Thesis for the Degree of Doctor of Philosophy

Controllability and optimal control for nonlinear differential equations

by

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Controllability and optimal control for nonlinear differential equations

(비선형 미분 방정식에 대한 제어가능성과 최적제어)

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Controllability and optimal control for nonlinear differential equations

A dissertation

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비선형 미분 방정식에 대한 제어가능성과 최적제어

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요　약

본 논문에서는 Hilbert 공간 H상에서 비선형항을 포함하는 다음과 같은 비선형 발전방정식:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) + Bu(t), \\ x(0) = x_0. \end{cases}$$

에 대한 해의 존재성과 유일성, 그리고 정칙성을 공간이론과 작용소이론을 이용하여 수학적 이론을 정립한다.

이러한 성질들을 이용하여 응용 상 중요한 제어이론을 유도하는데 목적이 있다. 여기서, 주 작용소 A가 Garding 조건을 만족하는 미분연산자로부터 정의된 선형작용소인 경우와 일반적인 비선형인 경우로 나누어 다루었다. 그리고 비선형항 $f(\bullet, x)$는 일반화된 Lipschitz 연속성을 만족한다.

얻어진 결과를 요약해보면

(1) 선형방정식에서 얻은 해의 정칙성 이론들이 비선형항이 포함된 준선형계에도 성립 가능성을 부등식 이론을 이용하여 증명하였다.

(2) $x(t,f,u)$는 비선형항 f와 제어 u에 대응하는 해(궤적)이라 두면 도달가능한 집합 $R_T(f) = x(T;f,u) : u \in L^2(0, T: U)$이 주어진 공간 H상에서 조밀한 공간이 되는 즉 가제어성을 증명하였다.

(3) 주어진 목적함수에 대한 최적제어의 존재성과 존재에 대한 필요조건을 유도하였다.

Chapter 1
Introduction

In control and system theory the fundamentally important concept of controllability arose naturally during the early development of optimal control theory in the 1960s and was developed by a number of mathematicians and engineers in the world.

In addition to making further contributions to control theory of discrete processes, the present paper gives a treatment of constrained control problems with emphasis on the controllability of dynamical discrete-time systems.

Control theory of nonlinear systems requires more sophisticated methods than those of linear systems. The difficulties increase to the same extent as passing from linear discrete-time systems to nonlinear discrete-time systems, especially when the constraints on both control and state are involved, the nonlinear controllability problem becomes to be considerably difficult.

We deal with the existence, uniqueness, and a variation of solutions of the nonlinear control system with nonlinear monotone hemicontinuous and coercive operator. Of caurse, we study the semilinear case with linear principal operator satisfying Gårding's inequality.

For general nonlinear control systems, we wiuse several approaches for the study of controllability problem:

(1) Fixed-point methods

(2) Methods based on functional analysis

(3) Approximate linearization methods based on the stability theorems.

This dissertation is organized as follows;

In Chapter 2, we obtain the regularity for semilinear equation by converting the problem into a fixed point problem with nonlinear monotone hemicontinuous and coercive operator. Moreover, We show the approximate cotrollability for semilinear equation.

The previous results on the approximate controllability of a semilinear control system have been studied as a particular case of sufficient conditions. In [1], Carrasco and Lebia give sufficient conditions for approximate controllability of parabolic equations with delay and in [2, 3, 4, 5, 6], the authors proved the approximate controllability under the range conditions of the controller B.

However, Triggian [7] proved that the abstract linear system is never exactly controllable in an infinite dimensional space when the semigroup generated by A is compact.

Our approximate controllability is an attempt to extend under more general conditions. We show that the input to solution map is compact by using the fact that $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ furnished with the usual topology is compactly embedded in $L^2(0, T, H)$ provided that the injection $V \subset H$ is compact.

In Chapter 3, as mentioned in chapter 2, the principal operator A is a monotone hemicontinuous operator from V to V^* and satisfies the coercive condition and B is a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$. If $Bu \in L^2(0, T; V^*)$, it is well known as the quasi-autonomous differential equation(see Theorem 2.6 of Chapter III in Barbu [8]).

The existence and the norm estimate of a solution was given in [8]. Based on the result, we intend to establish the approximate controllability for equation. Approximate controllability for a class of systems governed by a class of nonlinear evolution equations with nonlinear operator A have been studied in references by Naito [2] and Zhou [5]. As for the semilinear control system with the linear operator A generated C_0-semigroup, Naito [2] proved the approximate controllability under the range conditions of the controller B. The papers treating the controllability for systems with nonlinear principal operator A are not so many. We will prove the approximately controllability for (E) under a rather applicable assumption on the range of the control operator B.

In Chapter 4, we transform the variational inequality into nonlinear functional differential control problem according to the subdifferential operator $\partial\phi$ and deal with the existence for solution when the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R} \times V$ into H. In view of the monotonicity of $\partial\phi$, we show that the solution mapping is continuous. Thereafter, we obtain the approximate controllability for the control system governed by the variational inequality problem with the control term Bu instead of k. Sufficient conditions for approximate controllability of the system are discussed under the bounded condition on the controller operator B, which is that for any

$\varepsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\int_0^T S(T-s)\{p(s)-(Bu)(s)\}| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q_1 ||p||_{L^2(0,t;H)}, \quad 0 \le t \le T, \end{cases}$$

where q_1 is constant and independent of p. $S(t)$ is an analytic semigroup generated by A.

In Chapter 5, we deal with optimal control problems governed by semilinear parabolic type equations in Chapter 4. Let $\mathcal{U}$ be a Hilbert space of control variables, and B be a bounded linear operator from $\mathcal{U}$ into $L^2(0,T;H)$. Let admissible set $\mathcal{U}_{ad}$ be a closed convex subset of $\mathcal{U}$. Let $J = J(v)$ be a given quadratic cost function. We consider optimal control problems finding a control $\hat{u} \in U_{ad}$ for a given cost function. First of all, we study the regularity and a variational of constant formula for solutions of the nonlinear functional differential equation. Thereafter, we prove the existence and the uniqueness of optimal control for the problem. Consequently, in view of the monotonicity of $\partial\phi$, we show that the mapping $u \mapsto x_u$ is Lipschitz continuous in order to establish the necessary conditions of optimality of optimal controls for various observation cases.

We will also characterize the optimal controls by giving necessary conditions for optimality by proving the Gâteaux differentiability of solution mapping on control variables.

Chapter 2
Approximate controllability and regularity for nonlinear differential equations

2.1. Introduction

Let H and V be two real separable Hilbert spaces such that V is a dense subspace of H. We are interested in the following nonlinear differential control system on H:

$$\begin{cases} x'(t) + Ax(t) = g(t, x_t, \int_0^t k(t, s, x_s)ds) + (Bu)(t), \quad 0 < t, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s \le 0, \end{cases} \tag{SE}$$

where the nonlinear term, which is a Lipschitz continuous operator, is a semilinear version of the quasilinear form. The principal operator A is assumed to be a single valued, monotone operator, which is hemicontinuous and coercive from V to V^*. Here V^* stands for the dual space of V. Let U be a Banach space of control variables. The controller B is a linear bounded operator from a Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$ for any $T > 0$. Let the nonlinear mapping k be Lipschitz continuous from $\mathbb{R} \times [-h, 0] \times V$ into H. If the right side of the equation (SE) belongs to $L^2(0, T; V^*)$, it is well known as the quasi-autonomous differential equation(see Theorem 2.6 of Chapter III in [8]).

The problem of existence for solutions of semilinear evolution equations in Banach spaces has been established by several authors [8, 9, 10]. We refer to [9, 11, 12] to see the existence of solutions for a class of nonlinear evolution equations with monotone perturbations

First, we begin with the existence, and a variational constant formula for solutions of the equation (SE) on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$, which is also applicable to optimal control problem. We prove the existence and uniqueness for solution of the equation by converting the problem into a fixed point problem. Thereafter, based on the regularity results for solutions of (SE), we intend to establish the approximate controllability for (SE). The controllability results for linear control systems have been proved by many authors and several authors have extended these concepts to infinite dimensional semilinear system (see [3, 4, 5]). In recent years, as for the controllability for semilinear differential equations, Carrasco and Lebia [1] discussed sufficient conditions for approximate controllability of parabolic equations with delay, and Naito [2] and [3, 4, 5, 6] proved the approximate controllability under the range conditions of the controller B.

The previous results on the approximate controllability of a semilinear control system have been proved as a particular case of sufficient conditions for the approximate solvability of semilinear equations by assuming either that the semigroup generated by A is a compact operator or that the corresponding linear system (SE) when $g \equiv 0$ is approximately controllable. However, Triggian [7] proved that the abstract linear system is never exactly controllable in an infinite dimensional space when the semigroup generated by A is compact. Thus, we will establish the approximate controllability under more general conditions on the nonlinear term and the controller.

Our goal in this section is to establish the approximate controllablility for (SE) under a stronger assumption that $\{y : y(t) = (Bu)(t),\ \ u \in L^2(0,T;U)$ $\}$ is dense subspace of $L^2(0,T,H)$, which is reasonable and widely used in case of the nonlinear system. We show that the input to solution(control to state) map is compact by using the fact that $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ furnished with the usual topology is compactly embedded in $L^2(0,T,H)$ provided that the injection $V \subset H$ is compact.

Lastly we give a simple example to which the range conditions of the controller can be applied.

2.2. Nonlinear functional equations

Let H and V be two real Hilbert spaces. Assume and V is dense subspace in H and the injection of V into H is continuous. If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V(resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$). The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u||_* \leq |u| \leq ||u||, \quad u \in V$$

where $||\cdot||_*$ is the norm of the element of V^*. If an operator A is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{\frac{1}{2},2} = H$$

where $(V, V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between V and V^*.

We note that a nonlinear operator A is said to be hemicontinuous on V if

$$w - \lim_{t \to 0} A(x + ty) = Ax$$

for every $x,\ y \in V$ where "$w-$lim" indicates the weak convergence on V^*. Let $A : V \to V^*$ be given a single valued, monotone operator and hemicontinuous from V to V^* such that

(A1) $A(0) = 0, \quad (Au - Av, u - v) \geq \omega_1||u - v||^2 - \omega_2|u - v|^2,$

(A2) $||Au||_* \leq \omega_3(||u|| + 1)$

for every $u, v \in V$ where $\omega_2 \in \mathbb{R}$ and $\omega_1,\ \omega_3$ are some positive constants.

Here, we note that if $0 \neq A(0)$ we need the following assumption

$$(Au, u) \geq \omega_1||u||^2 - \omega_2|u|^2$$

for every $u \in V$. It is also known that A is maximal monotone and $R(A) = V^*$ where $R(A)$ denotes the range of A.

Let the controller B is a bounded linear operator from a Banach space $L^2(0,T;U)$ to $L^2(0,T;H)$ where U is a Banach space.

For each $t \in [0,T]$, we define $x_t : [-h, 0] \to H$ as

$$x_t(s) = x(t+s), \quad -h \leq s \leq 0.$$

We will set

$$\Pi = L^2(-h, 0; V) \quad \text{and} \quad \mathbb{R}^+ = [0, \infty).$$

Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0,\infty)$ and the Borel σ-field on $[-h, 0]$ respectively. Let $k : \mathbb{R}^+ \times \mathbb{R}^+ \times \Pi \to H$ be a nonlinear mapping satisfying the following:

(K1) For any $x. \in \Pi$ the mapping $k(\cdot,\cdot,x.)$ is strongly $\mathcal{L} \times \mathcal{B}$-measurable;

(K2) There exist positive constants K_0, K_1 such that

$$|k(t,s,x.) - k(t,s,y.)| \leq K_1 ||x. - y.||_\Pi,$$

$$|k(t,s,0)| \leq K_0$$

for all $(t,s) \in \mathbb{R}^+ \times [-h, 0]$ and $x., y. \in \Pi$.

Let $g : \mathbb{R}^+ \times \Pi \times H \to H$ be a nonlinear mapping satisfying the following:

(G1) For any $x \in \Pi$, $y \in H$ the mapping $g(\cdot, x., y)$ is strongly $\mathcal{L}$-measurable;

(G2) There exist positive constants L_0, L_1, L_2 such that

$$|g(t, x., y) - g(t, \hat{x}., \hat{y})| \leq L_1 ||x. - \hat{x}.||_\Pi + L_2|y - \hat{y}|,$$

$$|g(t, 0, 0)| \leq L_0$$

for all $t \in \mathbb{R}^+$, $x., \hat{x}. \in \Pi$, and $y, \hat{y} \in H$.

Remark 2.2.1 The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [13].

For $x \in L^2(-h, T; V)$, $T > 0$ we set

$$G(t, x) = g(t, x_t, \int_0^t k(t, s, x_s) ds).$$

Here as in [13] we consider the Borel measurable corrections of $x(\cdot)$.

Lemma 2.2.1 Let $x \in L^2(-h, T; V)$. Then the mapping $t \mapsto x_t$ belongs to $C([0, T]; \Pi)$ and

$$||x.||_{L^2(0.T;\Pi)} \leq \sqrt{T} ||x||_{L^2(-h,T;V)}. \tag{2.2.1}$$

Proof. It is easy to verify the first paragraph and (2.2.1) is a consequence of the estimate

$$||x.||^2_{L^2(0.T;\Pi)} \leq \int_0^T ||x_t||^2_\Pi dt \leq \int_0^T \int_{-h}^0 ||x(t+s)||^2 ds dt$$

$$\leq \int_0^T dt \int_{-h}^T ||x(s)||^2 ds \leq T ||x||^2_{L^2(-h,T;V)}.$$

□

Lemma 2.2.2 Let $x \in L^2(-h,T;V)$, $T > 0$. Then $G(\cdot,x) \in L^2(0,T;H)$ and

$$||G(\cdot,x)||_{L^2(0,T;H)} \le L_0\sqrt{T} + L_2K_0T^{3/2}/\sqrt{3} \tag{2.2.2}$$

$$+ (L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})||x||_{L^2(-h,T;V)}.$$

Moreover if $x_1,\ x_2 \in L^2(-h,T;V)$, then

$$||G(\cdot,x_1) - G(\cdot,x_2)||_{L^2(0,T;H)} \le (L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})||x_1 - x_2||_{L^2(-h,T;V)}. \tag{2.2.3}$$

Proof. It follows from (K2) and (2.2.1) that

$$\begin{aligned}
||\int_0^{\cdot} k(\cdot,s,x_s)ds||_{L^2(0,T;H)} &\le ||\int_0^{\cdot} k(\cdot,s,0)ds||_{L^2(0,T;H)} \\
&\quad + ||\int_0^{\cdot} (k(\cdot,s,x_s) - k(\cdot,s,0))ds||_{L^2(0,T;H)} \\
&\le K_0T^{3/2}/\sqrt{3} + \{\int_0^T |\int_0^t K_1||x_s||_\Pi ds|^2 dt\}^{1/2} \\
&\le K_0T^{3/2}/\sqrt{3} + \{\int_0^T K_1^2 t \int_0^t ||x_s||_\Pi^2 ds dt\}^{1/2} \\
&\le K_0T^{3/2}/\sqrt{3} + K_1T/\sqrt{2}||x_\cdot||_{L^2(0,T;\Pi)} \\
&\le K_0T^{3/2}/\sqrt{3} + K_1T^{3/2}/\sqrt{2}||x||_{L^2(-h,T;V)}
\end{aligned}$$

and hence, from (G2), (2.2.1) and the above inequality it is easily seen that

$$||G(\cdot,x)||_{L^2(0,T;H)} \le ||G(\cdot,0)|| + ||G(\cdot,x)-G(\cdot,0)||$$

$$\le L_0\sqrt{T} + L_1||x.||_{L^2(0,T;\Pi)} + L_2||\int_0^{\cdot} k(\cdot,s,x_s)ds||_{L^2(0,T;H)}$$

$$\le L_0\sqrt{T} + L_1\sqrt{T}||x||_{L^2(-h,T;V)}$$

$$+ L_2(K_0T^{3/2}/\sqrt{3} + K_1T^{3/2}/\sqrt{2}||x||_{L^2(-h,T;V)}).$$

Similarly, we can prove (2.2.3). □

Let us consider the quasi-autonomous differential equation

$$\begin{cases} x'(t) + Ax(t) = f(t), \quad 0 < t \le T, \\ x(0) = \phi^0 \end{cases} \tag{E}$$

where A satisfies the hypotheses mentioned above. The following result is from Theorem 2.6 of Chapter III in [8].

Proposition 2.2.1 Let $\phi^0 \in H$ and $f \in L^2(0,T;V^*)$. Then there exists a unique solution x of (E) belonging to

$$C([0,T];H) \cap L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$$

and satisfying

$$|x(t)|^2 + \int_0^t ||x(s)||^2 ds \le C_1(|\phi^0|^2 + \int_0^t ||f(s)||_*^2 ds), \tag{2.2.4}$$

$$\int_0^t ||\frac{dx(s)}{ds}||_*^2 dt \le C_1(|\phi^0|^2 + \int_0^t ||f(s)||_*^2 ds) \tag{2.2.5}$$

where C_1 is a constant.

Acting on both sides of (E) by $x(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 \leq \omega_2|x(t)|^2 + (f(t), x(t)).$$

As is seen Theorem 2.6 in [8], integrating from 0 to t we can determine the constant C_1 in Proposition 2.1.

We establish the following result on the solvability of the equation (SE).

