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On Conic Approximate Solutions of Nonlinear Multiobjective Optimization Problems

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On Conic Approximate Solutions of Nonlinear Multiobjective Optimization Problems
(비선형 다목적 최적화 문제의 추 근사해에 관한 연구)

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비선형 다목적 최적화 문제의 추 근사해에 관한 연구

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요 약

본 논문에서는 실 공간에서의 한 순서가 내부가 공집합이 아닌 폐 볼록 뾰족한 콘에 의해 유도되고 제약식이 있는 미분불가능 준 볼록성 다목적 최적화 문제를 연구하였다.
먼저 약 준 근사 유효해에 대한 Karush Kuhn Tucker 필요최적조건을 정립하였고 약 준 근사 유효해에 대한 Karush Kuhn Tucker 필요최적조건을 설명하기 위해 예를 제시하였다. 또한, 제약식이 없는 미분불가능 다목적 최적화 문제의 약 준 근사 유효해는 제약식이 없는 미분불가능 다목적 최적화 문제의 준 근사 임계점임을 보였다. 그리고 약 준 근사 유효해에 대한 Karush Kuhn Tucker 충분조건을 제시하였다. 아울러, Wolfe형 쌍대문제를 정식화하고 약 준 근사 유효해에 관한 약 쌍대정리와 강 쌍대정리를 증명하였다.

1 Introduction

It is well known that optimality conditions and objective functions properties play a key role in mathematical programming as well as its applications. One of the main tools here is to employ the separation theorem of convex sets (see e.g.,[19]) to establish necessary conditions for approximate weakly efficient solutions of a multiobjective optimization problem, and to use various kinds of (generalized) convexity of functions to formulate sufficient conditions for such approximate weakly efficient solutions. In this thesis, we establish necessary conditions for approximate weakly efficient solutions of a multiobjective optimization problem with inequality constraints. As usually, we use separation theorem, which is a useful tool as we mentioned, to establish our main results. Along with optimality conditions, we introduce Wolfe type dual problems and investigate weak and strong duality theorems under assumptions of C-convexity. It is worth to mentioning that in the middle of the nineteen eighties, Loridan [17] introduced a notion of ϵ-efficient solutions for multiobjective problems(MOPs), which was followed by White [25] who proposed several concepts of approximate solutions for MOPs and drafted methods for their generating. For the last two decades, approximate efficient solutions of MOPs have been examined in the literature by many authors from different points of view. Existence conditions were developed by Deng [11] and Dutta and Vetrivel [10] for convex MOPs while Karush-Kuhn-Tucker type conditions were derived by Dutta and Vetrivel [10] and Liu [16]. Yokoyama [26, 27] analyzed connections between different definitions of approximate solutions. Tammer [23], Tanaka [24], and others studied approximate solutions of vector optimization problems in general ordered vector spaces. In view of the literature, the current belief is that the concept of ϵ-efficient solutions accounts for modeling limitations or computational inaccuracies, and thus is tolerable rather than desirable. Consequently, methods purposely avoiding efficiency and guaranteeing ϵ-efficiency have not been well developed.

The aim of this thesis is to study nondifferentiable constrained multiobjective problems where the partial order in the image space is induced by a proper cone C (closed, convex and pointed solid cone). In Section 2, some basic definitions and several auxiliary results are presented. In Section 3, we show that a weakly C-ϵ-efficient solution is a C-ϵ-critical point. Moreover, we investigate necessary and sufficient optimality conditions for weakly C-ϵ-efficient solutions. In addition, we establish a Wolfe type dual model and state weak and strong duality theorems in Section 4. Throughout the present thesis, some examples are given to illustrate our results.

2 Preliminaries

Let us first recall some notations and preliminary results which will be used throughout this thesis; see e.g., [9, 19]. We denote by $\mathbb{R}^n$ the Euclidean space of dimension n. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0, \ i = 1, \ldots, n\}$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y \rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. We say that a set A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0, 1]$, $a_1, a_2 \in A$. Let ϕ be a function from $\mathbb{R}^n$ to $\bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = [-\infty, +\infty]$. Here, ϕ is said to be proper if for all $x \in \mathbb{R}^n$, $\phi(x) > -\infty$ and there exists $x_0 \in \mathbb{R}^n$ such that $\phi(x_0) \in \mathbb{R}$. We denote the domain of ϕ by domϕ, that is, dom$\phi := \{x \in \mathbb{R}^n \mid \phi(x) < +\infty\}$. A function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ is said to be proper convex if

$$\phi((1-\mu)x + \mu y) \leqq (1-\mu)\phi(x) + \mu\phi(y),$$

for all $\mu \in [0,1]$, for all $x, y \in \mathbb{R}^n$. Let $D \subseteq \mathbb{R}^p$ be a p-dimensional vector space and D^* be a dual space of D.

The cone C is said to be pointed if it contains no line(or equivalently, $x \in C,\ -x \in C \Rightarrow x = 0$, in other words, $C \cap (-C) = \{0\}$, see [14]). Let us consider a proper cone C in D, that is, C is a closed convex and pointed cone with nonempty interior [5]. The positive dual cone to C and the strict positive dual cone to C, denoted as

$$C^+ := \{d^* \in D^* \mid \langle d^*, d \rangle \geqq 0,\ \forall d \in C\},$$

and

$$C^{S+} := \{d^* \in D^* \mid \langle d^*, d \rangle > 0,\ \forall d \in C \backslash \{0\}\},$$

respectively. Since $D \subseteq \mathbb{R}^p$, each element in D^* can be represented as a p-dimensional vector.

Consider a set $F \subset \mathbb{R}^n$. The support function $\sigma_F : \mathbb{R}^n \to \bar{\mathbb{R}}$, to F at $\bar{x} \in \mathbb{R}^n$ is defined as

$$\sigma_F(\bar{x}) = \sup_{x \in F} \langle \bar{x}, x \rangle;$$

and the indicator function, $\delta_F : \mathbb{R}^n \to \bar{\mathbb{R}}$, to the set F is defined as

$$\delta_F(x) = \begin{cases} 0, & x \in F, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is worth to noting that if F is convex, then indicator function δ_F is also convex.

In order to establish optimality conditions, let us consider the proper cone $C \subset \mathbb{R}^p$, which induces a partial order on D. We define C^0 as $C \setminus \{0\}$. Thus,

$$\begin{aligned} x \leqq_C y \quad &\text{if and only if} \quad y - x \in C, \\ x \leq_C y \quad &\text{if and only if} \quad y - x \in C^0, \\ x <_C y \quad &\text{if and only if} \quad y - x \in \text{int}C. \end{aligned}$$

Now we give the notion of ϵ-subdifferential. It is worth to mention that firstly its idea can be found in the work of Brondsted and Rockafellar [3], but the theory of ϵ-subdifferential calculus was given by Hiriart-Urruty [13]. One can also refer to Dhara and Dutta [9] to understand this notion easily with the aid of some examples.

