Thesis for the Degree of Doctor of Philosophy

A study on classes of analytic functions and related topics

by

Hyo Jeong Lee
Department of Applied Mathematics
The Graduate School
Pukyong National University

February 2017

A study classes of analytic functions and related topics
(해석함수들의 족들과 관련 주제들에 관한 연구)

Advisor : Nak Eun Cho

by

Hyo Jeong Lee

A thesis submitted in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in the Department of Applied Mathematics, Graduate School,
Pukyong National University

February 2017

A study on classes of analytic functions and related topics

A dissertation

by

Hyo Jeong Lee

Approved by :

(Chairman) Tae Hwa Kim, Ph. D.

(Member) Jin Mun Jeong, Ph. D.

(Member) Oh Sang Kwon, Ph. D.

(Member) Young Jae Sim, Ph. D.

(Member) Nak Eun Cho, Ph. D.

February 24, 2017

Contents

Abstract (Korean) iv

1 Introduction **1**

2 Preliminaries **4**

2.1 Analytic and univalent functions 4

2.2 Several subclasses of univalent functions 5

2.3 Differential subordination 7

2.4 Uniformly convex and uniformly starlike functions . . 8

2.5 Circular domains 9

2.6 Convolution 10

2.7 Hypergeometric functions 11

2.8 Convolution operator related to hypergeometric functions 12

3 Bi-univalent functions associated with subordination **13**

3.1 Introduction 13

3.2 Main results 15

4 Convexity and univalence conditions for certain integral operators 20

4.1 Introduction 20

4.2 Main results 22

4.2.1 Convexity of $\mathcal{I}_1(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ . . . 22

4.2.2 Univalency of $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ . . . 24

5 Mapping properties for certain subclasses of analytic functions 27

5.1 Introduction 27

5.2 Main results 29

6 Applications of the first-order differential subordinations 35

6.1 Introduction 35

6.2 Main result 36

6.3 Some applications 40

7 Subclasses of starlike and convex functions associated with Bessel functions 44

7.1 Introduction 44
7.2 Main results 47

8 Geometric properties of normalized Lommel functions 53

8.1 Introduction 53
8.2 Main results 55

해석함수들의 족들과 관련 주제들에 관한 연구

이 효 정

부경대학교 대학원 응용수학과

요 약

기하함수이론은 복소해석함수들의 기하학적 성질들을 연구하는 복소해석학의 일부이며, 특히, 단엽함수들에 대한 성질은 기하함수이론에서 가장 중요한 주제 중의 하나이다.

본 논문에서는 해석함수들의 다양한 부분족들을 소개하고, 이 족들에 대하여 여러 가지 사상성질들을 조사하였으며 단엽함수들과 관련된 여러 주제들에 대하여 해석함수들의 기하학적 성질들을 연구하였다. 구체적인 내용들은 다음과 같다.

먼저 2장에서는 본 연구의 기본틀을 구성하기 위해 필요한 수학적 개념 및 용어, 그리고 중요한 보조정리들을 소개하였고, 중요한 두 가지 개념 즉, 미분종속원리와 대합에 대해서 간략하게 소개하였다.

제 3장에서는 단위 개원판에서 대합에 의해 정의된 양단엽함수들의 새로운 부분족들을 소개하였으며, 그 부분족들에 속한 함수들에 대한 테일러 계수들의 유계성에 관하여 연구하였다.

제 4장에서는 단위 개원판에서 볼록과 단엽이 되기 위한 적분연산자들의 충분조건을 제시하였다. 그리고 5장에서는 가우스 초기하 함수에 의해 정의된 선형연산자를 사용하여 해석함수들의 여러 가지 사상성질들을 얻었다.

제 6장에서는 일계 위수 미분종속의 응용과 기존의 알려진 여러 결과들을 확장하였다. 그리고 제 7장에서는 해석함수들의 부분족들 $\mathcal{S}_{\mathcal{P}}\mathcal{T}(v,\delta)$, $\mathcal{UCT}(v,\delta)$, $\mathcal{PT}(v)$, $\mathcal{CPT}(v)$ 을 소개하고, 이 족들에 관한 특성을 조사하였으며, 일반화된 Bessel 함수들과 관련된 적분연산자들의 특성에 관하여 연구하였다.

마지막으로, 제 8장에서는 미분종속원리와 부등식의 성질을 이용하여 표준화된 Lommel 함수들의 성형성과 볼록성과 같은 기하학적 성질을 밝혔다.

Chapter 1

Introduction

Geometric function theory is one of the most essential branch of complex analysis, which works the geometric properties of complex analytic functions. In 1851, Riemann stated impressive consequence in geometric function theory that is Riemann mapping theorem ([102]).

Theorem 1.0.1. ***(Riemann mapping theorem)****[102, 32] Let D be a simply connected domain which is a proper subset of the complex plane $\mathbb{C}$. Let ζ be a given point in D. Then there is a unique function f which maps D conformally onto the unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ and has the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

In geometric function theory, the theory of univalent functions is the essential topic, born around the turn of the century, yet it remains an active field of current research. Progress has been especially rapid in recent years. Above all, we consider the class $\mathcal{S}$ of functions f analytic and univalent in $\mathbb{E}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Hence, each $f \in \mathcal{S}$ is expressed by a

Taylor series expansion as:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ z \in \mathbb{E}.$$

The remarkable example of $f \in \mathcal{S}$ is the Koebe function

$$ķ(z) = z(1-z)^{-2} = z + 2z^2 + 3z^3 + \cdots .$$

One of the major problems of the field is the Biberbach conjecture as follows.

Theorem 1.0.2. ***(Bieberbach Conjecture)****[12, 32] The coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq n$ for $n = 2, 3, \cdots$. Strict inequality holds for all n unless f is the Koebe function or one of its rotations.*

For a number of years this conjecture has stood as a challenge to all mathematicians and has promoted the growth of important new methods in complex analysis. To this date it has been proved only for $n = 2, 3, 4, 5$ and 6 ([12, 59, 34, 93, 94, 86]). In 1985, Louis de Branges [24] proved this conjecture finally, for all coefficients n , that is now renowned as de Branges Theorem.

In present thesis, we establish various new subclasses of analytic functions as using certain linear operators. We concern some basic properties of these classes for instance, coefficient problems, convolution properties and some other topics. The details are as follows.

In Chapter 2, we review and assemble for later reference some of the general principles of complex analysis which underlie the theory of univalent functions. We do not underline proofs of the consequences but offer suitable references.

In Chapter 3, we present new subclass of bi-univalent functions defined by convolution in $\mathbb{E}$. Furthermore, we get estimates on the second and the third coefficients for functions belonging to the subclass.

In Chapter 4, we determined the order of convexity of a integral operator $\mathcal{I}_1(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$. Moreover, we obtain sufficient restrictions for the operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ to be univalent in $\mathbb{E}$

In Chapter 5, we obtain some mapping and inclusion relations for subclasses of analytic functions by using a linear operator defined by the Gaussian hypergeometric function. We derive a necessary restriction for the class $\mathfrak{R}^t(C, D, \varrho)$ and sufficient restrictions for the classes $\mathfrak{R}^t(C, D, \varrho)$, $\mathcal{UST}(\varrho)$ and $\mathcal{UCV}(\varrho)$, respectively.

In Chapter 6, we gives some applications of the first-order differential subordinations. We also extend and improve several previously known results.

In Chapter 7, we obtain some characterizations for the generalized Bessel functions of the first kind to be in the subclasses $\mathcal{S}_\mathcal{P}\mathcal{T}(\upsilon, \delta)$, $\mathcal{UCT}(\upsilon, \delta)$, $\mathcal{PT}(\upsilon)$, and $\mathcal{CPT}(\upsilon)$ of analytic functions. Furthermore, we find an integral operator associated with the generalized Bessel functions.

In Chapter 8, we obtain some geometric properties such as starlikeness and convexity, for normalized Lommel functions of the first kind. For the purpose of verifying our principal consequences, we apply the concept of differential subordinations and some inequalities.

Chapter 2

Preliminaries

The purpose of this preliminary chapter is to review and assemble for later reference some of the general principles of complex analysis which underlie the theory of univalent functions. We do not underline proofs of the consequences but offer suitable references.

2.1 Analytic and univalent functions

In present section, we briefly introduce the classes $\mathcal{A}$ and $\mathcal{S}$, consisting of normalized analytic functions and normalized univalent functions, respectively, and we also show few of their basic properties.
Analytic functions are defined on an open subset of $\mathbb{C}$, that are differentiable. Complex differentiability has much stronger consequences than real differentiability. The definition of analytic functions is as follows.

Definition 2.1.1. *(cf. [114]) A function f is called to be analytic at a point if it is differentiable everywhere in some neighborhood of the point. A function f*

is analytic in a domain $\mathcal{D}$ if it is analytic at every point in $\mathcal{D}$. Furthermore, a function analytic at every point in the complex plane is called an entire functions.

Definition 2.1.2. *A function f is called to be in the class $\mathcal{A}$, if it is analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The function $f \in A$ is expressed by the following power series representation*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; z \in \mathbb{E}. \tag{2.1.1}$$

Definition 2.1.3. *(cf. [32, 114]) A function h is called to be univalent in $\mathcal{D} \subset \mathbb{C}$, if it never takes the same value twice; that is, if $h(\zeta_1) \neq f(\zeta_2)$ for all pairs of distinct points ζ_1 and ζ_2 in $\mathcal{D}$ with $\zeta_1 \neq \zeta_2$. That is, h is a one-to-one(or injective) mapping of $\mathcal{D}$ onto another domain. Analytically, a univalent function has a nonvanishing derivative and geometrically, a univalent function maps simple curves onto simple curves.*

We shall consider firstly the class $\mathcal{S}$ as follows:

Definition 2.1.4. *(cf. [114]) The class of all functions f, denoted by $\mathcal{S}$, are analytic and univalent in $\mathbb{E}$, and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Hence, each $f \in \mathcal{S}$ is expressed by (2.1.1).*

2.2 Several subclasses of univalent functions

Definition 2.2.1. *(cf. [114]) A set $\mathcal{D} \subset \mathbb{C}$ is called to be starlike with respect to z_0, if the staight line segment connecting any point in $\mathcal{D}$ to z_0 is contained in $\mathcal{D}$. A function $f \in \mathcal{S}$ is called to be starlike with respect to the origin if the disk $\mathbb{E}$ by f onto a domain starlike with respect to the point $w = 0$. We shall denote this subclass of $f \in \mathcal{S}$ by $f \in \mathcal{S}^*$.*

Definition 2.2.2. *(cf. [114]) The set $\mathcal{D}$ is said to be convex if the straight line segment connecting any two points in $\mathcal{D}$ is contained $\mathcal{D}$. A function $f \in \mathcal{S}$ is called to be convex if the disk $\mathbb{E}$ is mapped by f onto convex domain. We shall denote this subclass of $f \in \mathcal{S}$ by $f \in \mathcal{C}$.*

Analytically, convex and starlike functions are described the following two definitions.

Definition 2.2.3. *(Study. [118]) Let $f \in \mathcal{S}$. Then*

$$h \in \mathcal{C} \iff \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > 0,\ z \in \mathbb{E}. \tag{2.2.1}$$

Definition 2.2.4. *(Nevanlinna. [73]) Let $h \in \mathcal{S}$. Then*

$$h \in \mathcal{S}^* \iff \Re\left(\frac{zh'(z)}{h(z)}\right) > 0,\ z \in \mathbb{E}. \tag{2.2.2}$$

These classes $\mathcal{C}, \mathcal{S}^*$ and $\mathcal{S}$ are related as follows:

$$\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S} \subset \mathcal{A}.$$

Alexander [1] revealed analytic relation between convex and starlike functions firstly in 1915.

Theorem 2.2.1. *[1](Alexander's Theorem) Let h be analytic in $\mathbb{E}$, with $h(0) = 0$ and $h'(0) = 1$. Then*

$$h \in \mathcal{C} \iff zh'(z) \in \mathcal{S}^*$$

The concept of the classes $\mathcal{C}(\varrho)$ and $\mathcal{S}^*(\varrho)$ of convex and starlike functions of order ϱ, $0 \le \varrho < 1$, respectively were given by Robertson [103] in 1936, and defined by as follows:

$$\mathcal{C}(\varrho) = \left\{h \in \mathcal{A} : \Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \varrho,\ z \in \mathbb{E}\right\}, \tag{2.2.3}$$

$$\mathcal{S}^*(\varrho) = \left\{ h \in \mathcal{A} : \Re\left(\frac{zh'(z)}{h(z)}\right) > \varrho,\ z \in \mathbb{E} \right\}. \tag{2.2.4}$$

The next properties are renowned.

(i) By taking $\varrho = 0$, we derive the classes $\mathcal{C}$ and $\mathcal{S}^*$.

(ii) $\mathcal{C}(\varrho) \subset \mathcal{C},\ \mathcal{S}^*(\varrho) \subset \mathcal{S}^*$ and $\mathcal{C} \subset \mathcal{S}^*\left(\frac{1}{2}\right)$.

Let us offer references [30, 40, 103], for details.

2.3 Differential subordination

Simply stated, a differential subordination in the complex plane is the generalization of a differential inequality on the real line (cf. [68]). The notion of differential subordination was intiated by Lindelöf [54] and basis of this theory were created by Miller and Mocanu (see, [63, 64]). We previously offer a definition of differential subordination, the following is needed.

Definition 2.3.1. *An analytic function $w \in \mathbb{E}$ is called to be Schwarz function if it satisfies following conditions*

$$w(0) = 0 \textit{ and } |w(z)| < 1 \textit{ for } z \in \mathbb{E}$$

Definition 2.3.2. *[68, 54] Let $q, Q \in \mathcal{A}$. Then the function q is called to be subordinate to Q written $q \prec Q$ or $q(z) \prec Q(z)$, if there exists a function w analytic in $\mathbb{E}$, with*

$$w(0) = 0 \textit{ and } |w(z)| < 1,$$

and such that

$$q(z) = Q\left(w(z)\right).$$

If Q is univalent in $\mathbb{E}$, then

$$q \prec Q \iff q(0) = Q(0) \text{ and } q(\mathbb{E}) \subset Q(\mathbb{E}).$$

2.4 Uniformly convex and uniformly starlike functions

Goodman [41, 42] presented the notions of uniform convexity and uniform starlikeness for analytic and univalent functions, and provided proper subclasses such as $\mathcal{C}$ and $\mathcal{S}^*$, which mean classes of convex and starlike functions, respectively. Uniform classes are defined by geometrical mapping properties.

Definition 2.4.1. *[41] A function $h \in \mathbb{E}$ is called as uniformly convex, if h is a normalized convex function and has the property that for every circular arc r contained in $\mathbb{E}$, with center ζ also in $\mathbb{E}$, the image arc $h(\zeta)$ is a convex.*

Definition 2.4.2. *[42] A function $h \in \mathbb{E}$ is called to be uniformly starlike, if h is a normalized starlike function and has the property that for every circular arc r contained in $\mathbb{E}$, with center ζ also in $\mathbb{E}$, the image arc $h(\zeta)$ is a starlike with respect to $h(\zeta)$.*

Goodman stated the classes of uniformly convex functions and uniformly starlike functions by $\mathcal{UCV}$ and $\mathcal{UST}$ respectively, and these are as follows.

Definition 2.4.3. *[41] Let $h \in \mathcal{A}$. Then*

$$h \in \mathcal{UCV} \iff \Re\left\{1 + (z-\zeta)\frac{zh''(z)}{h'(z)}\right\} > 0, \tag{2.4.1}$$

for every (z, ζ) in $\mathbb{E} \times \mathbb{E}$.

Definition 2.4.4. *[42] Let* $h \in \mathcal{A}$*. Then*

$$h \in \mathcal{UST} \iff \Re\left\{\frac{(z-\zeta)h'(z)}{h(z)-h(\zeta)}\right\} > 0, \tag{2.4.2}$$

for every (z,ζ) *in* $\mathbb{E} \times \mathbb{E}$*.*

By taking $\zeta = 0$ in (2.4.1) and (2.4.2) we obtain the conditions of $\mathcal{C}$ and $\mathcal{S}^*$. Ma and Minda [61] and Rønning [105] discovered their own result respectively and a more applied single variable property for $\mathcal{UCV}$ and $\mathcal{UST}$, as follows.

