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# Subsethood measures for hesitant fuzzy sets

by

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Graduate School of Education

Pukyong National University

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# (Hesitant 퍼지 집합의 포함 관계 측도)

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by
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Hesitant 퍼지 집합의 포함 관계 측도

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요 약

본 논문의 목적은 subsethood, 엔트로피 그리고 유사성을 hesitant 퍼지 요소들을 위한 통일된 공식을 확립하는 것이다. subsethood의 엔트로피 측도를 유도하기 위해 hesitant 퍼지 subsethood 측도의 식을 제시한다.

최근에 Xu와 Xia는 hesitant 퍼지 요소(HFEs)를 위한 엔트로피와 유사성의 공리적 정의를 했고 hesitant 퍼지 요소의 엔트로피와 유사한 정도는 각각 그들의 공리적 정의에 기초해 변환할 수 있는 것으로 나타났다. 본 논문에서 subsethood, 엔트로피와 유사성 정도의 개념 사이에 hesitant 퍼지 요소의 통합 프레임워크를 소개하고 있다. hesitant 퍼지 요소의 subsethood 측도의 공리적 정의를 제안하고, 이것들로부터 엔트로피 측도를 유도한다. 포함 관계 측도 정의에 근거하여, 엔트로피-subsethood 정리의 hesitant 퍼지 버전을 증명하고 hesitant 퍼지 요소의 엔트로피를 얻는다. 더 나아가, Kosko가 소개한 퍼지집합에서의 hesitant-subsethood 퍼지 설정에서 퍼지 엔트로피 정리를 확장한다. 마지막으로, hesitant 퍼지 subsethood와 hesitant 유사성 정도 사이의 관계를 조사한다.

# 1 Introduction

Since Zadeh [24] introduced fuzzy sets (FSs) as a tool treating imprecision and uncertainty, many its extensions such as intuitionistic fuzzy sets (IFSs) [1], interval-valued fuzzy sets(IVFSs) [25], interval-valued intuitionistic fuzzy sets (IVIFSs) [2] and hesitant fuzzy sets (HFSs) [19] allowed people to deal with uncertainty and information in much broader perspective.

Zadeh [24] defined the subsethood for two FSs $A$ and $B$ in a unverse X as $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ for all $x \in X$ where $\mu_A$ and $\mu_B$ are the membership functions of A and B, respectively. This definition implies that subsethood is described by a bivalent function. Kosko [10, 11, 12] generalized this concept and defined a multivalent subsethood measure, expressing the degree to which a set A is a subset of B. Moreover, Kosko related entropy with subsethood. A method for constructing fuzzy subsethood and entropy measures based on Young's set of axiomatic requirements [23] was introduced by Bustince et al. [5]. Various researchers [3, 7, 8, 17, 23] introduced alternative measures of subsethood for FSs and proposed different systems of axioms, a measure of subsethood should comply with, in order to express our intuition of subsethood. In intuitionistic fuzzy setting, many studies have been done about this issue. Liu and Xiong [13] proposed the definition of subsethood for IFSs and discussed its relationship with entropy and similarity measures. Grzegorzewski and Mrówka [9] introduced a hamming distance-based on subsethood measure for IFSs and presented a modified set of axiomatic requirements. Cornelis and Kerre [6] approached the notion of subsethood for IFSs in defferent way, and proposed a set of axioms as an extension of Sinha and Dougherty [17] axioms for FSs. Vlachos and Sergiadis [20] proposed an axiomatic skeleton for subsethood measures in interval-valued fuzzy setting, in order for subsethood to reduce to entropy. Based on this set of axioms, they proved the interbval-valued fuzzy version of entropy-subsethood theorem and derived new measures of subsethood and entropy for IVFSs.

Recently, Xu and Xia [22] presented the axiomatic definitions of entropy and similarity measure for hesitant fuzzy elements (HFEs), and showed that the entropy and the similarity measure for HFEs can be transformed by each other based on their axiomatic definitions. In this thesis, a unified framework between the concepts of subsethood, entropy and similarity measure for HTEs is presented. We propose axiomatic definition of subsethood measure for HFEs, and reduce an entropy measure from them. Based on this definition, we prove an hesitant fuzzy version of the entropy-subsethood theorem [10, 11, 12] and derive entropy for HFEs. Furthermore, we extend the fuzzy entropy theorem [10, 11, 12] in the hesitant fuzzy setting. Finally, we investigate the relationship between hesitant fuzzy subsethood and hesitant similarity measures.

# 2 Basic concepts for hesitant fuzzy elements

The hesitant fuzzy set, as a generalization of FS, permits the membership degree of an element to a set presented as several possible values between 0 and 1, which can better describe the situations where people have hesitancy in providing their preferences over objects in process of decision making.

**Definition 2.1** [19] Given a fixed set $X$, a hesitant fuzzy set (HFS) on $X$ in terms of function $\alpha$ is that when applied to $X$ returns a subset of [0,1], which can be represented as the following mathematical symbol:

$$A = \{\langle x, \alpha(x)\rangle | x \in X\}, \tag{1}$$

where $\alpha(x)$ is a set of the some values in $[0, 1]$, denoting the possible membership degrees of the element $x \in X$ to the set $A$. For convenience, Xia and Xu [21] called $\alpha(x)$ a hesitant fuzzy element (HFE) and the set of all HFEs is denoted by H. Especially, it there is only one value in $\alpha(x)$, then the HFS reduces to the FS, which indicates that FSs are special type of HFSs, therefore, the theory for HFSs can also be applied to FSs.

