1 Introduction

In this paper, we deal with the approximate controllability for the following nonlinear functional differential equation with time delays in a Hilbert space H:

$$\begin{cases} x^{'}(t) + Ax(t) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds) + (Bu)(t), \quad 0 < t \leq T, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h, 0). \end{cases} \tag{1.1}$$

Here, A_0 is the operator associated with a sesquilinear form defined on $V \times V$ and satisfying Gårding's inequality, where V is another Hilbert space such that $V \subset H \subset V^*$. The nonlinear term $G(t,x)$, which is a Lipschitz continuous operator from $L^2(-h,T;V)$ to $L^2(0,T;H)$, is a semilinear version of the quasilinear one considered in Yong and Pan [1]. The controller B is a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$, where U is some Banach space. Precise assumptions are given in the next section.

It is well known that the future state realistic models in the natural sciences, biology economics and engineering depends not only on the present but also on the past state. Such applications are used to study the stability, controllability and the time optimal control problems of hereditary systems. The regular problems the semilinear functional differential equations with unbounded delays has been surveyed in Vrabie [2] and Jeong et al. [3]. The approximate controllability for semilinear differential systems has been also studied in [3-8]. As for the regularity results for a class of nonlinear evolution equations with the nonlinear operator A were developed in many references [9-12]. Ahmed and Xiang [12] gave some existence results for the initial value problem in case where the nonlinear term is not monotone, which improved Hirano's result [13]. Recently, the controllability of neutral evolution integrodifferential systems with state dependent delay has been studied by [14, 15] , impulsive neutral functional evolution integrodifferential systems with infinite delay in [16], and the approximate controllability of stochastic equations by authors [17, 18, 19].

We will first establish a variation of constant formula for solutions of the given equation with a general condition of the local Lipschitz continuity of the nonlinear operator , which is reasonable and widely used in case of the nonlinear system. The main research direction is to find conditions on the nonlinear term such that the regularity result of (1.1) is preserved under perturbation. In order to prove the solvability of the initial value problem (1.1) in Section 3, we establish necessary estimates applying the result of Di Blasio et al. [20] to (1.1) considered as an equation in H as well as in V^* in Section 2.

Moreover in Section 4, we establish the approximate controllability of control system (1.1) with condition on a range condition of the controller and an inequality condition on the system parameters as in Naito [21]. In this paper, we no longer require the compact property of structural operators, and the uniform boundedness and the uniform continuity of nonlinear terms, but instead we need the regularity and a variation of solutions of the given equations with local Lipschitz continuity of nonlinear terms by using a successive approach method.

2 Preliminaries and Assumptions

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l,v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\mathrm{Re}\ a(u,u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u,\ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. From the following inequalities

$$\omega_1 ||u||^2 \leq \mathrm{Re}\, a(u,u) + \omega_2 |u|^2 \leq C|Au|\,|u| + \omega_2|u|^2 \leq \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [22]).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X,Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X,X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0,T;D(A)) \cap W^{1,2}(0,T;H),$$
$$\mathcal{W}_1(T) = L^2(0,T;V) \cap W^{1,2}(0,T;V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0,T];V), \quad \mathcal{W}_1(T) \subset C([0,T];H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \leq M_0 ||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \leq M_0 ||x||_{\mathcal{W}_1(T)}. \tag{2.5}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \leq M, \quad ||S(t)||_* \leq M.$$

The following Lemma is from Lemma 3.6.2 of [23].

Lemma 2.2. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:*

$$|S(t)x| \leq Mt^{-1/2}||x||_*, \quad ||S(t)x|| \leq Mt^{-1/2}|x|.$$

First of all, consider the following linear system

$$\begin{cases} x^{'}(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.6}$$

By virtue of Theorem 3.3 of [20](or Theorem 3.1 of [?], [23]), we have the following result on the corresponding linear equation of (2.6).

Lemma 2.3. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.6) belonging to $\mathcal{W}(T) \subset C([0,T];V)$ and satisfying

$$||x||_{\mathcal{W}(T)} \leq C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.7}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (2.6) belonging to $\mathcal{W}_1(T) \subset C([0,T];H)$ and satisfying

$$||x||_{\mathcal{W}_1(T)} \leq C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.