Definition 2.4.5. *Let* $h \in \mathcal{A}$*. Then*

$$h \in \mathcal{UCV} \iff \Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > \left|\frac{zh''(z)}{h'(z)}\right|, \ z \in \mathbb{E}. \tag{2.4.3}$$

Definition 2.4.6. *Let* $h \in \mathcal{A}$*. Then*

$$h \in \mathcal{UST} \iff \Re\left\{\frac{zh'(z)}{h(z)}\right\} > \left|\frac{zh'(z)}{h(z)} - 1\right|, \ z \in \mathbb{E}. \tag{2.4.4}$$

2.5 Circular domains

Janowski presented circular domain in 1973, as belows.

Definition 2.5.1. *[45] Let* P *be an analytic function with* $P(0) = 1$*. Then*

$$P \in P[C,D] \iff P(z) \prec \frac{1+Cz}{1+Dz}, \ -1 \leq D < C \leq 1. \tag{2.5.1}$$

Janowski also introduced $\mathcal{C}[C,D]$ and $\mathcal{S}^*[C,D]$ and these mean the classes of Janowski convex and Janowski starlike functions, respectively.

Definition 2.5.2. *[45] Let* $h \in \mathcal{A}$*. Then*

$$h \in \mathcal{C}[C,D] \iff 1 + \frac{zh''(z)}{h'(z)} \in P[C,D]. \tag{2.5.2}$$

Definition 2.5.3. *[45] Let $h \in \mathcal{A}$. Then*

$$h \in \mathcal{S}^*[C, D] \iff \frac{zh'(z)}{h(z)} \in P[C, D]. \tag{2.5.3}$$

Alexander type relation satisfies between $\mathcal{C}[C, D]$ and $\mathcal{S}^*[C, D]$.

Remark 2.5.1. *By taking $C = 1$ and $D = -1$, we obtain $\mathcal{C}[C, D] = \mathcal{C}$ and $\mathcal{S}^*[C, D] = \mathcal{S}^*$.*

Janowski functions are studied by many scholars like Noor [74, 78], Polatoglu [97, 98], Cho [16, 17] and Liu *et.el.* [55, 56, 57]

2.6 Convolution

The convolution (cf. [32]), or Hadamard product, of two power series

$$f(z) = \sum_{k=1}^{\infty} m_k z^k, \ z \in \mathbb{E}$$

and

$$g(z) = \sum_{k=1}^{\infty} n_k z^k, \ z \in \mathbb{E}$$

convergent in $\mathbb{E}$ is the function $h = f * g$ with power series

$$h(z) = \sum_{k=1}^{\infty} m_k n_k z^k, \ z \in \mathbb{E}.$$

The term "convolution" arises from the formula

$$h(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f\left(re^{i(\theta - t)}\right) g(e^{it}) dt, \ r < 1.$$

The geometry series $l(z) = \sum_{k=1}^{\infty} z^k = \frac{z}{1-z}$ roles as identity element under convolution such as,

$$(f * l)(z) = f(z) = (l * f)(z), \text{ for all } f \in \mathcal{A}.$$

2.7 Hypergeometric functions

Hypergeometric functions (cf. [68]) are known as special functions, because these functions are the solution of special types of differential equations. Recently, many authors (see, [17, 79]) applied hypergeometric functions to define various integral and convolution operator. Here we discuss two types of hypergeometric functions that is the Kummer and Guass hypergeometric functions.

Let $b, d \in \mathbb{C}$ with $d \neq 0, -1, -2, \cdots$. The function

$$\Psi(b, d; z) = {}_1F_1(b, d; z) = 1 + \frac{b}{d}\frac{z}{1!} + \frac{b(b+1)}{d(d+1)}\frac{z^2}{2!} + \cdots . \tag{2.7.1}$$

is called the confluent (or Kummer) hypergeometric functions, analytic in $\mathbb{E}$ and obey the Kummer's differential equation

$$zw''(z) + [d - z]w'(z) - cw(z) = 0.$$

The Pochhammer symbol denoted by $(\nu)_k$ is defined by

$$\begin{aligned} (\nu)_k &= \frac{\Gamma(\nu + k)}{\Gamma(\nu)} \qquad (2.7.2) \\ &= \begin{cases} 1 & if\ k = 0, \\ \nu(\nu+1)(\nu+2)\cdots(\nu+k-1) & if\ k \in \mathbb{N}, \end{cases} \end{aligned}$$

where $\Gamma(\nu)$ denote the gamma function. Then equation (2.7.1) can be express as

$$\begin{aligned} \Psi(b, d; z) &= \sum_{k=0}^{\infty} \frac{(b)_k}{(d)_k}\frac{z^k}{k!} \qquad (2.7.3) \\ &= \frac{\Gamma(d)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(b+k)}{\Gamma(d+k)}\frac{z^k}{k!}. \end{aligned}$$

Let $b, d, e \in \mathbb{C}$ with $e \neq 0, -1, -2, \cdots$. The function

$$\begin{aligned} F(b,d,e;z) &= {}_2F_1(b,d,e;z) \\ &= 1 + \frac{bd}{e}\frac{z}{1!} + \frac{b(b+1)d(d+1)}{e(e+1)}\frac{z^2}{2!} + \cdots , \end{aligned} \tag{2.7.4}$$

is said to be the (Gaussian) hypergeometric functions, is analytic in $\mathbb{E}$ and complies the hypergeometric differential equation

$$z(1-z)w''(z) + [e - (b+d+1)z]w'(z) - bdw(z) = 0 \tag{2.7.5}$$

As using the symbol offered in (2.7.2), we can rewrite F as

$$\begin{aligned} F(b,d,e;z) &= \sum_{k=0}^{\infty} \frac{(b)_k (d)_k}{(e)_k}\frac{z^k}{k!} \\ &= \frac{\Gamma(e)}{\Gamma(b)\Gamma(d)} \sum_{k=0}^{\infty} \frac{\Gamma(b+k)\Gamma(d+k)}{\Gamma(e+k)}\frac{z^k}{k!}. \end{aligned} \tag{2.7.6}$$

2.8 Convolution operator related to hypergeometric functions

Consider the linear operator $I_\varrho(b,d,e,q) : \mathcal{A}(q) \to \mathcal{A}(q)$ defined by

$$I_\varrho(b,d,e)h(z) = (z^q {}_2F_1(b,d,e;z))^{-1} * h(z), \tag{2.8.1}$$

where $b, d, e \in \mathbb{R}$, except $0, -1, -2, \cdots, \varrho > -q,\ z \in E$ and $({}_2F_1(b,d,e;z))^{-1}$ is given by

$$(z^q {}_2F_1(b,d,e;z)) * (z^q {}_2F_1(b,d,e;z))^{-1} = \frac{z^q}{(1+z)^{\varrho+q}}.$$

Simply, we express that

$$I_\varrho(b,d,e)h(z) = z^q + \sum_{k=1}^{\infty} \frac{(e)_k(\varrho+q)_k}{(b)_k(d)_k} z^{k+q}, \tag{2.8.2}$$

where $(\nu)_k$ is the Pochhammer symbol given by (2.7.2).

Chapter 3

Bi-univalent functions associated with subordination

3.1 Introduction

Let $\mathcal{A}$ denotes the class of functions defined by (2.1.1), which are analytic in $\mathbb{E}$. And $\mathcal{S}$ is the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{E}$. For $f(z)$ expressed by (2.1.1) and $\Psi(z)$ defined by

$$\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n \quad (\psi_n \geq 0\ ), \tag{3.1.1}$$

the Hadamard product $(f * \Psi)$ of the functions f and Ψ is defined as follows.

$$(f * \Psi)(z) = z + \sum_{n=2}^{\infty} a_n \psi_n z^n = (\Psi * f)(z). \tag{3.1.2}$$

A renowned concept that every function $q \in \mathcal{S}$ has an inverse q^{-1}, defined by

$$q^{-1}(q(\zeta)) = \zeta \quad (\zeta \in \mathbb{E})$$

and

$$q(q^{-1}(\omega)) = \omega \quad \left(|\omega| < r_0(q);\ r_0(q) \geq \frac{1}{4}\right),$$

where

$$q^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots . \tag{3.1.3}$$

A function $q \in \mathcal{A}$ is called to be bi-univalent in $\mathbb{E}$ if both q and q^{-1} are univalent in $\mathbb{E}$. Let Σ denotes the class of bi-univalent functions in $\mathbb{E}$ given by (2.1.1). To obtain our primary consequences, we shall consider the next lemma.

Lemma 3.1.1. *[99] Let $p \in \mathcal{P}$ the family of all functions p analytic in $\mathbb{E}$ for which $\mathrm{Re}p(z) > 0$ and have the form $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ for $z \in \mathbb{E}$. Then $|p_n| \leq 2$, for each n.*

Definition 3.1.1. *For $0 < \varrho \leq 1; \lambda \geq 1$, a function $f \in \Sigma$ given by (2.1.1) is called to be in the class $\mathcal{H}_\Sigma(h, \varrho, \lambda)$ if the next conditions are satisfied:*

$$\left|\arg\left(\frac{\zeta(f*h)'(\zeta)}{(1-\lambda)(f*h)(\zeta) + \lambda\zeta(f*h)'(\zeta)}\right)\right| < \frac{\varrho\pi}{2} \quad (\zeta \in \mathbb{E}) \tag{3.1.4}$$

and

$$\left|\arg\left(\frac{\omega((f*h)^{-1})'(\omega)}{(1-\lambda)(f*h)^{-1}(\omega) + \lambda\omega((f*h)^{-1})'(\omega)}\right)\right| < \frac{\varrho\pi}{2} \quad (\omega \in \mathbb{E}) \tag{3.1.5}$$

where the functions $h(\zeta)$ and $(f*h)^{-1}(\omega)$ are defined by

$$h(\zeta) = \zeta + \sum_{n=2}^{\infty} h_n\zeta^n \quad (h_n > 0) \tag{3.1.6}$$

and

$$(f*h)^{-1}(\omega) = \omega - a_2h_2\omega^2 + (2a_2^2h_2^2 - a_3h_3)\omega^3 - (5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4)\omega^4 + \cdots . \tag{3.1.7}$$

Definition 3.1.2. *For* $0 \leq \eta < 1; \lambda \geq 1,$ *a function* $f \in \Sigma$ *given by (2.1.1) is called to be in the class* $\mathcal{H}_\Sigma(h, \eta, \lambda)$ *if the next conditions are satisfied:*

$$\operatorname{Re}\left(\frac{\zeta(f*h)'(\zeta)}{(1-\lambda)(f*h)(\zeta)+\lambda\zeta(f*h)'(\zeta)}\right) > \eta \quad (\zeta \in \mathbb{E}) \tag{3.1.8}$$

and

$$\operatorname{Re}\left(\frac{\omega((f*h)^{-1})'(\omega)}{(1-\lambda)(f*h)^{-1}(\omega)+\lambda\omega((f*h)^{-1})'(\omega)}\right) > \eta \quad (\omega \in \mathbb{E}). \tag{3.1.9}$$

In this chapter, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in subclass of Σ by using the methods from Deniz [26].

3.2 Main results

Let φ be an analytic function with positive real part in $\mathbb{E}$ such that $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(\mathbb{E})$ is symmetric with respect to the real axis. This function is expressed by a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{3.2.1}$$

Now we present the class of bi-univalent functions as belows.

Consider the functions p and q by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots \tag{3.2.2}$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots . \tag{3.2.3}$$

It follows that

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left\{p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right\} \tag{3.2.4}$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left\{q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right\}. \tag{3.2.5}$$

Then p and q are analytic in $\mathbb{E}$ with $p(0) = q(0) = 1$. Since $u,\ v:\ \mathbb{E} \longrightarrow \mathbb{E}$, the functions p and q have a positive real part in $\mathbb{E}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i.

Definition 3.2.1. *Let $0 \leq \lambda < 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. A function $f \in \Sigma$ given by (2.1.1), is called to be in $\mathcal{H}_\Sigma(h, \varphi, \gamma, \lambda)$ if each of the next subordinate condition holds true:*

$$1 + \frac{1}{\gamma}\left(\frac{\zeta(f*h)'(\zeta)}{(1-\lambda)(f*h)(\zeta) + \lambda\zeta(f*h)'(\zeta)} - 1\right) \prec \varphi(\zeta) \quad (\zeta \in \mathbb{E}) \tag{3.2.6}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{\omega((f*h)^{-1})'(\omega)}{(1-\lambda)(f*h)^{-1}(\omega) + \lambda\omega((f*h)^{-1})'(\omega)} - 1\right) \prec \varphi(\omega) \quad (\omega \in \mathbb{E}) \tag{3.2.7}$$

Example 3.2.1. *For $\lambda = 0$ and $\gamma \in \mathbb{C} \setminus \{0\}$, a function $f \in \Sigma$ given by (2.1.1) is called to be in the class $\mathcal{H}_\Sigma(h, \varphi, \gamma)$ if the conditions are satisfied as belows:*

$$1 + \frac{1}{\gamma}\left(\frac{\zeta(f*h)'(\zeta)}{(f*h)(\zeta)} - 1\right) \prec \varphi(\zeta)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{\omega((f*h)^{-1})'(\omega)}{(f*h)^{-1}(\omega)} - 1\right) \prec \varphi(\omega)$$

where $\zeta, \omega \in \mathbb{E}$ and the function $h(\zeta)$ and $(f*h)^{-1}(\omega)$ are given by (3.1.6) and (3.1.7) respectively.

Theorem 3.2.1. *Let $f(z) \in \mathcal{H}_\Sigma(h, \varphi, \gamma, \lambda)$ be of the form (2.1.1). Then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{h_2(1-\lambda)\sqrt{\gamma B_1^2 - (B_2 - B_1)}} \quad (B_1 > 0) \tag{3.2.8}$$

and

$$|a_3| \leq \frac{|\gamma^2| B_1^2}{(1-\lambda)^2 h_3} + \frac{|\gamma| B_1|}{2(1-\lambda) h_3} \quad (B_1 > 0) \tag{3.2.9}$$

where the coefficients B_1 is given by (3.2.1).

Proof. From (3.2.6) and (3.2.7) that

$$1 + \frac{1}{\gamma}\left(\frac{\zeta(f*h)'(\zeta)}{(1-\lambda)(f*h)(\zeta) + \lambda\zeta(f*h)'(\zeta)} - 1\right) = \varphi(u(\zeta)) \quad (\zeta \in \mathbb{E}) \tag{3.2.10}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{\omega((f*h)^{-1})'(\omega)}{(1-\lambda)(f*h)^{-1}(\omega) + \lambda\omega((f*h)^{-1})'(\omega)} - 1\right) = \varphi(v(\omega)) \quad (\omega \in \mathbb{E}), \tag{3.2.11}$$

where u and $v : \mathbb{E} \longrightarrow \mathbb{E}$ are analytic. Substituting from (3.2.4) and (3.2.5) into (3.2.10) and (3.2.11), respectively and by using (3.2.1), we get

$$\begin{aligned} \varphi(u(\zeta)) &= \varphi\left(\frac{1}{2}\left[p_1\zeta + \left(p_2 - \frac{p_1^2}{2}\right)\zeta^2 + \cdots\right]\right) \\ &= 1 + \frac{1}{2}B_1 p_1 \zeta + \left[\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2 p_1^2\right]\zeta^2 + \cdots \end{aligned} \tag{3.2.12}$$

and

$$\begin{aligned} \varphi(v(\omega)) &= \varphi\left(\frac{1}{2}\left[q_1\omega + \left(q_2 - \frac{q_1^2}{2}\right)\omega^2 + \cdots\right]\right) \\ &= 1 + \frac{1}{2}B_1 q_1 \omega + \left[\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2 q_1^2\right]\omega^2 + \cdots . \end{aligned} \tag{3.2.13}$$

It follows from (3.2.12) and (3.2.13) that

$$\frac{1}{\gamma}(1-\lambda)a_2h_2 = \frac{1}{2}B_1p_1, \tag{3.2.14}$$

$$\frac{1}{\gamma}\{2(1-\lambda)a_3h_3 - (1-\lambda^2)a_2^2h_2^2\} = \frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4}B_2p_1^2, \tag{3.2.15}$$

$$-\frac{(1-\lambda)}{\gamma}a_2h_2 = \frac{B_1q_1}{2} \tag{3.2.16}$$

and

$$\frac{1}{\gamma}\left[(3-4\lambda+\lambda^2)a_2^2h_2^2 - 2(1-\lambda)a_3h_3\right] = \frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4}B_2q_1^2. \tag{3.2.17}$$

From (3.2.14) and (3.2.16), we find that

$$p_1 = -q_1 \tag{3.2.18}$$

and

$$8(1-\lambda)^2a_2^2h_2^2 = \gamma^2B_1^2(p_1^2+q_1^2). \tag{3.2.19}$$

By using Lemma 3.1.1, we get

$$|a_2| \le \frac{|\gamma|B_1\sqrt{B_1}}{h_2(1-\lambda)\sqrt{\gamma B_1^2 - (B_2-B_1)}} \quad (B_1 > 0) \tag{3.2.20}$$

By subtracting (3.2.17) from (3.2.15), we have

$$\frac{1}{\gamma}\left[4(1-\lambda)a_3h_3 - 4(1-\lambda)a_2^2h_2^2\right] = \frac{1}{2}B_1(p_2-q_2). \tag{3.2.21}$$

i.e.,

$$a_3 = \frac{\gamma^2 B_1^2 (p_1^2 + q_1^2)}{8(1-\lambda)^2 h_3} + \frac{\gamma B_1 (p_2 - q_2)}{8(1-\lambda) h_3}.$$

By using Lemma 3.1.1, we get

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2 h_3} + \frac{|\gamma| B_1}{2(1-\lambda) h_3} \quad (B_1 > 0)$$

□

If we take $\lambda = 0$ in Theorem 3.2.1, then we get the next consequence.