Some useful operations on HFEs are as follows:

**Definition 2.2** [15] Let $\alpha$, $\alpha_1$ and $\alpha_2$ be three HFEs, then

(1) $\alpha_1 \cup \alpha_2 = \cup_{\gamma_1 \in \alpha_1, \gamma_2 \in \alpha_2}\{\max\{\gamma_1, \gamma_2\}\}$;

(2) $\alpha_1 \cap \alpha_2 = \cup_{\gamma_1 \in \alpha_1, \gamma_2 \in \alpha_2}\{\min\{\gamma_1, \gamma_2\}\}$;

(3) $\alpha^c = \cup_{\gamma \in \alpha}\{1 - \gamma\}$.

It is noted that the number of values in different HFEs may be different, let $l_\alpha$ be the number of values in $\alpha$. Then we arrange the elements in $\alpha$ in increasing order, let $\alpha_{\sigma(i)}$ $(i = 1, 2, \ldots, l_\alpha)$ be the $i$th smallest value in $\alpha$. To operate correctly, we assume that the HFEs $\alpha$ and $\beta$ should have the same length $l$ when we compare them. If the one is shorter than the other, we should extend the shorter one until both of them have the same length. To extend the shorter one, the best way is to add the same value several times in it. So, if there is only one value in $\alpha$, we should extend it by repeating the value in it until it has the same length with $\beta$. Thus, we define the inclusion between two GHFEs $\tilde{\alpha}$ and $\tilde{\beta}$ as follows:

$$\alpha \subset \beta \text{ if and only if } \alpha_{\sigma(i)} \leq \beta_{\sigma(i)}, \text{ for } i = 1, 2, \ldots, l. \tag{2}$$

**Definition 2.3** [22] An entropy on HFE $\alpha$ is a real-valued function $E : H \to [0, 1]$, satisfying the following axiomatic requiements:

(1) $E(\alpha) = 0$ if and only if $\alpha = 0$ or $\alpha = 1$;

(2) $E(\alpha) = 1$ if and only if $\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} = 1$ for $i = 1, 2, \ldots, l_\alpha$;

(3) $E(\alpha) \leq E(\beta)$ if $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)}$ for $\beta_{\sigma(i)} + \beta_{\sigma(l-i+1)} \leq 1$, or if $\alpha_{\sigma(i)} \geq \beta_{\sigma(i)}$ for $\beta_{\sigma(i)} + \beta_{\sigma(l-i+1)} \geq 1$, $i = 1, 2, \ldots, l$;

(4) $E(\alpha) = E(\alpha^c)$.

Definition 2.3 is developed based on the axiomatic definition of fuzzy set. Motivated by the entropy measures for fuzzy sets, we can construct some entropy formulas based on Definition 2.3 as follows:

$$\begin{aligned} E_1(\alpha) = \frac{1}{l_\alpha(\sqrt{2}-1)} \sum_{i=1}^{l_\alpha} \Bigg( & \sin \frac{\pi\big(\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}\big)}{4} \\ & + \sin \frac{\pi\big(2 - \alpha_{\sigma(i)} - \alpha_{\sigma(l_\alpha - i+1)}\big)}{4} - 1 \Bigg) \end{aligned} \tag{3}$$

$$\begin{aligned} E_2(\alpha) = \frac{1}{l_\alpha(\sqrt{2}-1)} \sum_{i=1}^{l_\alpha} \Bigg( & \cos \frac{\pi\big(\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}\big)}{4} \\ & + \cos \frac{\pi\big(2 - \alpha_{\sigma(i)} - \alpha_{\sigma(l_\alpha - i+1)}\big)}{4} - 1 \Bigg) \end{aligned} \tag{4}$$

$$\begin{aligned} E_3(\alpha) = -\frac{1}{l_\alpha \ln 2} \sum_{i=1}^{l_\alpha} \Bigg( & \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \ln \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \\ + & \frac{2 - \alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \ln \frac{2 - \alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Bigg) \end{aligned} \tag{5}$$

$$E_4(\alpha) = \frac{1}{l_\alpha(2^{(1-s)t}-1)} \sum_{i=1}^{l_\alpha} \Bigg( \bigg( \Big( \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Big)^s + \Big( 1 - \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Big)^s \bigg)^t - 1 \Bigg), \ t \neq 0, s \neq 1, s > 0 \tag{6}$$

Moreover, with the change of the parameters in $E_4$, some special cases can be obtained:

If $t = 1$, then

$$E_4(\alpha) = \frac{1}{l_\alpha(2^{1-s}-1)} \sum_{i=1}^{l_\alpha} \bigg( \Big( \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Big)^s + \Big( 1 - \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Big)^s - 1 \bigg) \tag{7}$$

If $t = \frac{1}{s}$, then

$$E_4(\alpha) = \frac{s}{l_\alpha(2^{(1-s)/s}-1)} \sum_{i=1}^{l_\alpha} \Bigg( \bigg( \Big( \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Big)^s + \Big( 1 - \frac{\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)}}{2} \Big)^s \bigg)^{1/s} - 1 \Bigg) \tag{8}$$

Xu and Xia gave the hesitant fuzzy similarity measure defined as

**Definition 2.4** [22] For two HFEs $\alpha$ and $\beta$, the similarity measure between $\alpha$ and $\beta$, denoted as $S(\alpha, \beta)$, should satisfy the following properties:

