$$

Hence, $\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\lambda_t g_t(\cdot,v_t))^* + C^*\times\mathbb{R}_+ = \{(a_1,a_2,\alpha) \mid \max\{0,2a_1\} \leqq \alpha,\ a_2 \leqq 0\}$ is closed and convex.

We obtain the surrogate duality theorem for the semi-infinite optimization problem under the Slater condition:

Corollary 3.2.1. *Let $g_t : \mathbb{R}^n\times\mathbb{R}^q \to \mathbb{R}$, $t\in T$, be continuous functions and let C be a closed convex cone. Suppose that each $\mathcal{V}_t \subseteq \mathbb{R}^q$, $t\in T$, is convex, for all $v_t\in\mathbb{R}^q$, $g_t(\cdot,v_t)$ is a convex function, for each $x\in\mathbb{R}^n$ and $g_t(x,\cdot)$ is concave on $\mathcal{V}_t$. If the condition (C1) and the condition (C2) hold, then*

$$\inf\{f(x) \mid x\in F\} = \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{f(x) \mid \sum_{t\in T}\lambda_t g_t(x,v_t) \leqq 0,\ x\in C\}.$$

Proof. By Proposition 3.2.1 and Proposition 3.2.2, we know that the robust characteristic cone is convex and closed. So, by Theorem 3.2.1, the theorem holds. □

Now, we consider the following standard form of fractional semi-infinite optimization problem (FSIP) with a geometric constraint set:

$$\begin{aligned}
\text{(FSIP)} \quad \min \quad & \frac{p(x)}{q(x)} \\
\text{s.t.} \quad & g_t(x) \leqq 0, \ t \in T, \\
& x \in C,
\end{aligned}$$

where $p, g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are convex functions, C is a closed convex cone of $\mathbb{R}^n$ and $q : \mathbb{R}^n \to \mathbb{R}$ is a linear function such that for any $x \in C$, $p(x) \geqq 0$ and $q(x) > 0$. The fractional semi-infinite optimization problem (FSIP) in the face of data uncertainty in the constraints can be captured by the problem:

$$\begin{aligned}
\text{(UFSIP)} \quad \min \quad & \frac{p(x)}{q(x)} \\
\text{s.t.} \quad & g_t(x, v_t) \leqq 0, \ t \in T, \\
& x \in C,
\end{aligned}$$

where $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_t(\cdot, v_t)$ is convex and $v_t \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_t \subset \mathbb{R}^q$, $t \in T$. The robust counterpart of (UFSIP) is

$$\begin{aligned}
\text{(RFSIP)} \quad \min \quad & \frac{p(x)}{q(x)} \\
\text{s.t.} \quad & g_t(x, v_t) \leqq 0, \ \forall v_t \in \mathcal{V}_t, \ t \in T, \\
& x \in C.
\end{aligned}$$

Theorem 3.2.2. *Let C be a closed convex cone of $\mathbb{R}^n$. Let $p : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}$ be convex and linear respectively such that for any $x \in C$, $p(x) \geqq 0$ and $q(x) > 0$, and let $g_t : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be continuous functions such that for each $t \in T$ and $v_t \in \mathcal{V}_t$, $g_t(\cdot, v_t)$ is a convex function. Assume that the robust characteristic cone,*

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^* + C^* + \mathbb{R}_+$$

is closed and convex. Then

$$\inf\{\frac{p(x)}{q(x)} \mid x \in F\} = \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{\frac{p(x)}{q(x)} \mid \sum_{t\in T}\lambda_t g_t(x, v_t) \leqq 0,\ x \in C\}.$$

Proof. By Result 3 in [48], $\frac{p(x)}{q(x)}$ is a quasiconvex function. Thus, by Theorem 3.2.1, the conclusion holds. □

3.3 Application to Robust Linear Semi-infinite Optimization Problem

In this section, by using the results in Section 3.2, we induce characterizations of the robust moment cone of Goberna et al. [22]. Moreover, we present surrogate duality theorems for linear semi-infinite optimization problem under a closed and convex cone constraint qualification.

Now we consider the following linear semi-infinite optimization problem in the absence of data uncertainty [22]:

$$\begin{aligned} \text{(LSIP)} \quad \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & \langle a_t, x\rangle \geqq b_t,\ \forall t \in T, \\ & x \in C, \end{aligned}$$

where $c, a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$, $t \in T$. The semi-infinite optimization problem in the face of data uncertainty in the linear constraints can be captured by the problem [22]

$$\begin{aligned} \text{(ULSIP)} \quad \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & \langle a_t, x\rangle \geqq b_t,\ \forall t \in T, \\ & x \in C, \end{aligned}$$

where a_t and b_t are uncertain parameters, and (a_t, b_t) belongs to the set $\mathcal{V}_t \subset \mathbb{R}^{n+1}$ for all $t \in T$.

Let $(a_t, b_t) \in \mathcal{V}_t$, for $t \in T$. The set-valued mapping $\mathcal{V} : T \rightrightarrows \mathbb{R}^{n+1}$ is defined as $\mathcal{V}(t) := \mathcal{V}_t$ for all $t \in T$ [22].

The robust counterpart of (ULSIP) [22] is

$$\begin{aligned} \text{(RLSIP)} \quad \min \quad & \langle c, x\rangle \\ \text{s.t.} \quad & \langle a_t, x\rangle \geqq b_t,\ \forall (a_t, b_t) \in \mathcal{V}_t,\ \forall t \in T, \\ & x \in C. \end{aligned}$$

Clearly, $F_1 := \{x \in C \mid \langle a_t, x\rangle \leqq b_t,\ \forall (a_t, b_t) \in \mathcal{V}_t,\ \forall t \in T\}$ is the feasible set of (RLSIP). Goberna et al. [22] defined the robust moment cone of (RLSIP) for $x \in \mathbb{R}^n$ as

$$\bigcup_{(a_t,b_t)_{t\in T}\in\mathcal{V}} \operatorname{co}\operatorname{cone}\{(a_t,b_t),\ t\in T : (0_n,1)\}. \tag{3.2}$$

For $x \in C$, (3.2) is transformed into

$$\bigcup_{(a_t,b_t)_{t\in T}\in\mathcal{V}} \operatorname{co}\operatorname{cone}\{(a_t,b_t),\ t\in T\} + C^* \times \mathbb{R}_+.$$

It can be induced by our robust characteristic cone of (RSIP) as follows:

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^* + C^* \times \mathbb{R}_+,$$

where $g_t(x,u_t) = \langle a_t, x\rangle - bt,\ v_t = (a_t,b_t) \in \mathcal{V}_t,\ t\in T,\ x\in C.$

Proposition 3.3.1. *Let $g_t : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$, $t \in T$, be continuous functions. Let $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$, $\mathcal{V}_t \subset \mathbb{R}^{n+1}$, $t\in T$, and let C be a closed convex cone. Then, for $x \in C$,*

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^* + C^* + \mathbb{R}_+$$

$$= \bigcup_{(a_t,b_t)\in\mathcal{V}_t} \operatorname{co}\operatorname{cone}\{(a_t,b_t) \mid t\in T\} + C^* \times \mathbb{R}_+.$$

Proof. Define $g_t : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$g_t(x,v_t) = \langle a_t, x\rangle - bt,\ v_t = (a_t,b_t) \in \mathcal{V}_t,\ t\in T.$$

Then we have

$$\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^* + C^* \times \mathbb{R}_+$$

$$= \bigcup_{\substack{(a_t,b_t)_{t\in T}\in\mathcal{V}\\ \lambda\in\mathbb{R}_+^{(T)}}} \operatorname{epi}(\sum_{t\in T}\lambda_t\langle a_t,\cdot\rangle - \sum_{t\in T}\lambda_t b_t)^* + C^* \times \mathbb{R}_+.$$

Let $(w,\alpha) \in \bigcup_{\substack{(a_t,b_t)_{t\in T}\in\mathcal{V}\\ \lambda\in\mathbb{R}_+^{(T)}}} \operatorname{epi}(\sum_{t\in T}\lambda_t\langle a_t,\cdot\rangle - \sum_{t\in T}\lambda_t b_t)^* + C^* \times \mathbb{R}_+$. It means that there exist $v_t = (a_t, b_t) \in \mathcal{V}_t$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that

$$(w,\alpha) \in \operatorname{epi}(\sum_{t\in T}\lambda_t\langle a_t,\cdot\rangle - \sum_{t\in T}\lambda_t b_t)^* + C^* \times \mathbb{R}_+,$$

that is, there exist $c^* \in C^*$ and $r \in \mathbb{R}_+$ such that

$$\langle w - c^*, x\rangle - \sum_{t\in T}\lambda_t\langle a_t, x\rangle + \sum_{t\in T}\lambda_t b_t \leqq \alpha - r,\ \forall x \in C$$

$$\Leftrightarrow\ \langle w - c^* - \sum_{t\in T}\lambda_t a_t, x\rangle + \sum_{t\in T}\lambda_t b_t \leqq \alpha - r,\ \forall x \in C$$

$$\Leftrightarrow\ w = \sum_{t\in T}\lambda_t a_t + c^* \ \text{ and }\ \sum_{t\in T}\lambda_t b_t + r \leqq \alpha$$

$$\Leftrightarrow\ w = \sum_{t\in T}\lambda_t a_t + c^* \ \text{ and }\ \alpha \in \sum_{t\in T}\lambda_t b_t + \mathbb{R}_+.$$