Corollary 3.2.1. *Let $f(z) \in \mathcal{H}_\Sigma(\varphi, \gamma, \lambda)$ be of the form (2.1.1). Then*

$$|a_2| \leq \frac{|\gamma| B_1 \sqrt{B_1}}{h_2 \sqrt{\gamma B_1^2 - (B_2 - B_1)}}$$

and

$$|a_3| \leq \frac{|\gamma^2| B_1^2}{h_3} + \frac{|\gamma| B_1}{2h_3}.$$

where the coefficients B_1 is given by (3.2.1).

Chapter 4

Convexity and univalence conditions for certain integral operators

4.1 Introduction

Let $\mathcal{A}, \mathcal{S}, \mathcal{S}^*(\delta)$ and $\mathcal{C}(\delta)$ denote the functions classes defined by (2.1.1), univalent, starlike and convex functions of order δ, respectively.

Silverman [110] researched an representation related to the quotient of the analytic expression of convex and starlike functions. For $0 < \mu \leq 1$, he considered the class

$$\mathcal{G}_\mu = \left\{ h \in \mathcal{A} : \left| \frac{1 + vh''(v)/h'(v)}{vh'(v)/h(v)} - 1 \right| < \mu, \ v \in \mathbb{E} \right\},$$

and proved that

$$\mathcal{G}_\mu \subset \mathcal{S}^* \left(\frac{2}{1 + \sqrt{1 + 8\mu}} \right).$$

Moreover, Tuneski [121] proved that if $h \in \mathcal{G}_\mu$ $(0 < \mu < 1)$, then

$$\left|\frac{\upsilon h'(\upsilon)}{h(\upsilon)} - 1\right| < \frac{\mu}{1-\mu} \ (\upsilon \in \mathbb{E}). \tag{4.1.1}$$

For the parameter $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for all $i \in \bar{\mathbb{N}} = \{1, 2, \cdots, n\}$ and $\nu \in \mathbb{C}$ with $Re\{\nu\} > 0$, we define a integral operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}}) : \mathcal{A} \longrightarrow \mathcal{A}$ as follows:

$$\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})(z) \tag{4.1.2}$$
$$:= \left\{\int_0^z \nu t^{\nu-1} \Pi_{i=1}^n (f_i'(t))^{\alpha_i} \left(\frac{g_i(t)}{t}\right)^{\beta_i} (e^{h_i(t)})^{\gamma_i} dt\right\}^{\frac{1}{\nu}}.$$

We note that for some special real or complex parameters $\alpha_i, \beta_i, \gamma_i$ and ν, the integral operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ defined by (4.1.2) have been extensively studied by many authors(see [13, 14, 28, 33, 39, 65, 91, 92]). In present chapter, we determined the order of convexity of a integral operator $\mathcal{I}_1(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$. Moreover, we obtain sufficient restrictions for the operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ to be univalent in $\mathbb{E}$.

To investigate present study, we need to recall following lemmas.

Lemma 4.1.1. *[90] Let $\nu \in \mathbb{C}$ with $\mathrm{Re}\{\nu\} > 0$. If $f \in \mathcal{A}$ complies*

$$\frac{1 - |z|^{2\mathrm{Re}\{\nu\}}}{\mathrm{Re}\{\nu\}} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad (z \in \mathbb{E}),$$

then the integral operator

$$F_\nu(z) = \left\{\nu \int_0^z t^{\nu-1} f'(t) dt\right\}^{\frac{1}{\nu}}$$

is in $\mathcal{S}$.

Lemma 4.1.2. *[92] Let $\zeta \in \mathbb{C}$ with $\mathrm{Re}\{\zeta\} > 0$ and $c \in \mathbb{C}$ with $|c| \leq 1$ and $c \neq -1$. If $f \in \mathcal{A}$ complies*

$$\left| c|z|^{2\zeta} + (1 - |z|^{2\zeta}) \frac{zf''(z)}{\zeta f'(z)} \right| \leq 1 \quad (z \in \mathbb{E}),$$

then the integral operator

$$F_\zeta(z) = \left\{ \zeta \int_0^z t^{\zeta-1} f'(t) dt \right\}^{\frac{1}{\zeta}}$$

is in $\mathcal{S}$.

4.2 Main results

4.2.1 Convexity of $\mathcal{I}_1(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$

We begin by investigating the order of convexity of the integral operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ defined by (4.1.2) with $\nu = 1$.

Theorem 4.2.1. *Let $f_i, g_i, h_i \in \mathcal{G}_{\mu_i}$ for all $i \in \bar{\mathbb{N}}$ with $|h_i(z)| \leq M_i \quad (M_i > 0)$ and satisfy*

$$0 < \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i| M_i}{1 - \mu_i} \leq 1,$$
$$(\alpha_i, \beta_i, \gamma_i \in \mathbb{C};\ 0 < \mu_i < 1; i \in \bar{\mathbb{N}}).$$

Then the integral operator defined by

$$\begin{aligned} q(z) : &= \mathcal{I}_1(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})(z) \\ &= \int_0^z \Pi_{i=1}^n (f'(t))^{\alpha_i} \left(\frac{g_i(t)}{t} \right)^{\beta_i} \left(e^{h_i(t)} \right)^{\gamma_i} dt \end{aligned} \tag{4.2.1}$$

is convex of order δ given by

$$\delta = 1 - \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i| M_i}{1 - \mu_i}$$

Proof. From (4.2.1), we obtain simply following two equations, that are

$$q'(z) = \Pi_{i=1}^{n} (f_i'(z))^{\alpha_i} \left(\frac{g_i(z)}{z}\right)^{\beta_i} \left(e^{h_i(z)}\right)^{\gamma_i}, \quad (z \in \mathbb{E}) \tag{4.2.2}$$

and

$$q(0) = q'(0) - 1 = 0.$$

By using logarithmic differentiation to both sides of (4.2.2), we get

$$\frac{zq''(z)}{q'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i''(z)}{f_i'(z)}\right) + \sum_{i=1}^{n} \beta_i \left(\frac{zg_i'(z)}{g_i(z)} - 1\right) + \sum_{i=1}^{n} \gamma_i z h_i'(z). \tag{4.2.3}$$

According to the General Schwarz lemma, we have $|h_i(z)| \leq M_i$ $(z \in \mathbb{E})$ for all $i = 1, 2, \cdots, n$. Hence from the definition of $\mathcal{G}_i$ and (4.1.1), we obtain

$$\begin{aligned}
\left|\frac{zq''(z)}{q'(z)}\right| &\leq \sum_{i=1}^{n} |\alpha_i| \left(\mu_i \left|\frac{zf_i'(z)}{f_i(z)}\right| + \left|\frac{zf_i'(z)}{f_i(z)} - 1\right|\right) \\
&\qquad + \sum_{i=1}^{n} |\beta_i| \left|\frac{zg_i'(z)}{g_i(z)} - 1\right| + \sum_{i=1}^{n} |\gamma_i| \left|\frac{zh_i'(z)}{h_i(z)}\right| M_i \\
&\leq \sum_{i=1}^{n} |\alpha_i| \left[\mu_i \left(\frac{\mu_i}{1-\mu_i} + 1\right) + \frac{\mu_i}{1-\mu_i}\right] \\
&\qquad + \sum_{i=1}^{n} |\beta_i| \left(\frac{\mu_i}{1-\mu_i}\right) + \sum_{i=1}^{n} |\gamma_i| \left(\frac{\mu_i}{1-\mu_i} + 1\right) M_i \\
&= \sum_{i=1}^{n} |\alpha_i| \frac{2\mu_i}{1-\mu_i} + \sum_{i=1}^{n} |\beta_i| \frac{\mu_i}{1-\mu_i} + \sum_{i=1}^{n} |\gamma_i| \frac{M_i}{1-\mu_i} \\
&= \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i| M_i}{1-\mu_i} \\
&= 1 - \delta.
\end{aligned} \tag{4.2.4}$$

Therefore, the function q is convex of order

$$\delta = 1 - \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i| M_i}{1-\mu_i}.$$

□

If we take $n = 1,\ \alpha_1 = \alpha,\ \beta_1 = \beta,\ \gamma_1 = \gamma,\ f_1 = f,\ g_1 = g,\ h_1 = h$ and $M_1 = M$ in Theorem 4.2.1, we derive the consequence as follows.

Corollary 4.2.1. *Let $f, g, h \in \mathcal{G}_\mu$ with $|h(z)| \leq M \quad (M > 0)$ and satisfy*

$$0 < \frac{(2|\alpha| + |\beta|)\mu + |\gamma|M}{1-\mu} \leq 1 \quad (\alpha, \beta, \gamma \in \mathbb{C};\ 0 < \mu < 1).$$

Then the integral operator $\mathcal{I}_1(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ defined by (4.1.2) with $n = 1$ is convex order of δ given by

$$\delta = 1 - \frac{(2|\alpha| + |\beta|)\mu + |\gamma|M}{1-\mu}.$$

4.2.2 Univalency of $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$

Next, applying Lemma 4.1.1 and Lemma 4.1.2 we obtain some sufficient conditions for the integral operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i; i \in \bar{\mathbb{N}})$ defined by (4.1.2) to be univalent in $\mathbb{E}$.

Theorem 4.2.2. *Let $\nu \in \mathbb{C}$ with*

$$\mathrm{Re}\{\nu\} \geq \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i|M_i}{1-\mu_i} \tag{4.2.5}$$
$$(\alpha_i, \beta_i, \gamma_i \in \mathbb{C};\ 0 < \mu_i < 1; i \in \bar{\mathbb{N}}).$$

If $f_i, g_i, h_i \in \mathcal{G}_i$ for all $i \in \mathbb{N}$, then $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i;\ i \in \bar{\mathbb{N}})$ defined by (4.1.2) is univalent in $\mathbb{E}$.

Proof. Let us define the function q as in Theorem 4.2.1. Then we have (4.2.3). By using the same method as in (4.2.4) and the assumption (4.2.5), we get

$$\begin{aligned} \frac{1-|z|^{2\mathrm{Re}\{\nu\}}}{\mathrm{Re}\{\nu\}} \left| \frac{zq''(z)}{q'(z)} \right| &\leq \frac{1-|z|^{2\mathrm{Re}\{\nu\}}}{\mathrm{Re}\{\nu\}} \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i|M_i}{1-\mu_i} \\ &\leq \frac{1}{\mathrm{Re}\{\nu\}} \sum_{i=1}^{n} \frac{(2|\alpha_i| + |\beta_i|)\mu_i + |\gamma_i|M_i}{1-\mu_i} \\ &\leq 1. \end{aligned}$$

Therefore, by applying Lemma 4.1.1 for the function q, we prove that the integral operator $\mathcal{I}_\nu(\alpha_i,\beta_i,\gamma_i;f_i,g_i,h_i;\ i\in\bar{\mathbb{N}})$ is univalent in $\mathbb{E}$. $\square$

Theorem 4.2.3. *Let $c\in\mathbb{C}$ is satisfied with following condition:*

$$|c|\le 1-\frac{1}{\mathrm{Re}\{\zeta\}}\sum_{i=1}^{n}\frac{(2|\alpha_i|+|\beta_i|)\mu_i+|\gamma_i|M_i}{1-\mu_i} \qquad (\alpha_i,\beta_i,\gamma_i\in\mathbb{C};\ 0<\mu_i<1;\ i\in\bar{\mathbb{N}}) \tag{4.2.6}$$

where $\zeta\in\mathbb{C}$ with

$$\mathrm{Re}\{\zeta\}\ge\sum_{i=1}^{n}\frac{(2|\alpha_i|+|\beta_i|)\mu_i+|\gamma_i|M_i}{1-\mu_i} \qquad (\alpha_i,\beta_i,\gamma_i\in\mathbb{C};\ 0<\mu_i<1;\ i\in\bar{\mathbb{N}}). \tag{4.2.7}$$

If $f_i,g_i,h_i\in\mathcal{G}_i$ for all $i\in\bar{\mathbb{N}}$, then $\mathcal{I}_\nu(\alpha_i,\beta_i,\gamma_i;f_i,g_i,h_i;\ i\in\bar{\mathbb{N}})$ defined by (4.1.2) is univalent in $\mathbb{E}$.

Proof. Let us define the function q as in Theorem 4.2.1. Then from (4.2.4), (4.2.6) and (4.2.7), we derive

$$\begin{aligned}
&\left|c|z|^{2\zeta}+(1-|z|^{2\zeta})\frac{zq''(z)}{\zeta q'(z)}\right|\\
&\le\left|c|z|^{2\zeta}+\frac{(1-|z|^{2\zeta})}{\zeta}\sum_{i=1}^{n}\frac{(2|\alpha_i|+|\beta_i|)\mu_i+|\gamma_i|M_i}{1-\mu_i}\right|\\
&\le|c|+\frac{1}{\mathrm{Re}\{\zeta\}}\sum_{i=1}^{n}\frac{(2|\alpha_i|+|\beta_i|)\mu_i+|\gamma_i|M_i}{1-\mu_i}\\
&\le 1.
\end{aligned}$$

Therefore, by applying Lemma 4.1.2 for the function q, we close that the integral operator $\mathcal{I}_\nu(\alpha_i,\beta_i,\gamma_i;f_i,g_i,h_i;1\le i\le n)$ defined by (4.1.2) is univalent in $\mathbb{E}$. $\square$

If we take $n = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, $\gamma_1 = \gamma$, $f_1 = f$, $g_1 = g$, $h_1 = h$ and $M_1 = M$ in Theorem 4.2.2, and Theorem 4.2.3, respectively, we get the next two corollaries.

Corollary 4.2.2. *Let* $f, g, h \in \mathcal{G}_\mu$ *and* $\nu \in \mathbb{C}$ *with*

$$\mathrm{Re}\{\nu\} \geq \frac{(2|\alpha| + |\beta|)\mu + |\gamma| M}{1-\mu} \quad (\alpha, \beta, \gamma \in \mathbb{C};\ 0 < \mu < 1).$$

Then the integral operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i;\ i \in \bar{\mathbb{N}})$ *defined by (4.1.2) with* $n = 1$ *is univalent in* $\mathbb{E}$.

Corollary 4.2.3. *Let* $f, g, h \in \mathcal{G}_\mu$ *and* $c \in \mathbb{C}$ *with*

$$|c| \leq 1 - \frac{1}{\mathrm{Re}\{\zeta\}} \frac{(2|\alpha| + |\beta|)\mu + |\gamma| M}{1-\mu} \quad (\alpha, \beta, \gamma \in \mathbb{C};\ 0 < \mu < 1)$$

where $\zeta \in \mathbb{C}$ *with*

$$\mathrm{Re}\{\zeta\} \geq \frac{(2|\alpha| + |\beta|)\mu + |\gamma| M}{1-\mu} \quad (\alpha, \beta, \gamma \in \mathbb{C};\ 0 < \mu < 1).$$

Then the integral operator $\mathcal{I}_\nu(\alpha_i, \beta_i, \gamma_i; f_i, g_i, h_i;\ i \in \bar{\mathbb{N}})$ *defined by (4.1.2) with* $n = 1$ *is univalent in* $\mathbb{E}$.