(1) $S(\alpha, \beta) = 0$ if $\alpha = 0$, $\beta = 1$ or $\alpha = 1$, $\beta = 0$;

(2) $S(\alpha, \beta) = 1$ if and only if $\alpha = \beta$, i.e. $\alpha_{\sigma(i)} = \beta_{\sigma(i)}$, $i = 1, 2, \dots, l$;

(3) $S(\alpha, \gamma) \leq S(\alpha, \beta)$, $S(\alpha, \gamma) \leq S(\beta, \gamma)$, if $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)} \leq \gamma_{\sigma(i)}$, or if $\alpha_{\sigma(i)} \geq \beta_{\sigma(i)} \geq \gamma_{\sigma(i)}$, $i = 1, 2, \dots, l$;

(4) $S(\alpha, \beta) = S(\beta, \alpha)$.

Based on Definition 2.4, some hesitant fuzzy similarity measures can be constructed as:

$$S_1(\alpha, \beta) = 1 - \frac{1}{l} \sum_{i=1}^{l} |\alpha_{\sigma(i)} - \beta_{\sigma(i)}| \tag{9}$$

$$S_2(\alpha,\beta) = 1 - \sqrt{\frac{1}{l}\sum_{i=1}^{l}(\alpha_{\sigma(i)} - \beta_{\sigma(i)})^2} \tag{10}$$

$$S_3(\alpha,\beta) = 1 - \sqrt[p]{\frac{1}{l}\sum_{i=1}^{l}|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^p} \tag{11}$$

$$S_4(\alpha,\beta) = \max_i\{|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|\} \tag{12}$$

$$S_5(\alpha,\beta) = \max_i\{|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^2\} \tag{13}$$

$$S_6(\alpha,\beta) = \max_i\{|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^p\} \tag{14}$$

$$S_7(\alpha,\beta) = 1 - \frac{1}{2}\left(\frac{1}{l}\sum_{i=1}^{l}|\alpha_{\sigma(i)} - \beta_{\sigma(i)}| + \max_i\{|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|\}\right) \tag{15}$$

$$S_8(\alpha,\beta) = 1 - \frac{1}{2}\left(\sqrt{\frac{1}{l}\sum_{i=1}^{l}|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^2} + \max_i\{|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^2\}\right) \tag{16}$$

$$S_9(\alpha,\beta) = 1 - \frac{1}{2}\left(\sqrt[p]{\frac{1}{l}\sum_{i=1}^{l}|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^p} + \max_i\{|\alpha_{\sigma(i)} - \beta_{\sigma(i)}|^p\}\right) \tag{17}$$

By analyzing these similarity measures, we can find that $S_1$ and $S_2$ are based on the Hamming distance and the Euclidean distance; $S_4$ and $S_5$ apply the Hausdorff metric to $S_1$ and $S_2$; $S_7$ combines $S_1$ with $S_4$; $S_8$ combines $S_2$ with $S_5$; $S_3$, $S_6$, and $S_9$ are further generalizations of $S_1$ and $S_2$, $S_4$, and $S_5$, $S_7$, and $S_8$, respectively; when $p = 1$, then $S_3$ becomes $S_1$, $S_6$ becomes $S_4$, and $S_9$ becomes $S_7$; when $p = 2$, then $S_3$ reduces to $S_2$, $S_6$ reduces to $S_5$, and $S_9$ reduces to $S_8$.

Many authors have investigated the relationships between similarity measures and entropy formulas under different environments, such as interval-valued fuzzy sets, interval-valued intuitionistic fuzzy sets. In what follow, we study the relationships between hesitant fuzzy similarity measures and hesitant fuzzy entropy formulas:

**Theorem 2.5** [22] *Let $\alpha$ be an HFE, then $S(\alpha, \alpha^c)$ is an entropy of $\alpha$.*

**Proof** (1) $S(\alpha, \alpha^c) = 0 \Leftrightarrow \alpha = 0$ and $\alpha^c = 1$ or $\alpha^c = 0$ and $\alpha = 1$;

(2) $S(\alpha, \alpha^c) = 1 \Leftrightarrow \alpha = \alpha^c \Leftrightarrow \alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} = 1$ for $i = 1, 2, \ldots, l$;

(3) Suppose that $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)}$, for $\beta_{\sigma(i)} + \beta_{\sigma(l_\alpha - i+1)} \leq 1$, $i = 1, 2, \ldots, l$, then $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)} \leq 1 - \beta_{\sigma(l_\alpha - i+1)} \leq 1 - \alpha_{\sigma(l_\alpha - i+1)}$. Therefore, by the definition of the similarity measure of HFE, we have $S(\alpha, \alpha^c) \leq S(\beta, \alpha^c) \leq S(\beta, \beta^c)$. With the same reason, we can prove it when $\alpha_{\sigma(i)} \geq \beta_{\sigma(i)}$, for $\beta_{\sigma(i)} + \beta_{\sigma(l_\alpha - i+1)} \geq 1$, $i = 1, 2, \ldots, l_\alpha$;