So, there exist $v_t = (a_t, b_t) \in \mathcal{V}_t$, $\lambda \in \mathbb{R}_+^{(T)}$ and $c^* \in C^*$ such that

$$\begin{aligned} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* \times \mathbb{R}_+ &= \operatorname{epi}(\sum_{t\in T} \lambda_t \langle a_t, \cdot\rangle - \sum_{t\in T} \lambda_t b_t)^* + C^* \times \mathbb{R}_+ \\ &= \{\sum_{t\in T} \lambda_t a_t + c^*\} \times \{\sum_{t\in T} \lambda_t b_t + \mathbb{R}_+\} \\ &= (\sum_{t\in T} \lambda_t a_t, \sum_{t\in T} \lambda_t b_t) + \{c^*\} \times \mathbb{R}_+ \\ &= \operatorname{co}\operatorname{cone}\{(a_t, b_t) \mid t \in T\} + \{c^*\} \times \mathbb{R}_+ \\ &\subseteq \operatorname{co}\operatorname{cone}\{(a_t, b_t) \mid t \in T\} + C^* \times \mathbb{R}_+. \end{aligned}$$

Hence,

$$\begin{aligned} &\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* + \mathbb{R}_+ \\ \subseteq &\bigcup_{(a_t,b_t)\in\mathcal{V}_t} \operatorname{co}\operatorname{cone}\{(a_t, b_t) \mid t \in T\} + C^* \times \mathbb{R}_+. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} &\bigcup_{(a_t,b_t)\in\mathcal{V}_t} \operatorname{co}\operatorname{cone}\{(a_t, b_t) \mid t \in T\} + C^* \times \mathbb{R}_+ \\ \subseteq &\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* + \mathbb{R}_+. \end{aligned}$$

Thus, we obtain the desired result. □

From Theorem 3.2.1, we obtain the following theorem:

Theorem 3.3.1. *Let $c, a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, $t \in T$, and C be a closed convex cone. Assume that the robust moment cone,*

$$\bigcup_{(a_t,b_t)_{t\in T}\in\mathcal{V}} \operatorname{co}\operatorname{cone}\{(a_t, b_t) : t \in T\} + C^* \times \mathbb{R}_+,$$

is closed and convex. Then

$$\inf\{\langle c, x\rangle \mid \langle a_t, x\rangle - b_t \leqq 0, \forall v_t = (a_t, b_t) \in \mathcal{V}_t, \forall t \in T, \ x \in C\}$$

$$= \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{\langle c, x\rangle \mid \sum_{t\in T} \lambda_t(\langle a_t, x\rangle - b_t) \leqq 0, \ x \in C\}.$$

Proof. Let $f(x) := \langle c, x\rangle$, $x \in C$. Define $g_t : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$g_t(x, v_t) = \langle a_t, x\rangle - bt, \ v_t = (a_t, b_t) \in \mathcal{V}_t, \ t \in T..$$

Then, by Proposition 3.3.1,

$$\bigcup_{(a_t,b_t)\in\mathcal{V}_t} \operatorname{cocone}\{(a_t, b_t) \mid t \in T\} + C^* \times \mathbb{R}_+$$

$$= \bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^* + C^* + \mathbb{R}_+.$$

By assumption, $\bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T} \lambda_t g_t(\cdot, v_t))^*$ is closed and convex, by Theorem 3.2.1,

$$\inf\{f(x) \mid g_t(x, v_t) \leqq 0, \forall v_t \in \mathcal{V}_t, \forall t \in T, \ x \in C\}$$

$$= \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{f(x) \mid \sum_{t\in T} \lambda_t g_t(x, v_t) \leqq 0, \ x \in C\}.$$

Thus, we see that

$$\inf\{\langle c,x\rangle \mid \langle a_t,x\rangle - b_t \leqq 0,\ \forall v_t=(a_t,b_t)\in\mathcal{V}_t,\ \forall t\in T,\ x\in C\}$$

$$= \max_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \inf\{\langle c,x\rangle \mid \sum_{t\in T}\lambda_t(\langle a_t,x\rangle - b_t)\leqq 0,\ x\in C\}.$$

□

Proposition 3.3.2. (cf. [22]) *Let $c, a_t\in\mathbb{R}^n$, $b_t\in\mathbb{R}$, $t\in T$, and C be a closed convex cone. Then*

$$\bigcup_{(a_t,b_t)_{t\in T}\in\mathcal{V}} \operatorname{co}\operatorname{cone}\{(a_t,b_t)\mid t\in T\}+C^*\times\mathbb{R}_+$$

is convex.

Proof. Define $g_t:\mathbb{R}^n\times\mathbb{R}^{n+1}\to\mathbb{R}$ by

$$g_t(x,v_t)=\langle a_t,x\rangle - b_t,\ v_t=(a_t,b_t)\in\mathcal{V}_t,\ t\in T.$$

Let $v_t=(a_t,b_t)\in\mathcal{V}_t$, for $t\in T$. Then $g_t:\mathbb{R}^n\times\mathbb{R}^{n+1}\to\mathbb{R}$, $t\in T$, are continuous, and $\mathcal{V}_t$ is convex. For all $v_t\in\mathbb{R}^{n+1}$, $g_t(\cdot,v_t)$ is convex, and for each $x\in\mathbb{R}^n$, $g_t(x,\cdot)$ is affine on $\mathcal{V}_t$. Since, by Proposition 3.3.1,

$$\bigcup_{(a_t,b_t)\in\mathcal{V}_t} \operatorname{cocone}\{(a_t,b_t)\mid t\in T\}+C^*\times\mathbb{R}_+$$

$$= \bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \operatorname{epi}(\sum_{t\in T}\lambda_t g_t(\cdot,v_t))^*+C^*+\mathbb{R}_+,$$

from Proposition 3.2.1,

$$\bigcup_{(a_t,b_t)\in\mathcal{V}_t} \text{cocone}\{(a_t,b_t) \mid t\in T\} + C^* \times \mathbb{R}_+$$

is convex. $\square$

Proposition 3.3.3. (cf. [*22*]) *Let T be a compact metric space and let $\mathcal{V}$ be compact-valued and uniformly upper semicontinuous on T. Let $c, a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, $t \in T$, and C be a closed convex cone. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\langle a_t, x_0\rangle < b_t$ for all $(a_t,b_t)\in\mathcal{V}_t$, $t\in T$. Then*

$$\bigcup_{(a_t,b_t)_{t\in T}\in\mathcal{V}} \text{co}\,\text{cone}\{(a_t,b_t) : t\in T\} + C^* \times \mathbb{R}_+$$

is closed.

Proof. Define $g_t : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$g_t(x, v_t) = \langle a_t, x\rangle - b_t,\ v_t = (a_t,b_t)\in\mathcal{V}_t,\ t\in T.$$

Let $v_t = (a_t,b_t) \in \mathcal{V}_t$, for $t\in T$. Then $g_t : \mathbb{R}^n\times\mathbb{R}^{n+1}\to\mathbb{R}$, $t\in T$ are continuous functions such that for all $v_t\in\mathbb{R}^{n+1}$, $g_t(\cdot, v_t)$ is a convex function. By Proposition 3.3.1,

$$\bigcup_{(a_t,b_t)\in\mathcal{V}_t} \text{cocone}\{(a_t,b_t) \mid t\in T\} + C^*\times\mathbb{R}_+$$

$$= \bigcup_{(v,\lambda)\in\mathcal{V}\times\mathbb{R}_+^{(T)}} \text{epi}(\sum_{t\in T}\lambda_t g_t(\cdot, v_t))^* + C^* + \mathbb{R}_+,$$

and by assumption, there exists $x_0 \in \mathbb{R}^n$ such that $g_(x, v_t) = \langle a_t, x_0 \rangle - b_t < 0$ for all $v_t = (a_t, b_t) \in \mathcal{V}_t$, $t \in T$. So, from Proposition 3.2.2,

$$\bigcup_{(a_t, b_t) \in \mathcal{V}_t} \text{cocone}\{(a_t, b_t) \mid t \in T\} + C^* \times \mathbb{R}_+$$

is closed. □

From Proposition 3.3.2 and 3.3.3, we obtain the following theorem:

Theorem 3.3.2. *Let T be a compact metric space and let $\mathcal{V}$ be compact-valued and uniformly upper semicontinuous on T. Let $c, a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, $t \in T$, and C be a closed convex cone. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\langle a_t, x_0 \rangle < b_t$ for all $(a_t, b_t) \in \mathcal{V}_t$, $t \in T$. Then*

$$\begin{aligned} & \inf\{\langle c, x \rangle \mid \langle a_t, x \rangle - b_t \leqq 0, \forall v_t = (a_t, b_t) \in \mathcal{V}_t, \forall t \in T, \ x \in C\} \\ = & \max_{(v,\lambda) \in \mathcal{V} \times \mathbb{R}_+^{(T)}} \inf\{\langle c, x \rangle \mid \sum_{t \in T} \lambda_t(\langle a_t, x \rangle - b_t) \leqq 0, \ x \in C\}. \end{aligned}$$

Chapter 4
Solving Robust SOS-convex Polynomial Optimization Problems with a SOS-concave Matrix Polynomial Constraint

4.1 Introduction

In this chapter, the tractable containments of a convex semi-algebraic set, defined by a SOS-concave matrix polynomial constraint, in a non-convex semi-algebraic set, defined by difference between a SOS-convex and a support function, are considered. Moreover, using our set containment characterizations, we derive a zero duality gap result for a robust SOS-convex polynomial problem (RP), where the dual problem $(\mathrm{D})^{\mathrm{sos}}$ can be represented by a sum of squares relaxation problem and other dual problem (SDP) and its dual problem (SDD) can be represented by a semidefinite program and which can be easily solved by interior-point methods. Also, we present the relations of the optimal solution of (RP) and the optimal solution of (SDD), and the optimal solution of $(\mathrm{D})^{\mathrm{sos}}$ and (SDP). Finally, we illustrate our results through a simple numerical example.