Chapter 5

Mapping properties for certain subclasses of analytic functions

5.1 Introduction

Let $\mathbf{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ (a_n \neq 0) \tag{5.1.1}$$

which are analytic in $\mathbb{E}$. We also denote by $\mathcal{S}$ the class of all functions in $\mathbf{A}$ which are univalent in $\mathbb{E}$. A function $h \in \mathbf{A}$ is called to be in the class $\mathfrak{R}^t(C, D, \varrho)$ if

$$\left| \frac{h'(v) - 1}{t(C - D) - D(h'(v) - 1)} \right| < \varrho \ (v \in \mathbb{E}), \tag{5.1.2}$$

where C and D are complex numbers with $C \neq D, t \in \mathbb{C}\backslash\{0\}$ and ϱ is a positive real number. Particularly, for some real numbers C and D with $-1 \leq D < C \leq 1$ and $\varrho = 1$ without any restriction of the coefficients a_n $(n \in \mathbb{N})$ the class $\mathfrak{R}^t(C, D, \varrho)$ was presented by Dixit and Pal [29]. Furthermore, as giving specific values t, C, D and ϱ in (5.1.2), we derive subclasses investigated by many researchers in earlier studies(see, [15, 50, 31, 89, 100]). A function $h \in \mathbf{A}$ is an element of the class $\mathcal{UST}(\eta)$ if

$$\text{Re}\left\{\frac{h(\upsilon)-h(\zeta)}{(\upsilon-\zeta)h'(\upsilon)}\right\} > \eta \quad (\upsilon,\zeta \in \mathbb{E}; 0 \le \eta < 1).$$

Moreover, a function $h \in \mathbf{A}$ is an element of the class $\mathcal{UCV}(\eta)$ if

$$\text{Re}\left\{1+\frac{(\upsilon-\zeta)h''(\upsilon)}{h'(\upsilon)}\right\} > \eta \quad (\upsilon \in \mathbb{E}; 0 \le \eta < 1).$$

The classes $\mathcal{UST}(0) \equiv \mathcal{UST}$ and $\mathcal{UCV}(0) \equiv \mathcal{UCV}$ are presented by Goodman [37, 38], which are named the classes of uniformly starlike and uniformly convex functions, respectively. The classes of uniformly starlike and uniformly convex functions have been widely investigated by Ma and Minda [61] and Rønning [106]. Now, we turn to the Gaussian hypergeometric function defined by (2.7.4), and present that $F(b,d;e;z) = F(d,b;e;z)$ and

$$F(b,d;e;1) = \frac{\Gamma(e-b-d)\Gamma(e)}{\Gamma(e-b)\Gamma(e-d)} \quad (\text{Re}\{e-b-d\} > 0).$$

We additionaly consider (see, [84, 123]) that the function $F(b,d;e;z)$ is bounded if $\text{Re}\{e-b-d\} > 0$, and has a pole at $z=1$ if $\text{Re}\{e-b-d\} \le 0$. Furthermore, univalence, starlikeness and convexity properties of $zF(b,d;e;z)$ have been extensively studied by Ponnusamy and Vuorinen [101] and Ruscheweyh and Singh [107].

We state the operator $I_{b,d;e}f$ by

$$I_{b,d;e}f(z) = zF(b,d;e;z) * f(z), \quad f \in \mathbf{A} \tag{5.1.3}$$

where $*$ denote the convolution of power series defined by (3.1.2). For a particular case of the operator $I_{b,d;e}f$, we can refer to the result by Swaminathan [119]. In present chapter, we derive a necessary restriction for the class $\mathfrak{R}^t(C,D,\varrho)$ and sufficient restrictions for the classes $\mathfrak{R}^t(C, D, \varrho)$, $\mathcal{UST}(\varrho)$ and $\mathcal{UCV}(\varrho)$, respectively. Furthermore, we investigate a restriction for univalency of the operator $I_{b,d;e}f$ defined by (5.1.3). Also, we note that the contents of present chapter have been published by Journal of Inequalities and Applications [51].

5.2 Main results

Theorem 5.2.1. *Let $f \in \mathfrak{R}^t(C, D, \varrho)$ defined by (5.1.1), with $a_n = |a_n|e^{i\frac{(3n+1)\pi}{2}}\,(n \in \mathbb{N}\backslash\{1\})$. Then*

$$\sum_{n=2}^{\infty} n(1-\varrho|D|)|a_n| \leq \varrho|t||C-D|. \tag{5.2.1}$$

Proof. From the definition of $\mathfrak{R}^t(C, D, \varrho)$, we get

$$|f'(z)-1| < \varrho|t(C-D) - D(f'(z)-1)| \ (z \in \mathbb{E}).$$

and so

$$\left|\sum_{n=2}^{\infty} n a_n z^{n-1}\right| < \varrho\left|t(C-D) - D\sum_{n=2}^{\infty} n a_n z^{n-1}\right| \tag{5.2.2}$$

If we take $z = re^{i\frac{\pi}{2}}$, then we see that

$$a_n z^{n-1} = |a_n| r^{n-1} \ \ (0 \leq r < 1). \tag{5.2.3}$$

Then, by using (5.2.3) to (5.2.2), we have

$$\begin{aligned}\sum_{n=2}^{\infty} n|a_n|r^{n-1} &< \varrho\left|t(C-D) - D\sum_{n=2}^{\infty} n|a_n|r^{n-1}\right| \\ &< \varrho\,|t(C-D)| + \varrho|D|\sum_{n=2}^{\infty} n|a_n|r^{n-1},\end{aligned}$$

or equivalently,

$$\sum_{n=2}^{\infty} n(1-\varrho|D|)|a_n|r^{n-1} < \varrho|t||C-D|. \tag{5.2.4}$$

If we let $r \to 1^-$ in (5.2.4), then we have the inequality (5.2.1). $\square$

Theorem 5.2.2. *Let $f \in \mathbf{A}$ defined by (5.1.1). If*

$$\sum_{n=2}^{\infty} n(1+\varrho|D|)|a_n| \leq \varrho|t||C-D|, \tag{5.2.5}$$

where C and D are complex numbers with $C \neq D, t \in \mathbb{C}\setminus\{0\}$ and ϱ is a positive real number, then $f \in \mathfrak{R}^t(C,D,\varrho)$. This consequence is sharp for the function defined as

$$f(z) = z + \sum_{n=2}^{\infty} \frac{\varrho t(C-D)\epsilon}{n^2(n-1)(1+\varrho|D|)} z^n$$

$$(C, D \in \mathbb{C}; C \neq D; t \in \mathbb{C}\setminus\{0\}; |\epsilon| = 1; z \in \mathbb{E}).$$

Proof. According to the definition of $\mathfrak{R}^t(C,D,\varrho)$, it is enough to show that

$$|f'(z)-1| < \varrho\,|t(C-D) - D(f'(z)-1)| \quad (z \in \mathbb{E}). \tag{5.2.6}$$

From (5.2.6), we obtain

$$\left|\sum_{n=2}^{\infty} na_n z^{n-1}\right| < \varrho\left|t(C-D) - D\sum_{n=2}^{\infty} na_n z^{n-1}\right|.$$

Thus, it satisfies to prove following result.

$$\sum_{n=2}^{\infty} n|a_n|r^{n-1} < \varrho\left(|t||C-D| - |D|\sum_{n=2}^{\infty} n|a_n|r^{n-1}\right),$$

which is equivalent to the relationship

$$\sum_{n=2}^{\infty} n(1+\varrho|D|)|a_n|r^{n-1} < \varrho|t||C-D|. \tag{5.2.7}$$

If we let $r \to 1^-$ in (5.2.7), then we get

$$\sum_{n=2}^{\infty} n(1+\varrho|D|)|a_n| \leq \varrho|t||C-D|.$$

□

Theorem 5.2.3. *Let $f \in \mathbf{A}$ defined by (5.1.1). If*

$$\sum_{n=2}^{\infty}((3-\eta)n-2)|a_n| \leq 1-\eta \quad (0 \leq \eta < 1), \tag{5.2.8}$$

then $f \in \mathcal{UST}(\eta)$. This consequence is sharp for the function defined as

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\eta)\epsilon}{n(n-1)((3-\eta)n-2)} z^n \quad (0 \leq \eta < 1; |\epsilon| = 1).$$

Proof. It is enough to prove that following condition.

$$\left| \frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} - 1 \right| < 1-\eta \quad (0 \leq \eta < 1; (z,\zeta) \in \mathbb{E} \times \mathbb{E}).$$

Then again, we get

$$\begin{aligned} &\left| \frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)} - 1 \right| \\ &= \left| \frac{\sum_{n=2}^{\infty} a_n(\zeta^{n-1} + z\zeta^{n-2} + \cdots + z^{n-1}) - \sum_{n=2}^{\infty} na_n z^{n-1}}{1 - \sum_{n=2}^{\infty} na_n z^{n-1}} \right| \\ &< \frac{\sum_{n=2}^{\infty} 2(n-1)|a_n|}{1 - \sum_{n=2}^{\infty} n|a_n|}, \end{aligned}$$

which is bounded by $1-\eta$ if the condition (5.2.8) is satisfied. □

Theorem 5.2.4. *Let $f \in \mathbf{A}$ defined by (5.1.1). If*

$$\sum_{n=2}^{\infty} n(2n-1-\eta)|a_n| \leq 1-\eta \quad (0 \leq \eta < 1), \tag{5.2.9}$$

then $f \in \mathcal{UCV}(\eta)$. This consequence is sharp for the function defined as

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\eta)\epsilon}{n^2(n-1)(2n-1-\eta)} z^n \quad (0 \leq \eta < 1; |\epsilon| = 1; z \in \mathbb{E})$$

Proof. It is enough to show that following condition.

$$\left|\frac{(z-\zeta)f''(z)}{f'(z)}\right| < 1-\eta \quad (0 \le \eta < 1; (z,\zeta) \in \mathbb{E}\times\mathbb{E}).$$

We derive

$$\begin{aligned}\left|\frac{(z-\zeta)f''(z)}{f'(z)}\right| &= \left|\frac{(z-\zeta)\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \\ &< \frac{2\sum_{n=1}^{\infty} n(n-1)|a_n|}{1-\sum_{n=2}^{\infty} n|a_n|},\end{aligned}$$

which is bounded by $1-\eta$ if the condition (5.2.9) is satisfied. □

Theorem 5.2.5. *Let $c, d \in \mathbb{C}\setminus\{0\}$ and $e > |c|+|d|$. If $f \in \mathfrak{R}^t(C, D, \varrho)$ with $c_n = |c_n|e^{i\frac{(3n+1)\pi}{2}}, 0 < |D| < 1$ and*

$$\frac{\Gamma(e-|c|-|d|)\Gamma(e)}{\Gamma(e-|c|)\Gamma(e-|d|)} \le \frac{1-\varrho|D|}{1+\varrho|D|} + 1.$$

Then $I_{c,d;e}f \in \mathfrak{R}^t(C, D, \varrho)$, where the operator $I_{c,d;e}f$ is defined by (5.1.3).

Proof. From Theorem 5.2.2, we want to show that

$$T_1 := \sum_{n=2}^{\infty} n(1+\varrho|D|)|C_n| \le \varrho|t||C-D|,$$

where

$$C_n = \frac{(c)_{n-1}(d)_{n-1}}{(e)_{n-1}(1)_{n-1}} c_n.$$

By using Theorem 5.2.1, we obtain

$$|c_n| \le \frac{\varrho|t||C-D|}{n(1-\varrho|D|)},$$

$$
\begin{aligned}
T_1 &\le \frac{\varrho|t||C-D|(1+\varrho|D|)}{1-\varrho|D|}\sum_{n=2}^{\infty}\frac{(|c|)_{n-1}(|d|)_{n-1}}{(e)_{n-1}(1)_{n-1}} \\
&= \frac{\varrho|t||C-D|(1+\varrho|D|)}{1-\varrho|D|}\left(\sum_{n=0}^{\infty}\frac{(|c|)_{n}(|d|)_{n}}{(e)_{n}(1)_{n}}-1\right) \\
&= \frac{\varrho|t||C-D|(1+\varrho|D|)}{1-\varrho|D|}\left(\frac{\Gamma(e-|c|-|d|)\Gamma(e)}{\Gamma(e-|c|)\Gamma(e-|d|)}-1\right) \\
&\le \varrho|t||C-D|.
\end{aligned}
$$

□

Now, we recall the next lemma which is required to prove Theorem 5.2.6.

Lemma 5.2.1. *[44] Let ω be regular in $\mathbb{E}$ with $\omega(0)=0$. Then, if $|\omega(z)|$ reaches a maximum value on the circle $|z|=r$ $(0 \le r < 1)$ at a point z_0, we denote that*

$$
z_0\omega'(z_0) = k\omega(z_0) \quad (k \ge 1).
$$

Theorem 5.2.6. *Let $f \in \mathbf{A}$ defined by (5.1.1). If*

$$
\left|\frac{(I_{c,d;e}f(z))'-1}{1-\varrho}\right|^{\beta}\left|\frac{z(I_{c,d;e}f(z))''}{(I_{c,d;e}f(z))'-\varrho}\right|^{\gamma} < \frac{1}{2^{\gamma}} \quad (z \in \mathbb{E}) \tag{5.2.10}
$$

for some real ϱ $(0 \le \varrho < 1)$, $\beta > 0$ and $\gamma > 0$. Then

$$
|(I_{c,d;e}f(z))'-1| < 1-\varrho \quad (z \in \mathbb{E}). \tag{5.2.11}
$$

Proof. Let us state ω as

$$
\omega(z) = \frac{(I_{c,d;e}f(z))'-1}{1-\varrho} \quad (z \in \mathbb{E}).
$$

Thus, it satisfies that ω is analytic in $\mathbb{E}$ with $\omega(0)=0$.

By using (5.2.10),

$$
\begin{aligned}
|\omega(z)|^{\beta}\left|\frac{z\omega'(z)}{\omega(z)+1}\right|^{\gamma} &= |\omega(z)|^{\beta+\gamma}\left|\frac{z\omega'(z)}{\omega(z)}\frac{1}{\omega(z)+1}\right|^{\gamma} \\
&< \frac{1}{2^{\gamma}} \quad (z \in \mathbb{E}).
\end{aligned} \tag{5.2.12}
$$

Here, we consider that there exists a point $z_0 \in \mathbb{E}$ as

$$\max_{|z|\le|z_0|}|\omega(z)| = |\omega(z_0)| = 1.$$

By using Lemma 5.2.1, we can put

$$\frac{z_0\omega'(z_0)}{\omega(z_0)} = k \ge 1.$$

Hence, we get

$$\begin{aligned} |\omega(z_0)|^\beta \left|\frac{z_0\omega'(z_0)}{\omega(z_0)+1}\right|^\gamma &= \left|\frac{z_0\omega'(z_0)}{\omega(z_0)}\frac{1}{\omega(z_0)+1}\right|^\gamma \\ &\ge \left(\frac{k}{2}\right)^\gamma \ge \frac{1}{2^\gamma}, \end{aligned}$$

which is contradiction to the condition (5.2.12). This proves that

$$|\omega(z)| = \left|\frac{(I_{c,d;e}f(z))' - 1}{1-\varrho}\right| < 1 \quad (z \in \mathbb{E}).$$

□

Remark 5.2.1. *From the restriction (5.2.12) in Theorem 5.2.6 we obtain*

$$\mathrm{Re}\{(I_{c,d;e}f(z))'\} > 0 \quad (z \in \mathbb{E}).$$

Thus, the function $I_{c,d;e}f$ *is univalent in* $\mathbb{E}$*, under the conditions of Theorem 5.2.6 by the Noshiro-Warschawski theorem [32].*

Chapter 6

Applications of the first-order differential subordinations

6.1 Introduction

Let $\mathcal{A}$ denotes the class defined by Definition 2.1.2. A function $l \in \mathcal{A}$ is known as strongly starlike of order ϱ $(0 < \varrho \leq 1)$ if and only if

$$\frac{zl'(z)}{l(z)} \prec \left(\frac{1+z}{1-z}\right)^{\varrho} \qquad (z \in \mathbb{E}). \tag{6.1.1}$$

We also note that the conditions (6.1.1) can be written by

$$\left|\arg \frac{zl'(z)}{l(z)}\right| < \frac{\pi}{2}\varrho \qquad (z \in \mathbb{E}),$$

where the notation $\prec$ denote, the subordination defined in Section 2.4. We express the subclass of $\mathcal{A}$ comprised of all strongly starlike functions of order ϱ $(0 < \varrho \leq 1)$ as $\mathcal{S}[\varrho]$ and also denote that $\mathcal{S}[1] \equiv \mathcal{S}^*$. This class is renowned as the class of all normalized starlike functions in $\mathbb{E}$. The class $\mathcal{S}[\varrho]$ and the related classes have been widely investigated by Mocanu [71] and Nunokawa [80].

