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Controllability and regularity for parabolic and hyperbolic equations

by

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Controllability and regularity for parabolic and hyperbolic equations
(방물형과 쌍곡형방정식의 정칙성과 제어성)

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A dissertation

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요 약

이 논문은 공학적, 물리학적으로 응용 가능한 1계 방물형 방정식과 2계 쌍곡형 방정식에 대한 해의 정칙성과 제어이론을 다루었다. 먼저, 시간지연을 가진 준선형 방물형방정식:

(E) $$\begin{cases} \dfrac{\partial}{\partial t}u(t,x)+A(x,D_x)u(t,x)+A_1(x,D_x)u(t-h,x) \\ \quad +\displaystyle\int_{-h}^{0}a(s)A_{2(x,D_x)u(t+s,x)ds}) \\ =F(t,u(t-h,x),\displaystyle\int_0^t k(t,s,u(s-h,x))ds)+f(t,x), \quad (t,x)\in[0,T]\times\Omega. \end{cases}$$

의 해의 존재성과 정칙성을 밝혔다. 이에 사용된 기본해를 이용하여 국소조건이 있는 준선형 방물형방정식을 수학적으로 해석하였다. 2계 비선형 변분부등식의 쌍곡형 함수미분방정식:

$$\begin{cases} (u''(t)+Au(t),u(t)-z)+\phi(u(t))-\phi(z)\le(f(t,u(t))+k(t),u(t)-z),a.e.,\forall z\in V \\ u(0)=u^0, \quad u'(0)=u^1 \end{cases}$$

의 선형성을 가진 변분부등식으로 변환을 통하여 해의 성질과 수학적 해석으로 일반적인 방물형의 변분부등식을 일반화하였다. (E) 방정식의 제어계(외압 $f(t,u)$ 대신에 제어기 와 제어로 이루어진 항으로 이루어진 방정식)가 포함된 제어이론으로서 $A(x,D_x)=A_0, \quad A_1(x,D_x)=\gamma A_0, \quad A_2(x,D_x)=A_0$ 인경우의 가제어성이 되기 위한 충분조건을 유도하여 기존 이론을 일반화하여 시간지연과 비선형 항을 포함한 비선형계의 수학적 해석과 응용가능성을 가능하게 하였다. 그리고 가제어성이 되지 않는 경우에는 주어진 목표에 도달 가능한 것 중 최적제어의 존재성과 최적시간을 규명하였다.

Chapter 1
Introduction and Preliminaries

the purpose of this paper is to give a systematic presentation of the theory of partial differential equations based mainly on the results from semigroups of linear operators. A semigroup theoretic development of a theory for the initial and mixed problems of parabolic hyperbolic equations is both powerful and beneficial since it enables one to investigate a broad class of various evolution functional differential equations. There have been two main objects of work in this paper. One of these is based on the retarded semilinear differential equations, which contain unbounded operators, nonlocal conditions, or nonlinear part involving integrodifferetial terms. Moreover, some results on the control problems for retarded functional differential equations of parabolic type with unbounded principal operators are constructed. The other is the regularity for nonlinear variational inequalities of second order in Hilbert spaces

In this paper, with semigroup theory, we study the wellposedness and control problems as linear or semilinear parabolic type and nonliner hyperbolic type equations on Hilbert spaces. Through this paper, we study for problems of differential system on two Hilbert space H and V such that V is a dense subspace of H.

Identifying the antidual of H with H we may consider $V \subset H \subset V^*$. and the injection of V into H is continous

The subject of Chapter 2 is concerned with the existence, uniqueness and norm estimations of solutions for a class of partial functional integrodifferential systems with delay terms:

$$\frac{\partial}{\partial t}u(t,x) + \mathcal{A}(x,D_x)u(t,x) + \mathcal{A}_1(x,D_x)u(t-h,x) \tag{ILE}$$

$$+ \int_{-h}^{0} a(s)\mathcal{A}_2(x,D_x)u(t+s,x)ds$$

$$= F(t, u(t-h,x), \int_0^t k(t,s,u(s-h,x))ds) + f(t,x), \quad (t,x) \in [0,T]\times\Omega.$$