Theorem 2.2.1 Let A and the nonlinear mapping g be given satisfying the assumptions mentioned above. Then for any $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $f \in L^2(0, T; V^*)$, $T > 0$, the following nonlinear equation

$$\begin{cases} x'(t) + Ax(t) = G(t, x) + f(t), & 0 < t \leq T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) & -h \leq s \leq 0 \end{cases} \tag{2.2.6}$$

has a unique solution x belonging to

$$L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying that there exists a constant C_2 such that

$$||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_2(1 + |\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}). \tag{2.2.7}$$

Proof. Let $y \in L^2(0, T; V)$. Then we extend it to the interval $(-h, 0)$ by setting $y(s) = \phi^1(s)$ for $s \in (-h, 0)$ and hence, $G(\cdot, y(\cdot)) \in L^2(0, T; H)$ from

Lemma 2.2.2. Thus, in virtue of Proposition 2.2.1 we know that the problem

$$\begin{cases} x'(t) + Ax(t) = G(t,y) + f(t), \quad 0 < t, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \leq s \leq 0 \end{cases} \tag{2.2.8}$$

has a unique solution $x_y \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ corresponding to y. Let us fix $T_0 > 0$ so that

$$\omega_1^{-1} e^{\omega_2 T_0}(L_1\sqrt{T}_0 + L_2 K_1 T_0^{3/2}/\sqrt{2}) < 1. \tag{2.2.9}$$

Let x_i, $i = 1,\ 2$, be the solution of (2.2.8) corresponding to y_i. Multiplying by $x_1(t) - x_2(t)$, we have that

$$(\dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)) + (Ax_1(t) - Ax_2(t), x_1(t) - x_2(t))$$

$$= (G(t,y_1) - G(t,y_2), x_1(t) - x_2(t)),$$

and hence it follows that

$$\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2$$

$$\leq \omega_2|x_1(t) - x_2(t)|^2 + ||G(t,y_1)) - G(t,y_2)||_*||x_1(t) - x_2(t)||.$$

From Lemma 2.2.2 and integrating over [0,t], it follows

$$\frac{1}{2}|x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds$$

$$\leq \frac{1}{2c}\int_0^t ||G(s,y_1) - G(s,y_2)||_*^2 ds$$

$$+ \frac{c}{2}\int_0^t ||x_1(s) - x_2(s)||^2 ds + \omega_2 \int_0^t |x_1(s)ds - x_2(s)ds|^2 ds,$$

where c is a positive constant satisfying $2\omega_1 - c > 0$. Here we used that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad p^{-1} + q^{-1} = 1(1 < p < \infty)$$

for any pair of nonnegative numbers a and b. Thus, from (2.2.3) it follows that

$$|x_1(t) - x_2(t)|^2 + (2\omega_1 - c)\int_0^t ||x_1(s)ds - x_2(s)ds||^2 ds$$

$$\leq c^{-1}(L_1\sqrt{T}_0 + L_2K_1T_0^{3/2}/\sqrt{2})^2 \int_0^t ||y_1(s) - y_2(s)||^2 ds$$

$$+ 2\omega_2 \int_0^t |x_1(s) - x_2(s)|^2 ds.$$

By using Gronwall's inequality, we get

$$|x_1(T_0) - x_2(T_0)|^2 + (2\omega_1 - c)\int_0^{T_0} ||x_1(s) - x_2(s)||^2 ds$$

$$\leq c^{-1}(L_1\sqrt{T}_0 + L_2K_1T_0^{3/2}/\sqrt{2})^2 e^{2\omega_2 T_0} \int_0^{T_0} ||y_1(s) - y_2(s)||^2 ds.$$

Taking $c = \omega_1$, it holds that

$$||x_1 - x_2||_{L^2(0,T_0;V)} \leq \omega_1^{-1} e^{\omega_2 T_0}(L_1\sqrt{T}_0$$

$$+ L_2K_1T_0^{3/2}/\sqrt{2})||y_1 - y_2||_{L^2(0,T_0;V)}.$$

Hence we have proved that $y \mapsto x$ is a strictly contraction from $L^2(0, T_0; V)$ to itself if the condition (2.2.9) is satisfied. It gives the equation (2.2.6) has a unique solution in $[0, T_0]$.

From now on, we derive the norm estimates of solution of the equation (2.2.6). Let y be the solution of

$$\begin{cases} y'(t) + Ay(t) = f(t), \quad 0 < t \le T_0, \\ y(0) = \phi^0. \end{cases} \tag{2.2.10}$$

Then

$$\frac{d}{dt}(x(t) - y(t)) + (Ax(t) - Ay(t)) = G(t, x),$$

by multiplying by $x(t) - y(t)$ and using the assumption (A1), we obtain

$$\frac{1}{2}\frac{d}{dt}|x(t) - y(t)|^2 + \omega_1||x(t) - y(t)||^2$$

$$\le \omega_2|x(t) - y(t)|^2 + ||G(t, x)||_*||x(t) - y(t)||.$$

By integrating over $[0, t]$ and using Gronwall's inequality, we have

$$||x - y||_{L^2(0,T_0;V)} \le \omega_1^{-1}e^{\omega_2 T_0}||G(\cdot, x)||_{L^2(0,T_0;V^*)}$$

$$\le \omega_1^{-1}e^{\omega_2 T_0}\{L_0\sqrt{T}_0 + L_2K_0T_0^{3/2}/\sqrt{3}$$

$$+ (L_1\sqrt{T}_0 + L_2K_1T_0^{3/2}/\sqrt{2})(||x||_{L^2(0,T_0;V)} + ||\phi^1||_{L^2(-h,0;V)})\},$$

and hence, putting

$$N = \omega_1^{-1}e^{\omega_2 T_0} \text{ and } L = L_1\sqrt{T}_0 + L_2K_1T_0^{3/2}/\sqrt{2},$$

it holds

$$||x||_{L^2(0,T_0;V)} \le \frac{N}{1-NL}(L_0\sqrt{T}_0 + L_2K_0T_0^{3/2}/\sqrt{3}) \tag{2.2.11}$$

$$+\frac{1}{1-NL}||y||_{L^2(0,T_0;V)} + \frac{NL}{1-NL}||\phi^1||_{L^2(-h,0;V)}$$

$$\le \frac{N}{1-NL}(L_0\sqrt{T}_0 + L_2K_0T_0^{3/2}/\sqrt{3})$$

$$+\frac{C_1}{1-NL}(|\phi^0| + ||f||_{L^2(0,T_0;V^*)})$$

$$+\frac{NL}{1-NL}||\phi^1||_{L^2(-h,0;V)}$$

$$\le C_2(1 + |\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T_0;V^*)})$$

for some positive constant C_2. Since the condition (2.2.9) is independent of initial values, the solution of (2.2.6) can be extended to the internal $[0, nT_0]$ for natural number n, i.e., for the initial value $(x(nT_0), x_{nT_0})$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (2.2.11) holds for the solution in $[0, (n+1)T_0]$. □

Theorem 2.2.2 If $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V))$ and $f \in L^2(0, T; V^*)$, then $x \in L^2(-h, T; V) \cap W^{1,2}(0, T; V^*)$, and the mapping

$$H \times L^2(-h, 0; V) \times L^2(0, T; V^*) \ni (\phi^0, \phi^1, f) \mapsto x \in L^2(-h, T; V) \cap W^{1,2}(0, T; V^*)$$

is continuous.

Proof. It is easy to show that if $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $f \in L^2(0, T; V^*)$ for every $T > 0$, then x belongs to $L^2(-h, T; V) \cap W^{1,2}(0, T; V^*)$.

Let

$$(\phi_i^0, \phi_i^1, f_i) \in H \times L^2(-h, 0; V) \times L^2(0, T_1; V^*)$$

and x_i be the solution of (2.2.6) with $(\phi_i^0, \phi_i^1, f_i)$ in place of (ϕ^0, ϕ^1, f) for $i = 1,\ 2$. Then in view of Proposition 2.2.1 and Lemma 2.2.2 we have

$$\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2 \tag{2.2.12}$$

$$\leq \omega_2|x_1(t) - x_2(t)|^2 + ||G(t, x_1) - G(t, x_2)||_*||x_1(t) - x_2(t)||$$

$$+ ||f_1(t) - f_2(t)||_*||x_1(t) - x_2(t)||$$

If $\omega_1 - c/2 > 0$, we can choose a constant $c_1 > 0$ so that

$$\omega_1 - c/2 - c_1/2 > 0$$

and

$$||f_1(t) - f_2(t))||_*||x_1(t) - x_2(t)|| \leq \frac{1}{2c_1}||f_1(t) - f_2(t)||_*^2$$

$$+ \frac{c_1}{2}||x_1(t) - x_2(t)||^2.$$

Let $T_1 < T$ be such that

$$2\omega_1 - c - c_1 - c^{-1}e^{2\omega_2 T_1}(L_1\sqrt{T}_1 + L_2K_1T_1^{3/2}/\sqrt{2})^2 > 0.$$

Integrating on (2.2.12) over $[0, T_1]$ and as is seen in the first part of proof, it follows

$$(2\omega_1 - c - c_1)||x_1 - x_2||^2_{L^2(0,T_1;V)} \le e^{2\omega_2 T_1}\{|\phi_1^0 - \phi_2^0|^2$$

$$+ \frac{1}{c}||G(t,x_1) - G(t,x_2)||^2_{L^2(0,T_1;V^*)} + \frac{1}{c_1}||f_1 - f_2||^2_{L^2(0,T_1;V^*)}\}$$

$$\le e^{2\omega_2 T_1}\{|\phi_1^0 - \phi_2^0|^2$$

$$+ \frac{1}{c}(L_1\sqrt{T}_1 + L_2K_1T_1^{3/2}/\sqrt{2})^2||x_1 - x_2||^2_{L^2(-h,T_1;V)}$$

$$+ \frac{1}{c_1}||f_1 - f_2||^2_{L^2(0,T_1;V^*)}\}.$$

Putting that

$$N_1 = 2\omega_1 - c - c_1 - c^{-1}e^{2\omega_2 T_1}(L_1\sqrt{T}_1 + L_2K_1T_1^{3/2}/\sqrt{2})^2$$

we have

$$||x_1 - x_2||_{L^2(0,T_1;V)} \le \frac{e^{\omega_2 T_1}}{N_1^{1/2}}(|\phi_1^0 - \phi_2^0| + \frac{1}{c_1}||f_1 - f_2||_{L^2(0,T_1;V^*)}) \qquad (2.2.13)$$

$$+ \frac{c^{-1/2}e^{\omega_2 T_1}(L_1\sqrt{T_1} + L_2K_1T_1^{3/2}/\sqrt{2})}{N_1^{1/2}}||\phi_1^1 - \phi_2^1||_{L^2(-h,0;V)}.$$

Suppose that

$$(\phi_n^0, \phi_n^1, f_n) \to (\phi^0, \phi^1, f) \text{ in } H \times L^2(-h,0;V) \times L^2(0,T;V^*),$$

and let x_n and x be the solutions (2.2.6) with $(\phi_n^0, \phi_n^1, f_n)$ and (ϕ^0, ϕ^1, f) respectively. By virtue of (2.2.13) with T replaced by T_1 we see that

$$x_n \to x \quad \text{in} \quad L^2(-h,T_1;V) \cap W^{1,2}(0,T_1;V^*) \subset C([0,T_1];H).$$

This implies that $(x_n(T_1), (x_n)_{T_1}) \to (x(T_1), x_{T_1})$ in $H \times L^2(-h, 0; V)$. Hence the same argument shows that

$$x_n \to x \quad \text{in} \quad L^2(T_1, \min\{2T_1, T\}; V) \cap W^{1,2}(T_1, \min\{2T_1, T\}; V^*).$$

Repeating this process we conclude that

$$x_n \to x \ \ \text{in} \ \ L^2(-h, T; V) \cap W^{1,2}(0, T; V^*).$$

□

Remark 2.2.2 For $x \in L^2(0, T; V)$ we set

$$G(t, x) = \int_0^t k(t-s) g(s, x(s)) ds$$

where k belongs to $L^2(0, T)$ and $g : [0, T] \times V \longrightarrow H$ be a nonlinear mapping satisfying

$$|g(t, x) - g(t, y)| \leq L||x - y||$$

for a positive constant L. Let $x \in L^2(0, T; V)$, $T > 0$. Then $G(\cdot, x) \in L^2(0, T; H)$ and

$$||G(\cdot, x)||_{L^2(0,T;H)} \leq L||k||_{L^2(0,T)} \sqrt{T} ||x||_{L^2(0,T;V)}.$$

Moreover if $x_1,\ x_2 \in L^2(0, T; V)$, then

$$||G(\cdot, x_1) - G(\cdot, x_2)||_{L^2(0,T;H)} \leq L||k|| \sqrt{T} ||x_1 - x_2||_{L^2(0,T;V)}.$$

Then with the condition that

$$\omega_1^{-1} e^{\omega_2 T_0} L||k|| \sqrt{T_0} < 1$$

in place of the condition (2.2.9), we can obtain the results of Theorem 2.2.1.

2.3. Approximate controllability

In what follows we assume that the embedding $V \subset H$ is compact and A is a continuous operator from V to V^* satisfying (A1) and (A2). For $h \in L^2(0,T;H)$ and let x_h be the solution of the following equation with $B = I$:

$$\begin{cases} x'(t) + Ax(t) = G(t,x) + h(t), \quad 0 < t, \\ x(0) = 0, \quad x(s) = 0 \quad -h \le s \le 0, \end{cases} \tag{2.3.1}$$

where

$$G(t,x) = g(t, x_t, \int_0^t k(t,s,x_s)ds).$$

We define the solution mapping S from $L^2(0,T;V^*)$ to $L^2(0,T;V)$ by

$$(Sh)(t) = x_h(t), \quad h \in L^2(0,T;V^*). \tag{2.3.2}$$

Let $\mathcal{A}$ and $\mathcal{G}$ be the Nemitsky operators corresponding to the maps A and G, which are defined by $\mathcal{A}(x)(\cdot) = Ax(\cdot)$ and $\mathcal{G}(h)(\cdot) = G(\cdot, x_h)$, respectively. Then since the solution x belongs to $L^2(-h,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$, it is represented by

$$x_h(t) = \int_0^t ((I + \mathcal{G} - \mathcal{A}S)h)(s)ds, \tag{2.3.3}$$

and with aid of Lemma 2.2.2 and Proposition 2.2.1

$$||Sh||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = ||x_h|| \le C_1||G(\cdot, x_h) + h||_{L^2(0,T;V^*)} \tag{2.3.4}$$

$$\le C_1\{L_0\sqrt{T} + L_2K_0T^{3/2}/\sqrt{3} + (L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})||x||_{L^2(0,T;V)}$$

$$+ ||h||_{L^2(0,T;V^*)}\}$$

$$\le C_1\{L_0\sqrt{T} + L_2K_0T^{3/2}/\sqrt{3}$$

$$+ (L_1\sqrt{T} + L_2K_1T^{3/2}/\sqrt{2})(1 + ||h||_{L^2(0,T;V^*)}) + ||h||_{L^2(0,T;V^*)}\}.$$

Hence if h is bounded in $L^2(0,T;V^*)$, then so is x_h in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is compact in view of Theorem 2 of Aubin [14]. Hence, the mapping $h \mapsto Sh = x_h$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$. Therefore, $\mathcal{G}$ is a compact mapping from $L^2(0,T;V^*)$ to $L^2(0,T;H)$ and so is $\mathcal{A}S$ from $L^2(0,T;V^*)$ to itself. The solution of (SE) is denoted by $x(T;g,u)$ associated with the nonlinear term g and control u at time T.

Definition 2.3.1 The system (SE) is said to be approximately controllable at time T if $Cl\{x(T;g,u) : u \in L^2(0,T;U)\} = V^*$ where Cl denotes the closure in V^*.

We assume

(T) $1 - \omega_1^{-1}\omega_3 e^{\omega_2 T} > 0$

(B) $Cl\{y : y(t) = (Bu)(t),\ \text{a.e. } u \in L^2(0,T;U)\} = L^2(0,T;H)$. Here Cl is the closure in $L^2(0,T;H)$.

Theorem 2.3.1 Let the assumptions (T) and (B) be satisfied. Then

$$Cl\{(I - \mathcal{A}S)h : h \in L^2(0,T;V^*)\} = L^2(0,T;V^*). \tag{2.3.5}$$

Therefore, the following nonlinear differential control system

$$\begin{cases} \frac{dx(t)}{dt} + Ax(t) = (Bu)(t), \quad 0 < t \le T, \\ x(0) = x_0 \end{cases} \tag{2.3.6}$$

is approximately controllable at time T.