If ψ is analytic in $\mathbb{D} \subset \mathbb{C}^2$, h is univalent in $\mathbb{E}$ and p is analytic in $\mathbb{E}$ with $(p(z), zp'(z)) \in \mathbb{D}$ for $z \in \mathbb{E}$, then p is called to satisfy the first-order differential subordination if

$$\psi(p(z), zp'(z)) \prec h(z) \ (z \in \mathbb{E}). \tag{6.1.2}$$

A function $q \in \mathcal{S}$ is called a dominant of the differential subordination, if $p \prec q$ for all p satisfying (6.1.2). If $\tilde{q}$ is a dominant of (6.1.2) and $\tilde{q} \prec q$ for all dominants of (6.1.2), then $\tilde{q}$ is said to be the best dominant of the differential subordination (6.1.2).

The general theory of the first-order differential subordinations, with many interesting applications, especially in the theory of univalent functions, was developed by Miller and Mocanu ([67, 68]). For several applications of the principle of differential subordinations in the investigations of various interesting subclasses of analytic and univalent functions. We here offer references of the recent works, for more detail(see, [109, 115, 116, 124, 125]).

In this chapter, we propose to derive some applications of the first-order differential subordinations. We also extend and improve the results proven earlier by Cho and Kim [18], Miller *et al.* [66], and Nunokawa *et al.* [80, 81, 82, 83]. We note that the contents of this chapter have been published by Filomat [21].

6.2 Main result

To verify our consequence, we recall the next Lemma by Miller and Mocanu [67].

Lemma 6.2.1. *[67] Let $q \in \mathcal{S}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$ with*

$$\varphi(\omega) \neq 0 \quad \textit{when} \quad \omega \in q(\mathbb{E}).$$

Set

$$Q(z) = zq'(z)\varphi(q(z)),\ h(z) = \theta(q(z)) + Q(z)$$

and consider that

(i) *Q is starlike in $\mathbb{E}$*

(ii) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \mathbb{E}).$

If p is analytic in $\mathbb{E}$ with

$$p(0) = q(0), \quad p(\mathbb{E}) \subset \mathbb{D}$$

and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) \quad (z \in \mathbb{E}), \tag{6.2.1}$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{E})$$

and q is the best dominant of (6.2.1).

With the help of the above Lemma 6.2.1, we get the next Theorem 6.2.1.

Theorem 6.2.1. *Let p be nonzero analytic in $\mathbb{E}$ with $p(0) = 1$. If*

$$\left|\arg\left(\beta p^{\gamma}(z) + \alpha z p'(z) p^{\gamma-1}(z)\right)\right| < \frac{\pi}{2}\delta(\alpha, \beta, \varrho, \gamma) \tag{6.2.2}$$

$$(\alpha, \beta > 0;\ \gamma \geq 0;\ 0 < \varrho \leq 1; -1 \leq \varrho\gamma \leq 1; z \in \mathbb{E}),$$

where $\delta(\alpha,\beta,\varrho,\gamma)$ $(0<\delta(\alpha,\beta,\varrho,\gamma)<1)$ *is the solution of the equation*:

$$\delta(\alpha,\beta,\varrho,\gamma)=\gamma\varrho+\frac{2}{\pi}\tan^{-1}\frac{\alpha\varrho}{\beta}, \tag{6.2.3}$$

then

$$|\arg\ p(z)|<\frac{\pi}{2}\varrho \qquad (z\in\mathbb{E}).$$

Proof. Let

$$q(z)=\left(\frac{1+z}{1-z}\right)^{\varrho}, \quad \theta(\omega)=\beta\omega^{\gamma} \quad \text{and} \quad \varphi(\omega)=\alpha\omega^{\gamma-1}$$

in Lemma 6.2.1. Then q is univalent(convex) in $\mathbb{E}$ and

$$\operatorname{Re}\{q(z)\}>0 \qquad (z\in\mathbb{E}).$$

Further, θ and φ are analytic in $q(\mathbb{E})$ and

$$\varphi(\omega)\neq 0 \qquad (\omega\in q(\mathbb{E})).$$

Set

$$Q(z)=zq'(z)\varphi(q(z))=\left(\frac{1+z}{1-z}\right)^{\varrho\gamma}\frac{2\alpha\varrho z}{1-z^2}$$

and

$$h(z)=\theta(q(z))+Q(z)=\left(\frac{1+z}{1-z}\right)^{\varrho\gamma}\left(\beta+\frac{2\alpha\varrho z}{1-z^2}\right).$$

Then we can see easily that the conditions (i) and (ii) of Lemma 6.2.1 are satisfied. We also note that $h(0)=\beta$ and

$$\begin{aligned} h(\mathrm{e}^{\mathrm{i}\theta}) &= \left(\tfrac{1+\mathrm{e}^{i\theta}}{1-\mathrm{e}^{i\theta}}\right)^{\varrho\gamma}\left(\beta+\tfrac{2\alpha\varrho\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{2i\theta}}\right) \\ &= \left(i\cot\tfrac{\theta}{2}\right)^{\varrho\gamma}\left(\beta+i\tfrac{\alpha\varrho}{\sin\theta}\right) \\ &= \left|\cot\tfrac{\theta}{2}\right|\mathrm{e}^{\pm\frac{\pi\varrho}{2}}\left(\beta+i\tfrac{\alpha\varrho}{\sin\theta}\right), \end{aligned} \tag{6.2.4}$$

where we take " + " for $0<\theta<\pi$, and " − " for $-\pi<\theta<0$. In view of the previous relation (6.2.4), we can see that the real and imaginary part of $h(\mathrm{e}^{i\theta})$ is an even and odd function of θ, respectively. Without loss of generality, we suppose that $0<\theta<\pi$. Hence, from (6.2.4), we obtain

$$\begin{aligned}\arg h(\mathrm{e}^{i\theta}) &= \frac{\pi}{2}\varrho\gamma + \arg\left(\beta + i\frac{\alpha\varrho}{\sin\theta}\right)\\ &= \frac{\pi}{2}\varrho\gamma + \tan^{-1}\frac{\alpha\varrho}{\beta\sin\theta}\\ &\geq \frac{\pi}{2}\varrho\gamma + \tan^{-1}\frac{\alpha\varrho}{\beta}\\ &= \frac{\pi}{2}\delta(\alpha,\beta,\varrho,\gamma),\end{aligned}$$

where $\delta(\alpha,\beta,\varrho,\gamma)$ is the solution of the equation given by (6.2.3). Therefore, we conclude that the condition (6.2.2) implies that

$$\beta p^{\gamma}(z) + \alpha z p'(z) p^{\gamma-1}(z) \prec h(z) \qquad (z\in\mathbb{E}).$$

Thus, by Lemma 6.2.1, we obtain

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\varrho} \qquad (z\in\mathbb{E}),$$

or equivalently,

$$|\arg\ p(z)| < \frac{\pi}{2}\varrho \qquad (z\in\mathbb{E}).$$

□

Remark 6.2.1. *To take $\gamma=0$ in Theorem 6.2.1, we derive the condition which is $p(z)\neq 0$ for $z\in\mathbb{E}$. If p has a zero $z_0\in\mathbb{E}$ of order m, then we may write*

$$p(z) = (z-z_0)^m p_1(z) \qquad (m\in\mathbb{N}),$$

where p_1 *is analytic in* $\mathbb{E}$ *with* $p_1(z_0) \neq 0$. *Thus*

$$\beta + \alpha \frac{zp'(z)}{p(z)} = \beta + \alpha \frac{z{p_1}'(z)}{p_1(z)} + \frac{\alpha m z}{z - z_0}. \tag{6.2.5}$$

Therefore, choosing $z \to z_0$, *suitably the argument of the right-hand of (6.2.5) can take any value between* 0 *and* 2π, *which contradicts the hypothesis (6.2.2).*

6.3 Some applications

If we take

$$\alpha = 1 \qquad \text{and} \qquad \gamma = 0$$

in Theorem 6.2.1, then we get the next Corollary by Nunokawa *et al.* [82].

Corollary 6.3.1. *Let* p *be analytic in* $\mathbb{E}$ *with* $p(0) = 1$. *If*

$$\left| \arg \left(\beta + \frac{zp'(z)}{p(z)} \right) \right| < \tan^{-1} \frac{\varrho}{\beta} \quad (\beta > 0;\ 0 < \varrho \leq 1;\ z \in \mathbb{E}),$$

then

$$|\arg\ p(z)| < \frac{\pi}{2} \varrho \quad (z \in \mathbb{E}).$$

Letting

$$\beta = 1 \qquad \text{and} \qquad p(z) = \frac{f(z)}{z} \quad (z \in \mathbb{E})$$

in Corollary 6.3.1, we derive the consequence as belows.

Corollary 6.3.2. *Let* $f \in \mathcal{A}$. *If*

$$\left| \arg\ \frac{zf'(z)}{f(z)} \right| < \tan^{-1} \varrho \quad (0 < \varrho \leq 1;\ z \in \mathbb{E}),$$

then

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\varrho \quad (z \in \mathbb{E}).$$

Making

$$\alpha = \beta = 1 \qquad \text{and} \qquad p(z) = \frac{f(z)}{z} \quad (z \in \mathbb{E})$$

in Theorem 6.2.1, we have the next consequence.

Corollary 6.3.3. *Let* $f \in \mathcal{A}$. *If*

$$\left|\arg \frac{zf'(z)f^{\gamma-1}(z)}{z^{\gamma}}\right| < \frac{\pi}{2}\delta(\varrho,\gamma) \quad (\gamma \geq 0;\ 0 < \varrho \leq 1;\ z \in \mathbb{E}),$$

where $\delta(\varrho,\gamma)$ $(0 < \delta(\varrho,\gamma) < 1)$ *is the solution of the equation*

$$\delta(\varrho,\gamma) = \varrho\gamma + \frac{2}{\pi}\tan^{-1}\varrho, \tag{6.3.1}$$

then

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\varrho \quad (z \in \mathbb{E}).$$

Remark 6.3.1. *If we take*

$$\gamma = 2 \qquad \text{and} \qquad \delta(\varrho, 2) = 1,$$

in Corollary 6.3.3, then we have the result obtained by Lee and Nunokawa [52].

Taking $\gamma = 1$ in Corollary 6.3.3, we derive the next consequence.

Corollary 6.3.4. *Let* $f \in \mathcal{A}$. *If*

$$|\arg f'(z)| < \frac{\pi}{2}\delta(\varrho) \quad (0 < \varrho \leq 1;\ z \in \mathbb{E}),$$

where $\delta(\varrho)$ *is the solution* $\delta(\varrho, 1)$ *of the equation given by (6.3.1) with* $\gamma = 1$*, then*

$$\left|\arg \frac{f(z)}{z}\right| < \frac{\pi}{2}\varrho \quad (z \in \mathbb{E}).$$

Applying Corollary 6.3.4, we have the following result immediately.

Corollary 6.3.5. *Let* $f \in \mathcal{A}$*. If*

$$|\arg f'(z)| < \frac{\pi}{2}\delta(\varrho) \quad (0 < \varrho \leq 1;\ \in \mathbb{E}),$$

where $\delta(\varrho)$ *is given by Corollary 6.3.4, then*

$$|\arg F'(z)| < \frac{\pi}{2}\varrho \quad (z \in \mathbb{E}),$$

where F *is defined by*

$$F(z) = \int_0^z \frac{f(t)}{t} dt \quad (z \in \mathbb{E}).$$

Furthermore, from Theorem 6.2.1, we have the next consequence.

Corollary 6.3.6. *Let* $f \in \mathcal{A}$*. If*

$$\left|\arg \frac{zf'(z)f^{\gamma-1}(z)}{z^{\gamma}}\right| < \frac{\pi}{2}\delta(\varrho, \gamma, c) \qquad (0 < \varrho \leq 1;\ c > -\gamma;\ \gamma > 0;\ z \in \mathbb{E}),$$

where $\delta(\varrho, \gamma, c)$ $(0 < \delta(\varrho, \gamma, c) < 1)$ *is the solution of the equation:*

$$\delta(\varrho, \gamma, c) = \varrho + \frac{2}{\pi}\tan^{-1}\frac{\varrho}{c+\gamma},$$

then

$$\left|\arg \frac{zF'(z)F^{\gamma-1}(z)}{z^\gamma}\right| < \frac{\pi}{2}\varrho \qquad (z \in \mathbb{E}),$$

where F is the integral operator defined by

$$F(z) = \left(\frac{c+\gamma}{z^c}\int_0^z t^{c-1}f^\gamma(t)dt\right)^{\frac{1}{\gamma}} \qquad (z \in \mathbb{E}).$$

Proof. According to the definition of F that

$$cF^\gamma(z) + \gamma zF'(z)F^{\gamma-1}(z) = (c+\gamma)f^\gamma(z).$$

Let

$$p(z) = \frac{zF'(z)F^{\gamma-1}(z)}{z^\gamma} \qquad (z \in \mathbb{E}).$$

Then, after a simple calculation, we find that

$$(c+\gamma)p(z) + zp'(z) = (c+\gamma)\frac{zf'(z)f^{\gamma-1}(z)}{z^\gamma}.$$

Therefore, by applying Theorem 6.2.1, we have Corollary 6.3.6. □

Chapter 7

Subclasses of starlike and convex functions associated with Bessel functions

7.1 Introduction

Let $\mathcal{A}$ denote the class of functions defined by (2.1.1) and $\mathcal{T}$ denote the subclass of $\mathcal{A}$ of the functions defined by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{7.1.1}$$

Let $\mathcal{T}^*(\upsilon)$ and $\mathcal{C}(\upsilon)$ denote the subclasses of $\mathcal{T}$ consisting of starlike and convex functions of order υ $(0 \leq \upsilon < 1)$ (see [111]), respectively. In 1997, Bharati *et al.* [11] presented the subclasses of starlike and convex functions as follows.

Definition 7.1.1. *Let* $f \in \mathcal{A}$ *defined by (2.1.1). A function* $f \in \mathcal{S}_{\mathcal{P}}\mathcal{T}(\upsilon, \delta)$, *if it fulfills the condition:*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \upsilon\left|\frac{zf'(z)}{f(z)}-1\right|+\delta\ (\upsilon \geq 0;\ 0 \leq \delta < 1)$$

and

$$f \in \mathcal{UCV}(\upsilon,\delta) \iff zf' \in \mathcal{S_P}(\upsilon,\delta)$$

.

Definition 7.1.2. *Let $f \in \mathcal{A}$ defined by (2.1.1). A function $f \in \mathcal{P}(\upsilon)$, if it fulfills the condition:*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)+\upsilon \geq \left|\frac{zf'(z)}{f(z)}-\upsilon\right|\ (0 < \upsilon < \infty)$$

and

$$f \in \mathcal{CP}(\upsilon) \iff zf' \in \mathcal{P}(\upsilon).$$

Denote

$$\mathcal{PT}(\upsilon) = \mathcal{P}(\upsilon) \cap \mathcal{T} \quad \text{and} \quad \mathcal{CPT}(\upsilon) = \mathcal{CP}(\upsilon) \cap \mathcal{T}.$$

Bharati *el al.* [11] show that

$$\mathcal{S_P T}(\upsilon,\delta) = \mathcal{T}^*((\upsilon+\delta)/(1+\upsilon)),$$

$$\mathcal{UCT}(\upsilon,\delta) = \mathcal{C}((\upsilon+\delta)/(1+\upsilon)),$$

$$\mathcal{PT}(\upsilon) = \mathcal{T}^*(1-\upsilon)\ \ (1/2 < \upsilon < 1)$$

and

$$\mathcal{CPT}(\upsilon) = \mathcal{C}(1-\upsilon)\ \ (1/2 < \upsilon < 1).$$

Particularly, we state that $\mathcal{UCV}(1,0)$ is the class of uniformly convex functions given by Goodman [37]. For more interesting developments of some related subclasses of $\mathcal{UCV}(\upsilon,\delta)$, for more details, we refer to the works of Goodman [38], Ma-Minda [61] and Rønning [105, 106].