Lemma 2.4. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \leq t \leq T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;D(A))} \leq C_1||k||_{L^2(0,T;H)}, \tag{2.9}$$
$$||x||_{L^2(0,T;H)} \leq C_2T||k||_{L^2(0,T;H)}, \tag{2.10}$$

and

$$||x||_{L^2(0,T;V)} \leq C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.11}$$

Proof. The assertion (2.9) is immediately obtained by (2.7). Since

$$\begin{aligned}||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \le M\int_0^T (\int_0^t |k(s)|ds)^2 dt\\ &\le M\int_0^T t\int_0^t |k(s)|^2 ds dt \le M\frac{T^2}{2}\int_0^T |k(s)|^2 ds\end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.3), (2.9), and (2.10) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

3 Semilinear differential equations

In this Section, we consider the maximal regularity of the following nonlinear functional differential equation

$$\begin{cases} x'(t) + Ax(t) = \int_{-h}^0 g(t,s,x(t),x(t+s))\mu(ds) + k(t), & 0 < t \le T,\\ x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h, 0), \end{cases} \tag{3.1}$$

where A is the operator mentioned in Section 2. We need to impose the following conditions.

Assumption (F). Let $\mathcal{L}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0,\infty)$ and the Borel σ-field on $[-h, 0]$, respectively. Let μ be a Borel measure on $[-h, 0]$ and $g : [0,\infty) \times [-h, 0] \times V \times V \to H$ be a nonlinear mapping satisfying the following:

(i) For any $x, y \in V$ the mapping $g(\cdot,\cdot,x,y)$ is strongly $\mathcal{L}\times\mathcal{B}$-measurable.

(ii) $g(t,s,x,y)$ is locally Lipschitz continuous in x and y, uniformly in $(t,s) \in [0,\infty)\times[-h,0]$, i.e., there exist positive constants $L_0, L_1(r)$ and L_2 such that

$$|g(t,s,x,y) - g(t,s,\hat{x},\hat{y})| \le L_1(r)|x-\hat{x}| + L_2||y-\hat{y}||,$$

for all $(t,s) \in [0,\infty)\times[-h,0]$, $y,\ \hat{y} \in V$, $|x| \le r$ and $|\hat{x}| \le r$.

(iii) There exists a real number L_0 such that

$$|g(t,s,x,y)| \le L_0(1+|x|+|y|), \quad |g(t,s,0,0)| \le L_0,$$

for any $(t,s) \in [0,\infty) \times [-h,0]$, $x \in H$, and $y \in V$.

Remark 3.1. *The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [1].*

For $x \in L^2(-h,T;V),\ T>0$ we set

$$G(t,x) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds). \tag{3.2}$$

Here, as in [1] we consider the Borel measurable corrections of $x(\cdot)$.

Let U be a Banach space and the controller operator B be a bounded linear operator from the Banach space $L^2(0,T;U)$ to $L^2(0,T;H)$.

Lemma 3.1. *Let $x \in L^2(-h,T;V), T>0$ and $||x||_{C([0,T],H)} \le r$. Then the nonlinear term $G(\cdot,x)$ defined by (3.2) belongs to $L^2(0,T;H)$ and*

$$\|G(\cdot,x)\|_{L^2(0,T;H)} \le \mu([-h,0])\big\{L_0\sqrt{T} + (L_1(r)+L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \tag{3.3}$$

Moreover, if $x_1, x_2 \in L^2(-h,T;V)$, then

$$\begin{aligned}\|G(\cdot,x_1)-G(\cdot,x_2)\|_{L^2(0,T;H)} &\le \mu([-h,0]) \\ &\times \big\{(L_1(r)+L_2)\|x_1-x_2\|_{L^2(0,T;V)} + L_2\|x_1-x_2\|_{L^2(-h,0;V)}\big\}\end{aligned} \tag{3.4}$$