Proof. Let $z \in L^2(0,T;V^*)$ and r be a constant such that

$$z \in U_r = \{x \in L^2(0,T;V^*) : ||x||_{L^2(0,T;V^*)} < r\}.$$

Take a constant $d > 0$ such that

$$(r + \omega_3 + N_2|x_0|)(1 - N_2)^{-1} < d, \tag{2.3.7}$$

where

$$N_2 = \omega_1^{-1}\omega_3 e^{\omega_2 T}.$$

Taking scalar product on both sides of (2.3.1) with $G = 0$ by $x(t)$

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 + \omega_1||x(t)||^2 \le \omega_2|x(t)|^2 + ||h(t)||_*||x(t)||$$

$$\le \omega_2|x(t)|^2 + \frac{1}{2c}||h(t)||_*^2 + \frac{c}{2}||x(t)||^2$$

where c is a positive constant satisfying $2\omega_1 - c > 0$. Integrating on $[0, t]$, we get

$$\frac{1}{2}|x(t)|^2 + \omega_1 \int_0^t ||x(s)||^2 ds \le \frac{1}{2}|x_0|^2 + \frac{1}{2c}\int_0^t ||h(s)||_*^2 ds$$
$$+ \frac{c}{2}\int_0^t ||x(s)||^2 ds + \omega_2 \int_0^t |x(s)|^2 ds,$$

and hence,

$$|x(t)|^2 + (2\omega_1 - c)\int_0^t ||x(s)||^2 ds \le |x_0|^2 + \frac{1}{c}\int_0^t ||h(s)||_*^2 ds$$
$$+ 2\omega_2 \int_0^t |x(s)|^2 ds.$$

By using Gronwall's inequality, it follows that

$$|x(T)|^2 + (2\omega_1 - c)\int_0^T ||x(s)||^2 ds \le e^{2\omega_2 T}(|x_0|^2 + \frac{1}{c}\int_0^T ||h(s)||_*^2 ds),$$

that is,

$$||Sh||_{L^2(0,T;V)} = ||x||_{L^2(0,T;V)} \tag{2.3.8}$$
$$\le e^{\omega_2 T}(2\omega_1 - c)^{-1/2}(|x_0| + c^{-1/2}||h||_{L^2(0,T;V^*)}).$$

Let us consider the equation

$$z = (I - \mathcal{A}S)w. \tag{2.3.9}$$

Let w be the solution of (2.3.9). Then $z \in U_d$ and taking $c = \omega_1$, from (2.3.7) and (2.3.8)

$$||w||_{L^2(0,T;V^*)} \le ||z||_{L^2(0,T;V^*)} + ||\mathcal{A}Sw||_{L^2(0,T;V^*)}$$

$$\le r + \omega_3(||Sw||_{L^2(0,T;V^*)} + 1)$$

$$\le r + \omega_3\{\omega_1^{-1/2}e^{\omega_2 T}(|x_0| + \omega_1^{-1/2}||w||) + 1\},$$

and hence

$$||w|| \le (r + \omega_3 + N_2|x_0|)(1 - N_2)^{-1} < d$$

it follows that $w \notin \partial U_d$ where ∂U_d stands for the boundary of U_d. Thus the homotopy property of topological degree theory there exists $w \in L^2(0,T;V^*)$ such that the equation (2.3.9) holds. Since the assumption (B), there exists a sequence $\{u_n\} \in L^2(0,T;U)$ such that $Bu_n \mapsto w$ in $L^2(0,T;V^*)$. Then by the last paragraph of Theorem 2.1 we have that $x(\cdot; g, u_n) \mapsto x_w$ in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$. Hence we have proved (2.3.5). Let $y \in V^*$. Then there exists an element $u \in L^2(0,T;U)$ such that

$$||\frac{y}{T} - (I - \mathcal{A}S)Bu||_{L^2(0,T;V^*)} < \frac{\epsilon}{\sqrt{T}}.$$

Thus

$$||y - x(T)||_* = ||y - \int_0^T ((I - \mathcal{A}S)Bu)(s)ds||_*$$

$$\le \int_0^T ||\frac{y}{T} - ((I - \mathcal{A}S)Bu)(s)||_* ds$$

$$\le \sqrt{T}||\frac{y}{T} - (I - \mathcal{A}S)Bu||_{L^2(0,T;V^*)} < \epsilon.$$

Therefore, the system (2.3.6) is approximately controllable at time T. $\square$

In order to investigate the controllability of the nonlinear control system, we need to impose the following condition.

(F) g is uniformly bounded: there exists a constant M_g such that

$$|g(t,x,y)| \le M_g,$$

for all $x,\ y \in V$.

By (F) it holds that

$$||G(\cdot,x)||_{L^2(0,T;H)} \le M_g\sqrt{T},$$

and for every $h \in L^2(0,T;V^*)$

$$||\mathcal{G}(h)||_{L^2(0,T;H)} \le M_g\sqrt{T} \tag{2.3.10}$$

Theorem 2.3.2 Let the assumptions (T), (B) and (F) be satisfied. Then we have

$$Cl\{(\mathcal{G}+I-\mathcal{A}S)h : h \in L^2(0,T;V^*)\} = L^2(0,T;V^*). \tag{2.3.11}$$

Thus the system (SE) is approximately controllable at time T.

Proof. Let U_r be the ball with radius r in $L^2(0,T;V^*)$ and $z \in U_r$. To prove (2.3.11) we will also use the degree theory for the equation

$$z = \lambda(\mathcal{G}-\mathcal{A}S)w + w, \quad 0 \le \lambda \le 1 \tag{2.3.12}$$

in open ball U_d where the constant d satisfies

$$(r + \omega_3 + N_2|x_0| + M_g\sqrt{T})(1 - N_2)^{-1} < d \tag{2.3.13}$$

where the constant N_2 is in Theorem 2.3.1. If w is the solution of (2.3.12) then $z \in U_d$ and from Lemma 2.2.1

$$\begin{aligned}||w||_{L^2(0,T;V^*)} &\leq ||z|| + ||\mathcal{A}Sw|| + ||\mathcal{G}w|| \\ &\leq r + \omega_3(||Sw|| + 1) + M_g\sqrt{T}) \\ &\leq r + \omega_3\{\omega_1^{-1/2}e^{\omega_2 T}(|x_0| + \omega_1^{-1/2}||w||) + 1\} + M_g\sqrt{T},\end{aligned}$$

and hence

$$||w|| \leq (r + \omega_3 + N_2|x_0| + M_g\sqrt{T})(1 - N_2)^{-1} < d$$

it follows that $w \notin \partial U_d$. Hence, there exists $w \in L^2(0,T;V^*)$ such that the equation (2.3.12) holds. Using the similar way to the last part of Theorem 2.3.1 and the assumption (B) there exists a sequence $\{u_n\} \in L^2(0,T;U)$ such that $Bu_n \mapsto w$ in $L^2(0,T;V^*)$ and $x(\cdot;g,u_n) \mapsto x_w$ in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$. Thus, we have proved (2.3.11) and the system (2.1.1) is approximately controllable at time T. □

2.4. Example

Let $-A$ be an operator associated with a bounded sesquilinear form $a(u,v)$ defined in $V \times V$ and satisfying Gårding's inequality

$$\operatorname{Re} a(u,v) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0$$

for any $u \in V$. It is known that A generates an analytic semigroup in both H and V^*. In virtue of the Riesz-Schauder theorem, if the embedding $V \subset H$ is compact then the operator A has discrete spectrum

$$\sigma(A) = \{\mu_n : n = 1,\ 2,\ \dots\}$$

which has no point of accumulation except possibly $\mu = \infty$. Let μ_n be a pole of the resolvent of A of order k_n and P_n the spectral projection associated with μ_n

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - A)^{-1} d\mu,$$

where Γ_n is a small circle centered at μ_n such that it surrounds no point of $\sigma(A)$ except μ_n. Then the generalized eigenspace corresponding to μ_n is given by

$$H_n = P_n H = \{P_n u : u \in H\},$$

and we have that from $P_n^2 = P_n$ and $H_n \subset V$ it follows that

$$P_n V = \{P_n u : u \in V\} = H_n.$$

Definition 2.4.1 The system of the generalized eigenspaces of A is complete in H if $\mathrm{Cl}\{\mathrm{span}\{H_n : n = 1,\ 2,\ \dots\}\} = H$ where Cl denotes the closure in H.

We need the following hypotheses:

(B1) The system of the generalized eigenspaces of A is complete.

(B2) There exists a constant $d > 0$ such that

$$||v|| \leq d||Bv||, \quad v \in L^2(0,T;U).$$

We can see many examples which satisfy (B2)(cf. [5, 6]).

Consider about the intercept controller B defined by

$$(Bu)(t) = \sum_{n=1}^{\infty} u_n(t), \tag{2.4.1}$$

where

$$u_n = \begin{cases} 0, & 0 \leq t \leq \frac{T}{n} \\ P_n u(t), & \frac{T}{n} < t \leq T. \end{cases}$$

Hence we see that $u_1(t) \equiv 0$ and $u_n(t) \in \operatorname{Im} P_n$.

First of all, for the meaning of the condition (B) in section 2.3, we need to show the existence of controller satisfying $\mathrm{Cl}\{Bu : u \in L^2(0,T;U)\} \neq L^2(0,T;H)$. In fact, by completion of the generalized eigenspaces of A we may write that $f(t) = \sum_{n=1}^{\infty} P_n f(t)$ for $f \in L^2(0,T;H)$. Let us choose $f \in L^2(0,T;H)$ satisfying

$$\int_0^T ||P_1 f(t)||^2 dt > 0.$$

Then since

$$\int_0^T ||f(t) - Bu(t)||^2 dt = \int_0^T \sum_{n=1}^{\infty} ||P_n(f(t) - Bu(t))||^2 dt$$

$$\geq \int_0^T ||P_1(f(t) - Bu(t))||^2 dt = \int_0^T ||P_1 f(t)||^2 dt > 0,$$

the statement mentioned above is reasonable.

Let $f \in L^2(0,T;H)$ and $\alpha = T/(T-T/n)$. Then we know

$$f(\cdot) \equiv \alpha K_{[T,T/n]} f(\alpha(\,\cdot\, - T/n)) \quad \text{in} \quad L^2(0,T;H),$$

where $K_{[T,T/n]}$ is the characteristic function of $[T,T/n]$. Define

$$w(s) = \sum_{n=1}^{\infty} w_n(s), \quad w_n(s) = \alpha K_{[T,T/n]} B^{-1} P_n f(\alpha(s-T/n)).$$

Thus $(Bw)(t) = \sum_{n=1}^{\infty} P_n f(s)$, a.e.. Since the system of the generalized eigenspaces of A is complete, it holds that for every $f \in L^2(0,T;H)$ and $\epsilon > 0$

$$||f(\cdot) - \sum_{n=1}^{\infty} P_n f(\cdot)||_{L^2(0,T;H)} = ||f(\cdot) - Bw||_{L^2(0,T;H)} < \epsilon.$$

Thus, the intercept controller B defined by (2.4.1) satisfies the condition (B).

Chapter 3
Approximate controllability for nonlinear differential equations with quasi-autonomous operator

3.1. Introduction

Let H and V be two real separable Hilbert spaces such that V is a dense subspace of H. We are interested in the approximate controllability for the following nonlinear functional control system on H:

$$\begin{cases} \frac{dx(t)}{dt} & +Ax(t) \ni (Bu)(t), \quad 0 < t \leq T, \\ x(0) & = x_0. \end{cases} \tag{E}$$

Assume that A is a monotone hemicontinuous operator from V to V^* and satisfies the coercive condition. Here V^* stands for the dual space of V. Let U be a Banach space and the controller operator B be a bounded linear operator from the Banach space $L^2(0,T;U)$ to $L^2(0,T;H)$. If $Bu \in L^2(0,T;V^*)$, it is well known as the quasi-autonomous differential equation(see Theorem 2.6 of Chapter III in Barbu [8]). In [8], the existence and the norm estimate of a solution of the above equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ was given, and results similar to this case were obtained by many authors(see bibliographical notes of [5, 8, 12, 15, 16]), which is also applicable to an optimal control problem.

The optimal control problems for a class of systems governed by a class of nonlinear evolution equations with nonlinear operator A have been studied by Ahmed, Teo and Xiang [9, 11, 17]. The condition equivalent to the approximate controllability for semilinear control system have been obtained in by Naito [2] and Zhou [5]. As for the semilinear control system with the linear operator A generated C_0-semigroup, Naito [2] proved the approximate controllability under the range conditions of the controller B. The papers treating the controllability for systems with nonlinear principal operator A are not many.

In the present section, we will prove the approximately controllable for (E) under a rather applicable assumption on the range of the control operator B, namely that $\{y : y(t) = Bu(t), \quad u \in L^2(0,T;U)\}$ is dense subspace of $L^2(0,T,H)$, which is reasonable and widely used in case of the nonlinear system(refer to [2, 5, 18]).

3.2. Quasi-autonomous differential equations

Let H and V be two real separable Hilbert spaces forming Gelfand triple $V \subset H \subset V^*$ with pivot space H as mentioned in Chapter 2. Let $h \in L^2(0,T;V^*)$ and x be the solution of the following quasi-autonomous differential equation with forcing term $h(t)$:

$$\begin{cases} \frac{dx(t)}{dt} + Ax(t) \ni h(t), \quad 0 < t \leq T, \\ x(0) = x_0 \end{cases} \tag{3.2.1}$$

where A is given satisfying the hypotheses mentioned above. The following result is from Theorem 2.6 of Chapter III in [8].

Proposition 3.2.1 Let $x_0 \in H$ and $h \in L^2(0,T;V^*)$. Then there exists a unique solution x of (3.2.1) belonging to

$$C([0,T];H) \cap L^2(0,T;H) \cap W^{1,2}(0,T;V^*)$$

and satisfying

$$|x(t)|^2 + \int_0^t ||x(s)||^2 ds \le C_1(|x_0|^2 + \int_0^t ||h(s)||_*^2 ds + 1), \tag{3.2.2}$$

$$\int_0^t ||\frac{dx(s)}{ds}||_*^2 dt \le C_1(|x_0|^2 + \int_0^t ||h(s)||_*^2 ds + 1) \tag{3.2.3}$$

where C_1 is a constant.

Lemma 3.2.1 Let x_h and x_k be the solutions of (3.2.1) corresponding to h and k in $L^2(0,T;V^*)$. Then we have that

$$\frac{1}{2}|x_h(t) - x_k(t)|^2 + \omega_1 \int_0^t ||x_h(s) - x_k(s)||^2 ds$$
$$\le \int_0^t e^{2\omega_2(t-s)} ||x_h(s) - x_k(s)|| \, ||h(s) - k(s)||_* ds, \tag{3.2.4}$$

and

$$\frac{1}{2}|x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds$$
$$\le \frac{e^{2\omega_2 t}}{2}|x_0|^2 + \int_0^t e^{2\omega_2(t-s)} ||x_h(s)|| \, ||h(s)||_* ds. \tag{3.2.5}$$

Proof. In order to prove (3.2.5), taking scalar product on both sides of (3.2.1) by $x(t)$,

$$\frac{1}{2}\frac{d}{dt}|x_h(t)|^2 + \omega_1||x_h(t)||^2 \leq \omega_2|x_h(t)|^2 + ||x_h(t)||\,||h(t)||_*.$$

Integrating on $[0,t]$, we get

$$\frac{1}{2}|x_h(t)|^2 + \omega_1\int_0^t ||x_h(s)||^2 ds$$

$$\leq \frac{1}{2}|x_0|^2 + \omega_2\int_0^t |x_h(s)|^2 ds + \int_0^t ||x_h(s)||\,||h(s)||_* ds. \tag{3.2.6}$$

From (3.2.6) it follows that

$$\frac{d}{dt}\{e^{-2\omega_2 t}\int_0^t |x_h(s)|^2 ds\} = 2e^{-2\omega_2 t}\{\frac{1}{2}|x_h(t)|^2 - \omega_2\int_0^t |x_h(s)|^2 ds\}$$

$$\leq 2e^{-2\omega_2 t}\{\frac{1}{2}|x_0|^2 + \int_0^t ||x_h(s)||\,||h(s)||_* ds\}. \tag{3.2.7}$$

Integrating (3.2.7) over $(0,t)$ we have

$$e^{-2\omega_2 t}\int_0^t |x_h(s)|^2 ds \leq 2\int_0^t e^{-2\omega_2\tau}\int_0^\tau ||x_h(s)||\,||h(s)||_* ds d\tau + \frac{1-e^{-2\omega_2 t}}{2\omega_2}|x_0|^2$$

$$= 2\int_0^t\int_s^t e^{-2\omega_2\tau}d\tau ||x_h(s)||\,||h(s)||_* ds + \frac{1-e^{-2\omega_2 t}}{2\omega_2}|x_0|^2$$

$$= 2\int_0^t \frac{e^{-2\omega_2 s} - e^{-2\omega_2 t}}{2\omega_2}||x_h(s)||\,||h(s)||_* ds + \frac{1-e^{-2\omega_2 t}}{2\omega_2}|x_0|^2$$

$$= \frac{1}{\omega_2}\int_0^t (e^{-2\omega_2 s} - e^{-2\omega_2 t})||x_h(s)||\,||h(s)||_* ds + \frac{1-e^{-2\omega_2 t}}{2\omega_2}|x_0|^2,$$

and hence,

$$\omega_2 \int_0^t |x_h(s)|^2 ds \le \int_0^t (e^{2\omega_2(t-s)} - 1)||x_h(s)||\, ||h(s)||_* ds + \frac{e^{2\omega_2 t} - 1}{2} |x_0|^2. \tag{3.2.8}$$

Combining (3.2.6) with (3.2.8) it follows that

$$\frac{1}{2}|x_h(t)|^2 + \omega_1 \int_0^t ||x_h(s)||^2 ds \le \frac{e^{2\omega_2 t}}{2}|x_0|^2 + \int_0^t e^{2\omega_2(t-s)}||x_h(s)||\, ||h(s)||_* ds.$$

We also obtain (3.2.4) by the similar argument in the proof of (3.2.5). □

Theorem 3.2.1 If $(x_0, h) \in H \times L^2(0,T;V^*)$, then $x \in L^2(0,T;V) \cap C([0,T];H)$ and the mapping

$$H \times L^2(0,T;V^*) \ni (x_0, h) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$$

is continuous.