In recent, Baricz [6] defined a generalized Bessel function $\omega_{p,b,c} \equiv \omega$ as follows:

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \ c^n}{n! \ \Gamma\left(p+n+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, \tag{7.1.2}$$

which is the special solution of the second order linear homogeneous differential equation

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)]\omega(z) = 0 \ (b, p, c \in \mathbb{C}), \tag{7.1.3}$$

which is a natural generalization of Bessel's equation. Solutions of (7.1.3) are considered the generalized Bessel function of order p. The particular solution given by (7.1.2) is called the generalized Bessel function of the first kind of order p. We also note that the function $\omega_{p,b,c}$ is commonly not univalent in $\mathbb{E}$, even though the series defined above is convergent everywhere.

Now, we consider the function $u_{p,b,c}(z)$ defined by

$$u_{p,b,c}(z) = 2^p\Gamma\left(p+\frac{b+1}{2}\right) \ z^{-\frac{p}{2}} \ \omega_{p,b,c} \ (\sqrt{z}), \quad \sqrt{1} = 1.$$

As using the renowned Pochhammer symbol defined by (2.7.2) in Section 2.7, we can present $u_{p,b,c}(z) \equiv u$ as

$$u_p(z) = u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p+\frac{b+1}{2}\right)_n} \frac{z^n}{n!} \tag{7.1.4}$$

$$\left(p+\frac{b+1}{2} \notin \ \mathbb{N}^- \cup \{0\}; \mathbb{N}^- = \{-1, -2, \cdots\}\right).$$

Then the function $u_{p,b,c}$ is analytic on $\mathbb{C}$ and fulfills the second-order linear differential equation

$$4z^2u''(z) + 2(2p+b+1)zu'(z) + cu(z) = 0.$$

The research of the generalized Bessel function is a intriguing subject in geometric function theory recently and we refer to the works of Baricz [6, 5, 3, 4] and Cho *et al.* [22] and Mondal and Swaminathan [72] and Deniz [25] and so on (see, [27, 23, 126]).

In this chapter, we establish sufficient restrictions for zu_p to be in $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\upsilon,\delta)$ and $\mathcal{UCV}(\upsilon,\delta)$ and also give necessary and sufficient conditions for $z(2-u_p)$ to be $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\upsilon,\delta)$, $\mathcal{UCT}(\upsilon,\delta)$, $\mathcal{PT}(\upsilon)$, $\mathcal{CPT}(\upsilon)$. Furthermore, we investigate an integral operator associated with the function u_p. Throughout this chapter, we will use in (7.1.4) the following notation for convenience:

$$m = p + \frac{b+1}{2}.$$

We remark that the contents of this chapter have been published by Filomat [20], recently.

7.2 Main results

To prove our principal consequences, we recall the next Lemmas by Bharati *et al* [11].

Lemma 7.2.1. *[11] (i) A sufficient condition for f defined by (2.1.1) to be in $\mathcal{S}_{\mathcal{P}}(\upsilon,\delta)$ is that*

$$\sum_{n=2}^{\infty}(n(1+\upsilon)-(\upsilon+\delta))|a_n| \le 1-\delta \qquad (\upsilon \ge 0\ ; 0 \le \delta < 1) \tag{7.2.1}$$

and a necessary and sufficient condition for f defined by (7.1.1) to be in $\mathcal{S}_{\mathcal{P}}\mathcal{T}(\upsilon,\delta)$ is that the condition (7.2.1) is satisfied.

(ii) A sufficient condition for f defined by (2.1.1) to be in $\mathcal{UCV}(\upsilon,\delta)$ is that

$$\sum_{n=2}^{\infty}n(n(1+\upsilon)-(\upsilon+\delta))|a_n| \le 1-\delta \qquad (\upsilon \ge 0\ ; 0 \le \delta < 1) \tag{7.2.2}$$

and a necessary and sufficient condition for f defined by (7.1.1) to be in $\mathcal{UCT}(\upsilon,\delta)$ is that the condition (7.2.2) is satisfied.

Lemma 7.2.2. *[11] (i) A a necessary and sufficient condition for f defined by (7.1.1) to be in $\mathcal{PT}(\upsilon)$ is that*

$$\sum_{n=2}^{\infty}(n-1-\upsilon)a_n \leq \upsilon \qquad (1/2<\upsilon\leq 1) \tag{7.2.3}$$

(ii) A a necessary and sufficient condition for f defined by (7.1.1) to be in $\mathcal{CPT}(\upsilon)$ is that

$$\sum_{n=2}^{\infty} n(n-1-\upsilon)a_n \leq \upsilon \qquad (1/2<\upsilon\leq 1) \tag{7.2.4}$$

Lemma 7.2.3. *[4] Let $b,p,c\in\mathbb{C}$ and $m\notin\mathbb{N}^-\cup\{0\}$. Then the function u_p defined by (7.1.4) complies the following recursive relation:*

$$4mu_p'(z) = -cu_{p+1}(z)\ (z\in\mathbb{C}). \tag{7.2.5}$$

Theorem 7.2.1. *Let $c<0$ and $m>0$. Then $zu_p\in\mathcal{S}_{\mathcal{P}}(\upsilon,\delta)$ if*

$$(1+\upsilon)u_p'(1)+(1-\delta)[u_p(1)-1]\leq 1-\delta \qquad (\upsilon\geq 0\ ;0\leq\delta<1). \tag{7.2.6}$$

Proof. Since

$$zu_p(z) = z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}z^n,$$

by using (i) in Lemma 7.2.1, it enough to show that

$$\mathcal{L}(c,m,\upsilon,\delta) := \sum_{n=2}^{\infty}[n(1+\upsilon)-(\upsilon+\delta)]\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\leq 1-\delta.$$

By brief calculation, we have

$$\begin{aligned}\mathcal{L}(c,m,\upsilon,\delta) &= \sum_{n=2}^{\infty}[(n-1)(1+\upsilon)+(1-\delta)]\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \qquad (7.2.7)\\ &= (1+\upsilon)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!}+(1-\delta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}\\ &= (1+\upsilon)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ n!}+(1-\delta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}\\ &= (1+\upsilon)\frac{(-c/4)}{m}\sum_{n=0}^{\infty}\frac{(-c/4)^{n}}{(m+1)_{n}\ n!}+(1-\delta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!}\\ &= (1+\upsilon)\frac{(-c/4)}{m}u_{p+1}(1)+(1-\delta)[u_p(1)-1]\\ &= (1+\upsilon)u_p'(1)+(1-\delta)[u_p(1)-1].\end{aligned}$$

Therefore, we know that the last expression (7.2.7) is bounded above by $1-\delta$ if (7.2.6) is fulfilled. □

Corollary 7.2.1. *Let $c<0$ and $m>0$. Then $z(2-u_p(z))\in\mathcal{S}_{\mathcal{P}}\mathcal{T}(\upsilon,\delta)$ if and only if the condition* (7.2.6) *is satisfied.*

Proof. Since

$$z(2-u_p(z)) = z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!}z^n,$$

As using analogous methods given in the proof of Theorem 7.2.1, we derive immediately Corollary 7.2.1. □

Theorem 7.2.2. *Let $c<0$ and $m>0$. Then $zu_p\in\mathcal{UCV}(\upsilon,\delta)$ if*

$$(1+\upsilon)u_p''(1)+(3+2\upsilon-\delta)u_p'(1)+(1-\delta)[u_p(1)-1]\le 1-\delta \qquad (\upsilon\ge 0\ ;0\le\delta<1). \tag{7.2.8}$$

Proof. Since

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} z^n,$$

by using (ii) in Lemma 7.2.1, it is enough to show that

$$\mathcal{P}(c,m,\upsilon,\delta) := \sum_{n=2}^{\infty} n[n(1+\upsilon) - (\upsilon+\delta)] \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \leq 1-\delta.$$

By using $n^2 = (n-1)(n-2) + 3(n-1) + 1$ and $n = (n-1)+1$, we can expatiate on the above terms as follows:

$$\begin{aligned}
&\mathcal{P}(c,m,\upsilon,\delta) \\
&\quad = (1+\upsilon)\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\qquad\qquad + (3+2\upsilon-\delta)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} + (1-\delta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad = (1+\upsilon)\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-3)!} \\
&\qquad\qquad + (3+2\upsilon-\delta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-2)!} + (1-\delta)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (n-1)!} \\
&\quad = (1+\upsilon)\sum_{n=2}^{\infty}\frac{(-c/4)^{n}}{(m)_{n}\ (n-2)!} \\
&\qquad\qquad + (3+2\upsilon-\delta)\sum_{n=1}^{\infty}\frac{(-c/4)^{n}}{(m)_{n}\ (n-1)!} + (1-\delta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!} \\
&\quad = (1+\upsilon)\frac{(-c/4)^2}{m(m+1)}\sum_{n=1}^{\infty}\frac{(-c/4)^{n-1}}{(m+2)_{n-1}\ (n-1)!} \\
&\qquad\qquad + (3+2\upsilon-\delta)\frac{(-c/4)}{m}\sum_{n=1}^{\infty}\frac{(-c/4)^{n-1}}{(m+1)_{n-1}\ (n-1)!} + (1-\delta)\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(m)_{n+1}\ (n+1)!} \\
&\quad = (1+\upsilon)\frac{(-c/4)^2}{m(m+1)}u_{p+2}(1) + (3+2\upsilon-\delta)\frac{(-c/4)}{m}u_{p+1}(1) + (1-\delta)[u_p(1)-1] \\
&\quad = (1+\upsilon)u_p''(1) + (3+2\upsilon-\delta)u_p'(1) + (1-\delta)[u_p(1)-1].
\end{aligned}$$

Therefore, we know that the last expression is bounded above by $1-\delta$ if (7.2.8) is fulfilled. $\square$

Similarly, by using a analogous method like in the proof of Corollary 7.2.1, we derive the next consequence.

Corollary 7.2.2. *Let* $c < 0$ *and* $m > 0$. *Then* $z(2-u_p) \in \mathcal{UCT}(\upsilon, \delta)$ *if and only if the condition* (7.2.8) *is satisfied.*

The proofs of Theorem 7.2.3 and Theorem 7.2.4 are much akin to those of Theorem 7.2.1 or Theorem 7.2.2 and so the particulars are omitted.

Theorem 7.2.3. *Let* $c < 0$ *and* $m > 0$. *Then*

$$z(2-u_p) \in \mathcal{PT}(\upsilon) \iff u_p'(1) + \upsilon u_p(1) \leq 2\upsilon \qquad (1/2 < \upsilon \leq 1). \tag{7.2.9}$$

Theorem 7.2.4. *Let* $c < 0$ *and* $m > 0$. *Then*

$$z(2-u_p) \in \mathcal{CPT}(\upsilon) \iff u_p''(1) + (2+\upsilon)u_p'(1) + \upsilon u_p(1) \leq 2\upsilon \qquad (1/2 < \upsilon \leq 1). \tag{7.2.10}$$

In the next theorems, we derive consequences of analogous types associated with a special integral operator $\mathcal{I}(c, m; z)$ as belows:

$$\mathcal{I}(c, m; z) = \int_0^z (2 - u_p(t))dt \tag{7.2.11}$$

Theorem 7.2.5. *Let* $c < 0$ *and* $m > 0$. *Then* $\mathcal{I}(c, m; z) \in \mathcal{UCT}(\upsilon, \delta)$ *if and only if the the condition* (7.2.6) *is fulfilled.*

Proof. Since

$$\mathcal{I}(c, m; z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\, n!} z^n,$$

by using (ii) in Lemma 7.2.1, it is enough to prove that

$$\sum_{n=2}^{\infty}(n(1+\upsilon)-(\upsilon+\delta))\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}\leq 1-\delta.$$

The rests of the proof of Theorem 7.2.5 is analogous to those of Theorem 7.2.1. Thus the particulars are omitted. □

Similarly, by using (ii) in Lemma 7.2.2 and Theorem 7.2.3, we derive the next theorem.

Theorem 7.2.6. *Let* $c<0$ *and* $m>0$. *Then* $\mathcal{I}(c,m;z)\in\mathcal{CPT}(\upsilon)$ *if and only if the the condition* (7.2.9) *is satisfied.*

Chapter 8

Geometric properties of normalized Lommel functions

8.1 Introduction

Let $\mathcal{S}$ denote the class of functions that are analytic and univalent in $\mathbb{E}$, with $\mathcal{S}^*(\varrho)$ and $\mathcal{C}(\varrho)$ designating the subclasses of $\mathcal{S}$ that are, respectively, starlike of order ϱ and convex of order ϱ, $0 \le \varrho < 1$. According to the definition of subordination, we present the following classes. That is a function q given by (2.1.1) is called to be in $\mathcal{S}^*[C, D]$ if

$$\frac{zq'(z)}{q(z)} \prec \frac{1+Cz}{1+Dz} \quad (z \in \mathbb{E},\ -1 \le D < C \le 1) \tag{8.1.1}$$

and in $\mathcal{C}[C, D]$ if

$$1 + \frac{zq''(z)}{q'(z)} \prec \frac{1+Cz}{1+Dz} \quad (z \in \mathbb{E},\ -1 \le D < C \le 1)\,. \tag{8.1.2}$$

The family $\mathcal{S}^*[C,D]$ was investigated in [35], [36] and [46]. We say that a function $q(z)$ given by (2.1.1) is in $\mathcal{S}^*(c,d)$ if

$$\left|\frac{zq'(z)}{q(z)} - c\right| < d \quad (z \in \mathbb{E}, c \geq d) \tag{8.1.3}$$

and in $\mathcal{C}(c,d)$ if

$$\left|\left(1 + \frac{zq''(z)}{q'(z)}\right) - c\right| < d, \quad (z \in \mathbb{E}, c \geq d). \tag{8.1.4}$$

The family $\mathcal{S}^*(c,d)$ was introduced in [113]. In addition to the condition $c \geq d$ for the families $\mathcal{S}^*(c,d)$ and $\mathcal{C}(c,d)$, at the origin we have

$$|1 - c| < d. \tag{8.1.5}$$

Observe that $(1+z)/(1-z)$ is mapped by $\mathbb{E}$ onto the right half plane so that $\mathcal{S}^*[-1,1]$ and $\mathcal{C}[-1,1]$ are the families of starlike and convex functions, respectively. Note that functions in $\mathcal{S}^*[C,D]$ and $\mathcal{S}^*(c,d)$ are starlike, that functions in $\mathcal{C}[C,D]$ and $\mathcal{C}(c,d)$ are convex, and that $q \in \mathcal{C}[C,D]$ ($q \in \mathcal{C}(c,d)$) if and only if $zq' \in \mathcal{S}^*[C,D]$ ($zq' \in \mathcal{S}^*(c,d)$).

Lemma 8.1.1. *[112] (i) If* $-1 < D < C \leq 1$, *then*

$$\mathcal{S}^*[C,D] \equiv \mathcal{S}^*\left(\frac{1-CD}{1-D^2}, \frac{C-D}{1-D^2}\right).$$

(ii) If $c \geq d$, *then*

$$\mathcal{S}^*(c,d) \equiv \mathcal{S}^*\left[\frac{d^2-c^2+c}{d}, \frac{1-c}{d}\right].$$

In this chapter, we get some geometric properties of the function $h_{\mu,v}$ which is a normalized Lommel functions of the first kind $s_{\mu,v}$.