(4) $S(\alpha, \alpha^c) = S(\alpha^c, \alpha)$. □

**Example 2.6** [22] For two HFEs $\alpha$ and $\beta$, we can construct the following entropy formulas based on the similarity measures $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, $S_6$, $S_7$, $S_8$ and $S_9$:

$$S_1(\alpha, \alpha^c) = 1 - \frac{1}{l_\alpha} \sum_{i=1}^{l_\alpha} |\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} - 1| \tag{18}$$

$$S_2(\alpha, \alpha^c) = 1 - \sqrt{\frac{1}{l_\alpha} \sum_{i=1}^{l_\alpha} |\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} - 1|^2} \tag{19}$$

$$S_3(\alpha, \alpha^c) = 1 - \sqrt[p]{\frac{1}{l_\alpha} \sum_{i=1}^{l} |\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} - 1|^p} \tag{20}$$

$$S_4(\alpha, \alpha^c) = \max_i \{|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} - 1|\} \tag{21}$$

$$S_5(\alpha, \alpha^c) = \max_i \{|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} - 1|^2\} \tag{22}$$

$$S_6(\alpha, \alpha^c) = \max_i \{|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} - 1|^p\} \tag{23}$$

$$S_7(\alpha,\alpha^c) = 1 - \frac{1}{2}\bigg(\frac{1}{l_\alpha}\sum_{i=1}^{l_\alpha}|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha-i+1)} - 1|$$
$$+ \max_i\{|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha-i+1)} - 1|\}\bigg) \tag{24}$$

$$S_8(\alpha,\alpha^c) = 1 - \frac{1}{2}\bigg(\sqrt{\frac{1}{l_\alpha}\sum_{i=1}^{l_\alpha}|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha-i+1)} - 1|^2}$$
$$+ \max_i\{|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha-i+1)} - 1|^2\}\bigg) \tag{25}$$

$$S_9(\alpha,\alpha^c) = 1 - \frac{1}{2}\bigg(\sqrt[p]{\frac{1}{l_\alpha}\sum_{i=1}^{l_\alpha}|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha-i+1)} - 1|^p}$$
$$+ \max_i\{|\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha-i+1)} - 1|^p\}\bigg) \tag{26}$$

In this thesis, we let $[l/2]$ denote the largest integer no bigger than $1/2$, and $\overline{[l/2]}$ denote the smallest integer no smaller than $1/2$, then we get the following theorem:

**Theorem 2.7** [22] *For an HFE $\alpha$, let $\eta_\alpha = \{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(\overline{[l/2]})}\}$ and $\gamma_\alpha = \{1-\alpha_{\sigma(l_\alpha)}, 1-\alpha_{\sigma(l_\alpha-1)}, \ldots, 1-\alpha_{\sigma([l_\alpha/2]+1)}\}$, then $S(\eta_\alpha, \gamma_\alpha)$ is an entropy of $\alpha$.*

**Proof** (1) $S(\eta_\alpha,\gamma_\alpha) = 0 \Leftrightarrow \eta_\alpha = 0$, $\gamma_\alpha = 1$ or $\eta_\alpha = 1$, $\gamma_\alpha = 0 \Leftrightarrow \alpha = 0$ or $\alpha = 1$;

(2) $S(\eta_\alpha,\gamma_\alpha) = 1 \Leftrightarrow \eta_\alpha = \gamma_\alpha \Leftrightarrow$ for $i = 1, 2, \ldots, l_\alpha$;

(3) Assume $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)}$, for $\beta_{\sigma(i)} + \beta_{\sigma(l-i+1)} \leq 1$, $i = 1, 2, \ldots, l$, then we have $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)} \leq 1 - \beta_{\sigma(l-i+1)} \leq 1 - \alpha_{\sigma(l-i+1)}$. Therefore, from the definition of the similarity measure of HFEs, we have $S(\eta_\alpha,\gamma_\alpha) \leq S(\eta_\beta,\gamma_\alpha) \leq S(\eta_\beta,\gamma_\beta)$. Similarly, we can prove it is also true when $\alpha_{\sigma(i)} \geq \beta_{\sigma(i)}$, for $\beta_{\sigma(i)} + \beta_{\sigma(l-i+1)} \geq 1$, $i = 1, 2, \ldots, l$;

(4) $S(\eta_\alpha,\gamma_\alpha) = S(\gamma_\alpha,\eta_\alpha)$. □

**Example 2.8** [22] For an HFE $\alpha$, we can construct the following entropy formulas based on the similarity measures $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, $S_6$, $S_7$, $S_8$ and $S_9$:

$$S_1(\eta_\alpha,\gamma_\alpha)=1-\overline{\left[\frac{2}{l_\alpha}\right]}\sum_{i=1}^{\overline{[l_\alpha/2]}}|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1| \tag{27}$$

$$S_2(\eta_\alpha,\gamma_\alpha)=1-\sqrt{\overline{\left[\frac{2}{l_\alpha}\right]}\sum_{i=1}^{\overline{[l_\alpha/2]}}|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^2} \tag{28}$$

$$S_3(\eta_\alpha,\gamma_\alpha)=1-\sqrt[p]{\overline{\left[\frac{2}{l_\alpha}\right]}\sum_{i=1}^{\overline{[l_\alpha/2]}}|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^p} \tag{29}$$

$$S_4(\eta_\alpha,\gamma_\alpha)=\max_i\{|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|\} \tag{30}$$

$$S_5(\eta_\alpha,\gamma_\alpha)=\max_i\{|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^2\} \tag{31}$$

$$S_6(\eta_\alpha,\gamma_\alpha)=\max_i\{|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^p\} \tag{32}$$

$$S_7(\eta_\alpha,\gamma_\alpha)=1-\overline{\left[\frac{1}{l_\alpha}\right]}\Bigg(\sum_{i=1}^{\overline{[l_\alpha/2]}}|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1| + \max_i\{|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|\}\Bigg) \tag{33}$$