Proof. From (ii) of Assumption (F), it is easily seen that

$$\begin{aligned}\|G(\cdot,x)\|_{L^2(0,T;H)} &\le \mu([-h,0])\big\{L_0\sqrt{T} + L_1(r)\|x\|_{L^2(0,T,V)} + \|x\|_{L^2(-h,T,V)}\big\} \\ &\le \mu([-h,0])\big\{L_0\sqrt{T} + (L_1(r)+L_2)\|x\|_{L^2(0,T,V)} + L_2\|x\|_{L^2(-h,0;v)}\big\}.\end{aligned}$$

The proof of (3.4) is similar. □

From now on, we establish the following results on the local solvability of (3.1) represented by

$$\begin{cases} x'(t) + Ax(t) = G(t,x) + k(t), & t \in (0,T] \\ x(0) = g^0, x(s) = g^1(s), & s \in [-h,0]. \end{cases}$$

Theorem 3.1. *Let Assumption (F) be satisfied. Assume that $(g^0, g^1) \in H \times L^2(-h, 0; V)$, $k \in L^2(0, T; V^*)$. Then, there exists a time $T_0 \in (0, T)$ such that the equation (3.1) admits a solution*

$$x \in L^2(-h, T_0; V) \cap W^{1,2}(0, T_0; V^*) \subset C([0, T_0]; H). \tag{3.5}$$

Proof. For a solution of (3.1) in the wider sense, we are going to find a solution of the following integral equation

$$x(t) = S(t)g^0 + \int_0^t S(t-s)\{G(s, x) + k(s)\}ds. \tag{3.6}$$

To prove a local solution, we will use the successive iteration method. First, put

$$x_0(t) = S(t)g^0 + \int_0^t S(t-s)k(s)ds$$

and define $x_{j+1}(t)$ as

$$x_{j+1}(t) = x_0(t) + \int_0^t S(t-s)G(\cdot, x_j)ds. \tag{3.7}$$

By virtue of Lemma 2.3, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0||_{\mathcal{W}_1(t)} \leq C_1(|x_0| + ||k||_{L^2(0,t;V^*)}),$$

where C_1 is a constant in Lemma 2.3. Choose $r > C_1 M_0^{-1}(|x_0| + ||k||_{L^2(0,t;V^*)})$, where M_0 is the constant of (2.5). Putting $p(t) = \int_0^t S(t-s)G(\cdot, x_0)ds$, by (2.11) of Lemma 2.4, we have

$$\begin{aligned}
&||p||_{L^2(0,t;V)} \leq C_2\sqrt{t}||G(\cdot, x_0)||_{L^2(0,t;H)} \\
&\leq C_2\sqrt{t}\big\{\mu([-h, 0])L_0\sqrt{t} + (L_1(r) + L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\
&= C_2\mu([-h, 0])L_0 t + C_2\mu([-h, 0])\big[(L_1(r) + L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big]\sqrt{t}.
\end{aligned} \tag{3.8}$$

So that, from(3.5) and (3.6),

$$\begin{aligned}
&||x_1||_{L^2(0,t;V)} \\
&\leq r + C_2\mu([-h, 0])t + C_2\mu([-h, 0])\{(L_1(r) + L_2)\|x\|_{L^2(0,T;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\}\sqrt{t} \\
&\leq 3r
\end{aligned}$$

for any

$$m = \min\{r(C_2\mu([-h,0]))^{-1},\\ r\{(C_2\mu([-h,0]))\big((L_1(r)+L_2)\|x\|_{L^2(0,T;v)} + \|g^1\|_{L^2(-h,0;V)}\big)\}^{-2}\},$$

$0 \le t \le m$. By induction, it can be shown that for all $j = 1, 2, ...$

$$||x_j||_{L^2(0,t;V)} \le 3r, \quad 0 \le t \le m. \tag{3.9}$$

Hence, from the equation

$$x_{j+1}(t) - x_j(t) = \int_0^t S(t-s)\{G(t,x_j) - G(t,x_{j-1})\}ds$$

From (2.11), (3.7) and Assumption (F), we can observe that the inequality

$$\begin{aligned} ||x_{j+1} - x_j||_{L^2(0,t;V)} &\le C_2\sqrt{t}||G(\cdot,x_j) - G(\cdot,x_{j-1})||_{L^2(0,t;H)} \\ &\le \frac{\{C_2\mu([-h,0])(L_1(3r)+L_2)\sqrt{t}\}^j}{j!}||x_1 - x_0||_{L^2(0,t;V)} \end{aligned}$$