Proof. By virtue of Proposition 3.2.1 for any $(x_0, h) \in H \times L^2(0,T;V^*)$, the solution x of (3.2.1) belongs to $L^2(0,T;V) \cap C([0,T];H)$. Let $(x_{0i}, h_i) \in H \times L^2(0,T;V^*)$ and x_i be the solution of (3.2.1) with (x_{0i}, h_i) instead of (x_0, h) for $i = 1,\ 2$. Multiplying on (3.2.1) by $x_1(t) - x_2(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2$$

$$\le \omega_2|x_1(t) - x_2(t)|^2 + ||x_1(t) - x_2(t)||\, ||h_1(t) - h_2(t)||_*.$$

By the similar process of the proof of (3.2.5) it holds

$$\frac{1}{2}|x_1(t) - x_2(t)|^2 + \omega_1 \int_0^t ||x_1(s) - x_2(s)||^2 ds$$

$$\leq \frac{e^{2\omega_2 t}}{2}|x_{01} - x_{02}|^2 + \int_0^t e^{2\omega_2(t-s)}||x_1(s) - x_2(s)||\,||h_1(s) - h_2(s)||_* ds.$$

We can choose a constant $c > 0$ such that

$$\omega_1 - e^{2\omega_2 T}\frac{c}{2} > 0$$

and, hence

$$\int_0^T e^{2\omega_2(t-s)}||x_1(s) - x_2(s)||\,||h_1(s) - h_2(s)||_* ds$$

$$\leq e^{2\omega_2 T}\int_0^T \{\frac{c}{2}||x_1(s) - x_2(s)||^2 + \frac{1}{2c}||h_1(s) - h_2(s)||_*^2\} ds.$$

Thus, there exists a constant $C > 0$ such that

$$||x_1 - x_2||_{L^2(0,T,V)\cap C([0,T];H)} \leq C(|x_{01} - x_{02}| + ||h_1 - h_2||_{L^2(0,T;V^*)}). \quad (3.2.9)$$

Suppose $(x_{0n}, h_n) \to (x_0, h)$ in $H \times L^2(0, T; V^*)$, and let x_n and x be the solutions (E) with (x_{0n}, h_n) and (x_0, h), respectively. Then, by virtue of (3.2.9), we see that $x_n \to x$ in $L^2(0, T, V) \cap C([0, T]; H)$. □

3.3. Approximate controllability

In what follows we assume that the embedding $V \subset H$ is compact. Let x_h be the solution of (3.2.1) corresponding to h in $L^2(0, T; V^*)$. We define the solution mapping S from $L^2(0,T;V^*)$ to $L^2(0,T;V)$ by

$$(Sh)(t) = x_h(t), \quad h \in L^2(0,T;V^*).$$

Let $\mathcal{A}$ be the Nemitsky operator corresponding to the map A, which is defined by $\mathcal{A}(x)(\cdot) = Ax(\cdot)$.

Then

$$x_h(t) = \int_0^t ((I - \mathcal{A}S)h)(s)ds,$$

and with the aid of Proposition 3.2.1

$$\begin{aligned} &||Sh||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = ||x_h||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \\ &\le C_1(|x_0| + ||h||_{L^2(0,T;V^*)} + 1). \end{aligned} \tag{3.3.1}$$

Hence if h is bounded in $L^2(0,T;V^*)$, then so is x_h in $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0,T;V)\cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is compact in view of Theorem 2 of Aubin [14]. Hence, since the embedding $L^2(0,T;H) \subset L^2(0,T;V^*)$ is continuous, the mapping $h \mapsto Sh = x_h$ is compact from $L^2(0,T;V^*)$ to itself.

The solution of (E) is denoted by $x(T;u)$ associated with the control u at time T. The system (E) is said to be *approximately controllable* at time T if $Cl\{x(T;u) : u \in L^2(0,T;U)\} = H$ where Cl denotes the closure in H.

We assume

(B) $\quad Cl\{y : y(t) = (Bu)(t), \quad \text{a.e.} \quad u \in L^2(0,T;U)\} = L^2(0,T;H)$

where Cl denotes also the closure in $L^2(0,T;H)$.

The main results of this section is the following.

Theorem 3.3.1 Let the assumption (B) be satisfied. If our constants condition in (A1), (A2) contains the following inequality: $\omega_3 < \omega_1$, then

$$Cl\{(I - \mathcal{A}S)h : h \in L^2(0, T; V^*)\} = L^2(0, T; V^*). \tag{3.3.2}$$

Therefore, the nonlinear differential control system (E) is approximately controllable at time T.

Proof. Let us fix $T_0 > 0$ so that

$$N = \omega_1^{-1}\omega_3 e^{\omega_2 T_0} < 1. \tag{3.3.3}$$

Let $z \in L^2(0, T_0; V^*)$ and r be a constant such that

$$z \in U_r = \{x \in L^2(0, T_0; V^*) : ||x||_{L^2(0,T_0;V^*)} < r\}.$$

Take a constant $d > 0$ such that

$$(r + \omega_3 + \omega_3\omega_1^{-1/2} e^{\omega_2 T_0}|x_0|)(1 - N)^{-1} < d. \tag{3.3.4}$$

(3.2.5) in Lemma 3.2.1 implies

$$\omega_1 ||x_h||^2_{L^2(0,T_0;V)} \le \frac{e^{2\omega_2 T_0}}{2}|x_0|^2 + \frac{\omega_1}{2}||x_h||^2_{L^2(0,T_0;V)} + \frac{e^{2\omega_2 T_0}}{2\omega_1}||h||^2_{L^2(0,T_0;V^*)},$$

that is,

$$\begin{aligned} ||Sh||_{L^2(0,T_0;V)} &= ||x_h||_{L^2(0,T_0;V)} \\ &\le e^{\omega_2 T_0}(\omega_1^{-1/2}|x_0| + \omega_1^{-1}||h||_{L^2(0,T_0;V^*)}). \end{aligned} \tag{3.3.5}$$

Let us consider the equation

$$z = (I - \lambda \mathcal{A} S)h, \quad 0 \leq \lambda \leq 1. \tag{3.3.6}$$

Let h be the solution of (3.3.5). Then, for the element $z \in U_r$, from (3.3.4) and (3.3.5), it follows that

$$\begin{aligned} ||h||_{L^2(0,T_0;V^*)} &\leq ||z|| + ||\mathcal{A}Sh|| \leq r + \omega_3(||Sh|| + 1) \\ &\leq r + \omega_3\{e^{\omega_2 T_0}(\omega_1^{-1/2}|x_0| + \omega_1^{-1}||h||_{L^2(0,T_0;V^*)}) + 1\}, \end{aligned}$$

and hence

$$\begin{aligned} ||h|| &\leq (r + \omega_3 + \omega_1^{-1/2}\omega_3 e^{\omega_2 T_0}|x_0|)(1 - N)^{-1} \\ &< d. \end{aligned}$$

It follows that $h \notin \partial U_d$ where ∂U_d stands for the boundary of U_d. Thus the homotopy property of topological degree theory there exists $h \in U_d$ such that the equation

$$z = (I - \mathcal{A}S)h$$

holds. Since the assumption (B), there exists a sequence $\{u_n\} \in L^2(0, T_0; U)$ such that $Bu_n \mapsto h$ in $L^2(0, T_0; V^*)$. Then by Theorem 3.2.1 we have that $x(\cdot; u_n) \mapsto x_h$ in $L^2(0, T_0; V) \cap C([0, T_0]; H)$. Let $y \in H$. We can choose $g \in W^{1,2}(0, T_0; V^*)$ such that $g(0) = x_0$ and $g(T_0) = y$ and from the equation (3.3.6) there is $h \in L^2(0, T_0; V^*)$ such that $g' = (I - \mathcal{A}S)h$. By the assumption (B) there exists $u \in L^2(0, T_0; U)$ such that

$$||h - Bu||_{L^2(0,T_0;V^*)} \leq \frac{\sqrt{2}\omega_1}{e^{\omega_2 T_0}}\epsilon$$

for every $\epsilon > 0$. From (3.2.4)

$$\frac{1}{2}|x_h(t) - x_{Bu}(t)|^2 + \omega_1 \int_0^t ||x_h(s) - x_{Bu}(s)||^2 ds \leq$$

$$\int_0^t e^{2\omega_2(t-s)} ||x_h(s) - x_{Bu}(s)|| \, ||h(s) - (Bu)(s)||_* ds$$

$$\leq \omega_1 \int_0^t ||x_h(s) - x_{Bu}(s)||^2 ds + \frac{e^{2\omega_2 t}}{4\omega_1} \int_0^t ||h(s) - (Bu)(s)||^2 ds,$$

it holds

$$||x_h - x_{Bu}||_{C([0,T_0];H)} \leq \frac{e^{\omega_2 T_0}}{\sqrt{2\omega_1}} ||h - Bu||_{L^2(0,T_0;V^*)},$$

thus, we have

$$|y - x_h(T)| = |\int_0^{T_0} ((I - \mathcal{A}S)h)(s)ds - \int_0^{T_0} ((I - \mathcal{A}S)Bu)(s)ds|$$

$$\leq \frac{e^{\omega_2 T_0}}{\sqrt{2\omega_1}} ||h - Bu||_{L^2(0,T_0;V^*)} \leq \epsilon.$$

Therefore, the system (E) is approximately controllable at time T_0. Since the condition (3.3.3) is independent of initial values, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$. By repeating this process, the approximate controllability for (E) can be extended the interval $[0, nT_0]$ for natural number n, i.e., for the initial $x(nT_0)$ in the interval $[nT_0, (n+1)T_0]$. □

Chapter 4
Controllability for nonlinear variational inequalities of parabolic type

4.1. Introduction

Let H and V be two complex Hilbert spaces. Assume that V is a dense subspace in H and the injection of V into H is continuous. The norms on V and H will be denoted by $||\cdot||$ and $|\cdot|$, respectively. Let A be a continuous linear operator from V into V^* which is assumed to satisfy

$$(Au, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2$$

where $\omega_1 > 0$ and ω_2 is a real number and let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Consider the following variational inequality problem with nonlinear term:

$$\begin{cases} (x'(t) + Ax(t), x(t) - z) + \phi(x(t)) - \phi(z) \\ \quad \leq (f(t, x(t)) + k(t), x(t) - z), \ \text{a.e.}, \ 0 < t \leq T, \quad z \in V \\ x(0) = x_0. \end{cases} \tag{VIP}$$

According to the subdifferential operator $\partial\phi$, the problem (VIP) is represented by the following nonlinear functional differential problem on H:

$$\begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + k(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{NDE}$$

The existence and regularity for the parabolic variational inequality in the linear case($f \equiv 0$), which was first investigated by Brézis [15], has been developed as seen in section 4.3.2 of Barbu [19](also see section 4.3.1 in [8]).

First, in Section 4.2 we will deal with the existence for solutions of (NDE) when the nonlinear mapping f is a Lipschitz continuous from $\mathbb{R} \times V$ into H and the norm estimate of a solution of the above nonlinear equation on $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \cap C([0, T]; H)$ as seen in [20]. Consequently, in view of the monotonicity of $\partial\phi$, we show that the mapping

$$H \times L^2(0, T; V^*) \ni (x_0, k) \mapsto x \in L^2(0, T; V) \cap C([0, T]; H)$$

is continuous. Thereafter, we can obtain the approximate controllability for the nonlinear functional differential control problem governed by the variational inequality in Section 4.4. Let U be a complex Banach space and B be a bounded linear operator from $L^2(0, T; U)$ to $L^2(0, T; H)$. Let us consider the following control system governed by the variational inequality problem with the control term Bu instead of k:

$$\begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + (Bu)(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{NCE}$$

For every $\epsilon > 0$, we define the Moreau-Yosida approximation of ϕ as

$$\phi_\epsilon(x) = \inf\{||x-y||_*^2/2\epsilon + \phi(y) : y \in H\}.$$

Then the function ϕ_ϵ is Fréchet differentiable on H. By using the facts that its Fréchet differential $\partial\phi_\epsilon$ is a single valued and Lipschitz continuous on H, we investigate the control problem of (NCE) by transforming onto the semilinear differential equation with $\partial\phi_\epsilon$ in place of $\partial\phi$ and obtain the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \cap C([0,T];H)$ in section 4.3.

In recent years, as for the controllability for semilinear differential equations, Carrasco and Lebia [1] discussed sufficient conditions for approximate controllability of a system of parabolic equations with delay, Mahmudov [21] in case the semilinear equations with nonlocal conditions with condition on the uniform boundedness of the Frechet derivative of nonlinear term, and Sakthivel et al. [22] on impulsive and neutral functional differential equations.

In this section, in order to show the investigate the approximate controllability problem for (NCE), we assume range conditions of the controller B, which is that for any $\varepsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\int_0^T S(T-s)\{p(s)-(Bu)(s)\}| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q_1 ||p||_{L^2(0,t;H)}, \quad 0 \le t \le T, \end{cases}$$

where q_1 is a constant independent of p and $S(t)$ is an analytic semigroup generated by A.

Here, we remark that the quantity condition of the constant q_1 as seen in Zhou [[5]; (3.3)] is not necessary. Some examples to which main result can be applied are given in [2, 5].

If $D(A)$ is compactly embedded in V (or the semigroup operator $S(t)$ is compact), the following embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V))$$

is compact in view of Theorem 2 of Aubin [14]. Hence, the mapping $u \mapsto x$ is compact from $L^2(0,T;U)$ to $L^2(0,T;V)$. From these results we can obtain the approximate controllability for the equation (NCE), which is the extended result of Naito [2] to the equation (NCE). Finally, a simple examples which our main result can be applied is given.

4.2. Preliminaries

Forming Gelfand triple $V \subset H \subset V^*$ with pivot space H, for the sake of simplicity, we may consider

$$||u||_* \leq |u| \leq ||u||, \quad u \in V$$

where $|| \cdot ||_*$ is the norm of the element of V^*. We also assume that there exists a constant C_1 such that

$$||u|| \leq C_1 ||u||_{D(A)}^{1/2} |u|^{1/2} \tag{4.2.1}$$

for every $u \in D(A)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2, \quad u \in V \tag{4.2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with the sesquilinear form $a(\cdot,\cdot)$:

$$(Au, v) = a(u,v), \quad u,\ v \in V.$$

Then A is a bounded linear operator from V to V^* and $-A$ generates an analytic semigroup in both of H and V^* as is seen in [[16]; Theorem 3.6.1]. The realization for the operator A in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

be also denoted by A.

The following L^2-regularity for the abstract linear parabolic equation

$$\begin{cases} x'(t) + Ax(t) = k(t), & 0 < t \leq T, \\ x(0) = x_0 \end{cases} \tag{LE}$$

has a unique solution x in $[0,T]$ for each $T > 0$ if $x_0 \in (D(A), H)_{1/2,2}$ and $k \in L^2(0,T;H)$ where $(D(A),H)_{1/2,2}$ is the real interpolation space between $D(A)$ and H. Moreover, we have

$$||x||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T,H)} \leq C_2(||x_0||_{(D(A),H)_{1/2,2}} + ||k||_{L^2(0,T;H)}) \tag{4.2.3}$$

where C_2 depends on T and M (see Theorem 2.3 of [24], [38]).

Let $0 < \theta < 1$, $1 < p < \infty$. Then by considering an intermediate method between the initial Banach space and the domain of the infinitesimal generator A of the analytic semigroup $T(t)$ is represented by

$$(V, V^*)_{\theta,p} = \{x \in V^* : \int_0^\infty (t^\theta ||Ae^{tA}x||_*)^p \frac{dt}{t} < \infty\}$$

(see Theorem 3.5.3 of [25]).

Proposition 4.2.1 Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (LE) belonging to

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{4.2.4}$$

where C_2 is a constant depending on T.

Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then the subdifferential operator $\partial\phi$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \le \phi(y) + (x^*, x - y), \quad y \in V\}.$$

First, let us concern with the following perturbation of subdifferential operator:

$$\begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni k(t), & 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{VE}$$

Using the regularity for the variational inequality of parabolic type as seen in [2; section 4.3] we have the following result on the equation (VE).

Proposition 4.2.2 1) Let $k \in L^2(0,T;V^*)$ and $x_0 \in \overline{D(\phi)}$ where $\overline{D(\phi)}$ is the closure in H of the set $D(\phi) = \{u \in V : \phi(u) < \infty\}$. Then the equation (VE) has a unique solution

$$x \in L^2(0,T;V) \cap C([0,T];H),$$

which satisfies

$$x'(t) = (k(t) - Ax(t) - \partial\phi(x(t)))^0$$

and

$$||x||_{L^2 \cap C} \leq C_3(1 + |x_0| + ||k||_{L^2(0,T;V^*)}) \qquad (4.2.5)$$

where C_3 is some positive constant and $L^2 \cap C = L^2(0,T;V) \cap c([0,T];H)$ and where $(\partial\phi)^0$ is the minimal segment of $\partial\phi$.