$$S_8(\eta_\alpha,\gamma_\alpha)=1-\frac{1}{2}\Bigg(\sqrt{\overline{\left[\frac{2}{l_\alpha}\right]}\sum_{i=1}^{\overline{[l_\alpha/2]}}|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^2} + \max_i\{|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^2\}\Bigg) \tag{34}$$

$$S_9(\eta_\alpha,\gamma_\alpha)=1-\frac{1}{2}\Bigg(\sqrt[p]{\overline{\left[\frac{2}{l_\alpha}\right]}\sum_{i=1}^{\overline{[l_\alpha/2]}}|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^p} + \max_i\{|\alpha_{\sigma(i)}+\alpha_{\sigma(l_\alpha-i+1)}-1|^p\}\Bigg) \tag{35}$$

# 3 Subsethood measures for HFEs

In this section, we shall present the axiomatic definition of subsethood measure for HFE motivated by Liu and Xiong [13], Vlachos and Sergiadis [20] and Park et al. [14], from which we can establish a connection between subsethood, entropy and similarity measures for HFEs.

**Definition 3.1** Let $\alpha$ and $\beta$ be two HFEs, then the subsethood measure of $\alpha$ to $\beta$, denoted as $C(\alpha,\beta)$, should satisfy the following conditions:

(1) $C(\alpha,\beta)=1$ if and only if $\alpha\subset\beta$, i.e., $\alpha_{\sigma(i)}\leq\beta_{\sigma(i)}$, $i=1,2,\ldots,l$;

(2) If $\alpha^c\subset\alpha$, then $C(\alpha,\alpha^c)=0$ if and only if $\alpha=1$;

(3) If $\alpha\subset\beta\subset\gamma$, $i=1,2,\ldots,l$, then $C(\beta,\alpha)\geq C(\gamma,\alpha)$, and if $\beta\subset\gamma$, then $C(\alpha,\beta)\leq C(\alpha,\gamma)$.

**Theorem 3.2** *Let $\alpha$ and $\beta$ be two HFEs, then*

$$C_1(\alpha,\beta)=1-\frac{\sum_{i=1}^{l}\max\{0,\alpha_{\sigma(i)}-\beta_{\sigma(i)}\}}{\sum_{i=1}^{l}\alpha_{\sigma(i)}} \tag{36}$$

*is subsethood measure of $\alpha$ to $\beta$.*

**Proof** (1) $C_1(\alpha,\beta)=1\Leftrightarrow\max\{0,\alpha_{\sigma(i)}-\beta_{\sigma(i)}\}=0, i=1,2,\ldots,l\Leftrightarrow\alpha_{\sigma(i)}\leq\beta_{\sigma(i)}, i=1,2,\ldots,l\Leftrightarrow\alpha\subset\beta$.

(2) Suppose that $\alpha^c\subset\alpha$, then $\alpha_{\sigma(i)}\geq 1-\alpha_{\sigma(l_\alpha-i+1)}$, $i=1,2,\ldots,l$. From Eq. (3), we obtain

$$\begin{aligned}C_1(\alpha,\alpha^c)&=1-\frac{\sum_{i=1}^{l}\max\{0,\alpha_{\sigma(i)}-1+\alpha_{\sigma(l_\alpha-i+1)}\}}{\sum_{i=1}^{l}\alpha_{\sigma(i)}}\\&=1-\frac{\sum_{i=1}^{l}(\alpha_{\sigma(i)}-1+\alpha_{\sigma(l_\alpha-i+1)})}{\sum_{i=1}^{l}\alpha_{\sigma(i)}}.\end{aligned} \tag{37}$$

Thus, we have $C_1(\alpha,\alpha^c)=0\Leftrightarrow\alpha=1$.

(3)Suppose that $\alpha\subset\beta\subset\gamma$. Since $\alpha_{\sigma(i)}\leq\beta_{\sigma(i)}\leq\gamma_{\sigma(i)}$, $i=1,2,\ldots,l$, we have

$$C_1(\beta,\alpha) = 1 - \frac{\sum_{i=1}^{l}(\beta_{\sigma(i)} - \alpha_{\sigma(i)})}{\sum_{i=1}^{l}\beta_{\sigma(i)}} = \frac{\sum_{i=1}^{l}\alpha_{\sigma(i)}}{\sum_{i=1}^{l}\beta_{\sigma(i)}}$$
$$\geq \frac{\sum_{i=1}^{l}\alpha_{\sigma(i)}}{\sum_{i=1}^{l}\gamma_{\sigma(i)}} = 1 - \frac{\sum_{i=1}^{l}(\gamma_{\sigma(i)} - \alpha_{\sigma(i)})}{\sum_{i=1}^{l}\gamma_{\sigma(i)}} = C_1(\gamma,\alpha). \tag{38}$$

Next, suppose that $\beta \subset \gamma$, then $\beta_{\sigma(i)} \leq \gamma_{\sigma(i)}$, $i = 1, 2, \ldots, l$. Due to the monotonicity of max operator, it follows that

$$C_1(\alpha,\beta) = 1 - \frac{\sum_{i=1}^{l}\max\{0, \alpha_{\sigma(i)} - \beta_{\sigma(i)}\}}{\sum_{i=1}^{l}\alpha_{\sigma(i)}}$$
$$\leq 1 - \frac{\sum_{i=1}^{l}\max\{0, \alpha_{\sigma(i)} - \gamma_{\sigma(i)}\}}{\sum_{i=1}^{l}\alpha_{\sigma(i)}} = C_1(\alpha,\gamma). \tag{39}$$

□

**Remark 3.3** Note that if $\alpha = 0$, Eq. (3) is undefined. However, since $0 \subset \beta$, for any HFE $\beta$, by the definition, we have $C_1(0,\beta) = 1$.