holds for any $0 \le t \le m$. Choose $T_0 > 0$ satisfying

$$T_0 < \min\{m\,,\, \{C_2\mu([-h,0])(L_1(3r)+L_2)\}^{-2}\}. \tag{3.10}$$

Then $\{x_j\}$ is strongly convergent to a function x in $L^2(0,T_0;V)$ uniformly on $0 \le t \le T_0$. By letting $j \to \infty$ in (3.7), we obtain (3.6). Next, we prove the uniqueness of the solution. Let $\epsilon > 0$ be given. For $\epsilon \le t \le T_0$, set

$$x^\epsilon(t) = S(t)g^0 + \int_0^{t-\epsilon} S(t-s)\{G(s,x^\epsilon) + k(s)\}ds. \tag{3.11}$$

Then we have $x^\epsilon \in \mathcal{W}_1(T_0)$ and for $x^\epsilon,\ y^\epsilon \in B_r(T_0)$ which is a ball with radius r in $L^2(0,T_0;V)$, since

$$\begin{aligned} x(t) - x^\epsilon(t) &= \int_0^t S(t-s)\{G(s,x) - G(s,x^\epsilon)\}ds \\ &\quad + \int_{t-\epsilon}^t S(t-s)\{G(s,x^\epsilon) + k(s)\}ds, \end{aligned}$$

with aid of Lemma 2.4,

$$\begin{aligned} &||x - x^\epsilon||_{L^2(0,T_0;V)} \le C_2\mu([-h,0])(L_1(r)+L_2)\sqrt{T_0}||x - x^\epsilon||_{L^2(0,T_0;V)} \\ &+ C_2\sqrt{\epsilon}\mu([-h,0])\{(L_0\sqrt{T}_0 + (L_1+L_2)||x||_{L^2(0,T_0;V)} + \sqrt{T}_0||k||_{L^2(0,T_0;H)}\}. \end{aligned}$$

we have $x^\epsilon \to x$ as $\epsilon \to 0$ in $L^2(0,T_0;V)$. Suppose y is another solution of (3.1) and y_ϵ is defined as (3.11) with the initial data (g^0,g^1). Let $x^\epsilon,\ y^\epsilon \in B_r$. Then From Lemma 2.2, it follows that

$$
\begin{aligned}
||x^\epsilon - y^\epsilon||_{L^2(0,T_0;V)} &\le \Big[\int_0^{T_0} || \int_0^{s-\epsilon} S(s-\tau)\{(G(\cdot,x^\epsilon)-G(\cdot,y^\epsilon))\}d\tau||^2 ds\Big]^{1/2}\\
&\le M\Big[\int_0^{T_0}\Big(\int_0^{s-\epsilon}(s-\tau)^{-1/2}|G(\cdot,x^\epsilon)-G(\cdot,y^\epsilon)|d\tau\Big)^2 ds\Big]^{1/2}\\
&\le M\mu([-h,0])L_1(r)\Big[\int_0^{T_0}\int_0^{s-\epsilon}(s-\tau)^{-1}d\tau\int_0^{s-\epsilon}||x^\epsilon(\tau)-y^\epsilon(\tau)||^2 d\tau ds\Big]^{1/2}\\
&\le M\mu([-h,0])L_1(r)\log\frac{T_0}{\epsilon}\int_0^{T_0}||x^\epsilon - y^\epsilon||_{L^2(0,s;V)}ds,
\end{aligned}
$$

so that by using Gronwall's inequality, independently of ϵ, we get $x^\epsilon = y^\epsilon$ in $L^2(0,T_0;V)$, which proves the uniqueness of solution of (3.1) in $\mathcal{W}_1(T_0)$. □

From now on, we give a norm estimation of the solution of (3.3) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.2. *Under the Assumption (F) for the nonlinear mapping G, there exists a unique solution x of (3.1) such that*

$$x \in \mathcal{W}_1(T) \subset C([0,T];H). \tag{3.12}$$

for any $(g^0,g^1) \in H \times L^2(0,T;V)$, $k \in L^2(0,T;V^*)$. Moreover, there exists a constant C_3 such that

$$||x||_{\mathcal{W}_1} \le C_3(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{3.13}$$

where C_3 is a constant depending on T.