Definition 2.1 *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a convex function. For $\epsilon \geqq 0$, the ϵ-subdifferential of ϕ at $\bar{x} \in \text{dom}f$ is given by*

$$\partial_\epsilon \phi(\bar{x}) = \{\xi \in \mathbb{R}^n \mid \phi(x) - \phi(\bar{x}) \geqq \langle \xi, x - \bar{x} \rangle - \epsilon, \ \forall x \in \mathbb{R}^n\}.$$

Definition 2.2 *Consider a convex set $Q \subset \mathbb{R}^n$. Then for $\epsilon \geqq 0$, the ϵ-subdifferential of the indicator function at $\bar{x} \in Q$ is*

$$\begin{aligned} \partial_\epsilon \delta_Q(\bar{x}) &= \{\xi \in \mathbb{R}^n \mid \delta_Q(x) - \delta_Q(\bar{x}) \geqq \langle \xi, x - \bar{x} \rangle - \epsilon, \forall x \in R^n\} \\ &= \{\xi \in \mathbb{R}^n \mid \epsilon \geqq \langle \xi, x - \bar{x} \rangle, \forall x \in Q\}, \end{aligned}$$

which is also called the ϵ-normal set and denoted as $N_{Q,\epsilon}(\bar{x})$.

Definition 2.3 [9] *The relative interior of a convex set $F \subset \mathbb{R}^n$, riF, is the interior of F relative to the affine hull of F, that is,*

$$riF = \{x \in \mathbb{R}^n \mid \text{there exists } \epsilon > 0 \text{ such that } (x + \epsilon\mathbb{B}) \cap aff\, F \subset F\},$$

where $\mathbb{B}$ stands for the unit ball in $\mathbb{R}^n$, and $aff\, F$ is the affine hull of F.

Definition 2.4 *Consider a function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$. The conjugate of ϕ, $\phi^* : \mathbb{R}^n \to \bar{\mathbb{R}}$, is defined as*

$$\phi^*(\xi) = \sup_{x \in \mathbb{R}^n} \{\langle \xi, x \rangle - \phi(x)\}.$$

It is worth to mentioning for any function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$, the conjugate function ϕ^* is always lower semicontinuous convex. In addition, if ϕ is proper convex, then ϕ^* is also proper convex.

Definition 2.5 [28] *Let Γ be a convex subset of $\mathbb{R}^n$. Then the function f is said to be C-convex on convex set Γ if for any $x, y \in \Gamma$ and $t \in [0, 1]$*

$$tf(x) + (1 - t)f(y) - f(tx + (1 - t)y) \in C.$$

Definition 2.6 [28] *Let Γ be a convex subset of $\mathbb{R}^n$. Then the function f is said to be C-convexlike on Γ if for any $x, y \in \Gamma$ and $t \in [0, 1]$, there exists $z \in \Gamma$ such that*

$$tf(x) + (1 - t)f(y) - f(z) \in C.$$

Let us consider the following nondifferentiable unconstrained multiobjective optimization problem:

$$(\mathrm{MP})_U \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \mathbb{R}^n, \end{array}$$

where $f = (f_1, ..., f_p) : \mathbb{R}^n \to \mathbb{R}^p$ is a C-convex function.

Since it is usual to request the incorporation of some constraints to our multiobjective optimization problem. In such a way, we are going to focus the optimization study on the following one:

$$(\mathrm{MP}) \qquad \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leqq 0,\ i = 1, \ldots, m, \end{array}$$

where $f = (f_1, ..., f_p) : \mathbb{R}^n \to \mathbb{R}^p$ is a C-convex function, and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are convex functions. The feasible set of (MP) is defined by

$$F_P := \{x \in \mathbb{R}^n \mid g_i(x) \leqq 0,\ i = 1, \ldots, m\}.$$

Definition 2.7 *Let $\epsilon \in C$ be given. Then $\bar{x} \in F_P$ is said to be a C-ϵ-efficient solution of* (MP) *if there does not exist another feasible point x such that $f(x) \leq_C f(\bar{x}) - \epsilon$, which is equivalent to that $f(x) - f(\bar{x}) + \epsilon \notin -C^0$, $\forall x \in F_P$.*

Definition 2.8 *Let $\epsilon \in C$ be given. Then $\bar{x} \in F_P$ is said to be a weakly C-ϵ-efficient solution of* (MP) *if there does not exist another feasible point x such that $f(x) <_C f(\bar{x}) - \epsilon$, which is equivalent to that $f(x) - f(\bar{x}) + \epsilon \notin -\mathrm{int}C$, $\forall x \in F_P$.*

Remark 2.1 *We introduce some special cases about a weakly C-ϵ-efficient solution as follows:*

(i) *Let* $\epsilon \in C$ *be given. If* $C = \mathbb{R}^p_+$, *then* $\bar{x} \in F_P$ *is said to be a weakly* ϵ*-efficient solution of* (MP) *if there does not exist another feasible point* x *of* (MP) *such that* $f(x) < f(\bar{x}) - \epsilon$, *which is equivalent to that* $f(x) - f(\bar{x}) + \epsilon \notin -\text{int}\mathbb{R}^p_+$, $\forall x \in F_P$. *Many research papers studied weakly* ϵ*-efficient solutions in (finite) multiobjective programs* [6], *multiobjective semi-infinite programs* [21, 22].

(ii) *A feasible point,* $\bar{x}$ *is said to be a* C*-weakly efficient solution of* (MP) *if there does not exist another feasible point* x *of* (MP) *such that* $f(x) <_C f(\bar{x})$, *which is equivalent to that* $f(x) - f(\bar{x}) \notin -\text{int}C$, $\forall x \in F_P$. *Some results were obtained, one can see* [1, 2, 12].

(iii) *Let* $C = \mathbb{R}^p_+$. *A feasible point,* $\bar{x}$ *is said to be a weakly efficient solution of* (MP) *if there does not exist another feasible point* x *of* (MP) *such that* $f(x) < f(\bar{x})$, *which is equivalent to that* $f(x) - f(\bar{x}) \notin -\text{int}\mathbb{R}^p_+$, $\forall x \in F_P$. *A lot of results were obtained, one can refer to* [4, 7, 8, 18].

Now we give the following example to illustrate the mentioned solutions above.

Example 2.1 Consider the following multiobjective optimization problem:

$$\text{(MP)} \qquad \begin{array}{ll} \text{minimize} & \big(f_1(x), f_2(x)\big) \\ \text{subject to} & x \in F_P := \mathbb{R}, \end{array}$$

where $f_1(x) = x$ and $f_2(x) = \frac{1}{2}x^2$.