Now we give some definitions and preliminary results which will be used in this chapter. A semi-algebraic subset of $\mathbb{R}^n$ is a set of $\{x_i \mid i = 1, \ldots, n\}$ in $\mathbb{R}^n$ satisfying a Boolean combination of polynomial equations and inequalities with real coefficients [15]. We say that a real polynomial f is a sum of squares if there exist real polynomials f_j, $j = 1, \dots, r$, such that $f = \sum_{j=1}^{r} f_j^2$. The set consisting of all sums of squares of real polynomials is denoted by Σ^2. Moreover, the set consisting of all sum of squares of real polynomials with degree at most d is denoted by Σ_d^2. For a multi-index $\alpha \in \mathbb{N}^n$, let $|\alpha| := \sum_{i=1}^{n} \alpha_i$. x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Consider the vector $v_d(x) = (x^\alpha)_{|\alpha| \leqq d} = (1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_{n-1}x_n, x_n^2, \dots, x_1^d, \dots, x_n^d)^T$, of all the monomials x^α of degree less than or equal to d, which has a dimension $s(d) := \begin{pmatrix} n+d \\ n \end{pmatrix}$. An $n \times n$ symmetric matrix X is said to be a positive semidefinite (psd) matrix if for all $v \in \mathbb{R}^n$, $v^T X v \geq 0$. Similarly, an $n \times n$ symmetric matrix X is called a positive definite (pd) matrix if for all non-zero $v \in \mathbb{R}^n$, $v^T X v > 0$. Let S^n be a set of $n \times n$ symmetric matrices and let S_+^n be a set of $n \times n$ positive semidefinite symmetric matrices. Similarly, S_{++}^n denotes the set of positive definite $n \times n$ symmetric matrices. For $X, Y \in S^n$, $X \succeq Y$(resp. $X \succ Y$) if and only if $X - Y$ is positive semidefinite (resp. positive definite).

We now introduce a definition of SOS-convex polynomials.

Definition 4.1.1. [1, 2, 30] *A real polynomial f on $\mathbb{R}^n$ is called SOS-convex if a Hessian matrix function $H : x \mapsto \nabla_{xx}^2 f(x)$ is a SOS matrix polynomial, that is, there exists a matrix polynomial $F(x)$ such that $\nabla_{xx}^2 f(x) = F(x)F(x)^T$, equivalently, for all x, $y \in \mathbb{R}^n$ and for all $\lambda \in [0, 1]$,*

$$\lambda f(x) + (1 - \lambda) f(y) - f(\lambda x + (1 - \lambda) y)$$

is a sum of squares polynomial in $\mathbb{R}[x;y]$ *(with respect to variables* x *and* y *). Moreover, we say* f *is SOS-concave if* $-f$ *is SOS-convex.*

Clearly, a SOS-convex polynomial is convex. However, the converse is not true. Thus, there exists a convex polynomial which is not SOS-convex [1, 2]. The degree of a polynomial g is denoted by deg$g$. The set of convex polynomials in n variables of degree d and SOS-convex polynomials in n variables of degree d are denoted by $\tilde{C}_{n,d}$ and $\tilde{\Sigma}C_{n,d}$, respectively. Then $\tilde{C}_{n,d} = \tilde{\Sigma}C_{n,d}$ if and only if $n = 1$ or $d = 2$ or $(n, d) = (2, 4)$ [2].

Now we introduce a definition of concave matrix.

Definition 4.1.2. *A* $m \times m$ *symmetric matrix polynomial* $G(x)$ *is called concave matrix if for any* x, $y \in \mathbb{R}^n$ *and any* $\lambda \in [0, 1]$,

$$G((1-\lambda)x + \lambda y) \succeq (1-\lambda)G(x) + \lambda G(y).$$

Remark 4.1.1. *Let* $G(x)$ *be a* $m \times m$ *symmetric matrix polynomial. Then the following statements are equivalent:*

(i) $G(x)$ *is concave;*

(ii) *For all* $\Lambda \in S^m_+$, $-\langle \Lambda, G(x) \rangle$ *is convex, where* $\langle \Lambda, G(x) \rangle = \mathrm{tr}(\Lambda G(x))$*;*

(iii) *For all* $\xi \in \mathbb{R}^m$, $-\langle \xi\xi^T, G(x) \rangle$ *is convex;*

(iv) *For all* $\xi \in \mathbb{R}^m$, $-\xi^T G(x)\xi$ *is convex;*

(v) *For all* $\xi \in \mathbb{R}^m$, $-\nabla^2_{xx}(\xi^T G(x)\xi) \succeq 0$.

The definition of SOS-concave matrix is as follows:

Definition 4.1.3. [59] *A $m \times m$ symmetric matrix polynomial $G(x)$ is called SOS-concave if for every $\xi \in \mathbb{R}^m$, there exists a polynomial matrix $F_\xi(x)$ in x such that*

$$-\nabla^2_{xx}(\xi^T G(x)\xi) = F_\xi(x)^T F_\xi(x).$$

From Definition 4.1.3, we can obtain the following result.

Remark 4.1.2. *Let $G(x)$ be a $m \times m$ symmetric matrix polynomial. Then the following statements are equivalent:*

(i) *$G(x)$ is SOS-concave;*

(ii) *For all $\Lambda \in S^m_+$, $-\langle \Lambda, G(x) \rangle$ is SOS-convex;*

(iii) *For all $\xi \in \mathbb{R}^m$, $-\langle \xi\xi^T, G(x) \rangle$ is SOS-convex;*

(iv) *For all $\xi \in \mathbb{R}^m$, $-\xi^T G(x)\xi$ is SOS-convex.*

The following simple example illustrates a SOS-concave matrix polynomial.

Example 4.1.1. Consider the following polynomial matrix:

$$G(x_1, x_2) = \begin{pmatrix} -x_1^2 - 4x_1 - 3 - x_2^2 & x_2 \\ x_2 & -x_2 \end{pmatrix}.$$

Then, for all $\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2$,

$$-\xi^T G(x_1, x_2)\xi = \xi_1^2 x_1^2 + \xi_1^2 x_2^2 + 4\xi_1^2 x_1 + (\xi_2^2 - 2\xi_1\xi_2)x_2 + 3\xi_1^2.$$

and

$$-\nabla_{xx}^2(\xi^T G(x)\xi) = \begin{pmatrix} 2\xi_1^2 & 0 \\ 0 & 2\xi_1^2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\xi_1 & 0 \\ 0 & \sqrt{2}\xi_1 \end{pmatrix}^T \begin{pmatrix} \sqrt{2}\xi_1 & 0 \\ 0 & \sqrt{2}\xi_1 \end{pmatrix}.$$

So, $-\xi^T G(x)\xi$ is a SOS-convex polynomial. It follows from Remark 2.2 (iv) that $G(x)$ is a SOS-concave matrix polynomial.

Now we introduce a definition of support functions. Let K be a compact convex set in $\mathbb{R}^n$. The support function $s(x|K)$ of K [58] is defined by

$$s(x|K) := \max\{u^T x : u \in K\}.$$

The following useful existence result for solutions of convex polynomial programs will play an important role later.

Lemma 4.1.1. [6] *Let $f_0, f_1, \ldots, f_m$ be convex polynomials on $\mathbb{R}^n$. Let $C := \{x \in \mathbb{R}^n : f_i(x) \leqq 0,\ i = 1, \ldots, m\}$. Suppose that* $\inf_{x\in C} f_0(x) > -\infty$*. Then,* $\arg\min_{x\in C} f_0(x) \neq \emptyset$.

Proposition 4.1.1. [50] *A polynomial $g \in \mathbb{R}[x]_{2d}$ has a sum of squares decomposition if and only if there exists a real symmetric and positive semidefinite matrix $Q \in \mathbb{R}^{s(d)\times s(d)}$ such that $g(x) = v_d(x)^T Q v_d(x)$, for all $x \in \mathbb{R}^n$.*

Now we let $v_d(x)v_d(x)^T = \sum_{\alpha\in\mathbb{N}^n} x^\alpha B_\alpha$, where B_α are $s(d) \times s(d)$ real symmetric matrices, Then $g(x) = \sum_{\alpha\in\mathbb{N}^n} g_\alpha x^\alpha$ is a sum of squares if and only if solving the following semidefinite feasibility problem [50]:

$$\begin{aligned} &\text{Find } Q \in \mathbb{R}^{s(d)\times s(d)} \text{ such that} \\ &Q = Q^T,\ Q \succeq 0,\ \langle Q, B_\alpha\rangle = g_\alpha,\ \forall \alpha \in \mathbb{N}^n. \end{aligned}$$

Lasserre [49] established an extension of Jensen's inequality when one restricts its application to the class of SOS- convex polynomials.

Lemma 4.1.2. [49] *Let $f \in \mathbb{R}[x]$ be a SOS-convex polynomial, and let $y = (y_\alpha)_{\alpha \in \mathbb{N}^n_{2d}}$ satisfy $y_0 = 1$ and $\sum_{\alpha \in \mathbb{N}^n_{2d}} y_\alpha B_\alpha \succeq 0$. Let $L_y : \mathbb{R}[x] \to \mathbb{R}$ be a linear function defined by $L_y(f) = \sum_\alpha f_\alpha y_\alpha$, where $f = \sum_\alpha f_\alpha x^\alpha$. Then*

$$L_y(f(x)) \geqq f(L_y(x)),$$

where $L_y(x) = (L_y(x_1), \ldots, L_y(x_n))$.