2) Let A be symmetric and let us assume that there exists $h \in H$ such that for every $\epsilon > 0$ and any $y \in D(\phi)$

$$J_\epsilon(y + \epsilon h) \in D(\phi) \text{ and } \phi(J_\epsilon(y + \epsilon h)) \leq \phi(y)$$

where $J_\epsilon = (I + \epsilon A)^{-1}$. Then for $k \in L^2(0,T;H)$ and $x_0 \in \overline{D(\phi)} \cap V$ the equation (VE) has a unique solution

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \cap C([0,T];H),$$

which satisfies

$$||x||_{L^2 \cap W^{1,2} \cap C} \leq C_3(1 + ||x_0|| + ||k||_{L^2(0,T;H)}). \qquad (4.2.6)$$

Here, we remark that if $D(A)$ is compactly embedded in V and $x \in L^2(0,T;D(A))$(or the semigroup operator $S(t)$ generated by A is compact), the following embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V))$$

is compact in view of Theorem 2 of Aubin [14]. Hence, the mapping $k \mapsto x$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$, which is applicable to the control problem.

(F) Let f be a nonlinear single valued mapping from $[0,\infty) \times V$ into H. We assume that

$$|f(t,x_1) - f(t,x_2)| \leq L||x_1 - x_2||,$$

for every $x_1,\ x_2 \in V$.

The following result is from Jeong and Park [26].

Theorem 4.2.1[[26]] Let the assumption (F) be satisfied. Assume that $k \in L^2(0,T;V^*)$ and $x_0 \in \overline{D(\phi)}$. Then, the equation (NDE) has a unique solution

$$x \in L^2(0,T;V) \cap C([0,T];H)$$

and there exists a constant C_4 depending on T such that

$$||x||_{L^2 \cap C} \leq C_4(1 + |x_0| + ||k||_{L^2(0,T;V^*)}). \tag{4.2.7}$$

Furthermore, if $k \in L^2(0,T;H)$ then the solution x belongs to $W^{1,2}(0,T;H)$ and satisfies

$$||x||_{W^{1,2}(0,T;H)} \le C_4(1 + |x_0| + ||k||_{L^2(0,T;H)}). \tag{4.2.8}$$

If $(x_0, k) \in H \times L^2(0,T;H)$, then the solution x of the equation (NDE) belongs to $x \in L^2(0,T;V) \cap C([0,T];H)$ and the mapping

$$H \times L^2(0,T;H) \ni (x_0, k) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$$

is continuous.

4.3. Smoothing system corresponding to (NCE)

For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = \inf\{||x - y||_*^2/2\epsilon + \phi(y) : y \in H\}.$$

Then the function ϕ_ϵ is Fréchet differentiable on H and its Frećhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1} where $\partial\phi_\epsilon = \epsilon^{-1}(I - (I + \epsilon\partial\phi)^{-1}]$ as is seen in Corollary 2.2 in [[8]; Chapter II). It is also well known results that $\lim_{\epsilon\to 0} \phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0} \partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$.

Now, we introduce the smoothing system corresponding to (NCE) as follows.

$$\begin{cases} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = f(t, x(t)) + (Bu)(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{SCE}$$

Since $-A$ generates a semigroup $S(t)$ on H, the mild solution of (SCE) can be represented by

$$x_\epsilon(t) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x_\epsilon(s)) + (Bu)(s) - \partial\phi_\epsilon(x_\epsilon(s))\}ds.$$

In virtue of Theorem 4.2.1 we know that if the assumption (F) is satisfied then for every $x_0 \in H$ and every $u \in L^2(0, T; U)$ the equation (SCE) has a unique solution

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \cap C([0, T]; H)$$

and there exists a constant C_4 depending on T such that

$$||x||_{L^2 \cap W^{1,2} \cap C} \leq C_4(1 + |x_0| + ||u||_{L^2(0,T;U)}). \tag{4.3.1}$$

(A) We assume the hypothesis that $(\partial\phi)^0$ is uniformly bounded, i.e.,

$$|(\partial\phi)^0 x| \leq M_1, \quad x \in H.$$

Lemma 4.3.1 Let x_ϵ and x_λ be the solutions of (SCE) with same control u. Then there exists a constant C independent of ϵ and λ such that

$$||x_\epsilon - x_\lambda||_{C([0,T];H) \cap L^2(0,T;V)} \leq C(\epsilon + \lambda), \quad 0 < T.$$

Proof. For given ϵ, $\lambda > 0$, let x_ϵ and x_λ be the solutions of (SCE) corresponding to ϵ and λ, respectively. Then from the equation (SCE) we have

$$x_\epsilon'(t) - x_\lambda'(t) + A(x_\epsilon(t) - x_\lambda(t)) + \partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t))$$

$$= f(t, x_\epsilon(t)) - f(t, x_\lambda(t)),$$

and hence, from (4.2.2) and multiplying by $x_\epsilon(t) - x_\lambda(t)$, it follows that

$$\frac{1}{2}\frac{d}{dt}|x_\epsilon(t) - x_\lambda(t)|^2 + \omega_1||x_\epsilon(t) - x_\lambda(t)||^2 \tag{4.3.2}$$

$$+ (\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), x_\epsilon(t) - x_\lambda(t))$$

$$\leq (f(t, x_\epsilon(t)) - f(t, x_\lambda(t)), x_\epsilon(t) - x_\lambda(t)) + \omega_2|x_\epsilon(t) - x_\lambda(t)|^2.$$

Let us choose a constant $c > 0$ such that $2\omega_1 - cL > 0$. Noting that

$$(f(t, x_\epsilon(t)) - f(t, x_\lambda(t)), x_\epsilon(t) - x_\lambda(t))$$

$$\leq |f(t, x_\epsilon(t)) - f(t, x_\lambda(t))|\,|x_\epsilon(t) - x_\lambda(t)|$$

$$\leq \frac{cL}{2}||x_\epsilon(t) - x_\lambda(t)||^2 + \frac{1}{2c}|x_\epsilon(t) - x_\lambda(t)|^2,$$

by integrating (4.3.2) over $[0, T]$ and using the monotonicity of $\partial\phi$ we have

$$\frac{1}{2}|x_\epsilon(t) - x_\lambda(t)|^2 + (\omega_1 - \frac{cL}{2})\int_0^T ||x_\epsilon(t) - x_\lambda(t)||^2 dt$$

$$\leq \int_0^T (\partial\phi_\epsilon(x_\epsilon(t)) - \partial\phi_\lambda(x_\lambda(t)), \lambda\partial\phi_\lambda(x_\lambda(t) - \epsilon\partial\phi_\epsilon(x_\epsilon(t))dt$$

$$+ (\frac{1}{2c} + \omega_2)\int_0^T |x_\epsilon(t) - x_\lambda(t)|^2 dt.$$

Here, we used that

$$\partial\phi_\epsilon(x_\epsilon(t)) = \epsilon^{-1}(x_\epsilon(t) - (I + \epsilon\partial\phi)^{-1}x_\epsilon(t)).$$

Since $|\partial\phi_\epsilon(x)| \le |(\partial\phi)^0 x|$ for every $x \in D(\partial\phi)$ it follows from (A) and using Gronwall's inequality that

$$||x_\epsilon - x_\lambda||_{C([0,T];H)\cap L^2(0,T;V)} \le C(\epsilon + \lambda), \quad 0 < T.$$

□

Theorem 4.3.1 Let the assumptions (F) and (A) be satisfied. Then $x = \lim_{\epsilon\to 0} x_\epsilon$ in $L^2(0,T;V) \cap C([0,T];H)$ is a solution of the equation (NCE) where x_ϵ is the solution of (SCE).

Proof. In virtue of Lemma 4.3.1, there exists $x(\cdot) \in L^2(0,T;V)$ such that

$$x_\epsilon(\cdot) \to x(\cdot) \quad \text{in} \quad L^2(0,T;V) \cap C([0,T];H).$$

From (F) it follows that

$$f(\cdot, x_\epsilon) \to f(\cdot, x), \quad \text{strongly in } L^2(0,T;H) \tag{4.3.3}$$

and

$$Ax_n \to Ax, \quad \text{strongly in } L^2(0,T;V^*). \tag{4.3.4}$$

Since $\partial\phi_\epsilon(x_\epsilon)$ are uniformly bounded by assumption (A), from (4.3.3) and (4.3.4) we have that

$$\frac{d}{dt}x_\epsilon \to \frac{d}{dt}x, \quad \text{weakly in } L^2(0,T;V^*),$$

therefore

$$\partial\phi_\epsilon(x_\epsilon) \to f(\cdot, x) + k - x' - Ax, \quad \text{weakly in } L^2(0,T;V^*).$$

Note that $\partial\phi_\epsilon(x_\epsilon) = \partial\phi((I+\epsilon\partial\phi)^{-1}x_\epsilon)$. Since $(I+\epsilon\partial\phi)^{-1}x_\epsilon \to x$ strongly and $\partial\phi$ is demiclosed, we have that

$$f(\cdot, x) + k - x' - Ax \in \partial\phi(x) \text{ in } L^2(0,T;V^*).$$

Thus we have proved that $x(t)$ satisfies a.e. on $(0,T)$ the equation (NCE). □

4.4. Approximate controllability

In this section we show the approximate controllability for the equation (NCE) with the more general condition for the range of the control operator, which is the extended result of Zhou [[5]; section 3] and Naito [2] to the equation (SCE).

For the sake of simplicity we assume that the solution semigroup $S(t)$ is uniformly bounded:

$$|S(t)| \leq M \quad t \geq 0.$$

Lemma 4.4.1 Let $u_i \in L^2(0,T;U)$ and $x_{\epsilon i}$ be the solution of (SCE) with u_i in place of u for $i = 1,\ 2$. Then there exists a constant $C > 0$ such that

$$|x_{\epsilon 1}(t) - x_{\epsilon 2}(t)| \leq M\sqrt{t}\{C(\epsilon^{-1} + L) + 1\}||Bu_1 - Bu_2||_{L^2(0,t;H)}$$

for $0 < t \leq T$.

Proof. In virtue of Theorem 4.2.1 it holds that there exists a constant $C > 0$ such that

$$||x_{\epsilon 1} - x_{\epsilon 2}||_{L^2(0,t;V)} \leq C||Bu_1 - Bu_2||_{L^2(0,t;H)}, \quad t > 0.$$

The proof of Lemma 4.3.1 is a consequence of the estimate

$$|x_{\epsilon 1}(t) - x_{\epsilon 2}(t)| = |\int_0^t S(t-s)[\{f(s, x_{\epsilon 1}(s)) - f(s, x_{\epsilon 2}(t))\}$$

$$+ \{\partial\phi_\epsilon(x_{\epsilon 1}(s) - \partial\phi_\epsilon(x_{\epsilon 2}(s)))\} + \{(Bu_1)(s) - (Bu_2)(s)\}]ds|$$

$$\leq M\sqrt{t}(\epsilon^{-1} + L)||x_{\epsilon 1} - x_{\epsilon 2}||_{L^2(0,t;V)} + M\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,t;H)}$$

$$\leq M\sqrt{t}\{C(\epsilon^{-1} + L) + 1\}||Bu_1 - Bu_2||_{L^2(0,t;H)}.$$

□

We denote the linear operator $\hat{S}$ from $L^2(0, T; H)$ to H by

$$\hat{S}p = \int_0^T S(T-s)p(s)ds$$

for $p \in L^2(0, T; H)$. The system (SCE) is approximately controllable on $[0, T]$ if for any $\varepsilon > 0$ and $\xi_T \in H$ there exists a control $u \in L^2(0, T; U)$ such that

$$|\xi_T - S(T)x_0 - \hat{S}\{f(\cdot, x_\epsilon(\cdot; g)) - \partial\phi_\epsilon(x_\epsilon(\cdot)\} - \hat{S}Bu| < \varepsilon.$$

We need the following hypothesis:

(B) For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}Bu| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \le q_1 ||p||_{L^2(0,t;H)}, \quad 0 \le t \le T. \end{cases} \tag{1}$$

where q_1 is a constant independent of p.

Remark 4.4.1. If the range of B is dense in $L^2(0, T; H)$ then Hypothesis (B) is satisfied (Theorem 3.3 of [2]). Some examples to which main result can be applied are given in [2, 5]. Those examples will be given which show that even if the range of B is not dense in $L^2(0, T; H)$. In [5], Zhou proved that such a system is approximately controllable under Hypothesis (B) dependent of the time T.
In this section, sufficient conditions for the approximate controllability of the system (SCE) are no need to assume the condition on the length T of the time interval, which has a simple form and can be easily checked in many examples. So this sufficient condition is more general than previous ones. It is suitable not only for a nonlinear abstract control system in Hilbert space, but also for the finite dimensional ordinary differential equations by using the spectral projection operator with finite rank associated with the generalized eigenspace.

The solutions of (NCE) and (SCE) are denoted by $x(t; \phi, f, u)$ and $x_\epsilon(t; \phi_\epsilon, f, u)$, respectively.

Theorem 4.4.1 Under the assumptions (F) and (B), the system (SCE) is approximately controllable on $[0, T]$.

Proof. We shall show that

$$D(A) \subset \mathrm{Cl}\{x_\epsilon(T; \phi_\epsilon, f, u) : u \in L^2(0, T; U)\}$$

where Cl denotes the closure in H, i.e., for given $\varepsilon > 0$ and $\xi_T \in D(A)$ there exists $u \in L^2(0, T; U)$ such that

$$|\xi_T - x_\epsilon(T; \phi_\epsilon, f, u)| < \varepsilon,$$

where

$$\begin{aligned} x_\epsilon(t; \phi_\epsilon, f, u) =& S(T)x_0 + \int_0^T S(T-s)\{f(s, x_\epsilon(s; \phi_\epsilon, f, u)) \\ & - \partial\phi_\epsilon(x_\epsilon(s; \phi_\epsilon, f, u) + (Bu)(s)\}ds. \end{aligned}$$

As $\xi_T \in D(A)$ there exists a $p \in L^2(0, T; H)$ such that

$$\hat{S}p = \xi_T - S(T)x_0,$$

for instance, take $p(s) = (\xi_T + sA\xi_T - S(s)x_0)/T$.

Set

$$F(x_\epsilon(s; \phi_\epsilon, f, u)) = f(s, x_\epsilon(s; \phi_\epsilon, f, u)) - \partial\phi_\epsilon(x_\epsilon(s; \phi_\epsilon, f, u)).$$

Then

$$\begin{aligned} &|F(x_\epsilon(s; \phi_\epsilon, f, u_1)) - F(x_\epsilon(s; \phi_\epsilon, f, u_2))| \\ &\le (\epsilon^{-1} + L)||x_\epsilon(s; \phi_\epsilon, f, u_1) - x_\epsilon(s; \phi_\epsilon, f, u_2)||. \end{aligned}$$

Let $u_1 \in L^2(0,T;U)$ be arbitrary fixed. Since by the assumption (B) there exists $u_2 \in L^2(0,T;U)$ such that

$$|\hat{S}(p - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4},$$

it follows

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}. \tag{4.4.1}$$

We can also choose $w_2 \in L^2(0,T;U)$ by the assumption (B) such that

$$|\hat{S}(F(x_\epsilon(\cdot;\phi_\epsilon, f, u_2)) - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_1))) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{4.4.2}$$

and

$$\|Bw_2\|_{L^2(0,t;H)} \leq q_1||F(x_\epsilon(\cdot;\phi_\epsilon, f, u_1)) - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_2))||_{L^2(0,t;H)}$$

for $0 \leq t \leq T$. Therefore, in view of Lemma 4.4.1 and the assumption (B)

$$||Bw_2||_{L^2(0,t;H)} \leq q_1\{\int_0^t |F(x_\epsilon(\tau;\phi_\epsilon, f, u_2)) - F(x_\epsilon(\tau;\phi_\epsilon, f, u_1))|^2 d\tau\}^{\frac{1}{2}}$$

$$\leq q_1(\epsilon^{-1} + L)\{\int_0^t ||x_\epsilon(\tau;\phi_\epsilon, f, u_1) - x_\epsilon(\tau;\phi_\epsilon, f, u_2))||^2 d\tau\}^{\frac{1}{2}}$$

$$\leq q_1(\epsilon^{-1} + L)[\int_0^t M^2\{C(\epsilon^{-1} + L) + 1\}^2 \tau ||Bu_2 - Bu_1||^2_{L^2(0,\tau;H)} d\tau]^{\frac{1}{2}}$$

$$\leq q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}(\int_0^t \tau d\tau)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)}$$

$$= q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}(\frac{t^2}{2})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)}.$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$|\hat{S}(F(x_\epsilon(\cdot;\phi_\epsilon, f, u_3))) - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8},$$

$$||Bw_3||_{L^2(0,t;H)} \leq q_1||F(x_\epsilon(\cdot;\phi_\epsilon, f, u_3)) - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_2))||_{L^2(0,t;H)}$$

for $0 \leq t \leq T$. Hence, we have

$$\begin{aligned}
&||Bw_3||_{L^2(0,t;H)} \\
&\leq q_1\{\int_0^t |F(x_\epsilon(\tau;\phi_\epsilon, f, u_3)) - F(x_\epsilon(\tau;\phi_\epsilon, f, u_2))|^2 d\tau\}^{\frac{1}{2}} \\
&\leq q_1(\epsilon^{-1} + L)\{\int_0^t ||x_\epsilon(\tau;\phi_\epsilon, f, u_3) - x_\epsilon(\tau;\phi_\epsilon, f, u_2)||^2 d\tau\}^{\frac{1}{2}} \\
&\leq q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}\{\int_0^t \tau||Bu_3 - Bu_2||^2_{L^2(0,\tau:H)} d\tau\}^{\frac{1}{2}} \\
&\leq q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}\{\int_0^t \tau||Bw_2||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\leq q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\} \\
&\quad \{\int_0^t \tau[q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}]^2 \frac{\tau^2}{2}||Bu_2 - Bu_1||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\leq [q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}]^2 (\int_0^t \frac{\tau^3}{2} d\tau)^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)} \\
&= [q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}]^2 (\frac{t^4}{2 \cdot 4})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)}.
\end{aligned}$$