(a) Let $C = \mathbb{R}^2_+$, then the weakly efficient solution set is

$$\{x \in \mathbb{R} \mid f(x) \notin f(\bar{x}) - \text{int}\mathbb{R}^2_+\} = (-\infty, 0\,].$$

(b) Let $C = \mathbb{R}^2_+$ and $\epsilon = (\epsilon_1, \epsilon_2) \in C$ be given.

Case 1. If $2\epsilon_2 > (\epsilon_1)^2$, the weakly ϵ-efficient solution set is

$$\{x \in \mathbb{R} \mid f(x) \notin f(\bar{x}) - \epsilon - \text{int}\mathbb{R}^2_+\} \;=\; (-\infty, \sqrt{2\epsilon_2}\,].$$

Case 2. If $2\epsilon_2 \leq (\epsilon_1)^2$, the weakly ϵ-efficient solution set is $(-\infty, \frac{(\epsilon_1)^2 + 2\epsilon_2}{2\epsilon_1}\,]$.

(c) Let $C := \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 \geqq |t_1|\,\}$ be given, then the C-weakly efficient solution set is

$$\{x \in \mathbb{R} \mid f(x) \notin f(\bar{x}) - \text{int}C\} = [-1, 1\,].$$

Lemma 2.1 (*Sum Rule*) *Consider two proper convex functions* $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}$, $i = 1, 2$ *such that ri dom* $\phi_1 \cap$ *ri dom* $\phi_2 \neq \emptyset$. *Then for* $\epsilon > 0$,

$$\partial_\epsilon(\phi_1 + \phi_2)(\bar{x}) = \bigcup_{\epsilon_1 \geqq 0, \epsilon_2 \geqq 0, \epsilon_1 + \epsilon_2 = \epsilon} (\partial_{\epsilon_1}\phi_1(\bar{x}) + \partial_{\epsilon_2}\phi_2(\bar{x}))$$

for every $\bar{x} \in$ *dom* $\phi_1 \cap$ *dom* ϕ_2.

3 ϵ-Optimality Conditions

In this section, first we establish necessary optimality condition for a weakly C-ϵ-efficient solution of $(\mathrm{MP})_U$, and then under Slater type constraint qualification, necessary optimality condition for a weakly C-ϵ-efficient solution of (MP) is given. Moreover, we establish the sufficient optimality condition for a weakly C-ϵ-efficient solution of (MP).

First of all, let us consider the unconstrained multiobjective optimization problem $(\mathrm{MP})_U$ and give the following necessary optimality condition.

Theorem 3.1 *If $\bar{x}$ is a weakly C-ϵ-efficient solution of $(\mathrm{MP})_U$, then $\bar{x}$ is a C-ϵ-critical point for $(\mathrm{MP})_U$, that is, there exists $\lambda \in C^+ \setminus \{0\}$ such that*

$$0 \in \partial_{\lambda^T \epsilon}(\lambda^T f)(\bar{x}).$$

Proof. Since $\bar{x}$ is a weakly C-ϵ-efficient solution, we have $f(x) - f(\bar{x}) + \epsilon \notin -\mathrm{int}C,\ \forall x \in \mathbb{R}^n$. By a separation theorem(see [9]), there exists $\lambda \in C^+$ with $\lambda \neq 0$ such that $\langle f(x) - f(\bar{x}) + \epsilon,\ \lambda \rangle \geqq 0,\ \forall x \in \mathbb{R}^n$. So,

$$\lambda^T \epsilon + \lambda^T f(x) \geqq \lambda^T f(\bar{x}),\ \forall x \in \mathbb{R}^n.$$

Thus, $0 \in \partial_{\lambda^T \epsilon}(\lambda^T f)(\bar{x})$. □

Remark 3.1 *If $\bar{x}$ is a C-ϵ-efficient solution of $(\mathrm{MP})_U$, then $\bar{x}$ is a C-ϵ-critical point for $(\mathrm{MP})_U$, that is, there exists $\lambda \in C^+ \setminus \{0\}$ such that*

$$0 \in \partial_{\lambda^T \epsilon}(\lambda^T f)(\bar{x}).$$

Now we examine the ϵ-necessary optimality condition of (MP). First, consider the following constrained convex optimization problem:

$$\text{(CP)} \qquad \begin{array}{ll} \text{minimize} & h(x) \\ \text{subject to} & x \in Q, \end{array}$$

where $h : \mathbb{R}^n \to \mathbb{R}$ is a convex function and Q is a convex set in $\mathbb{R}^n$.
The ϵ-optimality condition of (CP) is given in the following.

Lemma 3.1 [*9*, *20*] *Consider the convex optimization problem* (CP). *Assume that the Slater constraint qualification holds, that is, ri Q is nonempty. Let $\epsilon \geqq 0$ be given. Then $\bar{x} \in Q$ is an ϵ-solution of* (CP) *if and only if there exist $\epsilon_i \geqq 0$, $i = 1, 2$ with $\epsilon_1 + \epsilon_2 = \epsilon$ such that*

$$0 \in \partial_{\epsilon_1} h(\bar{x}) + N_{Q,\epsilon_2}(\bar{x}).$$

We can easily show Lemma 3.1 by using Lemma 2.1 (Sum Rule of ϵ-subdifferential), along with Definition 2.2 (ϵ-normal set).

Note that for a nonempty convex set Q, riQ is nonempty and hence the Slater constraint qualification holds. From the Lemma 3.1, it is obvious that to obtain the approximate optimality conditions in terms of the constraint functions g_i, $i = 1, ..., m$, here g_i should be convex functions, $N_{Q,\epsilon}(x)$ must be explicitly expressed in their terms. Below we present the result from Strodiot *et al.* [20], which acts as the tool in establishing the approximate optimality conditions. First, we define the right scalar multiplication from Rockafellar [19].

Definition 3.1 *[9, 19] Let $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper convex function and $\mu \geqq 0$. The right scalar multiplication, $\phi\mu$, is defined as*

$$(\phi\mu)(x) = \begin{cases} \mu\phi(\mu^{-1}x), & \mu > 0, \\ \delta_{\{0\}}(x), & \mu = 0. \end{cases}$$

A positively homogeneous convex function ψ generated by ϕ, is defined as

$$\psi(x) = \inf\{(\phi\mu)(x) \mid \mu \geqq 0\}.$$

We state the following theorem from Rockafellar [19].

Theorem 3.2 *[9, 19] Let $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a proper lower semicontinuous convex function. The support function of the set $Q = \{x \in R^n \mid \phi(x) \leqq 0\}$ is then* cl ψ*, where ψ is the positively homogeneous convex function generated by ϕ^*. Dually, the closure of the positively homogeneous convex function ψ generated by ϕ is the support function of the set $\{\xi \in \mathbb{R}^n \mid \phi^*(\xi) \leqq 0\}$.*

Now we give an example as below in order to understand Theorem 3.2 clearly.