4.2 Set Containment Characterizations

Under the Slater condition, we can obtain the following set containment result:

Theorem 4.2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a SOS-convex polynomial and let $G(x)$ be a $m \times m$ symmetric SOS-concave matrix polynomial. Let $h(x) = \max_{(a,b)\in\mathcal{U}} (a^T x + b)$, where $\mathcal{U}$ is a compact convex subset of $\mathbb{R}^{n+1}$. Assume that $K := \{x \in \mathbb{R}^n : G(x) \succeq 0\} \neq \emptyset$. Assume that the Slater condition holds, i.e., there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \succ 0$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : G(x) \succeq 0\} \subset \{x \in \mathbb{R}^n : f(x) - h(x) \geqq 0\}$;

(ii) *For any $(a,b) \in \mathcal{U}$, there exists $\Lambda \in S^m_+$ such that*

$$f - a^T(\cdot) - b - \langle \Lambda, G(\cdot)\rangle \in \Sigma^2.$$

Proof. [(ii) ⇒ (i)] Suppose that (ii) holds. Then, for any $(a, b) \in \mathcal{U}$ and any $\epsilon > 0$, there exist $\Lambda \in S^m_+$ and $\sigma \in \Sigma^2$ such that

$$f - a^T(\cdot) - b - \langle \Lambda, G(\cdot) \rangle = \sigma.$$

So, if $G(x) \succeq 0$, then $\langle \Lambda, G(x) \rangle \geqq 0$ and for any $(a, b) \in \mathcal{U}$, $f(x) - a^T x - b \geqq 0$, and hence, $f(x) - h(x) \geqq 0$. Thus (i) holds.

[(i) ⇒ (ii)] Assume that (i) holds. Then, we have for any $x \in \mathbb{R}^n$, $G(x) \succeq 0$ implies that $f(x) - h(x) \geqq 0$. Let $(a, b) \in \mathcal{U}$. Then, we have for any $x \in \mathbb{R}^n$,

$$\{x \in \mathbb{R}^n : G(x) \succeq 0\} \subset \{x \in \mathbb{R}^n : f(x) - a^T x - b \geqq 0, \ \forall (a, b) \in \mathcal{U}\}$$

Moreover, it is well known that the Slater condition implies the closedness of the set $\bigcup_{\Lambda \in S^m_+} \text{epi}(\langle \Lambda, G(\cdot) \rangle^*)$ [39]. So, it follows from Theorem 2.2 in [17] that there exists $\Lambda \in S^m_+$ such that

$$f(x) - a^T x - b - \langle \Lambda, G(x) \rangle \geqq 0, \quad \forall x \in \mathbb{R}^n. \tag{4.1}$$

Let $\phi(x) = f(x) - a^T x - b - \langle \Lambda, G(x) \rangle$. Then, since f and $-\langle \Lambda, G(x) \rangle$ are SOS-convex, and $-a^T x - b$ is affine, ϕ is SOS-convex. From (4.1), $\phi(x) \geqq 0$ for all $x \in \mathbb{R}^n$. By Lemma 4.1.1, ϕ has a global minimizer $x^* \in \mathbb{R}^n$, that is, $\phi(x) \geqq \phi(x^*)$ for all $x \in \mathbb{R}^n$, and hence $\nabla \phi(x^*) = 0$. Since ϕ is SOS-convex, it follow from Theorem 3.1 in [2] that there exists $\sigma \in \Sigma^2$ such that

$$\phi(x) - \phi(x^*) - \nabla \phi(x^*)^T (x - x^*) = \sigma.$$

Since $\phi(x^*) \geqq 0$ and $\nabla \phi(x^*) = 0$, $\phi \in \Sigma^2$ and thus (ii) holds. □

Using the proof approach of Theorem 4.2.1, we can obtain the following set containment result.

Theorem 4.2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a SOS-convex polynomial and $G(x)$ be a $m\times m$ symmetric SOS-concave matrix polynomial. Let $h(x) = \max_{u\in \mathrm{co}\{v_1,\ldots,v_l\}} u^T x$, where $v_1, \ldots, v_l \in \mathbb{R}^n$. Assume that $K := \{x \in \mathbb{R}^n : G(x) \succeq 0\} \neq \emptyset$. Assume that there exists $\hat{x} \in \mathbb{R}^n$ such that $G(\hat{x}) \succ 0$. Then the following statements are equivalent:*

(i) $\{x \in \mathbb{R}^n : G(x) \succeq 0\} \subset \{x \in \mathbb{R}^n : f(x) - h(x) \geqq 0\}$;

(ii) *For each $i = 1, \ldots, l$, there exists $\Lambda_i \in S^m_+$ such that*

$$f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle \in \Sigma^2.$$

Proof. [(ii) $\Rightarrow$ (i)] Suppose that (ii) holds. Then, for any $x \in \mathbb{R}^n$, for each $i = 1, \ldots, l$, there exist $\Lambda_i \in S^m_+$ and $\sigma_i \in \Sigma^2$ such that

$$f(x) - v_i^T x - \langle \Lambda_i, G(x)\rangle = \sigma_i.$$

It implies that $f(x) - v_i^T x - \langle \Lambda_i, G(x)\rangle \geqq 0$ for any $x \in \mathbb{R}^n$. If $G(x) \succeq 0$, then $\langle \Lambda_i, G(x)\rangle \geqq 0$, and so $f(x) - v_i^T(x) \geqq 0$, and hence, $f(x) - h(x) \geqq 0$. Thus (i) holds.

[(i) $\Rightarrow$ (ii)] Assume that (i) holds. Since $h(x) = \max_{u\in \mathrm{co}\{v_1,\ldots,v_l\}} u^T x$, for any $x \in \mathbb{R}^n$, $G(x)(x) \succeq 0$ implies that $f(x) - \max_{u\in \mathrm{co}\{v_1,\ldots,v_l\}} u^T x \geqq 0$. It follow from the above inequality that

$$0 \leqq f(x) - \max_{u\in \mathrm{co}\{v_1,\ldots,v_l\}} u^T x = \min_{i=1,\ldots,l} \{f(x) - v_i^T x\} \leqq f(x) - v_i^T x,\ i = 1, \ldots, l.$$

Since the Slater condition holds, it implies the closedness of the set $\bigcup_{\Lambda\in S^m_+}$ $\mathrm{epi}(\langle\Lambda, G(\cdot)\rangle^*)$. So, it follows from Theorem 2.2 in [17] that for each $i = 1,\ldots,l$, there exists $\Lambda_i \in S^m_+$ such that

$$f(x) - v_i^T x - \langle \Lambda_i, G(x)\rangle \geqq 0, \quad \forall x \in \mathbb{R}^n. \tag{4.2}$$

Let $\phi_i(x) = f(x) - v_i^T x - \langle\Lambda_i, G(x)\rangle$, $i = 1,\ldots,l$. Then, since f and $-\langle\Lambda_i, G(x)\rangle$ are SOS-convex, and each $-v_i^T x$, $i = 1,\ldots,l$, is linear, each ϕ_i, $i = 1,\ldots,l$, is SOS-convex. Let $i \in \{1,\ldots,l\}$ be any fixed. From (4.2), $\phi_i(x) \geqq 0$, for all $x \in \mathbb{R}^n$. By Lemma 4.1.1, ϕ_i has a global minimizer $x^* \in \mathbb{R}^n$, that is, $\phi_i(x) \geqq \phi_i(x^*)$ for all $x \in \mathbb{R}^n$, and hence $\nabla\phi_i(x^*) = 0$. Since ϕ_i is SOS-convex, it follows from Theorem 3.1 in [2] that there exists $\sigma_i \in \Sigma^2$ such that

$$\phi_i(x) - \phi_i(x^*) - \nabla\phi_i(x^*)^T(x - x^*) = \sigma_i.$$

Since $\phi_i(x^*) \geqq 0$ and $\nabla\phi_i(x^*) = 0$, $\phi_i \in \Sigma^2$ and thus (ii) holds. $\square$

4.3 Exact SDP Relaxations

Consider the following SOS-convex polynomial optimization problem:

$$\begin{aligned} (\mathrm{P}) \qquad & \inf \;\; f(x) \\ & \text{s.t.} \;\; G(x) \succeq 0, \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a SOS-convex polynomial with degree $2d$, $G(x)$ is a $m \times m$ symmetric SOS-concave matrix polynomial.

The SOS-convex polynomial optimization problem (P) in the face of data uncertainty in the objective function can be captured by the problem:

$$
\begin{aligned}
\text{(UP)} \qquad & \inf \quad \varphi(x,u) \\
& \text{s.t.} \quad G(x) \succeq 0,
\end{aligned}
$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a SOS-convex polynomial defined by $\varphi(x,u) = f(x) - u^T x$, and $u \in \mathbb{R}^n$ is an uncertain parameter which belongs to the set $\text{co}\{v_1, \ldots, v_l\}$, $v_i \in \mathbb{R}^n$, $i = 1, \ldots, l$. Let $K := \{x \in \mathbb{R}^n : G(x) \succeq 0\}$.