Here, $\Omega \subset \mathcal{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathcal{A}(x,D_x)$ and $\mathcal{A}_\iota(x,D_x)(\iota = 1,\ 2)$ are second order linear differential operators with real coefficients, and $\mathcal{A}(x,D_x)$ is an elliptic operator in $\overline{\Omega}$. The function $a(s)$ is a real scalar function on $[-h,0]$, where $h > 0$ is a delay time and f is a forcing function. then, we can to establish a variation of constant formula and regularity property of solutions for the equation (ILE) with the aid of intermediate theory and the regularity for the corresponding linear equation(in case $F \equiv 0$). We can also see that the solution mapping $f \mapsto x_u$ is compact where x_f is a solution of (ILE) corresponding to the forcing term f which is an important rule to apply control and optimal problems.

In Chapter 3, we study the nonlocal initial value problem governed by retarded semilinear parabolic type equation in a Hilbert space as follows.

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds \\ \quad +f(t,x(t),x(b_1(t)),...,x(b_m(t))) + k(t),\ t \geq 0, \\ x(0) = g^0 - \phi(x), \quad x(s) = g^1(s) - e^s\phi(x), \quad -h \leq s < 0, \end{cases} \tag{NRE}$$

Let A_0 be the operator associated with a bounded sesquilinear form defined in $V \times V$ satisfying Gårding inequality. Then A_0 generates an analytic semigroup $S(t)$ in both H and V^* and so the equation (NRE) may be considered as an equation in both H and V^*. and equation (NRE) with unbounded principal operators and delay term. The operator A_1 is bounded linear from V to V^* The function $a(\cdot)$ is assumed to be a real valued and Hölder continuous in the interval $[-h, 0]$, and f, ϕ, $b_i (i = 1, \cdots, m)$ are given functions satisfying some assumptions. then, we obtain the regularity and existence of solutions of a retarded semilinear differential equation with nonlocal condition by applying Schauder's fixed point theorem. We construct the fundamental solution and establish the Hölder continuity results concerning the fundamental solution of its corresponding retarded linear equation and we prove the uniqueness of solutions of the given equation.

Chapter 4 is about the the initial value problem of the following nonlinear variational inequalities of second order in Hilbert spaces;

$$\begin{cases} (u''(t) + Au(t), u(t) - z) + \phi(u(t)) - \phi(z) \\ \quad \leq (f(t, u(t)) + k(t), u(t) - z), \text{ a.e., } \forall z \in V \\ u(0) = u^0, \quad u'(0) = u^1. \end{cases} \tag{NVE}$$

Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality, and let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. The nonlinear term $f(\cdot, u)$, which is a locally Lipschitz continuous operator with respect to u from V to H, and a forcing term $k \in L^2(0, T; V^*)$. we deal with the regularity for nonlinear variational inequalities of second order in Hilbert spaces with more general conditions on the nonlinear terms and without condition of the compactness of the principal operators. We also obtain the norm estimate of a solution of the given nonlinear equation on $C([0,T];V)\cap C^1((0,T];H)\cap C^2((0,T];V^*)$ by using the results of its corresponding the hyperbolic semilinear part.

In Chapter 5 is to construct some results on the control problems for the following retarded functional differential equation of parabolic type in a Hilbert space H:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t)=\mathcal{A}_0(x,D_x)u(x,t)+\mathcal{A}_1(x,D_x)u(x,t-h) \\ \quad +\int_{-h}^0 a(s)\mathcal{A}_2(x,D_x)u(x,t+s)ds+(B_0w(t))(x), \qquad (x,t)\in\Omega\times(0,T] \\ u(x,t)=0, \qquad x\in\partial\Omega,\ t\in(0,T], \\ u(x,0)=g^0(x),\ u(x,s)=g^1(x,s), \quad x\in\Omega,\ s\in[-h,0). \end{cases} \tag{CRE}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and h is some positive number. $\mathcal{A}_\iota(x,D_x)$, $\iota=0,\ 1,\ 2$, are second order linear differential operators with smooth coefficients in $\overline{\Omega}$, and $\mathcal{A}_0(x,D_x)$ is elliptic. We note that in order to guarantee the existence of fundamental solution of system (CRE), we must need the assumption that $a(\cdot)$ is Hölder continuous as seen in [7]. Let U be a Banach space of control variables and the controller B_0 be a bounded linear operator from U to $L^2(\Omega)$. then, we to establish relations between controllability of the given equation and observability of the adjoint system, we investigate the equivalent relation for the completeness of generalized eigenspaces of the infinitesimal generators. Finally, when the control space is a finite dimensional space, a necessary and sufficient for the approximate controllability of retarded equations is given as the so called Rank Condition.