Example 3.1 Consider $Q = \{x \in \mathbb{R} \mid \phi(x) \leqq 0\}$, where $\phi(x) = x^2 - x$ is a proper lower semicontinuous convex function for all $x \in \mathbb{R}$, obviously the support function of the set $Q = [0, 1]$ is

$$\sigma_Q(\xi) = \sigma_{[0,1]}(\xi) = \sup_{x\in[0,1]} \langle \xi, x \rangle = \begin{cases} 0, & \xi \leqq 0, \\ \xi, & \xi > 0. \end{cases}$$

On the other hand, calculating the conjugate function of $\phi(x)$, we have

$$\phi^*(\xi) = \sup_{x\in\mathbb{R}}\{\langle \xi, x\rangle - \phi(x)\} = \frac{(1+\xi)^2}{4}. \tag{3.1}$$

Now we deal with the positively homogeneous convex function ψ generated by ϕ^*, and

$$(\phi^*\mu)(\xi) = \begin{cases} \mu\phi^*(\mu^{-1}\xi), & \mu > 0, \\ \delta_{\{0\}}(\xi), & \mu = 0, \end{cases}$$

$$= \begin{cases} \frac{\xi^2}{4\mu} + \frac{\xi}{2} + \frac{\mu}{4}, & \mu > 0, \\ \delta_{\{0\}}(\xi), & \mu = 0, \end{cases}$$

and get

$$\mathrm{cl}\,\psi(\xi) = \inf\{(\phi^*\mu)(\xi) \mid \mu \geqq 0\} = \frac{|\xi|}{2} + \frac{\xi}{2} = \begin{cases} 0, & \xi \leqq 0, \\ \xi, & \xi > 0. \end{cases}$$

Therefore, the support function of the set $Q = \{x \in \mathbb{R} \mid \phi(x) \leqq 0\}$ is $\mathrm{cl}\psi$.

Dually let $Q^* = \{\xi \in \mathbb{R}^n \mid \phi^*(\xi) \leqq 0\}$, from (3.1) we know $\phi^*(\xi) = \frac{(1+\xi)^2}{4}$, when $\phi^*(\xi) \leqq 0$, then $\xi = -1$, and the set $Q^* = \{-1\}$.

Support function of the set Q^* is

$$\sigma_{Q^*}(x) = \sigma_{\{-1\}}(x) = \sup_{\xi\in\{-1\}} \langle x, \xi\rangle = -x.$$

We now deal with the closure of the positively homogeneous convex function

ψ generated by ϕ, and

$$(\phi\mu)(x) = \begin{cases} \mu\phi(\mu^{-1}x), & \mu > 0, \\ \delta_{\{0\}}(x), & \mu = 0, \end{cases}$$

$$= \begin{cases} \frac{x^2}{\mu} - x, & \mu > 0, \\ \delta_{\{0\}}(x), & \mu = 0, \end{cases}$$

and get

$$\mathrm{cl}\psi(x) = \inf\{(\phi\mu)(x) : \mu \geqq 0\} = -x.$$

Therefore, the closure of the positively homogeneous convex function ψ generated by ϕ is the support function of the set $\{\xi \in \mathbb{R}^n \mid \phi^*(\xi) \leqq 0\}$.

Lemma 3.2 [9] *Consider $\epsilon \geqq 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Let $\bar{x} \in \bar{Q} = \{x \in \mathbb{R}^n \mid g(x) \leqq 0\}$. Assume that the Slater constraint qualification holds, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g(\hat{x}) < 0$. Then $\xi \in N_{\bar{Q},\epsilon}(\bar{x})$ if and only if there exist $\mu \geqq 0$ and $\bar{\epsilon} \geqq 0$ such that*

$$\bar{\epsilon} \leqq \mu g(\bar{x}) + \epsilon \quad \text{and} \quad \xi \in \partial_{\bar{\epsilon}}(\mu g)(\bar{x}).$$

Up to now, we are ready to establish the ϵ-necessary optimality condition of (MP).

Theorem 3.3 *Let $\epsilon \in C$ and g_i, $i = 1, \ldots, m$, are convex functions. Assume that the Slater constraint qualification is satisfied, that is, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$ for every $i = 1, \ldots, m$. If $\bar{x}$ is a weakly C-ϵ-efficient* *solution of* (MP)*, then there exist* $\bar{\lambda} \in C^+ \setminus \{0\}$*,* $\bar{\epsilon}_0 \in C$*,* $\bar{\epsilon}_i \in C$*,* $i = 1, \dots, m$*, and* $\bar{\mu}_i \geqq 0$*,* $i = 1, \dots, m$*, such that*

$$0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x}),$$

$$\sum_{i=0}^{m} \bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) \leqq 0,$$

where $\bar{\lambda}^T \epsilon = \alpha$*,* $\bar{\lambda}^T \epsilon_i = \alpha_i$*,* $i = 0, \dots, m$*,* $\bar{\lambda}^T \bar{\epsilon}_i = \bar{\alpha}_i$*,* $i = 0, \dots, m$*.*

Proof. Let $\bar{x}$ be a weakly C-ϵ-efficient solution of (MP), equivalently, $\bar{x}$ is a weakly C-ϵ-efficient solution of the following unconstrained programming $(\overline{\text{MP}})$

$$(\overline{\text{MP}}) \qquad \text{minimize} \quad f(x) + \delta_{F_P}(x)e \qquad \text{subject to} \quad x \in \mathbb{R}^n.$$

Using Theorem 3.1, $\bar{x}$ is a C-ϵ-critical point of $(\overline{\text{MP}})$, that is, there exists $\bar{\lambda} \in C^+ \setminus \{0\}$ with $\bar{\lambda}^T e = 1$ such that

$$\bar{\lambda}^T f(\bar{x}) + \delta_{F_P}(\bar{x}) \leqq \bar{\lambda}^T f(x) + \delta_{F_P}(x) + \bar{\lambda}^T \epsilon, \ \ \forall x \in \mathbb{R}^n,$$

which means $\bar{x}$ is a $\bar{\lambda}^T \epsilon$-solution of the following scalar problem $(\text{P})^*$,

$$(\text{P})^* \qquad \text{minimize} \quad (\bar{\lambda}^T f + \delta_{F_P})(x) \qquad \text{subject to} \quad x \in \mathbb{R}^n,$$

without loss of generality, set $\bar{\lambda}^T \epsilon = \alpha$, obviously $\alpha \geqq 0$. Equivalently, $\bar{x}$ is a α-solution of the following unconstrained problem,