The robust counterpart (the worst case) of (UP):

$$
\begin{aligned}
\text{(RP)} \qquad & \inf \quad f(x) - \max_{u \in \text{co}\{v_1,\ldots,v_l\}} u^T x \\
& \text{s.t.} \quad G(x) \succeq 0.
\end{aligned}
$$

In the sequel, we assume that the optimal value of (RP) is finite. Moreover, the problem (RP) can be rewritten as follows:

$$
\min_{i=1,\ldots,l} \inf_{x \in K} \{f(x) - v_i^T x\}.
$$

The Lagrangian dual problem for (RP) is given by

$$
\text{(LD)} \qquad \min_{i=1,\ldots,l} \sup_{\Lambda_i \in S_+^m} \inf_{x \in \mathbb{R}^n} \{f(x) - v_i^T x - \langle \Lambda_i, G(x) \rangle\}.
$$

which can be written equivalently as

$$
\text{(D)} \quad \min_{i=1,\ldots,l} \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \in \mathbb{R} \mid f(x) - v_i^T x - \langle \Lambda_i, G(x) \rangle - \mu_i \geqq 0, \ \forall x \in \mathbb{R}^n\}.
$$

A sum of squares relaxation problem of (D) is as follows :

$$(\mathrm{D})^{\mathrm{sos}} \quad \min_{i=1,\ldots,l} \sup_{\mu_i\in\mathbb{R},\Lambda_i\in S_+^m} \{\mu_i \in \mathbb{R} \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma_{2d}^2\}.$$

Then, from Proposition 4.1.1, $(\mathrm{D})^{\mathrm{sos}}$ can be rewritten as the following semidefinite problem (SDP):

$$\begin{aligned}
(\mathrm{SDP}) \quad \min_{i=1,\ldots,l} \sup_{X,\Lambda_i} \quad & f_0 - (v_i^T(\cdot))_0 - \langle \Lambda_i, G_0\rangle - \langle X, B_0\rangle \\
\text{s.t.} \quad & \langle \Lambda_i, G_\alpha\rangle + \langle X, B_\alpha\rangle = f_\alpha - (v_i^T(\cdot))_\alpha, \\
& \alpha \neq 0,\ X \in S_+^{s(d)},\ \Lambda_i \in S_+^m.
\end{aligned}$$

The dual problem of (SDP) is the following semidefinite problem (SDD):

$$\begin{aligned}
(\mathrm{SDD}) \min_{i=1,\ldots,l} \inf_{y} \quad & (f - v_i^T(\cdot))_0 + \sum_{\alpha\neq 0} (f - v_i^T(\cdot))_\alpha y_\alpha \\
\text{s.t.} \quad & G_0 + \sum_{\alpha\neq 0} y_\alpha G_\alpha \succeq 0, \\
& B_0 + \sum_{\alpha\neq 0} y_\alpha B_\alpha \succeq 0.
\end{aligned}$$

Now, using the result of Theorem 4.2.2, we give a zero duality gap result for (RP), $(\mathrm{D})^{\mathrm{sos}}$, (SDP) and (SDD) under the Slater condition.

Theorem 4.3.1. (Zero duality gap) *Let* $K := \{x \in \mathbb{R}^n \mid G(x) \succeq 0\} \neq \emptyset$. *Assume that* $\inf(\mathrm{RP}) := f^*$ *is finite and the Slater condition holds, that is, there exists* $\hat{x} \in \mathbb{R}^n$ *such that* $G(\hat{x}) \succ 0$. *Then*

$$\mathrm{Val(RP)} = \mathrm{Val(D)}^{\mathrm{sos}} = \mathrm{Val(SDP)} = \mathrm{Val(SDD)}.$$

Proof. Let α be an optimal value of (RP). Then, we have

$$\{x \in \mathbb{R}^n : G(x) \succeq 0\} \subset \{x \in \mathbb{R}^n \mid f(x) - h(x) \geqq \alpha\}.$$

By Theorem 4.2.2, for each $i = 1, \ldots, l$, there exist $\sigma_i \in \Sigma^2$ and $\Lambda_i \in S_+^m$ such that

$$f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \alpha = \sigma_i \geqq 0,$$

and so, $\sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\} \geqq \alpha$. Thus, we have

$$\min_{i=1,\ldots,l} \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\} \geqq \alpha. \tag{4.3}$$

On the other hand, let $\bar{\mu} := \min_{i=1,\ldots,l} \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \in \mathbb{R} \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\}$. Then, we see that for all $i = 1, \ldots, l$,

$$\bar{\mu} \leqq \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \in \mathbb{R} \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\}.$$

Since for each $\mu_i \in \mathbb{R}$ and each $\Lambda_i \in S_+^m$, $f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2$ and $\langle \Lambda_i, G(\cdot)\rangle \geqq 0$, $f(x) - v_i^T x \geqq \mu_i$, for all $x \in K$, Hence for each $i = 1, \ldots, l$,

$$f(x) - v_i^T x \geqq \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \in \mathbb{R} \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\}, \ \forall x \in K,$$

and so, $f(x) - v_i^T x \geqq \bar{\mu}$, for all $x \in K$. Thus, $f(x) - \max_{i=1,\ldots,l} v_i^T x \geqq \bar{\mu}$, for all $x \in K$. So, we have $f(x) - h(x) \geqq \bar{\mu}$, for all $x \in K$. Hence,

$$\bar{\mu} = \min_{i=1,\ldots,l} \sup_{\mu_i \in \mathbb{R}, \Lambda_i \in S_+^m} \{\mu_i \mid f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\} \leqq \alpha. \tag{4.4}$$

Thus, from (4.3) and (4.4), Val(RP) = Val(D)$^{\text{sos}}$. Moreover, Val(D)$^{\text{sos}}$ = Val(SDP) obviously holds by the construction of (D)$^{\text{sos}}$ and (SDP).

Now, we will show that Val(SDP) $\leqq$ Val(SDD). Let for each $i = 1, \ldots, l$, (Λ_i, X) and y be any feasible for (SDP) and (SDD) respectively. Then we have

$$\begin{aligned}
& (f - v_i^T(\cdot))_0 + \langle G_0, \Lambda_i \rangle - \langle B_0, X \rangle \\
\leqq\ & (f - v_i^T(\cdot))_0 + \langle -\sum_{\alpha \neq 0} y_\alpha G_\alpha, \Lambda_i \rangle - \langle \sum_{\alpha \neq 0} y_\alpha B_\alpha, X \rangle \\
=\ & (f - v_i^T(\cdot))_0 + \sum_{\alpha \neq 0} y_\alpha (\langle -G_\alpha, \Lambda_i \rangle - \langle B_\alpha, X \rangle) \\
=\ & (f - v_i^T(\cdot))_0 + \sum_{\alpha \neq 0} y_\alpha (f - v_i^T(\cdot))_\alpha \\
=\ & \sum_{\alpha} y_\alpha (f - v_i^T(\cdot))_\alpha.
\end{aligned}$$

So, we have Val(SDP) $\leqq$ Val(SDD).

To finish the proof of the theorem, we will prove that Val(RP) $\geqq$ Val(SDD). Let $\tilde{x}$ be any feasible solution of (RP). Then $G(\tilde{x}) \succeq 0$. Let $\tilde{y} = (\tilde{y}_\alpha)_{\alpha \neq 0} = (\tilde{x}_1, \ldots, \tilde{x}_n, (\tilde{x}_1)^2, \tilde{x}_1\tilde{x}_2, \ldots, (\tilde{x}_1)^{2m}, \ldots, (\tilde{x}_n)^{2m})$. Then $0 \preceq G(\tilde{x}) = \sum_\alpha G_\alpha \tilde{x}^\alpha = G_0 + \sum_{\alpha \neq 0} G_\alpha \tilde{y}_\alpha$. Moreover, $\tilde{y}\tilde{y}^T = B_0 + \sum_{\alpha \neq 0} \tilde{y}_\alpha B_\alpha \succeq 0$. So, $\tilde{y}$ is feasible for (SDD). Moreover, since $\tilde{x}$ is feasible solution of (RP), we see that for each

$i = 1, \ldots, l,$

$$\begin{aligned} f(\tilde{x}) - v_i^T \tilde{x} = & \sum_{\alpha} (f - v_i^T(\cdot))_{\alpha} \tilde{x}^{\alpha} \\ = & (f - v_i^T(\cdot))_0 + \sum_{\alpha \neq 0} (f - v_i^T(\cdot))_{\alpha} \tilde{y}_{\alpha} \\ \geqq & \inf_y \sum_{\alpha} (f - v_i^T(\cdot))_{\alpha} y_{\alpha} \\ \geqq & \mathrm{Val(SDD)}. \end{aligned}$$

Hence,

$$\min_{i=1,\ldots,l} \{f(\tilde{x}) - v_i^T \tilde{x}\} = f(\tilde{x}) - \max_{u \in \mathrm{co}\{v_1,\ldots,v_l\}} u^T \tilde{x} = f(\tilde{x}) - h(\tilde{x}) \geqq \mathrm{Val(SDD)}.$$

Since $\tilde{x}$ is any feasible solution of (RP), we have

$$\mathrm{Val(RP)} = \inf_{x \in K} \{f(x) - h(x)\} \geqq \mathrm{Val(SDD)}.$$

Thus, we obtain the desired result. □

Now, we give the relations of the optimal solution of (RP) and the optimal solution of (SDD), and the optimal solution of $(\mathrm{D})^{\mathrm{sos}}$ and (SDP).

Theorem 4.3.2. *Assume that* $\inf \mathrm{(RP)} := f^*$ *is finite and the Slater condition holds, that is, there exists* $\hat{x} \in \mathbb{R}^n$ *such that* $G(\hat{x}) \succ 0$. *Let* $K := \{x \in \mathbb{R}^n \mid G(x) \succeq 0\} \neq \emptyset$. *Then the following statements hold:*

(i) $\bar{x}$ *is a minimizer of* (RP) *if and only if the vector*

$$\bar{y} := (\bar{x}_1, \ldots, \bar{x}_n, \bar{x}_1^2, \bar{x}_1\bar{x}_2, \ldots, \bar{x}_1^{2d}, \ldots, \bar{x}_n^{2d}) \tag{4.5}$$

is a minimizer of (SDD).