Proof. Let $y \in B_r$ be the solution of the following linear functional differential equation parabolic type;

$$
\begin{cases}
y'(t) + Ay(t) = k(t), \quad t \in (0,T_1].\\
y(0) = g^0.
\end{cases}
$$

Let the constant T_1 satisfy (3.10) and the following inequality:

$$C_0C_1\Big(\frac{T_1}{\sqrt{2}}\Big)^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2) < 1. \tag{3.14}$$

Then we have

$$\begin{cases} d(x-y)(t)/dt + A((x-y)(t)) = G(t,x), \quad t \in (0,T_1]. \\ (x-y)(0) = 0. \end{cases}$$

Hence, in view of (F) and Lemmas 2.3 and 3.1,

$$\begin{aligned}
&||x-y||_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)} \le C_1||G(\cdot,x)||_{L^2(0,T_1;H)} \\
&\le C_1\mu([-h,0])\big\{L_0\sqrt{T_1} + (L_1(r)+L_2)\|x\|_{L^2(0,T_1;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\} \\
&\le C_1\mu([-h,0])(L_1(r)+L_2)\big(||x-y||_{L^2(0,T_1:V)} + ||y||_{L^2(0,T_1;V)}\big) \\
&\quad + C_1\mu([-h,0])\big(L_0\sqrt{T_1} + L_2\|g^1\|_{L^2(-h,0;V)}\big).
\end{aligned}$$

Thus, by the above inequality and arguing and (2.3),

$$\begin{aligned}
||x-y||_{L^2(0,T_1;V)} &\le C_0||x-y||^{\frac{1}{2}}_{L^2(0,T_1;D(A))}||x-y||^{\frac{1}{2}}_{L^2(0,T_1;H)} \\
&\le C_0||x-y||^{\frac{1}{2}}_{L^2(0,T_1;D(A))}\{\frac{T_1}{\sqrt{2}}||x-y||_{W^{1,2}(0,T_1;H)}\}^{\frac{1}{2}} \\
&\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}||x-y||_{L^2(0,T_1;D(A))\cap W^{1,2}(0,T_1;H)} \\
&\le C_0(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\big\{C_1\mu([-h,0])(L_1(r)+L_2)||y||_{L^2(0,T_1;V)} \\
&\qquad + C_1\mu([-h,0])\big(L_0\sqrt{T_1} + L_2\|g^1\|_{L^2(-h,0;V)}\big)\big\} \\
&\quad + C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)||x-y||_{L^2(0,T_1:V)}.
\end{aligned}$$

Therefore, since

$$\begin{aligned}
||x-y||_{L^2(0,T_1;V)} \le & \frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}{1 - C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}||y||_{L^2(0,T_1;V)} \\
& + \frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])\big(L_0\sqrt{T_1} + L_2\|g^1\|_{L^2(-h,0;V)}\big)}{1 - C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)},
\end{aligned}$$

we have

$$\begin{aligned}
||x||_{L^2(0,T_1;V)} \le & \frac{1}{1 - C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)}||y||_{L^2(0,T_1;V)} \\
& \frac{C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])\big(L_0\sqrt{T_1} + L_2\|g^1\|_{L^2(-h,0;V)}\big)}{1 - C_0C_1(\frac{T_1}{\sqrt{2}})^{\frac{1}{2}}\mu([-h,0])(L_1(r)+L_2)},
\end{aligned}$$

and hence, with the aid of (2.8) in Lemma 2.3 and Lemma 3.1, we obtain

$$
\begin{aligned}
&||x||_{L^2(0,T_1;V)\cap W^{1,2}(0,T_1;V^*)} \qquad (3.15)\\
\leq &C_1(|g^0| + ||G(\cdot,x)||_{L^2(0,T_1;V^*)} + ||k||_{L^2(0,T_1:V^*)})\\
\leq &C_1\big[|g^0| + \mu([-h,0])\big\{L_0\sqrt{T_1} + (L_1(r)+L_2)\|x\|_{L^2(0,T_1;V)} + L_2\|g^1\|_{L^2(-h,0;V)}\big\}\\
&\quad + ||k||_{L^2(0,T_1:V^*)}\big]\\
\leq &C_3(|g^0| + ||k||_{L^2(0,T_1:V^*)}).
\end{aligned}
$$

for some constant C_3. Now from (2.5) and (3.15), it follows that

$$|x(T_1)| \leq ||x||_{C([0,T_1];H)} \leq M_0C_3(|g^0| + ||k||_{L^2(0,T_1;V^*)}). \qquad (3.16)$$