$$(\text{P}) \qquad \text{minimize} \quad (\bar{\lambda}^T f + \sum_{i=1}^{m} \delta_{Q_i})(x) \qquad \text{subject to} \quad x \in \mathbb{R}^n,$$

where $F_P = \bigcap_{i=1}^{m} Q_i$ and $Q_i = \{x \in \mathbb{R}^n \mid g_i(x) \leqq 0\}$, $i = 1, \ldots, m$. By the Slater constraint qualification, there exists $\hat{x} \in \mathbb{R}^n$ such that $g_i(\hat{x}) < 0$ for every $i = 1, \ldots, m$, which implies riQ_i, $i = 1, \ldots, m$, are nonempty. Since $\bar{x} \in F_P$ is an α-solution of (P), we have $0 \in \partial_\alpha \left(\bar{\lambda}^T f + \sum_{i=1}^{m} \delta_{Q_i} \right)(\bar{x})$. From Lemma 3.1, there exists $\alpha_i \geqq 0$, $i = 0, \ldots, m$, with $\alpha_0 + \sum_{i=1}^{m} \alpha_i = \alpha$ such that

$$0 \in \partial_{\alpha_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} N_{Q_i, \alpha_i}(\bar{x}).$$

Applying Lemma 3.2 to Q_i, $i = 1, \ldots, m$, there exist $\bar{\mu}_i \geqq 0$ and $\bar{\alpha}_i \geqq 0$, $i = 1, \ldots, m$, such that

$$0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x}),$$

and

$$\bar{\alpha}_i - \alpha_i \leqq \bar{\mu}_i g_i(\bar{x}) \leqq 0, \ i = 1, \ldots, m, \tag{3.2}$$

where $\bar{\alpha}_0 = \alpha_0$. Now summing (3.2) over $i = 1, \ldots, m$, and using the condition $\alpha_0 + \sum_{i=1}^{m} \alpha_i = \alpha$ leads to

$$\sum_{i=0}^{m} \bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) \leqq 0.$$

□

Now we give the following example to illustrate Theorem 3.3.

Example 3.2 Consider the following multiobjective optimization problem:

$$\text{(MP)} \qquad \text{minimize} \quad \Big(f_1(x), f_2(x)\Big)$$
$$\text{subject to} \quad g(x) \leqq 0.$$

Let $f_1(x) = x$, $f_2(x) = \frac{1}{2}x^2$, $g(x) = x^2 - x$, $C := \mathbb{R}^2_+$ and given $\epsilon = (\epsilon_1, \epsilon_2) = (\frac{1}{2}, 0) \in C$.

Observe that $F_P = [0, 1]$ is the feasible set and the weakly C-ϵ-efficient solution set of (MP) is $\{0\}$.

Since

$$0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x}),$$

$$\sum_{i=0}^{m} \bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) \leqq 0,$$

where $\bar{\lambda}^T \epsilon = \alpha$, $\bar{\lambda}^T \epsilon_i = \alpha_i$, $i = 0, \ldots, m$, $\bar{\lambda}^T \bar{\epsilon}_i = \bar{\alpha}_i$, $i = 0, \ldots, m$, then we can find $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2) \in C^+ \setminus \{0\}$ and $\bar{\epsilon}_0 = (\bar{\epsilon_0^1}, \bar{\epsilon_0^2}) \in C$, $\bar{\epsilon}_1 = (\bar{\epsilon_1^1}, \bar{\epsilon_1^2}) \in C$, $\bar{\mu} \geqq 0$.

By Theorem 3.3, we have

$$\bar{\lambda}_1 \epsilon_1 + \bar{\lambda} \epsilon_2 = \alpha = \frac{1}{2} \bar{\lambda}_1, \quad \bar{\lambda} \bar{\epsilon_0^1} + \bar{\lambda} \bar{\epsilon_0^2} = \bar{\alpha}_0 = \alpha_0,$$

and

$$\bar{\lambda} \bar{\epsilon_1^1} + \bar{\lambda} \bar{\epsilon_1^2} = \bar{\alpha}_1 = \alpha_1, \quad \bar{\alpha}_0 + \bar{\alpha}_1 - \alpha \leqq \bar{\mu} g(\bar{x}) \leqq 0,$$

as $0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x})$, then there exist $\xi_0 \in \partial_{\bar{\alpha}_0}(\lambda^T f)(\bar{x})$, $\xi_1 \in \partial_{\bar{\alpha}_1}(\bar{\mu} g)(\bar{x})$ with $\xi_1 + \xi_2 = 0$ and

$$(\bar{\lambda}_1 f_1 + \bar{\lambda}_2 f_2)(x) - (\bar{\lambda}_1 f_1 + \bar{\lambda}_2 f_2)(\bar{x}) \geqq \langle \xi_0, x - \bar{x} \rangle - \bar{\alpha}_0,$$

$$(\bar{\mu}g)(x) - (\bar{\mu}g)(\bar{x}) \geqq \langle \xi_1, x - \bar{x} \rangle - \bar{\alpha}_1.$$

After a calculation we have $\bar{\lambda}_1 = 0$, $\bar{\mu} = 0$, $\bar{\lambda}_2 > 0$. Take $\bar{\lambda}_2 = 1$ and use these conditions we get $\bar{\epsilon}_0^2 = 0$, $\bar{\epsilon}_1^2 = 0$, $\bar{\epsilon}_1^1 > 0$, take $\bar{\epsilon}_1^1 = \frac{1}{4}$, $\bar{\epsilon}_0^1 > 0$, take $\bar{\epsilon}_0^1 = \frac{1}{9}$, and we show the validity of Theorem 3.3.

Here after all, we explore the ϵ-sufficient condition for $(\mathrm{MP})_U$ and (MP), and before that, we give the following lemma that is a little bit different to [15, 28], which will be used to get the sufficient condition for $(\mathrm{MP})_U$ as the main tool.