By proceeding this process for $u_{n+1} = u_n - w_n (n = 1, 2, ...)$, and from that

$$||B(u_n - u_{n+1})||_{L^2(0,t;H)} = ||Bw_n||_{L^2(0,t;H)}$$

$$\leq [q_1 M(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}]^{n-1}$$

$$(\frac{t^{2n-2}}{2 \cdot 4 \cdots (2n-2)})^{\frac{1}{2}} ||Bu_2 - Bu_1||_{L^2(0,t;H)}$$

$$= [\frac{q_1 TM(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}}{\sqrt{2}}]^{n-1} \frac{1}{\sqrt{(n-1)!}} ||Bu_2 - Bu_1||_{L^2(0,t;H)},$$

it follows that

$$\sum_{n=1}^{\infty} ||Bu_{n+1} - Bu_n||_{L^2(0,T;H)}$$

$$\leq \sum_{n=0}^{\infty} [\frac{q_1 TM(\epsilon^{-1} + L)\{C(\epsilon^{-1} + L) + 1\}}{\sqrt{2}}]^n \frac{1}{\sqrt{n!}} ||Bu_2 - Bu_1||_{L^2(0,T;H)}$$

$$< \infty.$$

Therefore, there exists $u^* \in L^2(0, T; H)$ such that

$$\lim_{n \to \infty} Bu_n = u^* \quad \text{in} \quad L^2(0, T; H). \tag{4.4.3}$$

From (4.4.1) and (4.4.2) it follows that

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_2)) - \hat{S}Bu_3|$$

$$= |\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2$$

$$- \hat{S}[F(x_\epsilon(\cdot;\phi_\epsilon, f, u_2)) - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_1))]|$$

$$< (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.$$

By choosing choose $w_n \in L^2(0,T;U)$ by the assumption (B) such that

$$|\hat{S}(F(x_\epsilon(\cdot;\phi_\epsilon, f, u_n)) - F(x_\epsilon(\cdot;\phi_\epsilon, f, u_{n-1})) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}},$$

since $u_{n+1} = u_n - w_n$, we have

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_n)) - \hat{S}Bu_{n+1}|$$

$$< (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1\ 2, \ \cdots.$$

According to (4.4.3) for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Bu_{N+1} - \hat{S}Bu_N| < \frac{\varepsilon}{2}$$

and

$$|\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_N)) - \hat{S}Bu_N|$$

$$\leq |\xi_T - S(T)x_0 - \hat{S}F(x_\epsilon(\cdot;\phi_\epsilon, f, u_N)) - \hat{S}Bu_{N+1}|$$

$$+ |\hat{S}Bu_{N+1} - \hat{S}\Phi u_N|$$

$$< (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus the system (SCE) is approximately controllable on $[0, T]$ as N tends to infinity. □

From Theorem 4.3.1 and Theorem 4.4.1 we obtain the following results.

Theorem 4.4.2 Under the assumptions (A), (F) and (B), the system (NCE) is approximately controllable on $[0, T]$.

Example 4.4.1 Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and closure $\overline{\Omega}$. $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω. $C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω.

For $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p} = (\sum_{\alpha \leq m} ||D^\alpha f||_p^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad ||f||_{m,\infty} = \max_{\alpha \leq m} ||D^\alpha u||_\infty,$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $|| \cdot ||_p$. $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p = 2$, we denote $W^{m,2}(\Omega) = H^m(\Omega)$(simply, $W^{1,2}(\Omega) = H(\Omega)$), $W_0^{m,2}(\Omega) = H_0^m(\Omega)$. $H^{-1}(\Omega)$ stands for the dual space $W_0^{1,2}(\Omega)^*$ whose norm is denoted by $|| \cdot ||_{-1}$. From now on, we consider a Gelfand triple as $V = H_0(\Omega)$, $H = L^2(\Omega)$ and $V = H^{-1}(\Omega)$ to discuss our problems given in section 2.

We consider the control problem of the following variational inequality problem:

$$\begin{cases} (\partial/\partial t)u(x,t) + \mathcal{A}(x,D_x)u(x,t), u(x,t)-z) \\ + \int_\Omega |\mathrm{grad}\, u(t,x)|^2 dx - \int_\Omega |\mathrm{grad}\, z(t,x)|^2 dx \\ \leq \left(\int_0^t k(t-s)g(s,x(s))ds + (B_\alpha w(t))(x)), u(x,t)-z(x,t) \right), \\ \qquad\qquad\qquad (x,t) \in \Omega \times (0,T], \quad z(\cdot,t) \in H_0(\Omega), \\ u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,T]. \end{cases} \tag{4.4.4}$$

Here, $\mathcal{A}(x,D_x)$ is a second order linear differential operator with smooth coefficients in $\overline{\Omega}$, and $\mathcal{A}(x,D_x)$ is elliptic. If we put that $Au = \mathcal{A}(x,D_x)u$ then it is known that $-A$ generates an analytic semigroup in $H^{-1}(\Omega)$ as is seen in [26].

We denote the realization of $\mathcal{A}$ in $L^2(\Omega)$ under the Dirichlet boundary condition by $\hat{A}$:

$$D(\hat{A}) = H^2(\Omega) \cap H_0(\Omega),$$

$$\hat{A}u = \mathcal{A}u \quad \text{for} \quad u \in D(\hat{A}).$$

The operator $-\hat{A}$ generates analytic semigroup in $L^2(\Omega)$. From now on, both A and $\hat{A}$ are denoted simply by A. So, we may consider that $-A$ generates an analytic semigroup in both of $H = L^p(\Omega)$ and $V^* = H^{-1}(\Omega)$ as seen in section 4.2.

For every $u \in H_0(\Omega)$ define

$$\phi(u) = \begin{cases} \int_\Omega |\mathrm{grad}\, u)|^2 dx, & \text{if } u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to check if ϕ is proper and lower semicontinuous on V to $(-\infty, +\infty]$(see in section 2.3 of [8]).

Let $g : [0,T] \times V \longrightarrow H$ be a nonlinear mapping such that $t \mapsto g(t,x)$ is measurable and

$$|g(t,x) - g(t,y)| \le L||x - y||, \tag{4.4.5}$$

for a positive constant L. We assume that $g(t,0) = 0$ for the sake of simplicity.

For $x \in L^2(0,T;V)$ we set

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds$$

where k belongs to $L^2(0,T)$. By (4.4.5) it is easily seen that the nonlinear term f satisfies hypothesis (F) in section 2.

Let $U = H$, $0 < \alpha < T$ and define the intercept controller operator B_α on $L^2(0,T;H)$ by

$$B_\alpha u(t) = \begin{cases} 0, & 0 \le t < \alpha, \\ u(t), & \alpha \le t \le T \end{cases}$$

for $u \in L^2(0, T; H)$. For a given $p \in L^2(0, T; H)$ let us choose a control function u satisfying

$$u(t) = \begin{cases} 0, \quad 0 \le t < \alpha, \\ p(t) + \frac{\alpha}{T-\alpha} S(t - \frac{\alpha}{T-\alpha}(t-\alpha)) p(\frac{\alpha}{T-\alpha}(t-\alpha)), \ \alpha \le t \le T. \end{cases}$$

Then $u \in L^2(0, T; H)$ and $\hat{S}p = \hat{S}B_\alpha u$. From the following:

$$\begin{aligned} ||B_\alpha u||_{L^2(0,T;H)} &= ||u||_{L^2(\alpha,T;H)} \\ &\le ||p||_{L^2(\alpha,T;H)} + \frac{\alpha M}{T-\alpha} ||p(\frac{\alpha}{T-\alpha}(\cdot - \alpha))||_{L^2(\alpha,T;H)} \\ &\le (1 + M\sqrt{\frac{\alpha}{T-\alpha}}) ||p||_{L^2(0,T;H)}, \end{aligned}$$

it follows that the controller B_α satisfies hypothesis (B). Hence from Theorem 4.4.1 and Theorem 4.4.2, it follows that the system (4.4.4) is approximately controllable on $[0, T]$.

Chapter 5
Optimal Control Problems for Nonlinear Variational Evolution Inequalities

5.1. Introduction

In this section, we deal with optimal control problems governed by the following variational inequality in a Hilbert space H:

$$\begin{cases} (x'(t) + Ax(t), x(t) - z) + \phi(x(t)) - \phi(z) \\ \quad \le (f(t, x(t)) + Bu(t), x(t) - z), \text{ a.e., } 0 < t \le T, \quad z \in V \\ x(0) = x_0. \end{cases} \tag{VIP}$$

Here, A is a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality, where V is a dense subspace in H. Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Let $\mathcal{U}$ be a Hilbert space of control variables, and B be a bounded linear operator from $\mathcal{U}$ into $L^2(0, T; H)$. Let $\mathcal{U}_{ad}$ be a closed convex subset of $\mathcal{U}$, which is called the admissible set. Let $J = J(v)$ be a given quadratic cost function(see (5.3.3) or (5.4.6)). Then we will find an element $u \in \mathcal{U}_{ad}$ which attains minimum of $J(v)$ over $\mathcal{U}_{ad}$ subject to the equation (VIP).

Recently, initial and boundary value problems for permanent magnet technologies have been introduced via variational inequalities in [19, 27], and nonlinear variational inequalities of semilinear parabolic type in [20, 28]. The papers treating the variational inequalities with nonlinear perturbations are not so many. First of all, we deal with the existence and a variation of constant formula for solutions of the nonlinear functional differential equation (VIP) governed by the variational inequality in Hilbert spaces in Section 5.2.

Based on the regularity results for solution of (VIP), we intend to establish the optimal control problem for the cost problems in Section 5.3. For the optimal control problem of systems governed by variational inequalities, see [29, 19]. We refer to [30, 22] to see the applications of nonlinear variational inequalities. Necessary conditions for state constraint optimal control problems governed by semilinear elliptic problems have been obtained by Bonnans and Tida [32] using methods of convex analysis (see also [40]).

Let x_u stand for solution of (VIP) associated with the control $u \in \mathcal{U}$. When the nonlinear mapping f is Lipschitz continuous from $\mathbb{R} \times V$ into H, we will obtain the regularity for solutions of (VIP) and the norm estimate of a solution of the above nonlinear equation on desired solution space. Consequently, in view of the monotonicity of $\partial\phi$, we show that the mapping $u \mapsto x_u$ is continuous in order to establish the necessary conditions of optimality of optimal controls for various observation cases.

In Section 5.4, we will characterize the optimal controls by giving necessary conditions for optimality. For this, it is necessary to write down the necessary optimal condition due to the theory of Lions [40]. The most important objective of such a treatment is to derive necessary optimality conditions that are able to give complete information on the optimal control.

Since the optimal control problems governed by nonlinear equations are nonsmooth and nonconvex, the standard methods of deriving necessary conditions of optimality are inapplicable here. So we approximate the given problem by a family of smooth optimization problems and afterwards tend to consider the limit in the corresponding optimal control problems. An attractive feature of this approach is that it allows the treatment of optimal control problems governed by a large class of nonlinear systems with general cost criteria.

5.2. Regularity for solutions

Let H and V be two real separable Hilbert spaces forming Gelfand tripple $V \subset H \subset V^*$ with pivot space H as mentioned in Chapter 3. We have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{5.2.1}$$

where each space is dense in the next one which is continuous injection.

Lemma 5.2.1 With the notations (4.2.1) and (5.2.1), we have

$$(V, V^*)_{1/2,2} = H,$$

$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [34]).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X,Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X,X)$ is simply written as $\mathcal{L}(X)$. Here, we note that by using interpolation theory we have

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H). \tag{5.2.2}$$

First of all, consider the following linear system

$$\begin{cases} x^{'}(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{5.2.3}$$

By virtue of Theorem 3.3 of [38](or Theorem 3.1 of [18], [16]), we have the following result on the corresponding linear equation of (5.2.3).

Lemma 5.2.2 Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:

1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 5.2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (5.2.3) belonging to

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \leq C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{5.2.4}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (5.2.3) belonging to

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{5.2.5}$$

where C_1 is a constant depending on T.

Let Y be another Hilbert space of control variables and take $\mathcal{U} = L^2(0,T;Y)$ as stated in Introduction. Choose a bounded subset U of Y and call it a control set. Let us define an admissible control $\mathcal{U}_{ad}$ as

$$\mathcal{U}_{ad} = \{u \in L^2(0,T;Y) : u \text{ is a strongly measurable function satisfying}$$
$$u(t) \in U \text{ for almost all } t\}.$$

Noting that the subdifferential operator $\partial\phi$ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \leq \phi(y) + (x^*, x - y), \quad y \in V\},$$

the problem (VIP) is represented by the following nonlinear functional differential problem on H

$$\begin{cases} x'(t) + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + Bu(t), & 0 < t \le T, \\ x(0) = x_0. \end{cases}$$

Referring to Theorem 3.1 of [20], we establish the following results on the solvability of (VIP).

Proposition 5.2.1 1) Let the assumption (F) be satisfied. Assume that $u \in L^2(0,T;Y)$, $B \in \mathcal{L}(Y, V^*)$ and $x_0 \in \overline{D(\phi)}$ where $\overline{D(\phi)}$ is the closure in H of the set $D(\phi) = \{u \in V : \phi(u) < \infty\}$. Then, (VIP) has a unique solution

$$x \in L^2(0,T;V) \cap C([0,T];H)$$

which satisfies

$$x'(t) = Bu(t) - Ax(t) - (\partial\phi)^0(x(t)) + f(t, x(t)),$$

where $(\partial\phi)^0 : H \to H$ is the minimum element of $\partial\phi$ and there exists a constant C_2 depending on T such that

$$||x||_{L^2\cap C} \le C_2(1 + |x_0| + ||Bu||_{L^2(0,T;V^*)}), \tag{5.2.6}$$

where C_2 is some positive constant and $L^2 \cap C = L^2(0,T;V) \cap C([0,T];H)$.

Furthermore, if $B \in \mathcal{L}(Y, H)$ then the solution x belongs to $W^{1,2}(0,T;H)$ and satisfies

$$||x||_{W^{1,2}(0,T;H)} \le C_2(1 + |x_0| + ||Bu||_{L^2(0,T;H)}). \tag{5.2.7}$$

2) We assume

(A) A is symmetric and there exists $h \in H$ such that for every $\epsilon > 0$ and any $y \in D(\phi)$

$$J_\epsilon(y + \epsilon h) \in D(\phi) \text{ and } \phi(J_\epsilon(y + \epsilon h)) \le \phi(y),$$

where $J_\epsilon = (I + \epsilon A)^{-1}$.

Then for $u \in L^2(0,T;Y)$, $B \in \mathcal{L}(Y,H)$, and $x_0 \in \overline{D(\phi)} \cap V$ the equation (VIP) has a unique solution

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \cap C([0,T];H),$$

which satisfies

$$||x||_{L^2 \cap W^{1,2} \cap C} \le C_2(1 + ||x_0|| + ||Bu||_{L^2(0,T;H)}). \tag{5.2.8}$$

Remark 5.2.1 In terms of Lemma 5.2.1, the following inclusion

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

is well known as seen (4.2.1) and is an easy consequence of the definition of real interpolation spaces by the trace method(see [28, 16]).

The following Lemma is from Brézis [[15]; Lemma A.5].

Lemma 5.2.3 Let $m \in L^1(0,T;\mathbb{R})$ satisfying $m(t) \ge 0$ for all $t \in (0,T)$ and $a \ge 0$ be a constant. Let b be a continuous function on $[0,T] \subset \mathbb{R}$ satisfying the following inequality:

$$\frac{1}{2}b^2(t) \le \frac{1}{2}a^2 + \int_0^t m(s)b(s)ds, \quad t \in [0,T].$$

Then,

$$|b(t)| \le a + \int_0^t m(s)ds, \quad t \in [0,T].$$

For each $(x_0, u) \in H \times L^2(0,T;Y)$, we can define the continuous solution mapping $(x_0, u) \mapsto x$. Now, we can state the following theorem.

Theorem 5.2.1 1) Let the assumption (F) be satisfied, $x_0 \in H$, and $B \in \mathcal{L}(Y, V^*)$. Then the solution x of (VIP) belongs to $x \in L^2(0,T;V) \cap C([0,T];H)$ and the mapping

$$H \times L^2(0,T;Y) \ni (x_0, u) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$$

is Lipschtz continuous, i.e., suppose that $(x_{0i}, u_i) \in H \times L^2(0,T;Y)$ and x_i be the solution of (VIP) with (x_{0i}, u_i) in place of (x_0, u) for $i = 1,\ 2$,

$$||x_1 - x_2||_{L^2(0,T;V)\cap C([0,T];H)} \le C\{|x_{01} - x_{02}| + ||u_1 - u_2||_{L^2(0,T;Y)}\}, \quad (5.2.9)$$

where C is a constant.

2) Let the assumptions (A) and (F) be satisfied and let $B \in \mathcal{L}(Y, H)$ and $x_0 \in \overline{D(\phi)} \cap V$. Then $x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$, and the mapping

$$V \times L^2(0,T;Y) \ni (x_0, u) \mapsto x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \quad (5.2.10)$$

is continuous.