**Theorem 3.4** *For two HFEs $\alpha$ and $\beta$, we define*

$$C_2(\alpha,\beta) = \frac{1}{l}\sum_{i=1}^{l}\min\{1, g(\phi(\alpha_{\sigma(i)} - \beta_{\sigma(i)} + 1))\}, \tag{40}$$

*where $g : [0,2] \to [0,2]$ is real function with the properties: 1) $x > y \Rightarrow g(x) < g(y)$ for $x, y \in [0,2]$; 2) $g(x) = 0 \Leftrightarrow x = 2$; 3) $g(1) = 1$ and $\phi : [0,2] \to [0,2]$ are real functions with the following properties: 1) $x > y \Rightarrow \phi(x) > \phi(y)$ for $x, y \in [0,2]$; 2) $\phi(x) = 2 \Leftrightarrow x = 2$; 3) $\phi(1) = 1$.*

*Then $C_2(\alpha,\beta)$ is susethood measure of $\alpha$ to $\beta$.*

**Proof** (1) Suppose that $\alpha \subset \beta$, let $\alpha_{\sigma(i)} = \alpha_{\sigma(i)} - \beta_{\sigma(i)} + 1$, $i = 1, 2, \ldots, l$. Since $\alpha_i \leq 1$, we have $\phi(\alpha_i) \leq 1$ and then $g(\phi(\alpha_i)) \geq g(1)$=1. Thus, we have $C_2(\alpha,\beta) = \frac{1}{l}\sum_{i=1}^{l}\min\{1, g(\phi(\alpha_i))\} = 1$. Suppose that $C_2(\alpha,\beta) = 1$, then $g(\phi(\alpha_{\sigma(i)} - \beta_{\sigma(i)} + 1)) \geq 1$, $i = 1, 2, \ldots, l$. Thus, we get $\alpha_{\sigma(i)} - \beta_{\sigma(i)} + 1 \leq 1$, $i = 1, 2, \ldots, l$. In fact, suppose that there exists $j$ such that $\alpha_{\sigma(j)} - \beta_{\sigma(j)} + 1 > 1$. Since $\alpha = \alpha_{\sigma(j)} - \beta_{\sigma(j)} + 1 > 1$, then $\phi(\alpha) > 1$ and thus $g(\phi(\alpha)) < g(1) = 1$, which is a contradiction. So, we have $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)}$, $i = 1, 2, \ldots, l$ and thus $\alpha \subset \beta$.

(2) Suppose that $\alpha^c \subset \alpha$, then $\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} \geq 1$, $i = 1, 2, \ldots, l_\alpha$, and so we have

$$\begin{aligned} C_2(\alpha, \alpha^c) = 0 &\Leftrightarrow \phi(\alpha_{\sigma(i)} - (1 - \alpha_{\sigma(l_\alpha - i+1)}) + 1) = 2, i = 1, 2, \ldots, l_\alpha \\ &\Leftrightarrow \alpha_{\sigma(i)} = 1, i = 1, 2, \ldots, l_\alpha \\ &\Leftrightarrow \alpha = 1. \end{aligned}$$

(3) Suppose that $\alpha \subset \beta \subset \gamma$, let $\alpha_{1,i} = \beta_{\sigma(i)} - \alpha_{\sigma(i)} + 1$ and $\alpha_{2,i} = \gamma_{\sigma(i)} - \alpha_{\sigma(i)} + 1$, $i = 1, 2, \ldots, l$. Then $\phi(\alpha_{1,i}) \leq \phi(\alpha_{2,i})$, $i = 1, 2, \ldots, l$, which implies $g(\phi(\alpha_{1,i})) \geq g(\phi(\alpha_{2,i}))$, $i = 1, 2, \ldots, l$. Thus $C_2(\gamma, \alpha) \leq C_2(\beta, \alpha)$. With the same reason, we can prove that $C_2(\alpha, \beta) \leq C_2(\alpha, \gamma)$ is also true for $\beta \subset \gamma$. □

Now, we start with the following definition to generalize the fuzzy entropy theorem in the setting of HFEs.

**Definition 3.5** Let $\alpha$ and $\beta$ be two HFEs, then

(1) $\alpha \tilde{\cup} \beta = \cup_{i=1}^{l} \{\max\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}\}$;

(2) $\alpha \tilde{\cap} \beta = \cup_{i=1}^{l} \{\min\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}\}$.

**Theorem 3.6** *Let $\alpha$,$\beta$ and $\gamma$ be two HFEs, then*

(1) $\alpha \tilde{\cup} \beta = \beta \tilde{\cup} \alpha$;

(2) $\alpha \tilde{\cap} \beta = \beta \tilde{\cap} \alpha$;

(3) $\alpha^c \tilde{\cup} \beta^c = (\alpha \tilde{\cap} \beta)^c$;

(4) $\alpha^c \tilde{\cap} \beta^c = (\alpha \tilde{\cup} \beta)^c$;

(5) $\alpha \tilde{\cup} (\beta \tilde{\cap} \gamma) = (\alpha \tilde{\cup} \beta) \tilde{\cap} (\alpha \tilde{\cup} \gamma)$;

(6) $\alpha \tilde{\cap} (\beta \tilde{\cup} \gamma) = (\alpha \tilde{\cap} \beta) \tilde{\cup} (\alpha \tilde{\cap} \gamma)$.