Lemma 3.3 *Let Γ be a convex subset of $\mathbb{R}^n$, $\epsilon \in C$ and $f : \Gamma \to \mathbb{R}^p$ be a C-convexlike function. Then, exactly one of the following statements holds.*

(i) *There exists $x_0 \in \Gamma$ such that $f(x_0) - \epsilon \in -\mathrm{int}C$.*

(ii) *There exists $\lambda \in C^+ \setminus \{0\}$ such that $\langle \lambda, f(x) - \epsilon \rangle \geqq 0, \forall x \in \Gamma$.*

Proof. [(i) $\Rightarrow$ (ii)]Assume that (i) is impossible. This implies that there does not exist $x \in \Gamma$ such that

$$f(x) - \epsilon \in -\mathrm{int}C.$$

Write

$$f(\Gamma) = \{f(x) \mid x \in \Gamma\}.$$

Because

$$C + \mathrm{int}C \subseteq \mathrm{int}C,$$

we obtain for any $\alpha > 0$,

$$- f(x) + \epsilon \notin \text{int}C \Rightarrow -\alpha f(x) + \alpha\epsilon \notin \text{int}C$$

$$\Rightarrow - \alpha f(x) + \alpha\epsilon \notin \text{int}C + C \Rightarrow -\alpha(f(x) - \epsilon) - C \notin \text{int}C.$$

We have

$$\bigcup_{\alpha>0} -\alpha(f(\Gamma) - \epsilon) - C \cap \text{int}C = \emptyset.$$

Let

$$S = \bigcup_{\alpha>0} -\alpha(f(\Gamma) - \epsilon) - C.$$

Then,

$$S \cap \text{int}C = \emptyset.$$

Letting

$$\bar{S} = \bigcup_{\alpha>0} \alpha(f(\Gamma) - \epsilon) + C,$$

we will show that $\bar{S}$ is convex. Take

$$s_i = \alpha_i f(x^i) - \alpha_i\epsilon + c_i \in \bar{S},\ i = 1, 2,$$

where $c_i \in C,\ \alpha_i > 0,\ i = 1, 2$. For $t \in [0, 1]$, set

$$c_0 = tc_1 + (1 - t)c_2.$$

Due to the convexity of C, we have that $c_0 \in C$.

From the assumption of C-convexlike function, there exists $x^3 \in \Gamma$ such that

$$\frac{t\alpha_1}{t\alpha_1 + (1 - t)\alpha_2} f(x^1) + \frac{(1 - t)\alpha_2}{t\alpha_1 + (1 - t)\alpha_2} f(x^2) - f(x^3) := \acute{c} \in C.$$

But

$$
\begin{aligned}
& t\alpha_1 f(x^1) + (1-t)\alpha_2 f(x^2) + tc_1 + (1-t)c_2 \\
= \; & [t\alpha_1 + (1-t)\alpha_2][\tfrac{t\alpha_1}{t\alpha_1+(1-t)\alpha_2} f(x^1) + \tfrac{(1-t)\alpha_2}{t\alpha_1+(1-t)\alpha_2} f(x^2)] + c_0 \\
= \; & [t\alpha_1 + (1-t)\alpha_2][\acute{c} + f(x^3)] + c_0 \\
= \; & [t\alpha_1 + (1-t)\alpha_2] f(x^3) + [t\alpha_1 + (1-t)\alpha_2]\acute{c} + c_0 \\
\in \; & \bigcup_{\alpha>0} \alpha f(x^3) + C + C \subseteq \bigcup_{\alpha>0} \alpha f(x^3) + C.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& ts_1 + (1-t)s_2 = [t\alpha_1 + (1-t)\alpha_2]\hat{c} - [t\alpha_1 + (1-t)\alpha_2]\epsilon + c_0 \\
\in \; & \bigcup_{\alpha>0} \alpha(f(x^3) - \epsilon) + C \\
\subseteq \; & \bigcup_{\alpha>0} \alpha(f(\Gamma) - \epsilon) + C = \bar{S},
\end{aligned}
$$

where

$$
\hat{c} = \frac{t\alpha_1}{t\alpha_1 + (1-t)\alpha_2} f(x^1) + \frac{(1-t)\alpha_2}{t\alpha_1 + (1-t)\alpha_2} f(x^2).
$$

So, $\bar{S}$ is a convex set. Therefore, S is convex and $S \cap \text{int}C = \emptyset$. Hence, by the separation theorem(see [9]) of convex sets of $\mathbb{R}^p$, there exists $\lambda \in C^+ \backslash \{0\}$ such that

$$
\langle s, \lambda \rangle \leqq 0 \leqq \langle d, \lambda \rangle, \; \forall s \in S, d \in C.
$$

From this, we conclude that

$$
\langle -\alpha(f(x) - \epsilon) - v, \lambda \rangle \leqq 0, \; \forall \alpha > 0, \; \forall x \in \Gamma, \; \forall v \in C.
$$

Therefore,

$$\langle f(x) - \epsilon + v, \lambda \rangle \geqq 0, \ \forall x \in \Gamma, \ \forall v \in C.$$

Since C is a closed convex cone, this implies that

$$\langle f(x) - \epsilon, \lambda \rangle \geqq \langle -v, \lambda \rangle, \ \forall v \in C, \forall x \in \Gamma$$

$$\Rightarrow \ \langle f(x) - \epsilon, \lambda \rangle \geqq \sup_{v \in C}(-\langle v, \lambda \rangle) = 0, \ \forall x \in \Gamma,$$

which states that (ii) has a solution.

[(ii) $\Rightarrow$ $(\bar{\text{i}})$]Assume that (ii) holds, there exists $\lambda \in C^+ \setminus \{0\}$ such that

$$\langle \lambda, f(x_0) - \epsilon \rangle \geqq 0, \ \text{for some } x_0 \in \Gamma,$$

i.e.,

$$\langle \lambda, -f(x_0) + \epsilon \rangle \leqq 0. \tag{3.3}$$

Now we suppose (i) holds, in other words there exists $x_0 \in \Gamma$ such that

$$f(x_0) - \epsilon \in -\text{int}C,$$

i.e.,

$$-f(x_0) + \epsilon \in \text{int}C.$$

Since C^+ is the positive dual cone to C, then

$$\langle \lambda, -f(x_0) + \epsilon \rangle > 0,$$

which contradicts to (3.3), and (i) does not hold. □

Obviously, every C-convex function is C-convexlike. From Lemma 3.3, the following Corollary also holds.

Corollary 3.1 *Let* Γ *be a convex subset of* $\mathbb{R}^n$, $\epsilon \in C$ *and* $f : \Gamma \to \mathbb{R}^p$ *be a* C*-convex function. Then, exactly one of the following statements holds.*

(i) *There exists* $x_0 \in \Gamma$ *such that* $f(x_0) - \epsilon \in -\mathrm{int}C$.

(ii) *There exists* $\lambda \in C^+ \setminus \{0\}$ *such that* $\langle \lambda, f(x) - \epsilon \rangle \geqq 0, \forall x \in \Gamma$.

Now, we give the sufficient condition for $(\mathrm{MP})_U$ with the help of Corollary 3.1.

Theorem 3.4 *Assume that* f *is a* C*-convex function, then every* C*-*ϵ*-critical point* $\bar{x}$ *is a weakly* C*-*ϵ*-efficient solution of* $(\mathrm{MP})_U$.