Proof. Due to Proposition 5.2.1, we can infer that (VIP) possesses a unique solution $x \in L^2(0,T;V) \cap C([0,T];H)$ with the data condition $(x_0, u) \in$

$H \times L^2(0,T;Y)$. Now, we will prove the inequality (5.2.9). For that purpose, we denote $x_1 - x_2$ by X. Then

$$\begin{cases} X'(t) + AX(t) + \partial\phi(x_1(t)) - \partial\phi(x_2(t)) \\ \quad \ni f(t,x_1(t)) - f(t,x_2(t)) + B(u_1(t) - u_2(t)), \quad 0 < t \le T, \\ X(0) = x_{01} - x_{02}. \end{cases}$$

Multiplying on the above equation by $X(t)$, we have

$$\frac{1}{2}\frac{d}{dt}|X(t)|^2 + \omega_1||X(t)||^2$$

$$\le \omega_2|X(t)|^2 + \{|f(t,x_1(t)) - f(t,x_2(t))| + |B(u_1(t) - u_2(t))|\}|X(t)|.$$

Put

$$H(t) = (L||X(t)|| + |B(u_1(t) - u_2(t))|)|X(t)|.$$

By integrating the above inequality over $[0,t]$, we have

$$\frac{1}{2}|X(t)|^2 + \omega_1\int_0^t ||X(s)||^2 ds \tag{5.2.11}$$

$$\le \frac{1}{2}|x_{01} - x_{02}|^2 + \omega_2\int_0^t |X(s)|^2 ds + \int_0^t H(s)ds.$$

Note that

$$\frac{d}{dt}\{e^{-2\omega_2 t}\int_0^t |X(s)|^2 ds\} \le 2e^{-2\omega_2 t}\{\frac{1}{2}|X(t)|^2 - \omega_2\int_0^t |X(s)|^2 ds\}$$

$$\le 2e^{-2\omega_2 t}\{\frac{1}{2}|x_{01} - x_{02}|^2 + \int_0^t H(s)ds\},$$

integrating the above inequality over $(0,t)$, we have

$$e^{-2\omega_2 t}\int_0^t |X(s)|^2 ds \le 2\int_0^t e^{-2\omega_2\tau}\{\frac{1}{2}|x_{01}-x_{02}|^2+\int_0^\tau H(s)ds\}d\tau$$

$$=\frac{1-e^{-2\omega_2 t}}{2\omega_2}|x_{01}-x_{02}|^2+2\int_0^t\int_s^t e^{-2\omega_2\tau}d\tau H(s)ds$$

$$=\frac{1-e^{-2\omega_2 t}}{2\omega_2}|x_{01}-x_{02}|^2+\frac{1}{\omega_2}\int_0^t (e^{-2\omega_2 s}-e^{-2\omega_2 t})H(s)ds.$$

Thus, we get

$$\omega_2\int_0^t |X(s)|^2 ds \le \frac{1}{2}(e^{2\omega_2 t}-1)|x_{01}-x_{02}|^2+\int_0^t (e^{2\omega_2(t-s)}-1)H(s)ds.$$

Combining this with (5.2.11) it holds that

$$\frac{1}{2}|X(t)|^2+\omega_1\int_0^t ||X(s)||^2 ds \le \frac{1}{2}e^{2\omega_2 t}|x_{01}-x_{02}|^2+\int_0^t e^{2\omega_2(t-s)}H(s)ds. \tag{5.2.12}$$

By Lemma 5.2.3, the following inequality

$$\frac{1}{2}(e^{-\omega_2 t}|X(t)|)^2+\omega_1 e^{-2\omega_2 t}\int_0^t ||X(s)||^2 ds$$

$$\le \frac{1}{2}|x_{01}-x_{02}|^2+\int_0^t e^{-\omega_2 s}(L||X(s)||+|B(u_1(s)-u_2(s))|)e^{-\omega_2 s}|X(s)|ds$$

implies that

$$e^{-\omega_2 t}|X(t)| \le |x_{01}-x_{02}|+\int_0^t e^{-\omega_2 s}(L||X(s)||+|B(u_1(s)-u_2(s))|)ds. \tag{5.2.13}$$

From (5.2.12) and (5.2.13) it follows that

$$\frac{1}{2}|X(t)|^2 + \omega_1 \int_0^t ||X(s)||^2 ds \le \frac{1}{2} e^{2\omega_2 t}|x_{01} - x_{02}|^2 \tag{5.2.14}$$

$$+ \int_0^t e^{2\omega_2(t-s)}(L||X(s)|| + |B(u_1(s) - u_2(s))|)e^{\omega_2 s}|x_{01} - x_{02}|ds$$

$$+ \int_0^t e^{2\omega_2(t-s)}(L||X(s)|| + |B(u_1(s) - u_2(s))|)$$

$$\times \int_0^s e^{\omega_2(s-\tau)}(L||X(\tau)|| + |B(u_1(\tau) - u_2(\tau))|)d\tau ds.$$

$$= I + II + III.$$

Putting

$$G(s) = ||X(s)|| + |B(u_1(s) - u_2(s))|.$$

The third term of the right hand side of (5.2.14) is estimated as

$$III = L^2 e^{2\omega_2 t} \int_0^t e^{-\omega_2 s}||G(s)|| \int_0^s e^{-\omega_2 \tau}||G(\tau)||d\tau ds \tag{5.2.15}$$

$$= L^2 e^{2\omega_2 t} \int_0^t \frac{1}{2}\frac{d}{ds}\{\int_0^s e^{-\omega_2 \tau}||G(\tau)||d\tau\}^2 ds$$

$$= \frac{1}{2} L^2 e^{2\omega_2 t}\{\int_0^t e^{-\omega_2 \tau}||G(\tau)||d\tau\}^2$$

$$\le \frac{1}{2} L^2 e^{2\omega_2 t} \frac{1 - e^{-2\omega_2 t}}{2\omega_2} \int_0^t ||G(\tau)||^2 d\tau = \frac{L^2}{4\omega_2}(e^{2\omega_2 t} - 1) \int_0^t ||G(s)||^2 ds$$

$$\le \frac{L^2(e^{2\omega_2 t} - 1)}{2\omega_2} \int_0^t (||X(s)||^2 + |B(u_1(s) - u_2(s))|^2) ds.$$

The second term of the right hand side of (5.2.14) is estimated as

$$II = e^{2\omega_2 t}\int_0^t e^{-\omega_2 s}(L||X(s)|| + |B(u_1(s) - u_2(s))|)ds|x_{01} - x_{02}| \tag{5.2.16}$$

$$\leq \frac{1}{2}e^{2\omega_2 t}L^2\int_0^t (||X(s)||^2 + |B(u_1(s) - u_2(s))|^2)ds + \frac{1}{2}e^{2\omega_2 t}|x_{01} - x_{02}|^2.$$

Thus, from (5.2.15) and (5.2.16), we apply Gronwall's inequality to (5.2.5), and we arrive at

$$\frac{1}{2}|X(t)|^2 + \omega_1\int_0^t ||X(s)||^2 ds \leq C(|x_{01} - x_{02}|^2 + \int_0^{T_1} |B(u_1(s) - u_2(s))|^2 ds), \tag{5.2.17}$$

where $C > 0$ is a constant. Suppose $(x_{0n}, u_n) \to (x_0, u)$ in $H \times L^2(0, T; Y)$, and let x_n and x be the solutions (VIP) with (x_{0n}, u_n) and (x_0, u), respectively. Then, by virtue of (5.2.17), we see that $x_n \to x$ in $L^2(0, T, V) \cap C([0, T]; H)$.

2) It is easy to show that if $x_0 \in V$ and $B \in \mathcal{L}(Y, H)$, then x belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$. Let $(x_{0i}, u_i) \in V \times L^2(0, T; H)$, and x_i be the solution of (VIP) with (x_{0i}, u_i) in place of (x_0, u) for $i = 1,\ 2$. Then in view of Lemma 5.2.2 and Assumption (F), we have

$$||x_1 - x_2||_{L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)} \leq C_1\{||x_{01} - x_{02}|| \tag{5.2.18}$$

$$+ ||f(\cdot, x_1) - f(\cdot, x_2)||_{L^2(0,T;H)} + ||B(u_1 - u_2)||_{L^2(0,T;H)}\}$$

$$\leq C_1\{||x_{01} - x_{02}|| + ||B(u_1 - u_2)||_{L^2(0,T;H)}$$

$$+ L||x_1 - x_2||_{L^2(0,T:V)}\}.$$

Since

$$x_1(t) - x_2(t) = x_{01} - x_{02} + \int_0^t (\dot{x}_1(s) - \dot{x}_2(s))ds,$$

we get, noting that $|\cdot| \le ||\cdot||$,

$$||x_1 - x_2||_{L^2(0,T;H)} \le \sqrt{T}||x_{01} - x_{02}|| + \frac{T}{\sqrt{2}}||x_1 - x_2||_{W^{1,2}(0,T;H)}.$$

Hence arguing as in (4.2.1) we get

$$\begin{aligned} ||x_1 - x_2||_{L^2(0,T;V)} &\le C_0||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))}||x_1 - x_2||^{1/2}_{L^2(0,T;H)} \qquad (5.2.19)\\ &\le C_0||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))} \\ &\times \{T^{1/4}||x_{01} - x_{02}||^{1/2} + (\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||^{1/2}_{W^{1,2}(0,T;H)}\} \\ &\le C_0T^{1/4}||x_{01} - x_{02}||^{1/2}||x_1 - x_2||^{1/2}_{L^2(0,T;D(A))} \\ &+ C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \\ &\le 2^{-7/4}C_0||x_{01} - x_{02}|| + 2C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}. \end{aligned}$$

Combining (5.2.18) and (5.2.19) we obtain

$$||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1\{||x_{01} - x_{02}||\} + ||Bu_1 - Bu_2||_{L^2(0,T;H)} \qquad (5.2.20)$$

$$+ 2^{-7/4}C_0C_1L||x_{01} - x_{02}|| + 2C_0C_1(\frac{T}{\sqrt{2}})^{1/2}L||x_1 - x_2||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)}.$$

Suppose that

$$(x_{0n}, u_n) \mapsto (x_0, u) \in V \times L^2(0, T; Y),$$

and let x_n and x be the solutions (VIP) with (x_{0n}, u_n) and (x_0, u) respectively. Let $0 < T_1 \leq T$ be such that

$$2C_0C_1(T_1/\sqrt{2})^{1/2}L < 1.$$

Then by virtue of (5.2.23) with T replaced by T_1 we see that

$$x_n \to x \in L^2(0, T_1; D(A)) \cap W^{1,2}(0, T_1; H).$$

This implies that $(x_n(T_1), (x_n)_{T_1}) \mapsto (x(T_1), x_{T_1})$ in $V \times L^2(0, T; D(A))$. Hence the same argument shows that $x_n \mapsto x$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $x_n \mapsto x$ in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; H)$. □

5.3. Optimal control problems

In this section we study the optimal control problems for the quadratic cost function in the framework of Lions [40]. In what follows we assume that the embedding $D(A) \subset V \subset H$ is compact.

Let Y be another Hilbert space of control variables, and B be a bounded linear operator from Y into H, i.e.,

$$B \in \mathcal{L}(Y, H), \tag{5.3.1}$$

which is called a controller. By virtue of Theorem 5.2.1, we can define uniquely the solution map $u \mapsto x(u)$ of $L^2(0,T;Y)$ into $L^2(0,T;V) \cap C([0,T];H)$. We will call the solution $x(u)$ the state of the control system (VIP).

Let M be a Hilbert space of observation variables. The observation of state is assumed to be given by

$$z(u) = Gx(u), \quad G \in \mathcal{L}(C(0,T;V^*), M), \tag{5.3.2}$$

where G is an operator called the observer. The quadratic cost function associated with the control system (VIP) is given by

$$J(v) = ||Gx(v) - z_d||_M^2 + (Rv, v)_{L^2(0,T;Y)} \quad \text{for } v \in L^2(0,T;Y), \tag{5.3.3}$$

where $z_d \in M$ is a desire value of $x(v)$ and $R \in \mathcal{L}(L^2(0,T;Y))$ is symmetric and positive, i.e.,

$$(Rv, v)_{L^2(0,T;Y)} = (v, Rv)_{L^2(0,T;Y)} \geq d||v||^2_{L^2(0,T;Y)} \tag{5.3.4}$$

for some $d > 0$. Let $\mathcal{U}_{ad}$ be a closed convex subset of $L^2(0,T;Y)$, which is called the admissible set. An element $u \in \mathcal{U}_{ad}$ which attains minimum of $J(v)$ over $\mathcal{U}_{ad}$ is called an optimal control for the cost function (5.3.3).

Remark 5.3.1 The solution space $\mathcal{W}$ of strong solutions of (VIP) is defined by

$$\mathcal{W} = L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

endowed with the norm

$$|| \cdot ||_{\mathcal{W}} = \max\{|| \cdot ||_{L^2(0,T;V)}, || \cdot ||_{W^{1,2}(0,T;V^*)}\}.$$

We consider the following two types of observation G of distributive and terminal values(see [36, 37]).

(1) We take $M = L^2((0,T)\times\Omega)\times L^2(\Omega)$ and $G \in \mathcal{L}(\mathcal{W}, M)$ and observe

$$z(v) = Gx(v) = (x(v;\cdot), x(v,T)) \in L^2((0,T)\times\Omega)\times L^2(\Omega);$$

(2) We take $M = L^2((0,T)\times\Omega)$ and $G \in \mathcal{L}(\mathcal{W}, M)$ and observe

$$z(v) = Gx(v) = y'(v;\cdot) \in L^2((0,T)\times\Omega).$$

The above observations are meaningful in view of the regularity of the equation 1) by Proposition 5.2.1.

Theorem 5.3.1 1) Let the assumption (F) be satisfied. Assume that $B \in \mathcal{L}(Y, V^*)$ and $x_0 \in \overline{D(\phi)}$. Let $x(u)$ be the solution of (VIP) corresponding to u. Then the mapping $u \mapsto x(u)$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;H)$.

2) Let the assumptions (A) and (F) be satisfied. If $B \in \mathcal{L}(Y, H)$ and $x_0 \in \overline{D(\phi)} \cap V$, then the mapping $u \mapsto x(u)$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;V)$.

Proof. 1) We define the solution mapping S from $L^2(0,T;Y)$ to $L^2(0,T;H)$ by

$$Su = x(u), \quad u \in L^2(0,T;Y).$$

In virtue of Lemma 5.2.2, we have

$$||Su||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} = ||x(u)|| \le C_1\{|x_0| + ||Bu||_{L^2(0,T;V^*)}\}.$$

Hence if u is bounded in $L^2(0,T;Y)$, then so is $x(u)$ in $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding $L^2(0,T;V)\cap W^{1,2}(0,T;V^*) \subset L^2(0,T;H)$ is also compact in view of Theorem 2 of Aubin [14]. Hence, the mapping $u \mapsto Su = x(u)$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;H)$.

2) If $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is compact. Hence, the proof of 2) is complete. □

As indicated in Introduction we need to show the existence of an optimal control and to give the characterizations of them. The existence of an optimal control u for the cost function (5.3.3) can be stated by the following theorem.