**Proof** (1) $\alpha \tilde{\cup} \beta = \cup_{i=1}^{l} \{\max\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}\} = \cup_{i=1}^{l} \{\max\{\beta_{\sigma(i)}, \alpha_{\sigma(i)}\}\} = \beta \tilde{\cup} \alpha$.

(2) $\alpha \tilde{\cap} \beta = \cup_{i=1}^{l} \{\min\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}\} = \cup_{i=1}^{l} \{\min\{\beta_{\sigma(i)}, \alpha_{\sigma(i)}\}\} = \beta \tilde{\cap} \alpha$.

(3)

$$\begin{aligned}\alpha^c\tilde{\cup}\beta^c &= \cup_{i=1}^{l}\{\max\{1-\alpha_{\sigma(l-i+1)}, 1-\beta_{\sigma(l-i+1)}\}\} \\ &= \left(\cup_{i=1}^{l}\{\min\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}\}\right)^c \\ &= (\alpha\tilde{\cap}\beta)^c.\end{aligned}$$

(4)

$$\begin{aligned}\alpha^c\tilde{\cap}\beta^c &= \cup_{i=1}^{l}\{\min\{1-\alpha_{\sigma(l-i+1)}, 1-\beta_{\sigma(l-i+1)}\}\} \\ &= \left(\cup_{i=1}^{l}\{\max\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}\}\right)^c \\ &= (\alpha\tilde{\cup}\beta)^c.\end{aligned}$$

(5) From Definition 3.5, we have

$$\begin{aligned}\alpha\tilde{\cup}(\beta\tilde{\cap}\gamma) &= \cup_{i=1}^{l}\{\max\{\alpha_{\sigma(i)}, \min\{\beta_{\sigma(i)}, \gamma_{\sigma(i)}\}\}\} \\ &= \cup_{i=1}^{l}\{\min\{\max\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}, \max\{\alpha_{\sigma(i)}, \gamma_{\sigma(i)}\}\}\} \\ &= (\alpha\tilde{\cup}\beta)\tilde{\cap}(\alpha\tilde{\cup}\gamma).\end{aligned}$$

(6) From Definition 3.5, we have

$$\begin{aligned}\alpha\tilde{\cap}(\beta\tilde{\cup}\gamma) &= \cup_{i=1}^{l}\{\min\{\alpha_{\sigma(i)}, \max\{\beta_{\sigma(i)}, \gamma_{\sigma(i)}\}\}\} \\ &= \cup_{i=1}^{l}\{\max\{\min\{\alpha_{\sigma(i)}, \beta_{\sigma(i)}\}, \min\{\alpha_{\sigma(i)}, \gamma_{\sigma(i)}\}\}\} \\ &= (\alpha\tilde{\cap}\beta)\tilde{\cup}(\alpha\tilde{\cap}\gamma).\end{aligned}$$

□

The relationships between subsethood measures and entropy have been studied by many authors under different environments, such as fuzzy sets, interval-valued fuzzy sets and interval-valued intuitionistic fuzzy sets. In the following, we investigate the relationship between hesitant fuzzy subsethood measure and hesitant fuzzy entropy:

**Theorem 3.7** *Let $\alpha$ be a HFE, then $E(\alpha) = C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c)$ is an entropy for $\alpha$.*

**Proof** (1) Suppose that $\alpha = 1$ or $\alpha = 0$, then $\alpha\tilde{\cup}\alpha^c = 1$ and $\alpha\tilde{\cap}\alpha^c = 0$. Since $\alpha\tilde{\cap}\alpha^c = (\alpha\tilde{\cup}\alpha^c)^c$, we have $\alpha\tilde{\cap}\alpha^c \supset (\alpha\tilde{\cup}\alpha^c)^c$ and thus by (2) of Definition 3.1, $E(\alpha) = C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) = 0$. Suppose that $E(\alpha) = C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) = 0$, that is $C(\alpha\tilde{\cup}\alpha^c, (\alpha\tilde{\cup}\alpha^c)^c) = 0$. Then, since $\alpha\tilde{\cup}\alpha^c \supset \alpha\tilde{\cap}\alpha^c$, by (2) of Definition 3.1, we obtain $\alpha\tilde{\cup}\alpha^c = 1$. Hence $\alpha = 1$ or $\alpha = 0$.

(2) Suppose that $\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} = 1$ for $i = 1, 2, \ldots, l_{\tilde{\alpha}}$, then $\alpha\tilde{\cup}\alpha^c = \alpha\tilde{\cap}\alpha^c$ and thus by (1) of Definition 3.1, $E(\alpha) = C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) = 1$. Suppose that $E(\alpha) = C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) = 1$, then from (1) Definition 3.1, we deduce $\alpha\tilde{\cup}\alpha^c = \alpha\tilde{\cap}\alpha^c$, which implies $\alpha_{\sigma(i)} + \alpha_{\sigma(l_\alpha - i+1)} = 1$, $i = 1, 2, \ldots, l_\alpha$.