Proof. Since $\bar{x}$ is C-ϵ-critical point for $(\mathrm{MP})_U$, there exists $\lambda \in C^+ \setminus \{0\}$ such that $0 \in \partial_{\lambda^T\epsilon}(\lambda^T f)(\bar{x})$, which implies that there exists $\xi \in \partial_{\lambda^T\epsilon}(\lambda^T f)(\bar{x})$, such that

$$\xi = 0. \tag{3.4}$$

As well, since f is a C-convex function, and for all $\lambda \in C^+ \setminus \{0\}$, $\lambda^T f$ a is convex function. Obviously, there exists $\lambda \in C^+ \setminus \{0\}$ such that

$$\lambda^T f(x) - \lambda^T f(\bar{x}) \geqq \langle \xi, x - \bar{x} \rangle - \lambda^T \epsilon, \; \xi \in \partial_{\lambda^T\epsilon}(\lambda^T f)(\bar{x}).$$

By (3.4), we get

$$\lambda^T f(x) \geqq \lambda^T f(\bar{x}) - \lambda^T \epsilon. \tag{3.5}$$

On the other hand, suppose to the contrary that $\bar{x}$ is not a weakly C-ϵ-efficient solution of $(\mathrm{MP})_U$, then there exists $x^* \in \mathbb{R}^n$ with $x^* \neq \bar{x}$, such that

$$f(x^*) - (f(\bar{x}) - \epsilon) \in -\mathrm{int}C.$$

By Corollary 3.1, we have

$$\langle \lambda, f(x) - (f(\bar{x}) - \epsilon)\rangle < 0, \ \ \forall x \in \mathbb{R}^n,$$

which contradicts to (3.5). □

Theorem 3.5 *Suppose that f is a C-convex function and g_i, $i = 1, \ldots, m$, are convex functions. If there exist $\bar{\lambda} \in C^+ \backslash \{0\}$, $\bar{\epsilon}_0 \in C$, $\bar{\epsilon}_i \in C$, $i = 1, \ldots, m$, and $\bar{\mu}_i \geqq 0$, $i = 1, \ldots, m$, such that*

$$0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x}),$$

$$\sum_{i=0}^{m} \bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) \leqq 0,$$

where $\bar{\lambda}^T \epsilon = \alpha$, $\bar{\lambda}^T \epsilon_i = \alpha_i$, $i = 0, \ldots, m$, $\bar{\lambda}^T \bar{\epsilon}_i = \bar{\alpha}_i$, $i = 0, \ldots, m$, then $\bar{x}$ is a weakly C-ϵ-efficient solution of (MP).

Proof. Suppose to the contrary, $\bar{x}$ is not a weakly C-ϵ-efficient solution of (MP), then there exists another feasible point x^* such that

$$f(x^*) - (f(\bar{x}) - \epsilon) \in -\text{int}C.$$

By Corollary 3.1, we have for all feasible point x,

$$\langle \lambda, f(x) - (f(\bar{x}) - \epsilon)\rangle < 0. \tag{3.6}$$

On the other hand, since there exist $\bar{\lambda} \in C^+ \setminus \{0\}$, $\bar{\epsilon}_0 \in C$, $\bar{\epsilon}_i \in C$, $i = 1, \ldots, m$, and $\bar{\mu}_i \geqq 0$, $i = 1, \ldots, m$, such that

$$0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x}),$$

$$\sum_{i=0}^{m}\bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m}\bar{\mu}_i g_i(\bar{x}) \leqq 0, \tag{3.7}$$

where $\bar{\lambda}^T\epsilon = \alpha$, $\bar{\lambda}^T\epsilon_i = \alpha_i$, $i = 0, \ldots, m$, $\bar{\lambda}^T\bar{\epsilon}_i = \bar{\alpha}_i$, $i = 0, \ldots, m$, we have that there exist $\xi_0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x})$, $\xi_i \in \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x})$, $i = 1, \ldots, m$, such that

$$\xi_0 + \sum_{i=1}^{m}\xi_i = 0. \tag{3.8}$$

In addition, by C-convexity of f, following the argument on Theorem 3.4, we have

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) \geqq \langle \xi_0, x - \bar{x}\rangle - \bar{\alpha}_0,$$

as well

$$\bar{\mu}_i g_i(x) - \bar{\mu}_i g_i(\bar{x}) \geqq \langle \xi_i, x - \bar{x}\rangle - \bar{\alpha}_i,$$

then with the help of (3.8),

$$\begin{aligned} &\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) + \sum_{i=1}^{m}\bar{\mu}_i g_i(x) - \sum_{i=1}^{m}\bar{\mu}_i g_i(\bar{x}) \\ \geqq &\langle \xi_0 + \sum_{i=1}^{m}\xi_i, x - \bar{x}\rangle - (\bar{\alpha}_0 + \sum_{i=1}^{m}\bar{\alpha}_i) \\ = &- (\bar{\alpha}_0 + \sum_{i=1}^{m}\bar{\alpha}_i), \end{aligned}$$

since $g_i(x) \leqq 0$, $i = 1, \ldots, m$,

$$\begin{aligned}
\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) &\geqq \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) - (\bar{\alpha}_0 + \sum_{i=1}^{m} \bar{\alpha}_i) \\
&= \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) - \sum_{i=0}^{m} \bar{\alpha}_i \\
&\geqq \sum_{i=0}^{m} \bar{\alpha}_i - \alpha - \sum_{i=0}^{m} \bar{\alpha}_i = -\alpha, \ \text{(with the help of 3.7)}
\end{aligned}$$

which contradicts to (3.6). □

4 ϵ-Duality Relations

In this section, we establish a dual model in the sense of Wolfe and explore approximate weak and strong duality theorems.
Now we formulate the Wolfe type dual problem (MD) for (MP) as follows

$$\begin{array}{lll}
\text{(MD)} & \text{maximize}_{(y,\lambda,\mu)} & f(y) + \mu^T g(y) e \\
& \text{subject to} & 0 \in \partial_{\alpha_0}(\lambda^T f)(y) + \sum_{i=1}^{m} \partial_{\alpha_i}(\mu_i g_i)(y), \\
& & \lambda \in C^+ \setminus \{0\} \text{ with } \lambda^T e = 1, \\
& & \sum_{i=0}^{m} \alpha_i - \alpha \leqq 0, \\
& & (y, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m_+,
\end{array}$$

where $\alpha = \lambda^T \epsilon, \alpha_i = \lambda^T \epsilon_i, \epsilon_i \in C, i = 0, \ldots, m$. We denote by F_D is the feasible set of (MD). Let $L(y, \lambda, \mu) = f(y) + \mu^T g(y) e$.

Definition 4.1 A feasible point $(\bar{y}, \bar{\lambda}, \bar{\mu})$ is said to be a weakly C-ϵ-efficient solution of (MD) if there exists no another feasible point (y, λ, μ) such that $L(y, \lambda, \mu) >_C L(\bar{y}, \bar{\lambda}, \bar{\mu}) + \epsilon$, which is equivalent to, $L(y, \lambda, \mu) - L(\bar{y}, \bar{\lambda}, \bar{\mu}) - \epsilon \notin \text{int}C$.