(ii) $(\bar{\Lambda}_{i_0}, \bar{\mu}_{i_0}) \in S^m_+ \times \mathbb{R}$ *is a maximizer of* $(\mathrm{D})^{\mathrm{sos}}$ *if and only if* $(\bar{\Lambda}_{i_0}, \bar{X}) \in S^m_+ \times S^{s(d)}_+$ *is a maximizer of* (SDP) *for some* $\bar{X} = \sum_{k=1}^r \bar{q}_k^{i_0} \bar{q}_k^{i_0 T}$ *and* $\bar{q}_k^{i_0} \in \mathbb{R}^{s(d)}$.

Proof. (i) ($\Rightarrow$) Let $\bar{\mu}$ be an optimal value of (RP). It follows that for any $x \in K$, $f(x) - h(x) \geqq \bar{\mu}$. By Theorem 4.2.2, equivalently, for each $i = 1, \ldots, l$, there exists $\Lambda_i \in S^m_+$ such that

$$f - v_i^T(\cdot) - \langle \Lambda_i, G(\cdot) \rangle - \bar{\mu} \in \Sigma^2.$$

Letting $\bar{\mu}_i = \inf_{x \in K} \{f(x) - v_i^T x\}$, $i = 1, \ldots, l$, equivalently, there exists $i_0 \in \{1, \ldots, l\}$ such that $\bar{\mu} = \bar{\mu}_{i_0}$ and

$$f - v_{i_0}^T(\cdot) - \langle \Lambda_{i_0}, G(\cdot) \rangle - \bar{\mu}_{i_0} \in \Sigma^2,$$

for some $\Lambda_{i_0} \in S^m_+$. It means that there exist some polynomials $q_k^{i_0}(x) : \mathbb{R}^n \to \mathbb{R}$ with degree d and coefficient vectors $q_k^{i_0} \in \mathbb{R}^{s(d)}$, $k = 1, \ldots, r$, such that

$$f(x) - v_{i_0}^T x - \langle \Lambda_{i_0}, G(x) \rangle - \bar{\mu}_{i_0} = \sum_{k=1}^r q_k^{i_0}(x)^2, \quad \forall x \in \mathbb{R}^n.$$

From Proposition 4.1.1, equivalently, there exists a real symmetric and positive semidefinite matrix $X \in S^{s(d)}_+$ such that

$$f(x) - v_{i_0}^T x - \langle \Lambda_{i_0}, G(x) \rangle - \bar{\mu}_{i_0} = v_d(x)^T X v_d(x), \quad \forall x \in \mathbb{R}^n \tag{4.6}$$

with $X = \sum_{k=1}^{r} q_k^{i_0} q_k^{i_0 T}$. Notice that $v_d(x)^T X v_d(x) = \langle X, v_d(x)v_d(x)^T \rangle$. Let $v_d(x)v_d(x)^T = \sum_{\alpha \in \mathbb{N}^n} x^\alpha B_\alpha \succeq 0$, where B_α are $s(d) \times s(d)$ real symmetric matrices. It follows that from (4.6),

$$f(x) - v_{i_0}^T x - \langle \Lambda_{i_0}, G(x) \rangle - \bar{\mu}_{i_0} = \langle X, \sum_{\alpha \in \mathbb{N}^n} x^\alpha B_\alpha \rangle, \quad \forall x \in \mathbb{R}^n. \tag{4.7}$$

Moreover, since $\langle \Lambda_{i_0}, G(x) \rangle = \sum_{\alpha \in \mathbb{N}^n} \langle \Lambda_{i_0}, G_\alpha \rangle x^\alpha$,

$$f(x) - v_{i_0}^T x - \langle \Lambda_{i_0}, G(x) \rangle - \bar{\mu}_{i_0} = \sum_{\alpha} (f - v_{i_0}^T(\cdot))_\alpha x^\alpha - \sum_{\alpha \in \mathbb{N}^n} \langle \Lambda_{i_0}, G_\alpha \rangle x^\alpha - \bar{\mu}_{i_0}. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\sum_{\alpha} (f - v_{i_0}^T(\cdot))_\alpha x^\alpha - \bar{\mu}_{i_0} = \langle X, \sum_{\alpha \in \mathbb{N}^n} x^\alpha B_\alpha \rangle + \langle \Lambda_{i_0}, \sum_{\alpha \in \mathbb{N}^n} x^\alpha G_\alpha \rangle. \tag{4.9}$$

Let $y = (y_\alpha)_{\alpha \neq 0} = (x_1, \ldots, x_n, (x_1)^2, x_1 x_2, \ldots, (x_1)^{2m}, \ldots, (x_n)^{2m})$ and $y_0 = 1$. Then, (4.9) is equivalent to that

$$\begin{aligned} & (f - v_{i_0}^T(\cdot))_0 + \sum_{\alpha \neq 0} (f - v_{i_0}^T(\cdot))_\alpha y_\alpha - \bar{\mu}_{i_0} \\ = \; & \langle X, B_0 + \sum_{\alpha \neq 0} y_\alpha B_\alpha \rangle + \langle \Lambda_{i_0}, G_0 + \sum_{\alpha \neq 0} y_\alpha G_\alpha \rangle. \end{aligned}$$

So, if $\alpha = 0$, then we have

$$(f - v_{i_0}^T(\cdot))_0 - \langle G_0, \Lambda_{i_0} \rangle - \langle B_0, X \rangle = \bar{\mu}_{i_0} \tag{4.10}$$

and if $\alpha \neq 0$, then we have

$$\langle G_\alpha, \Lambda_{i_0}\rangle + \langle B_\alpha, X\rangle = (f - v_{i_0}^T(\cdot))_\alpha. \quad (4.11)$$

By (4.10) and (4.11), we see that (Λ_{i_0}, X) is feasible for (SDP) with value $\bar{\mu}_{i_0}$. Since $\bar{\mu}$ is a minimum of (RP) and $\bar{\mu}_{i_0} = \bar{\mu}$, (Λ_{i_0}, X) is a maximizer of (SDP). Notice that Val(RP) = Val(SDP) = Val(SDD) (by Theorem 4.3.1) and $\bar{x}$ is an optimal solution of (RP). Since for $\bar{y}$ in (4.5), $G_0 + \sum_{\alpha\neq 0} y_\alpha G_\alpha \succeq 0$ and $B_0 + \sum_{\alpha\neq 0} y_\alpha B_\alpha \succeq 0$, $\bar{y}$ is feasible for (SDD) with value $\bar{\mu}_{i_0}$. Moreover, since $(f - v_i^T(\cdot))_0 + \sum_{\alpha\neq 0}(f - v_i^T(\cdot))_\alpha \bar{y}_\alpha = \sum_\alpha (f - v_i^T(\cdot))_\alpha \bar{x}^\alpha = \bar{\mu}_{i_0}$, $\bar{y}$ is minimizer of (SDD).

($\Leftarrow$) Suppose that there exist $i_0 \in \{i, \ldots, l\}$ such that the vector $\bar{y}$ in (4.5) is a minimizer of (SDD). Let $\bar{\mu}_{i_0}$ is an optimal value of (SDD). Since $G(x)$ is a $m \times m$ SOS-concave symmetric matrix polynomial, by Remark 4.1.2 (ii), for any $\Lambda \in S_+^m$, $-\langle \Lambda, G(x)\rangle$ is a SOS-convex polynomial. It follow from Lemma 4.1.2 that

$$L_{\bar{y}}(-\langle \Lambda, G(x)\rangle) \geqq -\langle \Lambda, G(L_{\bar{y}}(x))\rangle = -\langle \Lambda, G(\bar{x})\rangle, \quad (4.12)$$

where $\bar{x} = L_{\bar{y}}(x) = (L_{\bar{y}}(x_1), \ldots, L_{\bar{y}}(x_n))$. Moreover, since $\bar{y}$ is a feasible solution of (SDD) satisfying $\bar{y}_0 = 1$, we see that

$$L_{\bar{y}}(-\langle \Lambda, G(x)\rangle) = \sum_\alpha (-\langle \Lambda, G_\alpha\rangle \bar{y}_\alpha) = -\langle \Lambda, G_0 + \sum_{\alpha\neq 0} \bar{y}_\alpha G_\alpha\rangle \leqq 0. \quad (4.13)$$

So, from (4.12) and (4.13), we see that $\langle \Lambda, G(\bar{x}) \rangle \geqq 0$. Since $\Lambda \in S^m_+$, we have $G(\bar{x}) \succeq 0$, i.e., $\bar{x}$ is feasible for (RP). Similarly, since f is a SOS-convex polynomial and $v_{i_0}^T(\cdot)$ is linear,

$$\begin{aligned}
\bar{\mu}_{i_0} &= \sum_{\alpha}(f - v_{i_0}^T(\cdot))_\alpha \bar{y}_\alpha \\
&= L_{\bar{y}}(f - v_{i_0}^T(\cdot)) \geqq (f - v_{i_0}^T(\cdot))(L_{\bar{y}}(x)) \\
&= (f - v_{i_0}^T(\cdot))(\bar{x}) \\
&\geqq \min_{i=1,\ldots,l}(f - v_i^T(\cdot))(\bar{x}).
\end{aligned}$$

Moreover, since Val(RP) = Val(SDD) = $\bar{\mu}_{i_0}$ (by Theorem 4.3.1), $\bar{\mu}_{i_0} = \min_{i=1,\ldots,l}(f - v_i^T(\cdot))(\bar{x})$. It means that $\bar{x}$ is an optimal solution of (RP).