(3) Suppose that $\alpha_{\sigma(i)} \le \beta_{\sigma(i)}$ for $\beta_{\sigma(i)} + \beta_{\sigma(l-i+1)} \le 1$, $i = 1, 2, \ldots, l$, then $\alpha_{\sigma(i)} \le \beta_{\sigma(i)} \le 1 - \beta_{\sigma(l-i+1)} \le 1 - \alpha_{\sigma(l-i+1)}$. By Definition 3.1, $C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) \le C(\alpha\tilde{\cup}\alpha^c, \beta\tilde{\cap}\beta^c) \le C(\beta\tilde{\cup}\beta^c, \beta\tilde{\cap}\beta^c)$ and thus $E(\alpha) \le E(\beta)$. With the same reason, when $\alpha_{\sigma(i)} \ge \beta_{\sigma(i)}$ for $\beta_{\sigma(i)} + \beta_{\sigma(l-i+1)} \ge 1$, $i = 1, 2, \ldots, l$, we can prove $E(\tilde{\alpha}) \le E(\tilde{\beta})$.

(4) $E(\alpha^c) = C(\alpha^c\tilde{\cup}\alpha, \alpha^c\tilde{\cap}\alpha) = C(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) = E(\alpha)$. □

**Remark 3.8** Theorem 3.7 describes an interesting relationship between the entropy and subsethood measure for HFEs. It states that the entropy $E(\alpha)$ expresses the degree to which the supset $\alpha\tilde{\cup}\alpha^c$ is a subset of its own subset $\alpha\tilde{\cap}\alpha^c$. Evaluating for the proposed subsethood measures (36) and (40), yields two new entropy measures for HFEs given by

$$\begin{aligned} E_1(\alpha) &= C_1(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) \\ &= \frac{\sum_{i=1}^{l_\alpha} \min\{\alpha_{\sigma(i)}, 1 - \alpha_{\sigma(l_\alpha - i+1)}\}}{\sum_{i=1}^{l_\alpha} \max\{\alpha_{\sigma(i)}, 1 - \alpha_{\sigma(l_\alpha - i+1)}\}}, \end{aligned} \tag{41}$$

$$\begin{aligned} E_2 &= C_2(\alpha\tilde{\cup}\alpha^c, \alpha\tilde{\cap}\alpha^c) \\ &= \frac{1}{l}\sum_{i=1}^{l_\alpha} \min\{1, g(\phi(\max\{\alpha_{\sigma(i)}, 1 - \alpha_{\sigma(l_\alpha - i+1)}\} \\ &\qquad - \max\{\alpha_{\sigma(i)}, 1 - \alpha_{\sigma(l_\alpha - i+1)}\} + 1))\}. \end{aligned} \tag{42}$$

In the following, we investigate the relationship between hesitant fuzzy subsethood measure and hesitant similarity measure:

**Theorem 3.9** *Let $\alpha$ and $\beta$ be two HFEs, then $S(\alpha,\beta) = C(\alpha,\beta) \wedge C(\beta,\alpha)$ is a similarity measure of $\alpha$ and $\beta$.*

**Proof** (1) If $\alpha = 0$ and $\beta = 1$, then $\alpha = \beta^c \subset \beta$. By (2) of Definition 3.1, we have $S(\alpha,\beta) = C(\alpha,\beta) \wedge C(\beta,\alpha) = 0 \wedge C(\beta,\alpha) = 0$. With the same reason, when $\alpha = 1$ and $\beta = 0$, we can prove $S(\alpha,\beta) = 0$.

(2) $S(\alpha,\beta) = 1 \Leftrightarrow C(\alpha,\beta) = C(\beta,\alpha) = 1 \Leftrightarrow \alpha = \beta$.

(3) If $\alpha_{\sigma(i)} \leq \beta_{\sigma(i)} \leq \gamma_{\sigma(i)}$, $i = 1,2,\ldots,l$, then $\alpha \subset \beta \subset \gamma$ and thus, by (1) and (3) of Definition 3.1, we have

$$\begin{aligned} S(\alpha,\beta) &= C(\alpha,\beta) \wedge C(\beta,\alpha) = C(\beta,\alpha) \\ &\geq C(\gamma,\alpha) = 1 \wedge C(\gamma,\alpha) = C(\alpha,\gamma) \wedge C(\gamma,\alpha) = S(\alpha,\gamma), \quad (43) \\ S(\alpha,\gamma) &= C(\alpha,\gamma) \wedge C(\gamma,\alpha) = C(\gamma,\alpha) \\ &\leq C(\gamma,\beta) = 1 \wedge C(\gamma,\beta) = C(\beta,\gamma) \wedge C(\gamma,\beta) = S(\beta,\gamma). \quad (44) \end{aligned}$$

With the same reason, when $\alpha_{\sigma(i)} \geq \beta_{\sigma(i)} \geq \gamma_{\sigma(i)}$, $i = 1,2,\ldots,l$, we can prove $S(\alpha,\beta) \geq S(\alpha,\gamma)$ and $S(\alpha,\gamma) \leq S(\beta,\gamma)$.

(4) Obviously, $S(\alpha,\beta) = S(\beta,\alpha)$. □

# 4 Conclusions

In this thesis we presented a unified framwork for subsethood, entropy, and similarity for HFEs. An axiomatic skeleton for subsethood was introduced and new subsethood and entropy meaures in the hesitant fuzzy setting were proposed. Furthermore, hesitant fuzzy version of the entropy and entropy-subsethood theorems were stated and proved, which generalized the works of Kosko for FSs and Liu and Xiong for IFSs. Finally, we investigated the relationship between hesitant fuzzy subsethood and hesitant similarity measures.

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