Theorem 4.1 (**Approximate weak duality**) *Let x and (y, λ, μ) be feasible solution of* (MP) *and* (MD)*, respectively. Then*

$$f(x) - L(y, \lambda, \mu) + \epsilon \notin -\text{int}C.$$

Proof. Suppose to the contrary that

$$f(x) - L(y, \lambda, \mu) + \epsilon \in -\text{int}C,$$

i.e.,

$$f(x) - f(y) - \mu^T g(y)e + \epsilon \in -\text{int}C,$$

then there exists $\lambda \in C^+ \setminus \{0\}$ with $\lambda^T e = 1$ such that

$$\lambda^T f(x) - \lambda^T f(y) - \mu^T g(y) + \lambda^T \epsilon < 0. \tag{4.1}$$

Since (y, λ, μ) is a feasible solution of (MD), then there exist $\xi_0 \in \partial_{\alpha_0}(\lambda^T f)(y)$ and $\xi_i \in \partial_{\alpha_i}(\mu_i g_i)(y)$, $i = 1, \ldots, m$, $\sum_{i=0}^{m} \alpha_i - \alpha \leqq 0$, such that

$$0 = \xi_0 + \sum_{i=1}^{m} \xi_i.$$

Since $\xi_0 \in \partial_{\alpha_0}(\lambda^T f)(y)$ and $\xi_i \in \partial_{\alpha_i}(\mu_i g_i)(y)$, $i = 1, \ldots, m$,

$$\lambda^T f(x) - \lambda^T f(y) \geqq \langle \xi_0, x - y \rangle - \alpha_0$$

and

$$\mu_i g_i(x) - \mu_i g_i(y) \geqq \langle \xi_i, x-y \rangle - \alpha_i, \; i =, \ldots, m.$$

Thus

$$\begin{aligned} & \lambda^T f(x) - \lambda^T f(y) - \mu^T g(y) + \lambda^T \epsilon \\ \geqq \;& \langle \xi_0, x-y \rangle - \alpha_0 - \mu^T g(y) + \lambda^T \epsilon \\ = \;& \langle \xi_0, x-y \rangle - \alpha_0 + \mu^T g(x) - \mu^T g(y) - \mu^T g(x) + \lambda^T \epsilon \\ \geqq \;& \langle \xi_0, x-y \rangle - \alpha_0 + \langle \sum_{i=1}^m \xi_i, x-y \rangle - \sum_{i=1}^m \alpha_i - \mu^T g(x) + \lambda^T \epsilon \\ = \;& -\sum_{i=0}^m \alpha_i - \mu^T g(x) + \alpha (\text{where } \lambda^T \epsilon = \alpha) \\ \geqq \;& 0, \end{aligned}$$

which stands in contradiction to (4.1). □

Theorem 4.2 (**Approximate strong duality**) *Suppose that Slater constraint qualification is satisfied for* (MP). *If* $\bar{x}$ *is a weakly* C-ϵ-*efficient solution of* (MP), *then there exist* $\bar{\lambda} \in C^+ \setminus \{0\}$ *with* $\bar{\lambda}^T e = 1$, $\bar{\epsilon}_i \in C$, $i = 0, \ldots, m$ *and* $\bar{\mu}_i \geqq 0$, $i = 1, \ldots, m$, *such that* $(\bar{x}, \bar{\lambda}, \bar{\mu})$ *is a weakly* C-2ϵ-*efficient solution of* (MD).

Proof. Let $\bar{x}$ be a weakly C-ϵ-efficient solution of (MP). From Theorem 3.3, there exist $\bar{\lambda} \in C^+ \setminus \{0\}$ with $\bar{\lambda}^T e = 1$, $\bar{\epsilon}_i \in C$, $i = 0, \ldots, m$, and $\bar{\mu}_i \geqq 0$,

$i = 1, \ldots, m$, such that

$$0 \in \partial_{\bar{\alpha}_0}(\bar{\lambda}^T f)(\bar{x}) + \sum_{i=1}^{m} \partial_{\bar{\alpha}_i}(\bar{\mu}_i g_i)(\bar{x}),$$

$$\sum_{i=0}^{m} \bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) \leqq 0,$$

where $\bar{\lambda}^T \epsilon = \alpha$, $\bar{\lambda}^T \epsilon_i = \alpha_i$, $i = 0, \ldots, m$, $\bar{\lambda}^T \bar{\epsilon}_i = \bar{\alpha}_i$, $i = 0, \ldots, m$. So, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible point for (MD).

Now we will show that a feasible point $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly C-2ϵ-efficient solution of (MD), that is,

$$f(y) + \sum_{i=1}^{m} \mu_i g_i(y) e - [f(\bar{x}) + \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) e] - 2\epsilon \notin \text{int}C, \ \forall (y, \lambda, \mu) \in F_D.$$

Otherwise, suppose that there exists $(y, \lambda, \mu) \in F_D$ such that

$$f(y) + \sum_{i=1}^{m} \mu_i g_i(y) e - [f(\bar{x}) + \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x}) e] - 2\epsilon \in \text{int}C. \tag{4.2}$$

Then, multiplying both sides of (4.2) by $\bar{\lambda} \in C^+ \backslash \{0\}$ with $\bar{\lambda}^T e = 1$,

$$\bar{\lambda}^T f(y) + \sum_{i=1}^{m} \mu_i g_i(y) - [\bar{\lambda}^T f(\bar{x}) + \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{x})] - 2\alpha > 0. \tag{4.3}$$

Since $\sum_{i=0}^{m}\bar{\alpha}_i - \alpha \leqq \sum_{i=1}^{m}\bar{\mu}_i g_i(\bar{x})$, we have

$$\begin{aligned}
& \bar{\lambda}^T f(y) + \sum_{i=1}^{m}\mu_i g_i(y) - [\bar{\lambda}^T f(\bar{x}) + \sum_{i=1}^{m}\bar{\mu}_i g_i(\bar{x})] - 2\alpha \\
\leqq \ & \bar{\lambda}^T f(y) + \sum_{i=1}^{m}\mu_i g_i(y) - \bar{\lambda}^T f(\bar{x}) - \sum_{i=0}^{m}\bar{\alpha}_i - \alpha \\
\leqq \ & \bar{\lambda}^T f(y) + \sum_{i=1}^{m}\mu_i g_i(y) - \bar{\lambda}^T f(\bar{x}) - \alpha \leqq 0, \text{ (by using Theorem 4.1)}
\end{aligned}$$

which contradicts (4.3). Thus, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a weakly C-2ϵ-efficient solution of (MD). □

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