So, we can solve the equation in $[T_1, 2T_1]$ with the initial data $(x(T_1), x_{T_1})$, and obtain an analogous estimate to (3.15). Since the condition (3.14) is independent of initial values, the solution of (3.1) can be extended the internal $[0, nT_1]$ for a natural number n, i.e., for the initial $u(nT_1)$ in the interval $[nT_1,(n+1)T_1]$, as analogous estimate (3.15) holds for the solution in $[0,(n+1)T_1]$. □

By the similar way to Theorems 3.1 and 3.2, we also obtain the following results for (3.1) under Assumption (F) corresponding to 1) of Lemma 2.3.

Corollary 3.1. *Let $(g^0, g^1) \in V \times L^2(-h,0;D(A))$ and $k \in L^2(0,T;H)$. Then there exists a unique solution x of (3.1) such that*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V).$$

Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;D(A)\cap W^{1,2}(0,T;H)} \leq C_3(||g^0|| + ||k||_{L^2(0,T;H)}),$$

where C_3 is a constant depending on T.

4 Controllability for retarded systems

In this paper, we are concerned with the approximate controllability for the following the semilinear control system with a control part Bu in place of k of (3.1):

$$
\begin{cases}
x'(t) + Ax(t) = \int_{-h}^{0} g(t,s,x(t),x(t+s))\mu(ds) + (Bu)(t), \quad 0 < t \leq T,\\
x(0) = g^0, \quad x(s) = g^1(s) \quad s \in [-h,0).
\end{cases} \qquad (4.1)
$$

Here, U is a Banach space of control variables, and B is an operator from U to H, called controller. Let $x(T;g,u)$ be a state value of the system (4.1) at time T corresponding to the nonlinear term g and the control u.

Definition 4.1. *The system (4.1) is said to be approximately controllable in the time interval* $[0, T]$ *if for every desired final state* $x_1 \in H$ *and* $\epsilon > 0$ *there exists a control function* $u \in L^2(0, T; U)$ *such that the solution* $x(T; g, u)$ *of (4.1) satisfies* $|x(T; g, u) - x_1| < \epsilon$.

In order to obtain results of controllability, we need the stronger hypotheses than Assumption (F) of Section 3:

Assumption (F1). $g(t, s, x, y)$ satisfies Assumption (F) instead of (ii) to

(ii') $g(t, s, x, y)$ is locally Lipschitz continuous in x and y, uniformly in $(t, s) \in [0, \infty) \times [-h, 0]$, i.e., there exists a constant $L_0, L_1 = L_1(r) > 0$ and L_2 such that

$$|g(t, s, x, y) - g(t, s, \hat{x}, \hat{y})| \leq L_1|x - \hat{x}| + L_2||y - \hat{y}||,$$

for all $(t, s) \in [0, \infty) \times [-h, 0]$, $y,\ \hat{y} \in V$, $|x| \leq r$ and $|\hat{x}| \leq r$.

We define the linear operator $\hat{S}$ from $L^2(0, T; H)$ to H by

$$\hat{S}p = \int_0^T S(T - s)p(s)ds \quad \text{for } \mathrm{p} \in \mathrm{L}^2(0, \mathrm{T}; \mathrm{H}).$$

Assumption (B). For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}Bu| < \varepsilon, \\ ||Bu||_{L^2(0,t;H)} \leq q||p||_{L^2(0,t;H)}, \quad 0 \leq t \leq T. \end{cases}$$

where q is a constant independent of p.