8.2 Main results

We ponder the Lommel function of the first kind $s_{\mu,v}$ which is a special solution of the inhomogeneous Bessel differential equation ([122, 8]):

$$z^2 w''(z) + zw'(z) + (z^2 - v^2)w(z) = z^{\mu+1} \tag{8.2.1}$$

and it can be represented by a hypergeometric series

$$s_{\mu,v}(z) = \frac{z^{\mu+1}}{(\mu - v + 1)(\mu + v + 1)} {}_1F_2\left(1; \frac{\mu - v + 3}{2}, \frac{\mu + v + 3}{2}; -\frac{z^2}{4}\right),$$

where $\mu \pm v$ is not negative odd integer. Since the Lommel function $s_{\mu,v}$ does not appertain to $\mathcal{A}$, it is regarded as ordinary to find normalization of the Lommel function of the first kind

$$\begin{aligned} h_{\mu,v}(z) &= (\mu - v + 1)(\mu + v + 1) z^{\frac{1-\mu}{2}} s_{\mu,v}(\sqrt{z}) \\ &= z + \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{4}\right)^n}{(X)_n (Y)_n} z^{n+1}, \end{aligned} \tag{8.2.2}$$

where $X = \frac{\mu - v + 3}{2}, Y = \frac{\mu + v + 3}{2}$ and $(\nu)_k$ presents Pochhammer symbol defined by (2.7.2). The function $h_{\mu,v}$ is an element of the class $\mathcal{A}$, obviously.
In geometric properties of special functions, Baricz and Ponnusamy ([6, 7, 9]) derived geometric properties of generalized Bessel functions. Geometric properties of generalized Struve functions are obtained by Yağmur and Orhan [85, 126], recently. Moreover, Baricz and Szász [10] presented the starlikeness and close-to-convexity of the derivatives of a normalized form of $s_{\mu-\frac{1}{2},\frac{1}{2}}$, most recently.
Here, we present the next theorem for starlikeness and convexity for $f \in h_{\mu,v}$.

Theorem 8.2.1. *Let $\mu, v \in \mathbb{R}$ where $\mu \pm v$ are not negative odd integers,*

$$J = 4(X + 1)(Y + 1) = (\mu + 5)^2 - v^2$$

and

$$L = 4XY = (\mu+3)^2 - v^2.$$

Then, for all $z \in \mathbb{E}$ the following assertion holds true:

If $\mu > -5 + \sqrt{2+v^2}$ and

$$\frac{|D|(C-D)(J-2)(LJ-L+J)+J(1-D^2)(J-1)}{(J-2)(LJ-L-J)} < C-D, \qquad (8.2.3)$$

then $h_{\mu,v}(z) \in \mathcal{S}^*[C,D]$.

Proof. We apply the inequality

$$\left|\frac{zh'_{\mu,v}(z)}{h_{\mu,v}(z)} - \frac{1-CD}{1-D^2}\right| < \frac{C-D}{1-D^2}$$

to prove $\frac{zh'_{\mu,v}(z)}{h_{\mu,v}(z)}$ is subordinate to $\frac{1+Cz}{1+Dz}$. So, as using inequalities

$$n \leq 2^{n-1} \quad (n \in \mathbb{N}) \qquad (8.2.4)$$

and

$$\begin{aligned}(X+1)_{n-1}(Y+1)_{n-1} &\geq (X+1)^{n-1}(Y+1)^{n-1} \\ &(n \in \mathbb{N},\ X+1>0,\ Y+1>0),\end{aligned} \qquad (8.2.5)$$

we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{4^n(X)_n(Y)_n} &\leq \sum_{n=1}^{\infty}\frac{2^{n-1}}{4^n(X)_n(Y)_n} \\ &= \frac{1}{4XY}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\ &\leq \frac{1}{4XY}\sum_{n=1}^{\infty}\left[\frac{1}{2(X+1)(Y+1)}\right]^{n-1} \\ &= \frac{J}{L(J-2)} \quad (J>2)\end{aligned} \qquad (8.2.6)$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{4^n(X)_n(Y)_n} &= \frac{1}{4XY}\sum_{n=1}^{\infty}\frac{1}{4^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\ &\le \frac{1}{4XY}\sum_{n=1}^{\infty}\left[\frac{1}{4(X+1)(Y+1)}\right]^{n-1} \\ &= \frac{J}{L(J-1)} \quad (J>1).\end{aligned} \tag{8.2.7}$$

Using the inequalities (8.2.6) and (8.2.7), we obtain

$$\begin{aligned}&\left|(1-D^2)h'_{\mu,v}(z)-(1-CD)\frac{h_{\mu,v}(z)}{z}\right| \\ &= \left|D(C-D)+\sum_{n=1}^{\infty}((1-D^2)n+D(C-D))\frac{\left(-\frac{1}{4}\right)^n}{(X)_n(Y)_n}z^n\right| \\ &\le |D|(C-D)+(1-D^2)\sum_{n=1}^{\infty}\frac{n}{4^n(X)_n(Y)_n}+|D|(C-D)\sum_{n=1}^{\infty}\frac{1}{4^n(X)_n(Y)_n} \\ &\le |D|(C-D)+(1-D^2)\frac{J}{L(J-2)}+|D|(C-D)\frac{J}{L(J-1)} \\ &\le \frac{|D|(C-D)(J-2)(LJ-L+J)+J(1-D^2)(J-1)}{L(J-1)(J-2)}.\end{aligned} \tag{8.2.8}$$

Moreover, if we apply inequalities

$$|z_1-z_2| \ge ||z_1|-|z_2||$$

and (8.2.5), then we get

$$\begin{aligned}\left|\frac{h_{\mu,v}(z)}{z}\right| &= \left|1+\sum_{n=1}^{\infty}\frac{\left(-\frac{1}{4}\right)^n}{(X)_n(Y)_n}z^n\right| \\ &\geq 1-\frac{1}{4XY}\sum_{n=1}^{\infty}\frac{1}{4^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\ &\geq 1-\frac{1}{4XY}\sum_{n=1}^{\infty}\left[\frac{1}{4(X+1)(Y+1)}\right]^{n-1} \\ &= 1-\frac{1}{L}\sum_{n=1}^{\infty}\left(\frac{1}{J}\right)^{n-1} \\ &= \frac{LJ-L-J}{L(J-1)} \quad (J>1).\end{aligned} \tag{8.2.9}$$

As using the inequalities (8.2.3), (8.2.8) and (8.2.9), we derive

$$\begin{aligned}&\left|\frac{zh'_{\mu,v}(z)}{h_{\mu,v}(z)}-\frac{1-CD}{1-D^2}\right| \\ &\leq \frac{1}{(1-D^2)}\left|\frac{z}{h_{\mu,v}(z)}\right|\left|(1-D^2)h'_{\mu,v}(z)-(1-CD)\frac{h_{\mu,v}(z)}{z}\right| \\ &\leq \frac{|D|(C-D)(J-2)(LJ-L+J)+J(1-D^2)(J-1)}{(1-D^2)(J-2)(LJ-L-J)} \\ &< \frac{C-D}{1-D^2}.\end{aligned}$$

We note that the inequalities $X+1>0$, $Y+1>0$ and $J>2$ hold if and only if $\mu>-5+\sqrt{2+v^2}$. □

Theorem 8.2.2. *Let μ, $v\in\mathbb{R}$ where $\mu\pm v$ are not negative odd integers,*

$$J=4(X+1)(Y+1)=(\mu+5)^2-v^2$$

and

$$L=4XY=(\mu+3)^2-v^2.$$

Then, for all $z \in \mathbb{E}$ the next assertion holds true:

If $\mu > -5 + \sqrt{3 + v^2}$ *and*

$$\frac{(2J-3)[|D|(C-D)(J-3)(LJ-2L+2J)+2J(1-D^2)(J-2)]}{(J-2)(J-3)(2LJ-3L-4J)} < C-D, \tag{8.2.10}$$

then $h_{\mu,v}(z) \in \mathcal{C}[C,D]$.

Proof. To prove $1 + \frac{zh''_{\mu,v}(z)}{h'_{\mu,v}(z)}$ is subordinate to $\frac{1+Cz}{1+Dz}$, we apply the inequality

$$\left|\frac{h'_{\mu,v}(z) + zh''_{\mu,v}(z)}{h'_{\mu,v}(z)} - \frac{1-CD}{1-D^2}\right| < \frac{C-D}{1-D^2}.$$

So, by using the inequalities (8.2.5),

$$3^{n-1} \geq \frac{n(n+1)}{2} \quad \text{and} \quad 2^n \geq n+1 \quad (n \in \mathbb{N}),$$

we derive

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n(n+1)}{4^n (X)_n (Y)_n} &= \frac{1}{2XY} \sum_{n=1}^{\infty} \frac{\frac{n(n+1)}{2}}{3^{n-1}\left(\frac{4}{3}\right)^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\
&\leq \frac{1}{2XY} \sum_{n=1}^{\infty} \left[\frac{3}{4(X+1)(Y+1)}\right]^{n-1} \\
&= \frac{2J}{L(J-3)} \quad (J>3)
\end{aligned} \tag{8.2.11}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n+1}{4^n (X)_n (Y)_n} &\leq \sum_{n=1}^{\infty} \frac{2^n}{4^n (X)_n (Y)_n} \\
&= \frac{1}{2XY} \sum_{n=1}^{\infty} \frac{1}{2^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\
&\leq \frac{1}{2XY} \sum_{n=1}^{\infty} \left[\frac{1}{2(X+1)(Y+1)}\right]^{n-1} \\
&= \frac{2J}{L(J-2)} \quad (J>2).
\end{aligned} \tag{8.2.12}$$

Using the inequalities (8.2.11) and (8.2.12), we obtain

$$
\begin{aligned}
&|(1-D^2)(h'_{\mu,v}(z)+zh''_{\mu,v}(z))-(1-CD)h'_{\mu,v}(z)| \\
&=\left|D(C-D)+\sum_{n=1}^{\infty}((1-D^2)n+D(C-D))\frac{(n+1)\left(-\frac{1}{4}\right)^n}{(X)_n(Y)_n}z^n\right| \\
&\leq |D|(C-D)+(1-D^2)\sum_{n=1}^{\infty}\frac{n(n+1)}{4^n(X)_n(Y)_n}+|D|(C-D)\sum_{n=1}^{\infty}\frac{n+1}{4^n(X)_n(Y)_n} \\
&\leq |D|(C-D)+(1-D^2)\frac{2J}{L(J-3)}+|D|(C-D)\frac{2J}{L(J-2)} \\
&=\frac{|D|(C-D)(J-3)(LJ-2L+2J)+2J(1-D^2)(J-2)}{L(J-2)(J-3)}.
\end{aligned}
\tag{8.2.13}
$$

Further, if we apply inequalities (8.2.5) and

$$
\left(\frac{3}{2}\right)^{n-1}\geq\frac{n+1}{2}\quad (n\in\mathbb{N}),
$$

then we have

$$
\begin{aligned}
\left|h'_{\mu,v}(z)\right| &=\left|1+\sum_{n=1}^{\infty}\frac{(n+1)\left(-\frac{1}{4}\right)^n}{(X)_n(Y)_n}z^n\right| \\
&\geq 1-\sum_{n=1}^{\infty}\frac{n+1}{4^n(X)_n(Y)_n} \\
&=1-\frac{1}{2XY}\sum_{n=1}^{\infty}\frac{\frac{n+1}{2}}{4^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\
&=1-\frac{1}{2XY}\sum_{n=1}^{\infty}\frac{\frac{n+1}{2}}{\left(\frac{3}{2}\right)^{n-1}\left(\frac{8}{3}\right)^{n-1}(X+1)_{n-1}(Y+1)_{n-1}} \\
&\geq 1-\frac{1}{2XY}\sum_{n=1}^{\infty}\left[\frac{3}{8(X+1)(Y+1)}\right]^{n-1} \\
&=1-\frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{3}{2J}\right)^{n-1} \\
&=\frac{2JL-4J-3L}{L(2J-3)}\quad (J>3/2).
\end{aligned}
\tag{8.2.14}
$$

By applying the inequalities (8.2.10), (8.2.13) and (8.2.14), we elicit that

$$\begin{aligned}
&\left|\frac{h'_{\mu,v}(z)+zh''_{\mu,v}(z)}{h'_{\mu,v}(z)}-\frac{1-CD}{1-D^2}\right| \\
&\leq \frac{1}{(1-D^2)|h'_{\mu,v}(z)|}|(1-D^2)(h'_{\mu,v}(z)+zh''_{\mu,v}(z))-(1-CD)h'_{\mu,v}(z)| \\
&\leq \frac{(2J-3)[|D|(C-D)(J-3)(LJ-2L+2J)+2J(1-D^2)(J-2)]}{(1-D^2)(J-2)(J-3)(2JL-4J-3L)} \\
&< \frac{C-D}{1-D^2}.
\end{aligned}$$

We note that the inequalities $X+1>0$, $Y+1>0$ and $J>3$ hold if and only if $\mu > -5+\sqrt{3+v^2}$. □

By taking $C=1, D\to -1^+$ in Theorem 8.2.1 and Theorem 8.2.2, we get the next corollaries, respectively.

Corollary 8.2.1. *Let $\mu, v \in \mathbb{R}$ where $\mu \pm v$ are not negative odd integers,*

$$J = 4(X+1)(Y+1) = (\mu+5)^2 - v^2$$

and

$$L = 4XY = (\mu+3)^2 - v^2.$$

Then, for all $z \in \mathbb{E}$ the following assertions holds true:

If $\mu > -5+\sqrt{2+v^2}$ and $\frac{LJ-L+J}{LJ-L-J} < 1$, then $h_{\mu,v}(z) \in \mathcal{S}^*[-1,1]$.

Corollary 8.2.2. *Let $\mu, v \in \mathbb{R}$ where $\mu \pm v$ are not negative odd integers,*

$$J = 4(X+1)(Y+1) = (\mu+5)^2 - v^2$$

and

$$L = 4XY = (\mu+3)^2 - v^2.$$

Then for all $z \in \mathbb{E}$ *the following assertions holds true:*

If $\mu > -5+\sqrt{3+v^2}$ *and* $\frac{(2J-3)[(J-3)(LJ-2L+2J)+2J(J-2)]}{(J-2)(J-3)(2LJ-3L-4J)} < 1$, *then* $h_{\mu,v}(z) \in \mathcal{C}[-1,1]$.

References

[1] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math. Ser. **17** (1915), 12-22.

[2] Á. Baricz, *Bessel transforms and Hardy space of generalized Bessel functions*, Mathematica **48** (71) (2006), 127-136.

[3] Á. Baricz, *Generalized Bessel functions of the first kind*, PhD Thesis, Babes-Bolyai University, Cluj-Napoca, 2008.

[4] Á. Baricz, *Generalized Bessel functions of the first kind*, Lecture Notes in Math., Vol. 1994, Springer-Verlag, 2010.

[5] Á. Baricz, *Geometric properties of generalized Bessel functions of complex order*, Mathematica(Cluj) **48** (2006), 13-18.

[6] Á. Baricz, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73** (2008), 155-178.

[7] Á. Baricz, *Generalized Bessel functions of the first kind, Lecture Notes in Mathematics*, vol. 1994, Springer-Verlag, Berlin, 2010.

[8] Á. Baricz and S. Koumandos, *Turan type inequalities for some lommel functions of the first kind*, Proc. Edinburgh Math. Soc. **59** (2016), 569-579

[9] Á. Baricz and S. Ponnusamy, *Starlikeness and convexity of generalized Bessel functions*, Integral Transforms Spec. Funct. **21** (2010), 641-653.

[10] Á. Baricz and R. Szász, *Close-to-convexity of some special functions and their derivatives*, Bull. Malays. Math. Sci. Soc. **39** (2016) 427-437

[11] R. Bharati, R. Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math. **28** (1997), 17-32.

[12] L. Bieberbach, *Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*, S.-B. Preuss. Akad. Wiss. 1916, 940-955.

[13] D. Breaz and N. Breaz, *Two intergral operators*, Stud. Univ. Babes-Bolyai Math. **47** (2002), 13-21.

[14] D. Breaz and S. Owa, N. Breaz, *A new integral univlent operator*, Acta Univ. Apulensis Math. Inform. **16** (2008), 11-16.

[15] T. R. Caplinger and W. M. Causey, *A class of univalent functions*, Proc. Am. Math Soc. **39** (1973), 357-361.

[16] N. E. Cho, *The Noor integral operator and strongly close-to-convex functions*, J. Math. Anal. Appl. **283** (2003), 202-212.

[17] N. E. Cho, and I. H. Kim, *Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function*, Appl. Math. Comput. **187** (2007), 115-121.