Theorem 5.3.2 Let the assumptions (A) and (F) be satisfied and $x_0 \in \overline{D(\phi)} \cap V$. Then there exists at least one optimal control u for the control problem (VIP) associated with the cost function (5.3.3), i.e., there exists $u \in \mathcal{U}_{ad}$ such that

$$J(u) = \inf_{v\in\mathcal{U}_{ad}} J(v) := J. \tag{5.3.5}$$

Proof. Since $\mathcal{U}_{ad}$ is non-empty, there is a sequence $\{u_n\} \subset \mathcal{U}_{ad}$ such that minimizing sequence for the problem (5.3.5), which satisfies

$$\inf_{v\in\mathcal{U}_{ad}} J(v) = \lim_{n\to\infty} J(u_n) = m.$$

Obviously, $\{J(u_n)\}$ is bounded. Hence by (5.3.4) there is a positive constant K_0 such that

$$d||u_n||^2 \le (Ru_n, u_n) \le J(u_n) \le K_0.$$

This shows that $\{u_n\}$ is bounded in $\mathcal{U}_{ad}$. So we can extract a subsequence(denote again by $\{u_n\}$) of $\{u_n\}$ and find a $u \in \mathcal{U}_{ad}$ such that $w - \lim u_n = u$ in U. Let $x_n = x(u_n)$ be the solution of the following equation corresponding to u_n:

$$\begin{cases} x_n'(t) + Ax_n(t) + \partial\phi(x_n(t)) \ni f(t, x_n(t)) + Bu_n(t), & 0 < t \le T, \\ x_n(0) = x_0. \end{cases} \tag{5.3.6}$$

By (5.2.4) and (5.2.5) we know $\{x_n\}$ and $\{x_n'\}$ are bounded in $L^2(0,T;V)$ and $L^2(0,T;V^*)$, respectively. Therefore, by the extraction theorem of Rellich's, we can find a subsequence of $\{x_n\}$, say again $\{x_n\}$ and find x such that

$$x_n(\cdot) \to x(\cdot) \quad \text{weakly} \quad \text{in} \quad L^2(0,T;V) \cap C([0,T];H),$$

and

$$x_n' \to x', \quad \text{weakly in } L^2(0,T;V^*). \tag{5.3.7}$$

But by Theorem 5.3.1, we know that

$$x_n(\cdot) \to x(\cdot), \quad \text{strongly} \quad \text{in} \quad L^2(0,T;V).$$

From (F) it follows that

$$f(\cdot, x_n) \to f(\cdot, x), \quad \text{strongly in } L^2(0,T;H). \tag{5.3.8}$$

By the boundedness of A we have

$$Ax_n \to Ax, \quad \text{strongly in } L^2(0,T;V^*). \tag{5.3.9}$$

Since $\partial\phi(x_n)$ are uniformly bounded from (5.3.6)-(5.3.9) it follows that

$$\partial\phi(x_n) \to f(\cdot,x) + Bu - x' - Ax, \quad \text{weakly in } L^2(0,T;V^*),$$

and noting that $\partial\phi$ is demiclosed, we have that

$$f(\cdot,x) + Bu - x' - Ax \in \partial\phi(x) \text{ in } L^2(0,T;V^*).$$

Thus we have proved that $x(t)$ satisfies a.e. on $(0,T)$ the following equation:

$$\begin{cases} x^{'}(t) + Ax(t) + \partial\phi(x(t)) \ni f(t,x(t)) + Bu(t), \text{ a.e.}, \quad 0 < \mathrm{t} \leq \mathrm{T}, \\ x(0) = x_0. \end{cases} \tag{5.3.10}$$

Since G is continuous and $||\cdot||_M$ is lower semicontinuous, it holds that

$$||Gx(u) - z_d||_M \leq \liminf_{n\to\infty} ||Gx(u_n) - z_d||_M.$$

It is also clear from $\liminf_{n\to\infty} ||R^{1/2}u_n||_{L^2(0,T;Y)} \geq ||R^{1/2}u||_{L^2(0,T;Y)}$ that

$$\liminf_{n\to\infty} (Ru_n, u_n)_{L^2(0,T;Y)} \geq (Ru, u)_{L^2(0,T;Y)}.$$

Thus,

$$m = \lim_{n\to\infty} J(u_n) \geq J(u).$$

But since $J(u) \geq m$ by definition, we conclude $u \in \mathcal{U}_{ad}$ is a desired optimal control. □

5.4. Necessary conditions for optimality

In this section we will characterize the optimal controls by giving necessary conditions for optimality. For this it is necessary to write down the necessary optimal condition

$$DJ(u)(v-u) \geq 0, \quad v \in \mathcal{U}_{ad} \tag{5.4.1}$$

and to analyze (5.4.1) in view of the proper adjoint state system, where $DJ(u)$ denote the Gâteaux derivative of $J(v)$ at $v = u$. Therefore, we have to prove that the solution mapping $v \mapsto x(v)$ is Gâteaux differentiable at $v = u$. Here we note that from Theorem 5.2.1 it follows immediately that

$$\lim_{\lambda \to 0} x(u + \lambda w) = x(u) \quad \text{strongly in} \quad L^2(0,T;V) \cap C([0,T];H). \tag{5.4.2}$$

The solution map $v \mapsto x(v)$ of $L^2(0,T;Y)$ into $L^2(0,T;V) \cap C([0,T];H)$ is said to be Gâteaux differentiable at $v = u$ if for any $w \in L^2(0,T;Y)$ there exists a $Dx(u) \in \mathcal{L}(L^2(0,T;Y), L^2(0,T;V) \cap C([0,T];H)$ such that

$$\left|\left|\frac{1}{\lambda}(x(u+\lambda w) - x(u)) - Dx(u)w\right|\right| \to 0 \quad \text{as} \quad \lambda \to 0.$$

The operator $Dx(u)$ denotes the Gâteaux derivative of $x(u)$ at $v = u$ and the function $Dx(u)w \in L^2(0,T;V) \cap C([0,T];H))$ is called the Gâteaux derivative in the direction $w \in L^2(0,T;Y)$, which plays an important part in the nonlinear optimal control problems.

First, as is seen in Corollary 2.2 of Chapter II of [39], let us introduce the regularization of ϕ as follows.

Lemma 5.4.1 For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = ||x - J_\epsilon x||_*^2/2\epsilon + \phi(J_\epsilon x),$$

where $J_\epsilon = (I + \epsilon\phi)^{-1}$. Then the function ϕ_ϵ is Fréchet differentiable on H and its Frećhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1}. In addition,

$$\lim_{\epsilon\to 0} \phi_\epsilon(x) = \phi(x), \quad \forall x \in H,$$

$$\phi(J_\epsilon x) \le \phi_\epsilon(x) \le \phi(x), \quad \forall \epsilon > 0,\ x \in H,$$

and

$$\lim_{\epsilon\to 0} \partial\phi_\epsilon(x) = (\partial\phi)^0(x), \quad \forall x \in H,$$

where $(\partial\phi)^0(x)$ is the element of minimum norm in the set $\partial\phi(x)$.

Now, we introduce the smoothing system corresponding to (VIP) as follows.

$$\begin{cases} x'(t) + Ax(t) + \partial\phi_\epsilon(x(t)) = f(t, x(t)) + Bu(t), \quad 0 < t \le T, \\ x(0) = x_0. \end{cases} \tag{5.4.3}$$

Lemma 5.4.2 Let the assumption (F) be satisfied. Then the solution map $v \mapsto x(v)$ of $L^2(0,T;Y)$ into $L^2(0,T;V) \cap C([0,T];H)$ is Lipschitz continuous.

Moreover, let us assume the condition (A) in Proposition 5.2.1. Then the map $v \mapsto \partial\phi_\epsilon(x(v))$ of $L^2(0,T;Y)$ into $L^2(0,T;H) \cap C([0,T];V^*)$ is also Lipschitz continuous.

Proof. We set $w = v - u$. From Theorem 5.2.1, it follows immediately that

$$||x(u+\lambda w) - x(u)||_{C([0,T];H)} \leq \text{const.}|\lambda|||w||_{L^2(0,T;Y)},$$

so the solution map $v \mapsto x(v)$ of $L^2(0,T;Y)$ into $L^2(0,T;V) \cap C([0,T];H)$ is Gâteaux differentiable at $v = u$. Moreover, since

$$\partial\phi_\epsilon(x(u;t)) - \partial\phi_\epsilon(x(u+\lambda w;t)) = x^{'}(u+\lambda w;t) - x^{'}(u;t)$$

$$+ A(x(u+\lambda w;t) - x(u;t)) - \{f(t,x(u+\lambda w;t)) - f(t,x(u;t)\} - \lambda Bw(t),$$

by the assumption (A) and 2) of Theorem 5.2.1, it holds

$$||\partial\phi_\epsilon(x(u+\lambda w)) - \partial\phi_\epsilon(x(u))||_{L^2(0,T;H)}$$

$$\leq ||x^{'}(u+\lambda w) - x^{'}(u)||_{L^2(0,T;H)} + ||x(u+\lambda w) - x(u)||_{L^2(0,T;D(A))}$$

$$+ L||x(u+\lambda w) - x(u)||_{L^2(0,T;V)} + |\lambda|||B|||w||_{L^2(0,T;U)}$$

$$\leq \text{const.}|\lambda|||w||_{L^2(0,T;Y)},$$

and, by the relation (5.2.1),

$$||\partial\phi_\epsilon(x(u+\lambda w;t)) - \partial\phi_\epsilon(x(u;t))||_*$$

$$\leq ||x^{'}(u+\lambda w;t) - x^{'}(u;t)||_* + ||A||_{\mathcal{L}(V,V^*)}||(x(u+\lambda w;t) - x(u;t))||$$

$$+ L||x(u+\lambda w;t) - x(u;t)|| + |\lambda|||B|||w(t)|$$

$$\leq const.|\lambda|||w||_{L^2(0,T;Y)}.$$

So we know that there exists the Gâteaux derivative of the mapping $v \mapsto \phi_\epsilon(x(v))$ of $L^2(0,T;Y) \cap C([0,T];V^*)$. □

Let the solution space $\mathcal{W}_1$ of (VIP) of strong solutions is defined by

$$\mathcal{W}_1 = L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$$

as stated in Remark 5.3.1.

In order to obtain the optimality conditions, we require the following assumptions.

(F1) The Gâteaux derivative $\partial_2 f(t,x)$ in the second argument for $(t,x) \in (0,T)\times V$ is measurable in $t \in (0,T)$ for $x \in V$ and continuous in $x \in V$ for a.e. $t \in (0,T)$, and further there exist functions $\theta_1 \in L^1(0,T;\mathbb{R})$, $\theta_2 \in C(\mathbb{R}^+;\mathbb{R})$ such that

$$||\partial_2 f(t,x)||_* \le \theta_1(t) + \theta_2(||x||), \quad \forall (t,x) \in (0,T) \times V.$$

(F2) The map $x \to \partial\phi_\epsilon(x)$ is Gâteaux differentiable, and the value $D\partial\phi_\epsilon(x)Dx(u)$ is the Gâteaux derivative of $\partial\phi_\epsilon(x)x(u)$ at $u \in L^2(0,T;U)$ such that there exist functions $\theta_3, \theta_4 \in L^2(\mathbb{R}^+;\mathbb{R})$ such that

$$\|D\partial\phi_\epsilon(x)Dx(u)\|_* \le \theta_3(t) + \theta_4(\|u\|_{L^2(0,T;Y)}), \forall u \in L^2(0,T;Y).$$

Theorem 5.4.1 Let the assumptions (A), (F1) and (F2) be satisfied. Let $u \in \mathcal{U}_{ad}$ be an optimal control for the cost function J in (5.3.3). Then the following inequality:

$$(C^*\Lambda_M(Cx(u) - z_d), y)_{\mathcal{W}_1} + (Ru, v-u)_{L^2(0,T;Y)} \ge 0, \quad \forall v \in \mathcal{U}_{ad} \quad (5.4.4)$$

holds, where $y = Dx(u)(v-u) \in C([0,T];V^*)$ is a unique solution of the following equation:

$$\begin{cases} y'(t) + Ay(t) + D(\partial\phi)^0(x)(y(t)) = \partial_2 f(t,x)y(t) + Bw(t), \quad 0 < t \le T, \\ y(0) = 0. \end{cases} \tag{5.4.5}$$

Proof. We set $w = v - u$. Let $\lambda \in (-1,1)$, $\lambda \neq 0$. We set

$$y = \lim_{\lambda\to 0} \lambda^{-1}(x(u+\lambda w) - x(u)) = Dx(u)w.$$

From (5.4.3), we have

$$x'(u+\lambda w) - x'(u) + A(x(u+\lambda w) - x(u)) + \partial\phi_\epsilon(x(u+\lambda w)) - \partial\phi_\epsilon(x(u))$$

$$= f(\cdot, x(u+\lambda w)) - f(\cdot, x(u)) + \lambda Bw.$$

Then as an immediate consequence of Lemma 5.4.2 one obtains

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\{\partial\phi_\epsilon(x(u+\lambda w;t)) - \partial\phi_\epsilon(x(u;t))\} = D\partial\phi_\epsilon(x)y(t),$$

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\{f(t,x(u+\lambda w;t)) - f(t,x(u;t))\} = \partial_2 f(t,x)y(t),$$

thus, in the sense of (F2), we have that $y = Dx(u)(v-u)$ satisfies (5.4.5) and the cost $J(v)$ is Gâteaux differentiable at u in the direction $w = v - u$. The optimal condition (5.4.1) is rewritten as

$$(Cx(u) - z_d, y)_M + (Ru, v-u)_{L^2(0,T;Y)}$$

$$= (C^*\Lambda_M(Cx(u) - z_d), y)_{\mathcal{W}_1} + (Ru, v-u)_{L^2(0,T;Y)} \ge 0, \quad \forall v \in \mathcal{U}_{ad}$$

□

With every control $u \in L^2(0,T;Y)$, we consider the following distributional cost function expressed by

$$J_1(u) = \int_0^T ||Cx_u(t) - z_d(t)||_X^2 dt + \int_0^T (Ru(t), u(t))dt, \tag{5.4.6}$$

where the operator C is bounded from H to another Hilbert space X and $z_d \in L^2(0,T;X)$. Finally we are given R is a self adjoint and positive definite:

$$R \in \mathcal{L}(X), \quad \text{and} \quad (Ru, u) \geq c||u||, \quad c > 0.$$

Let $x_u(t)$ stand for solution of (VIP) associated with the control $u \in L^2(0,T;Y)$. Let $\mathcal{U}_{ad}$ be a closed convex subset of $L^2(0,T;Y)$.

Theorem 5.4.2 Let the assumption in Theorem 5.4.1 be satisfied and let the operators C and N satisfy the conditions mentioned above. Then there exists a unique element $u \in \mathcal{U}_{ad}$ such that

$$J_1(u) = \inf_{v \in \mathcal{U}_{ad}} J_1(v). \tag{5.4.7}$$

Furthermore, it is holds the following inequality:

$$\int_0^T (\Lambda_Y^{-1} B^* p_u(t) + Ru(t), (v-u)(t))dt \geq 0, \quad \forall v \in \mathcal{U}_{ad} \tag{5.4.8}$$

holds, where Λ_Y is the canonical isomorphism Y onto Y^* and p_u satisfies the following equation:

$$\begin{cases} p_u'(t) - A^* p_u(t) - D(\partial\phi)^0(x)^* p_u(t) + \partial_2 f(t,x)^* p_u(t) = -C^*\Lambda_X(Cx_u(t) - z_d(t)), \\ \quad \text{for} \quad 0 < \mathrm{t} \leq \mathrm{T}, \\ P_u(T) = 0. \end{cases} \tag{5.4.9}$$

Proof. Let x_u be a solution of (VIP) associated with the control u. Then it holds that

$$\begin{aligned} J_1(v) &= \int_0^T ||Cx_v(t) - z_d(t)||_X^2 dt + \int_0^T (Rv(t), v(t))dt \\ &= \int_0^T ||C(x_v(t) - x(t)) + Cx(t) - z_d(t)||_X^2 dt + \int_0^T (Rv(t), v(t))dt \\ &= \pi(v,v) - 2L(v) + \int_0^T ||z_d(t) - Cx(t)||_X^2 dt, \end{aligned}$$

where

$$\begin{aligned} \pi(u,v) &= \int_0^T (C(x_u(t) - x(t)), C(x_v(t) - x(t)))_X dt \\ &\quad + \int_0^T (Ru(t), v(t))dt \\ L(v) &= \int_0^T (z_d(t) - Cx(t), C(x_v(t) - x(t)))_X dt. \end{aligned}$$

The form $\pi(u,v)$ is a continuous bilinear form in $L^2(0,T;Y) \times L^2(0,T;Y)$ and from assumption of the positive definite of the operator R, we have

$$\pi(v,v) \geq c||v||^2 \quad v \in L^2(0,T;Y).$$

If u is an optimal control, similarly for (5.4.4) and (5.4.1) is equivalent to

$$\int_0^T (C^*\Lambda_X(Cx_u(t) - z_d(t)), y(t))dt + \int_0^T (Ru(t), (v-u)(t))dt \geq 0. \quad (5.4.10)$$

Now we formulate the adjoint system to describe the optimal condition:

$$\begin{cases} p'_u(t) - A^*p_u(t) - D\partial\phi_\epsilon(x)^*p_u(t) + \partial_2 f(t,x)^*p_u(t) = -(C^*\Lambda_X Cx_u(t) - z_d(t)), \\ \quad \text{for} \quad 0 < \mathrm{t} \leq \mathrm{T}, \\ P_u(T) = 0. \end{cases} \quad (5.4.11)$$

Taking into account the regularity result of Proposition 5.2.1 and the observation conditions, we can assert that (5.4.11) admits a unique weak solution p_u reversing the direction of time $t \to T - t$ by referring to the wellposedness result of Dautray and Lions [[38], p. 558-570].

We multiply both sides of equation (5.4.11) by $y(t)$ of (5.4.5) and integrate it over $[0,T]$. Then we have

$$\int_0^T (C^*\Lambda_X(Cx_u(t) - z_d(t)), y(t))dt \quad (5.4.12)$$

$$= -\int_0^T (p'_u(t), y(t))dt + \int_0^T (A^*p_u(t), y(t))dt + \int_0^T (D\partial\phi_\epsilon(x)^*p_u(t), y(t))dt$$

$$-\int_0^T (\partial_2 f(t,x)^*p_u(t), y(t))dt.$$

By the initial value condition of y and the terminal value condition of p_u, the left hand side of (5.4.12) yields

$$- (p_u(T), y(T)) + (p_u(0), y(0)) + \int_0^T (p_u(t), y'(t))dt + \int_0^T (p_u(t), Ay(t))dt$$

$$+ \int_0^T (p_u(t), D\partial\phi_\epsilon(x)y(t))dt - \int_0^T (p_u(t), \partial_2 f(t,x)y(t))dt$$

$$= \int_0^T (p_u(t), B(v-u)(t))dt.$$

Let u be the optimal control subject to (5.4.6). Then (5.4.10) is represented by

$$\int_\Omega (p_u(t), B(v-u)(t))dt + \int_0^T (Ru(t), (v-u)(t))dt \geq 0, \qquad (5.4.13)$$

which is rewritten by (5.4.8). Note that $C^* \in B(X^*, H)$ and for ϕ and ψ in H we have $(C^*\Lambda_X C\psi, \phi) =< C\psi, C\phi >_X$, where duality pairing is also denoted by $(\cdot,\cdot)$. □

Remark 5.4.1 Identifying the antidual X with X we meed not use the canonical isomorphism Λ_X. However, in case where $X \subset V^*$ this leads to difficulties since H has already been identified with its dual.

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