Assumption (H). We assume the following inequality condition:

$$(q + 1)\sqrt{T}C_2\mu([-h, 0])(L_1 + L_2) < 1.$$

Lemma 4.1. *Let* $u_i (i = 1, 2)$ *be in* $L^2(0, T; U)$ *and* $||x(t; g, u_i)||_{C([0,T],H)} \leq r$. *Then under the assumptions (F1) and (H), we have*

$$\begin{aligned} &||x(t; g, u_1) - x(t; g, u_2)||_{L^2(0,T;V)} \\ &\leq \big(1 - \sqrt{T}M\mu([-h, 0])(L_1 + L_2)\big)^{-1}\sqrt{t}M||Bu_1 - Bu_2||_{L^2(0,T;H)}. \end{aligned} \tag{4.2}$$

for $0 \leq t \leq T$.

Proof. Putting $p(t) = \int_0^t S(t-s)\{G(\cdot, x(\cdot; g, f, u_1)) - G(\cdot, x(\cdot; g, f, u_2))\}ds$, by (2.11) and Lemma 3.1, we have

$$||p||_{L^2(0,t;V)} \leq \sqrt{t} C_2 \mu([-h, 0])(L_1 + L_2)\|x_1 - x_2\|_{L^2(0,T;V)}.$$

From that the solution of (4.1) is represented by

$$x(t; g, u) = S(t)g^0 + \int_0^t S(t-s)\{G(s, x(\cdot; g, u)) + Bu(s)\}ds,$$

it follows (4.2). □

Theorem 4.1. *Under the assumptions (F1), (B) and (H), the system (4.1) is approximately controllable on* $[0, T]$.

Proof. We will show that $D(A) \subset \overline{R_T(g)}$, i.e., for given $\varepsilon > 0$ and $\xi_T \in D(A)$ there exists $u \in L^2(0, T; U)$ such that

$$|\xi_T - x(T; g, u)| < \varepsilon,$$

where

$$x(T; g, u) = S(T)g^0 + \int_0^T S(T-s)\{G(s, x(\cdot; g, u)) + Bu(s)\}ds.$$

As $\xi_T \in D(A)$ there exists a $p \in L^2(0, T; H)$ such that

$$\hat{S}p = \xi_T - S(T)g^0,$$

for instance, take $p(s) = (\xi_T - sA\xi_T) - S(s)g^0/T$. Let $u_1 \in L^2(0, T; U)$ be arbitrary fixed. Since by Assumption (B) there exists $u_2 \in L^2(0, T; U)$ such that

$$|\hat{S}(p - G(\,\cdot\,, x(\,\cdot\,; g, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4},$$

it follows

$$|\xi_T - S(T)g^0 - \hat{S}G(\,\cdot\,, x(\,\cdot\,; g, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}. \tag{4.3}$$

We can also choose $w_2 \in L^2(0, T; U)$ by Assumption (B) such that

$$|\hat{S}(G(\,\cdot\, x(\,\cdot\,; g, u_2)) - G(\,\cdot\, x(\,\cdot\,; g, u_1)) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{4.4}$$

and

$$||Bw_2||_{L^2(0,T;H)} \leq q||G(\,\cdot\,, x(\,\cdot\,; g, u_1)) - G(\,\cdot\,, x(\,\cdot\,; g, u_2))||_{L^2(0,T;H)}.$$

Choose a constant r_1 satisfying

$$||x(\,\cdot\,; g, u_1)||_{C([0,t];H)} \le r_1, \quad ||x(\,\cdot\,; g, u_2)||_{C([0,t];H)} \le r_1.$$

For brevity, set

$$\hat{M} := \big(1 - \sqrt{T} C_2 \mu([-h, 0])(L_1 + L_2)\big)^{-1} M.$$

Therefore, in view of Lemma 3.1 and Assumption (B)