(ii) Let $(\bar{\Lambda}_{i_0}, \bar{\mu}_{i_0}) \in S^m_+ \times \mathbb{R}$ be a maximizer of $(\mathrm{D})^{\mathrm{sos}}$, for some $i_0 \in \{i, \ldots, l\}$. Since $(\bar{\Lambda}_{i_0}, \bar{\mu}_{i_0}) \in S^m_+ \times \mathbb{R}$ is feasible for $(\mathrm{D})^{\mathrm{sos}}$,

$$f - v_{i_0}^T(\cdot) - \langle \bar{\Lambda}_{i_0}, G(\cdot) \rangle - \bar{\mu}_{i_0} \in \Sigma^2.$$

It means that there exist some polynomials $\bar{q}_k^{i_0}(x) : \mathbb{R}^n \to \mathbb{R}$ with degree d and coefficient vectors $\bar{q}_k^{i_0} \in \mathbb{R}^{s(d)}$, $k = 1, \ldots, r$, such that

$$f(x) - v_{i_0}^T x - \langle \bar{\Lambda}_{i_0}, G(x) \rangle - \bar{\mu}_{i_0} = \sum_{k=1}^{r} \bar{q}_k^{i_0}(x)^2, \quad x \in \mathbb{R}^n.$$

From Proposition 4.1.1, equivalently, there exists a real symmetric and positive semidefinite matrix $\bar{X} \in S^{s(d)}_+$ such that

$$f(x) - v_{i_0}^T x - \langle \bar{\Lambda}_{i_0}, G(x) \rangle - \bar{\mu}_{i_0} = v_d(x)^T \bar{X} v_d(x), \quad \forall x \in \mathbb{R}^n \tag{4.14}$$

with $\bar{X} = \sum_{k=1}^{r} \bar{q}_k^{i_0} \bar{q}_k^{i_0 T}$. Since $v_d(x)^T \bar{X} v_d(x) = \langle v_d(x) v_d(x)^T, \bar{X}\rangle = \langle B_0 + \sum_{\alpha \neq 0} x^\alpha B_\alpha, \bar{X}\rangle$, (4.14) is equivalent to that

$$f(x) - v_{i_0}^T x - \langle \bar{\Lambda}_{i_0}, G(x)\rangle - \bar{\mu}_{i_0} = \langle B_0 + \sum_{\alpha \neq 0} x^\alpha B_\alpha, \bar{X}\rangle, \quad \forall x \in \mathbb{R}^n. \quad (4.15)$$

Notice that

$$f(x) - v_{i_0}^T x - \langle \bar{\Lambda}_{i_0}, G(x)\rangle - \bar{\mu}_{i_0} = \sum_{\alpha} (f_\alpha - (v_{i_0}^T(\cdot))_\alpha - \langle \bar{\Lambda}_{i_0}, G_\alpha\rangle) x^\alpha - \bar{\mu}_{i_0}. \quad (4.16)$$

From (4.15) and (4.16), it follows that

$$\langle B_0 + \sum_{\alpha \neq 0} x^\alpha B_\alpha, \bar{X}\rangle = \sum_{\alpha} (f_\alpha - (v_{i_0}^T(\cdot))_\alpha - \langle \bar{\Lambda}_{i_0}, G_\alpha\rangle) x^\alpha - \bar{\mu}_{i_0}, \quad \forall x \in \mathbb{R}^n.$$

It means that

$$(f - v_{i_0}^T(\cdot))_0 - \langle G_0, \bar{\Lambda}_{i_0}\rangle - \langle B_0, \bar{X}\rangle = \bar{\mu}_{i_0} \quad (4.17)$$

and

$$\langle G_\alpha, \bar{\Lambda}_{i_0}\rangle + \langle B_\alpha, \bar{X}\rangle = (f - v_{i_0}^T(\cdot))_\alpha, \quad \forall \alpha \neq 0. \quad (4.18)$$

By (4.17) and (4.18), we see that $(\bar{\Lambda}_{i_0}, \bar{X})$ is feasible for (SDP) with value $\bar{\mu}_{i_0}$. Moreover, since $\mathrm{Val(D)}^{\mathrm{sos}} = \mathrm{Val(SDP)}$ (by Theorem 4.3.1), $(\bar{\Lambda}_{i_0}, \bar{X})$ is a maximizer of (SDP). □

Remark 4.3.1. *When $G(x)$ is a diagonal matrix polynomial and $h(x) = 0$, then Lasserre* [49] *proved Theorem 4.3.2* (i) *using Lemma 4.1.2.*

The following example shows that a zero duality gap result for (RP), $(D)^{sos}$, (SDP) and (SDD) and the relations of the optimal solution of (RP) and the optimal solution of (SDD), and the optimal solution of $(D)^{sos}$ and (SDP).

Example 4.3.1. Consider the following problem:

$$\text{(RP)} \qquad \min \qquad x_1^8 + x_1x_2 + x_1^2 + x_2^2 - \max_{u_1\in[-1,1],u_2\in[-1,1]} (u_1,u_2)^T(x_1,x_2)$$

$$\text{subsect to } \begin{pmatrix} -x_1^2 - 4x_1 - 3 - x_2^2 & x_2 \\ x_2 & -x_2 \end{pmatrix} \succeq 0.$$

Let $f(x_1,x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2$, $G(x_1,x_2) = \begin{pmatrix} -x_1^2 - 4x_1 - 3 - x_2^2 & x_2 \\ x_2 & -x_2 \end{pmatrix}$ and $h(x) = \max_{u_1\in[-1,1],u_2\in[-1,1]} (u_1,u_2)^T(x_1,x_2) = \max_{(u_1,u_2)\in \text{co}M} (u_1,u_2)^T(x_1,x_2) = |x_1| + |x_2|$, where $M = \{(1,1),(1,-1),(-1,1),(-1,-1)\}$. Let $v_1 = (1,1)$, $v_2 = (1,-1)$, $v_3 = (-1,1)$ and $v_4 = (-1,-1)$. Then, clearly f is a SOS-convex polynomial. Moreover we already checked that G is SOS-concave matrix in Example 4.1.1. Let $K := \{(x_1,x_2) \mid G(x_1,x_2) \succeq 0\}$ be a feasible set of (P_0). Then we have $K = \{(x_1,x_2) \mid (x_1+2)^2 + (x_2 - \frac{1}{2})^2 \leqq \frac{5}{4},\ x_2 \leqq 0\}$. Moreover, let $(\hat{x}_1,\hat{x}_2) = (-2, -\frac{1}{100})$. Then $G(\hat{x}_1,\hat{x}_2) \succ 0$. So, the Slater condition holds for (P_0). So, the optimal solution and the optimal value for (RP) is $(-1,0)$ and 1, respectively (See Fig. 4.3.1).

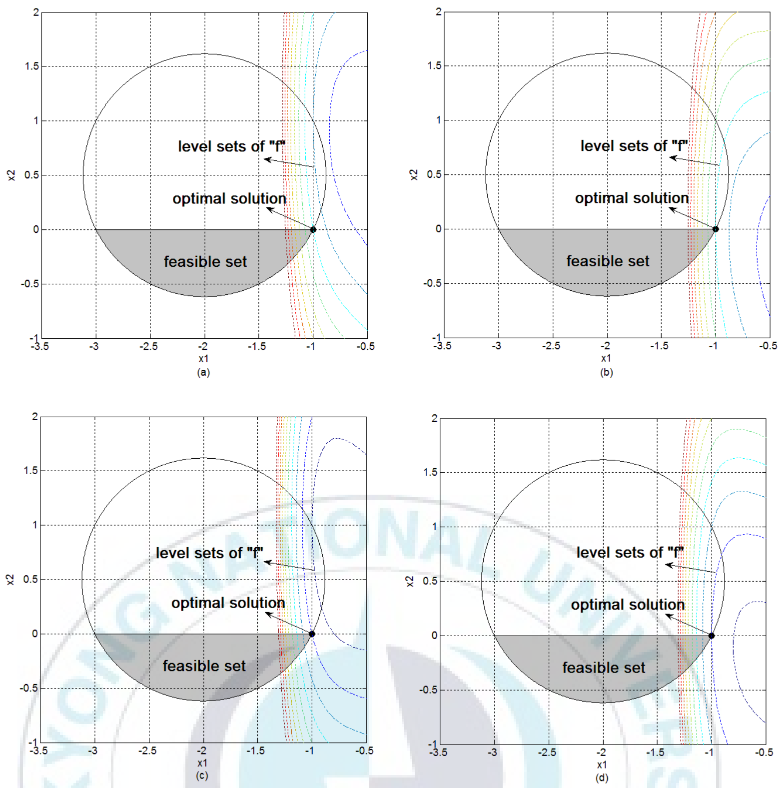

Figure 4.3.1 (a) $f(x_1,x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2 - x_1 - x_2$, (b) $f(x_1,x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2 - x_1 + x_2$, (c) $f(x_1,x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2 + x_1 - x_2$ and (d) $f(x_1,x_2) = x_1^8 + x_1x_2 + x_1^2 + x_2^2 + x_1 + x_2$. The feasible set of (P_0) is K(solid), an optimal solution of (P_0) is $(-1,0)$ (a dot) and level sets of the objective function f of (P_0) (dotted).