[18] N. E. Cho and J. A Kim, *On a sufficient condition and an angular estimation for ϕ-like functions*, Taiwanese J. Math. **2** (1998), 397-403.

[19] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl. **292** (2004), 470-483.

[20] N. E. Cho, H. J. Lee and R. Srivastava, *Characterizations for Certain Subclasses of Starlike and Convex Functions Associated with Bessel functions*, Filomat **30** (2016), 1911-1917

[21] N. E. Cho, H. J. Lee, J. H. Park and R. Srivastava, *Some Applications of the First-Order Differential Subordinations*, Filomat **30** (2016), 1465-1474.

[22] N. E. Cho, G. Murugusundaramoorthy and T.Janani, *Inclusion properties for certain subclasses of analytic functions associated with Bessel functions.*, J. Comput. Anal. Appl. **20** (2016), 81-90.

[23] J. S. Choi and P. Agarwal, *Certain unified integrals involving a product of Bessel functions of the first kind*, Honam Math. J. **35** (2013), 667-677.

[24] L. de-Branges, *A proof of the Bieberbach conjecture*, Acta. Math. **154** (1985), 137-152.

[25] E. Deniz, *Convexity of integral operators involving generalized Bessel functions*, Integral Transforms Spec. Funct. **24** (2013), 201-216.

[26] E. Deniz, *Certain subclasses of bi-univalent functions satisfying subordinate conditions*, Jour. Class. Anal. **2** (2013), 49-60.

[27] E. Deniz, Halit Orhan and H. M. Srivastava, *Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions*, Taiwanese J. Math. **15** (2011), 883-917.

[28] M. Dorf and J. Szynal, *Linear invariance and integral operators of univalent functions*, Demonstratio Math. **38** (2005), 47-57.

[29] K. K. Dixit and S. K. Pal, *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math. **26** (1995), 889-896.

[30] P. L. Duren,. *Univalent functions*, Grundlehren der Math. Wissenchanften, Springer-Verlag, New York-Berlin, 1983.

[31] Dashrath, *On some classes related to spiral-like univalent and multivalent functions*, Ph.D. Thesis. Kanpur University. Kanpur, 1984.

[32] P. L. Duren, *Univalent functions*, Springer. New York, 1983.

[33] B. A. Frasin, *Order of convexity and univalency of general integral operator*, J. Franklin Inst. **348** (2011), 1013-1019.

[34] P. R. Garabedian and M. A. Schiffer, *A proof of the Bieberbach conjecture for the fourth coefficient*, J. Rational Mech. Anal. **4** (1955), 427-465.

[35] R. M. Goel and B. S. Mehrok, *On the coefficient of a subclass of starlike functions*, Indian J. Pure Applied Math. **12** (1981), 634-647.

[36] R. M. Goel and B. S. Mehrok, *Some invariance properties of a subclass of close-to-convex fucntios*, Indian J. Pure Applied Math. **12** (1981), 1240-1249.

[37] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), 87-92.

[38] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155** (1991), 364-370.

[39] A. W. Goodman, *Univalent Functions II*, Mariner Pub. Co., Tampa, FL, 1983.

[40] A. W. Goodman, *Univalent Functions*, Vol. I and II, polygonal publishing house, Washington, New Jersey, 1983.

[41] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), 87-92.

[42] A. W. Goodman, *On uniformly starlike functions*, J. Math. Anal. Appl. **155** (1991), 364-370.

[43] G. Herglotz, *Uber potenzreiher mit positirem, reelen Teil Im einheitskreis*, S. B. Sachs, Akad. Wiss. Leipzig Math. Natur. K I. **63** (1911), 501-511.

[44] I. S. Jack, *functions starlike and convex of order* α, J. London. Math. Soc. **3** (1971), 469-474.

[45] W. Janowski, *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math. **28** (1973), 297-326.

[46] W. Janowski, *Some extremal problems fot certain families of analytic functions*, Bull. de L'Acad. Pol. des Sci. **21** (1973), 17-25.

[47] S. Kanas and W. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math. **105** (1999), 327-336.

[48] S. Kanas and W. Wisniowska, *Conic domains and starlike functions*, Rev. Roum. Math. Pures Appl. **45** (2000), 647-657.

[49] N. Khan, *On certain classes of analytic functions*, PhD Thesis, Comsats Institute of Information Technology, 2016.

[50] J. A. Kim and N. E. Cho, *Properties of convolutions for hypergeometric series with univalent functions*, Adv. Difference Equ. 2013, **2013:101**, 11 pp.

[51] H. J. Lee, N. E. Cho and S. Owa, *Properties for certain subclasses of analytic functions with nonzero coefficients*, J. Inequal. Appl. 2014, **2014:506**, 7 pp.

[52] S. K. Lee and M. Nunokawa, *On angular estimate of $f(z)/z$ certain analytic functions*, Math. Japon. **45** (1997), 47-49.

[53] J. E. Littlewood, *Lectures on the theory of functions*, Oxford University Press, Oxford and London, 1944.

[54] E. Lindelöf, *Mémoire sur certaines inégalitis dans la théorie des functions monogénses et sur quelques propriétés nouvelles de ces fonctions dans levoisinage d'un point singulier essentiel*, Ann. Soc. Sci. Fenn. **35** (1909), 1-35.

[55] J. L. Liu and J. Patel, *Certain properties of multivalent functions associated with an extended fractional differintegral operator*, Appl. Math. Comput. **203** (2004), 703-713.

[56] J. L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. **259** (2001), 566-581.

[57] J. L. Liu and K.I. Noor, *On subordinations for certain analytic functions associated with Noor integral operator*, Appl. Math. Comput. **187** (2007), 1453-1460.

[58] K. Löwner, *Üntersuchungen über die verzerrung bei konformen Abbildun gen des Eingeitschkreises* $|z| < 1$ *die durch functionen mit nicht verschwindender Ableingtung geliefert warden*, Ber. Verh. Sächs. Ges. Wiss. Leipzig. **69** (1917), 89-106.

[59] K. Löwner, *Üntersuchungen über schlichte konforme Abbildungen des Einheitskreises. I*, Math. Ann. **89** (1923), 103-121.

[60] W. Ma, and D. Minda, *A unified treatment of some special classes of univalent functions*, Proceedings of the conference on complex analysis (Tianjin), Conf. Proc. Lecture Notes Anal, International Press, Massachusetts. 157-169, 1994.

[61] W. Ma and D. Minda, *Uniformly convex funtions*, Ann. Polon. Math. **57** (1992), 166-175.

[62] A. Marx, *Untersuchungen über schlichte Abbildungen*, Math. Ann. **107** (1932/33), 40-65.

[63] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan. Math. J. **28** (1981), 157-171.

[64] S.S. Miller and P.T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** (1978), 289-305.

[65] S. S. Miller, P.T. Mocanu and M.O. Reade, *Starlike integral operators*, Pacific J. Math. **79** (1978), 157-168.

[66] S. S. Miller, P. T. Mocanu and M. O. Reade, *All alpha-convex functions are starlike*, Rev. Roumaine Math. Pures Appl. **17** (1972), 1395-1397.

[67] S. S. Miller and P. T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J. **32** (1985), 185-195.

[68] S. S. Miller and P. T. Mocanu, *Differential Subordinations*: *Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, **225**, Marcel Dekker Incorporated, New York and Basel, 2000.

[69] P. T. Mocanu, *Une propriete de convexite generlise dans la theorie de la representation conforme*, Mathematica (Cluj), **11** (1969), 127-133.

[70] P. T. Mocanu, T. Bulboaca and G. S. Salagean, *Teoria geometrica a functiilor univalente*, Casa Cartii de Stiinta, Cluj-Napoca, Romania, 2006.

[71] P. T. Mocanu, *Alpha-convex integral operators and strongly starlike functions,* Studia Univ. Babeş-Bolyai Math. **34** (1989), 18-24.

[72] S. R. Mondal and A. Swaminathan, *Geometric properties of generalized Bessel functions*, Bull. Malaysian Math. Sci. Soc. **35** (2012), 179-194.

[73] R. Nevanlinna, *Über die konforme abbildung sterngebieten*, oversikt av finska-vetenskaps societen forhandlingar, **63** (1920-21), 1-21.

[74] K. I. Noor, *Some classes of alpha-quasi-convex functions*, Int. J. Math. & Math. Sci. **11** (1988), 497-502.

[75] K. I. Noor, *On some classes of analytic functions involving Ruscheweyh derivatives*, J. Nat. Sci. & Math. **29** (1989), 47-60.

[76] K.I. Noor, *On some applications of the Ruscheweyh derivatives*, Math. Japonica. **36** (1991), 869-874.

[77] K.I. Noor, *On analytic functions defined by Ruscheweyh derivative*, J. Nat. Geometry. **10** (1996), 101-110.

[78] K. I. Noor, *Radius problems for subclass of close-to-convex univalent functions*, Int. J. Math. & Math. Sci. **15** (1992), 719-726.

[79] K. I. Noor and N. Khan, *Some classes of p-valent analytic functions associated with hypergeometric functions*, Filomat. **29** (2015), 1031-1038.

[80] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci. **69** (1993), 234-237.

[81] M. Nunokawa and S. Owa, *On certain differential subordinations*, Pan Amer. Math. J. **3** (1993), 35-38.

[82] M. Nunokawa, S. Owa, H. Saitoh and S. Fukui, *Arguments of certain analytic functions*, Chinese J. Math. **23** (1995), 41-48.

[83] M. Nunokawa and D. K. Thomas, *On convex and starlike functions in a sector*, J. Austral. Math. Soc. Ser. A **60** (1996), 363-368.

[84] S. Owa and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Canad. J. Math. **39** (1987), 1057-1077.

[85] H. Orhan and N. Yağmur, *Geometric properties of generalized struve functions*, An. Stiint. Univ. AI. I. Cuza Iasi. Mat. (N. S.) (2014) 1221-8421.

[86] M. Ozawa, *An elementary proof of the Bieberbach conjecture for the sixth coefficient*, Kodai Math. Sem. Rep. **21** (1969), 129-132.

[87] V. Paatero, *Uber die konforme abbildung von gebieten, deren Rander von beschrankter Drehung sind*, Ann. Acad. Sci. Fenn. Ser. A. **33** (1931), 77 pp.

[88] V. Paatero, *Uber Gebiete von beschrankter randdrehung*, Ann. Acad. Sci. Fenn. Ser. A. **37** (1933), 20 pp.

[89] K. S. Padmanabhan, *On a certain class of functions whose derivatives have a positive real part in the unit disc*, Ann. Pol. Math. **23** (1970/71), 73-81.

[90] N. N. Pascu, *An improvement of Becker's univalence criterion*, in: Proceedings of the Commemorative Session Simion Stolilow, University of Barsov, 1987, 43-48.

[91] N. Pascu and V. Pescar, *On the integral operators of Kim-Merkis and Pfaltzgraff*, Mathematica **32** (1990), 185-192.

[92] V. Pescar, *A new generalization of Ahlfor's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. **19** (1996), 53-54.

[93] R. Pederson and M. Schiffer, *A proof of the Bieberbach conjecture for the fifth coefficient*, Arch. Rational Mech. Anal. **45** (1972), 161-193.

[94] R. N. Pederson, *A proof of the Bieberbach conjecture for the sixth coefficient*, Arch. Rational Mech. Anal. **31** (1968/1969), 331-351.

[95] J. A. Pfaltzgraff, *Univalence of the integral of* $f'(z)^\lambda$, Bull. London Math. Soc. **7** (1975), 254-256.

[96] B. Pinchuk, *Functions with bounded boundary rotation*, Israel. J. Math. **10** (1971), 6-16.

[97] Y. Polatoglu, *Some results of analytic functions in the unit disk*, Publication de L'Institute Mathématique, Nouvelle série, tome. **78** (2005), 79-86.

[98] Y. Polatoglu, M. Bolcal, A. Sen and E. Yavuz, *A study on the generalization of Janowski function in the unit disk*, Acta Math. Acad. Paed. Nyíregyháziensis. **22** (2006), 27-31.

[99] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen. 1975.

[100] S. Ponnusamy and F. Rønning, *Starlikeness properties for convolutions involving hypergeometric series*, Ann. Univ. Mariae Curie-Skłodowska. Sect. A **52** (1998), 141-155.

[101] S. Ponnusamy and M. Vuorinen, *Univalence and convexity properties for Gaussian hypergeometric functions*, preprint 82, Department of Mathmatics, University of Helsinki (1995), 34 pp.

[102] G. Riemann, *Für eine allgemeine theorie der funktionen einer veränderlichen comlexen grosse*, inaugural dissertation, gottingen gesammetle mathematische werke 3rd ed., Springer-Verlg, Berlin. 3-48, 1851.

[103] M. S. Robertson, *On the theory of univalent functions*, Ann. Math. **37** (1936), 374-408. 1374-408.

[104] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. Lond. Math. Soc. **48** (1943), 48-82.

[105] F. Rønning, *On starlike functions associated with parabolic regions*, Ann Univ. Marie-Curie Skłodowska Sect. A **45** (1991), 117-122.

[106] F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 190-196.

[107] St. Ruscheweyh and V. Singh, *On the order of starlikeness of hypergeometric fucntions*, J. Math. Anal. Appl. **113** (1986), 1-11.

[108] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109-115.

[109] H. Shiraishi, S. Owa, T. Hayami, K. Kuroki and H. M. Srivastava, *Starlike problems for certain analytic functions concerned with subordinations*, Pan Amer. Math. J. **21** (2011), 63-77.

[110] H. Silverman, *Convex and starlike criteria*, Int. J. Math. Sci. **22** (1999), 75-79.

[111] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109-116.

[112] H. Silverman and E. M. Silvia, *Subclasses of starlike functions subordinate to convex functions*, Can. J. Math. **37** (1985), 48-61.

[113] H. Silverman, *Subclasses of starlike functions*, Rev. Roum. Math. Pures et Appl. **23** (1978), 1093-1099.

[114] H. Silverman, *Complex Variables*, Houghton Mifflin Co., Boston, Mass.-London, 1975. xi+415 pp. 30-01

[115] H. M. Srivastava and S. S. Eker, *Some applications of a subordination theorem for a class of analytic functions*, Appl. Math. Lett. **21** (2008), 394-399.

[116] H. M. Srivastava, D.-G. Yang and N-E. Xu, *Subordinations for multivalent analytic functions associated with the Dziok-Srivastava operator*, Integral Transforms Spec. Funct. **20** (2009), 581-606.

[117] E. Strohhacker, *Beitrage zür Theorie der schlichten Funktionen*, Math. Z. **37** (1933), 356-380.

[118] E. Study, *Konforme abbildung einfachzusammenhangender Bereiche*, B.C.Teubner, Leipzig und Berlin, 1913.

[119] A. Swaminathan, *Certain sufficiency condotions on Gaussian hypergeometric functions*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), Article 83, 10 pp.

[120] O. Tammi, *On the maximization of the coefficients of schlicht and related functions*, Ann, Acad. Sci. Fenn. Ser. AI, Math. Phys. **114** (1952) , 51 pp.

[121] N. Tuneski, *On the quotient of the representations of convexity and starlikeness*, Math. Nachr. **248/249** (2003), 200-203.

[122] G. N. Watson, *A Treatise on the Theory of Bessel Functions, Second edition*, Cambridge University Press, Cambridge, London and New York, 1944.

[123] E. T. Whitteaker and G. N. Watson, *A course of modern analysis: An introduction to the general theory of infinite processes and analytic functions; With an account of the principal transcendental functions*, Fourth ed., Cambridge University Press, Cambridge, London and New York, 1927.

[124] Q.-H. Xu, C.-B. Lv, N.-C. Luo and H. M. Srivastava, *Sharp coefficient estimates for a certain general class of spirallike functions by means of differential subordination*, Filomat **27** (2013), 1351-1356.

[125] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, *Some applications of differential subordination and the Dziok-Srivastava convolution operator*, Appl. Math. Comput. **230** (2014), 496-508.

[126] N. Yağmur and H. Orhan, *Starlikeness and convexity of generalized Struve functions*, Abstr. Appl. Anal. 2013, Art. ID 954513, 6 pp.

[127] N. Yağmur, *Hardy space of lommel functions*, Bull. Korean Math. Soc. **52** (2015), 1035-1046.