$$\begin{aligned} ||Bw_2||_{L^2(0,T;H)} &\le q||G(\,\cdot\,, x(\,\cdot\,; g, u_1)) - G(\,\cdot\,, x(\,\cdot\,; g, u_2))||_{L^2(0,T;H)} \\ &\le q\mu([-h, 0])(L_1 + L_2)||x(\cdot; g, u_2) - x(\cdot; g, u_1)||_{L^2(0,T;V)} \\ &\le q\mu([-h, 0])(L_1 + L_2)\hat{M}\sqrt{T}||Bu_2 - Bu_1||_{L^2(0,T;H)}. \end{aligned} \tag{4.5}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$\begin{aligned} &|\hat{S}(G(\,\cdot\,, x(\,\cdot\,; g, u_3)) - G(\,\cdot\,, x(\,\cdot\,; g, u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8}, \\ &||Bw_3||_{L^2(0,T;H)} \le q||G(\,\cdot\,, x(\,\cdot\,; g, u_3)) - G(\,\cdot\,, x(\,\cdot\,; g, u_2))||_{L^2(0,T;H)}. \end{aligned}$$

Let r_2 be a constant satisfying $r_2 \ge r_1$ and

$$||x(\,\cdot\,; g, u_3)||_{C([0,t];H)} \le r_2.$$

Then, in a similar way to (4.5) we have

$$\begin{aligned} ||Bw_3||_{L^2(0,T;H)} &\le q||G(\,\cdot\,, x(\,\cdot\,; g, u_3)) - G(\,\cdot\,, x(\,\cdot\,; g, u_2))||_{L^2(0,T;H)} \\ &\le q\mu([-h, 0])(L_1 + L_2)||x(\cdot; g, u_3) - x(\cdot; g, u_2)||_{L^2(0,T;V)} \\ &\le q\mu([-h, 0])(L_1 + L_2)\hat{M}\sqrt{T}||Bu_3 - Bu_2||_{L^2(0,T:H)} \\ &\le q\mu([-h, 0])(L_1 + L_2)\hat{M}\sqrt{T}||Bw_2||_{L^2(0,T;H)} \\ &\le (q\mu([-h, 0])(L_1 + L_2)\hat{M}\sqrt{T})^2||Bu_2 - Bu_1||_{L^2(0,T;H)}. \end{aligned}$$

By proceeding this process, it holds

$$\begin{aligned} ||B(u_n - u_{n+1})||_{L^2(0,T;H)} &= ||Bw_n||_{L^2(0,T;H)} \\ &\le (q\mu([-h, 0])(L_1 + L_2)\hat{M}\sqrt{T})^{n-1}||Bu_2 - Bu_1||_{L^2(0,T;H)}. \end{aligned}$$

Here, noting that Assumption (H) is equivalent to

$$q\mu([-h, 0])(L_1 + L_2)\hat{M}\sqrt{T} < 1,$$

it follows that there exists $u^* \in L^2(0,T;H)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;H).$$

From (4.3), (4.4) it follows that

$$\begin{aligned}
&|\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; g, u_2)) - \hat{S}Bu_3| \\
&= \big|\xi_T - S(T)g - \hat{S}G(\cdot, x(\cdot; g, u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2 \\
&\quad - \hat{S}[G(\cdot, x(\cdot; g, u_2)) - G(\cdot, x(\cdot; g, u_1))]\big| \\
&< (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.
\end{aligned}$$

By choosing $w_n \in L^2(0,T;U)$ by the assumption (B) such that

$$|\hat{S}\big(G(\,\cdot\,, x(\,\cdot\,; g, u_n)) - G(\,\cdot\,, x(\,\cdot\,; g, u_{n-1}))\big) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}},$$

putting $u_{n+1} = u_n - w_n$, we have

$$\begin{aligned}
&|\xi_T - S(T)g^0 - \hat{S}G(\cdot, x(\cdot; g, u_n)) - \hat{S}Bu_{n+1}| \\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1\ 2,\ \cdots.
\end{aligned}$$

Therefore, for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Bu_{N+1} - \hat{S}Bu_N| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned}
&|\xi_T - S(T)g^0 - \hat{S}G(\cdot, x(\cdot; g, u_N)) - \hat{S}Bu_N| \\
&\le |\xi_T - S(T)g^0 - \hat{S}G(\cdot, x(\cdot; g, u_N)) - \hat{S}Bu_{N+1}| + |\hat{S}Bu_{N+1} - \hat{S}Bu_N| \\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned}$$

Thus, System (4.1) is approximately controllable on $[0,T]$.

□

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