Now, we consider the dual problem of (RP) as follows:

$$(\mathrm{D})^{\mathrm{sos}} \quad \min_{i=1,\ldots,4} \max_{\mu_i\in\mathbb{R},\Lambda_i\in S_+^2} \{\mu_i \mid f - v_i^T(\cdot) + \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\}.$$

Then, the problem $(\mathrm{D})^{\mathrm{sos}}$ is equivalent to $\min_{i=1,\ldots,4} (\mathrm{D}_i)^{\mathrm{sos}}$, where $(\mathrm{D}_i)^{\mathrm{sos}} = \max_{\mu_i\in\mathbb{R},\Lambda_i\in S_+^2} \{\mu_i \mid f - v_i^T(\cdot) + \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2\}$. Since for each $i = 1, \ldots, 4$,

$f - v_i^T(\cdot) + \langle \Lambda_i, G(\cdot)\rangle - \mu_i \in \Sigma^2$, for each $i = 1, \ldots, 4$, there exist some polynomials $q_k^i(x) : \mathbb{R}^n \to \mathbb{R}$ with degree at most 4 and coefficient vectors $q_k^i \in \mathbb{R}^{s(d)}$, $k = 1, \ldots, r$, such that for any $x \in \mathbb{R}^2$,

$$f - v_i^T(x) + \langle \Lambda_i, G(x)\rangle - \mu_i = \sum_{k=1}^{r} q_k^i(x)^2 = v_4(x)^T X v_4(x),$$

where X is $s(4) \times s(4)$ real symmetric and positive semidefinite matrix. Notice that the dimension of $v_4(x)$ is 15. Then from Theorem 1 in [63], we can reduce the dimension of $v_8(x)$, that is, 6, and so $X \in S_+^6$. Actually, $v_4(x) = (1, x_1, x_2, x_1^2, x_1^3, x_1^4)$ in $(\mathrm{D})^{\mathrm{sos}}$. Then, by using the MATLAB optimization package SOSTOOLS [62] together with the SDP-solver SeDuMi [65], we can easily get the optimal value of $(\mathrm{D}_i)^{\mathrm{sos}}$, $i = 1, \ldots, 4$, that is, $\mu_i \approx 3$, $i = 1, 2$, and $\mu_i \approx 1$, $i = 3, 4$. So, the optimal value of $(\mathrm{D})^{\mathrm{sos}}$ is 1. We can not easily find optimal solutions of $(\mathrm{D})^{\mathrm{sos}}$, but actually optimal solutions of $(\mathrm{D})^{\mathrm{sos}}$ are $(\Lambda_3, \mu_3) \approx (\begin{pmatrix} 4.5003 & 1.0699 \\ 1.0699 & 4.1399 \end{pmatrix}, 1)$ and $(\Lambda_4, \mu_4) \approx (\begin{pmatrix} 4.5005 & 1.9507 \\ 1.9507 & 3.9016 \end{pmatrix}, 1)$. Now, we rewrite $(\mathrm{D})^{\mathrm{sos}}$ as the following semidefinite problem:

$$\begin{aligned} (\mathrm{SDP}) \quad \min_{i=1,\ldots,4} \ \sup_{X, \Lambda_i} \quad & (f - v_i^T(\cdot))_0 - \langle \Lambda_i, G_0\rangle - \langle X, B_0\rangle \\ \text{s.t.} \quad & \langle \Lambda_i, G_\alpha\rangle + \langle X, B_\alpha\rangle = (f - v_i^T(\cdot))_\alpha, \ \alpha \neq 0, \\ & X \in S_+^6, \ \Lambda_i \in S_+^2, \ i = 1, \ldots, 4. \end{aligned}$$

Then, by using the MATLAB optimization package VSDP [29] together with the SDP-solver SeDuMi [65], we can easily find the optimal solutions for

(SDP$_i$), $i = 1, \ldots, 4$, that is,

$$\bar{X}_1 = \begin{pmatrix} 13.5026 & 10.5017 & 0.5000 & -2.6382 & -0.4572 & -0.8196 \\ 10.5017 & 11.7773 & 0.5000 & 0.4572 & -1.5449 & -0.7265 \\ 0.5000 & 0.5000 & 6.5009 & 0.0000 & 0.0000 & -0.0000 \\ -2.6382 & 0.4572 & 0.0000 & 4.7290 & 0.7265 & -0.9070 \\ -0.4572 & -1.5449 & 0.0000 & 0.7265 & 1.8140 & -0.0000 \\ -0.8196 & -0.7265 & -0.0000 & -0.9070 & -0.0000 & 1.0000 \end{pmatrix},$$

$$\bar{\Lambda}_1 = \begin{pmatrix} 5.5009 & 1.4072 \\ 1.4072 & 4.8145 \end{pmatrix},$$

$$\bar{X}_2 = \begin{pmatrix} 13.5009 & 10.5006 & 0.5001 & -2.4328 & -0.2789 & -0.8464 \\ 10.5006 & 11.3658 & 0.5000 & 0.2789 & -1.2825 & -0.6962 \\ 0.5001 & 0.5000 & 6.5003 & -0.0000 & 0.0000 & 0.0000 \\ -2.4328 & 0.2789 & -0.0000 & 4.2577 & 0.6962 & -0.8498 \\ -0.2789 & -1.2825 & 0.0000 & 0.6962 & 1.6997 & 0.0000 \\ -0.8464 & -0.6962 & 0.0000 & -0.8498 & 0.0000 & 1.0000 \end{pmatrix},$$

$$\bar{\Lambda}_2 = \begin{pmatrix} 5.5003 & 2.1967 \\ 2.1967 & 4.3937 \end{pmatrix},$$

$$\bar{X}_3 = \begin{pmatrix} 12.5010 & 9.5007 & 0.5000 & -1.7733 & -0.0775 & -1.3042 \\ 9.5007 & 9.0470 & 0.5000 & 0.0775 & -0.2703 & -0.8012 \\ 0.5000 & 0.5000 & 5.5003 & 0.0000 & -0.0000 & 0.0000 \\ -1.7733 & 0.0775 & 0.0000 & 3.1490 & 0.8012 & -0.4970 \\ -0.0775 & -0.2703 & -0.0000 & 0.8012 & 0.9940 & -0.0000 \\ -1.3042 & -0.8012 & 0.0000 & -0.4970 & -0.0000 & 1.0000 \end{pmatrix},$$

$$\bar{\Lambda}_3 = \begin{pmatrix} 4.5003 & 1.0699 \\ 1.0699 & 4.1399 \end{pmatrix},$$

$$\bar{X}_4 = \begin{pmatrix} 13.5026 & 10.5017 & 0.5000 & -2.6382 & -0.4572 & -0.8196 \\ 10.5017 & 11.7773 & 0.5000 & 0.4572 & -1.5449 & -0.7265 \\ 0.5000 & 0.5000 & 6.5009 & 0.0000 & 0.0000 & -0.0000 \\ -2.6382 & 0.4572 & 0.0000 & 4.7290 & 0.7265 & -0.9070 \\ -0.4572 & -1.5449 & 0.0000 & 0.7265 & 1.8140 & -0.0000 \\ -0.8196 & -0.7265 & -0.0000 & -0.9070 & -0.0000 & 1.0000 \end{pmatrix} \text{ and}$$

$$\bar{\Lambda}_4 = \begin{pmatrix} 5.5009 & 1.4072 \\ 1.4072 & 4.8145 \end{pmatrix},$$

and the optimal values for (SDP$_i$), $i = 1, \ldots, 4$, that is, Val(SDP$_i$) ≈ 3.0000, $i = 1, 2$ and Val(SDP$_i$) ≈ 1.0000, $i = 3, 4$. So, the optimal solution and value for (SDP) are $(\bar{\Lambda}_3, \bar{X}_3)$ and $(\bar{\Lambda}_4, \bar{X}_4)$, and 1, respectively. Finally, we consider the dual problem (SDD) of (SDP) as follows:

$$\begin{aligned}
\text{(SDD)} \quad \min_{i=1,\ldots,4} \quad \inf_{y} \quad & \sum_{\alpha} (f - v_i^T(\cdot))_\alpha y_\alpha \\
\text{s.t.} \quad & G_0 + \sum_{\alpha} y_\alpha G_\alpha \succeq 0, \\
& B_0 + \sum_{\alpha} y_\alpha B_\alpha \succeq 0.
\end{aligned}$$

Then, by using the MATLAB optimization package OPTI Toolbox [16] together with the SDP-solver SeDuMi [65], we can easily find the optimal solutions for (SDD$_i$), $i = 1, \ldots, 4$, that is,

$$y_1 = (-1,0,1,0,0,-1,0,1,0,-1,0,1,-1,1), \quad y_2 = (-1,0,1,0,0,-1,0,1,0,-1,0,1,-1,1)$$

$$y_3 = (-1,0,1,0,0,-1,0,1,0,-1,0,1,-1,1), \quad y_4 = (-1,0,1,0,0,-1,0,1,0,-1,0,1,-1,1)$$

and the optimal values for (SDD$_i$), $i = 1, \ldots, 4$, that is, Val(SDD$_i$) ≈ 3, $i = 1, 2$, and Val(SDD$_i$) ≈ 1, $i = 3, 4$. So, the optimal solution and value for (SDD) are y_3, y_4, and 1, respectively. So, Val(RP) = Val(D)$^{\text{sos}}$ = Val(SDP) = Val(SDD). Thus, Theorem 4.3.1 holds. Moreover, Theorem 4.3.2 also holds.

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감사의 글

먼저 본 학위 논문이 완성되기까지 석사 및 박사 과정동안 늘 격려와 걱정을 해주시고 연구에 매진할 수 있도록 아낌없는 지도와 가르침을 주셨으며 부족한 저에게 국내외 학회에 참석 및 발표를 할 수 있게 기회를 주시어 저의 연구에 소중한 경험을 할 수 있게 도와주신 지도교수님이신 이규명 교수님께 깊은 감사의 말씀을 드립니다.

아울러 바쁘신 와중에도 저의 논문심사를 위해 소중한 조언과 격려를 해주신 경성대학교 이병수 교수님과 동서대학교 조경미 교수님께 진심으로 감사드립니다. 또한, 여러 국내외 학회에서 같이 동행하여 많은 도움을 주셨으며 대학원생활에서 많은 조언과 격려를 아낌없이 해주신 부경대학교 김도상 교수님과 김태화 교수님께 감사드립니다. 그리고 본 학위 논문 연구를 무사히 끝낼 수 있게 도와주신 부경대학교 응용수학과 교수님들께도 감사의 말씀드립니다.

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