Chapter 2
Regularity for semilinear retarded functional integrodifferential equations

2.1 Introduction

This paper is concerned with the existence, uniqueness and norm estimations of solutions for a class of partial functional integrodifferential systems with delay terms:

$$\frac{\partial}{\partial t}u(t,x) + \mathcal{A}(x,D_x)u(t,x) + \mathcal{A}_1(x,D_x)u(t-h,x) \tag{2.1.1}$$

$$+ \int_{-h}^{0} a(s)\mathcal{A}_2(x,D_x)u(t+s,x)ds$$

$$= F(t,u(t-h,x), \int_0^t k(t,s,u(s-h,x))ds) + f(t,x), \quad (t,x) \in [0,T]\times\Omega.$$

Here, $\Omega \subset \mathcal{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\mathcal{A}(x,D_x)$ and $\mathcal{A}_\iota(x,D_x)(\iota = 1,\ 2)$ are second order linear differential operators with real coefficients, and $\mathcal{A}(x,D_x)$ is an elliptic operator in $\overline{\Omega}$. The function $a(s)$ is a real scalar function on $[-h,0]$, where $h>0$ is a delay time and f is a forcing function. The boundary condition attached to (2.1.1) is given by Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \quad 0 < t \le T, \tag{2.1.2}$$

and the initial condition is given by

$$u(0,x) = g^0(x), \quad u(s,x) = g^1(s,x) \quad -h \le s \le 0. \tag{2.1.3}$$

Set

$$G(t,u) = F(t, u(t-h), \int_0^t k(t,s,u(s-h))ds).$$

The nonlinear term $G(t,\cdot)$, which is a Lipschitz continuous operator from $L^2(-h,T;V)$ to $L^2(-h,T;H)$, is a semilinear version of the quasilinear one considered in Yong and Pan [9]. Precise assumptions are given in the next section.

The abstract formulations of many partial integrodifferential equations arise in the mathematical description of the dynamical processes with heat flow in material with memory, viscoelasticity, and many physical phenomena (See [3, 4]). When $F \equiv 0$ in (2.1.1), this linear type of equations is studied extensively by Di Blasio et el. [2], Tanabe [7] and Jeong, Nakagiri [5, 6]. Most parts of previous results studied the regularity for nonlinear equations under conditions of the uniform boundedness of the nonlinear terms and the compactness of the principal operators.

The purpose of this paper is to establish a variation of constant formula and regularity property of solutions for the equation (2.1.1) with the aid of intermediate theory and the regularity for the corresponding linear equation(in case $F \equiv 0$). We can also see that the solution mapping $f \mapsto x_u$ is compact where x_f is a solution of (2.1.1) corresponding to the forcing term f which is an important rule to apply control and optimal problems.

In order to prove the solvability of the initial value problem (2.1.1) we establish necessary estimates applying the result of [2] to (2.1.1) considered as an equation in a Hilbert space. In this paper, we give preliminaries on linear equations, and then prove the local existence and uniqueness for solution of (2.1.1)-(2.1.3) by using the contraction principle. Finally, we establish the norm estimation of solutions by using the regularity for solutions associated with the linear part of the given equations and the global existence of solutions by the step by step method.

2.2 Preliminaries and local solutions

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. The norm of H(resp. V) is denoted by $|\cdot|$ (resp. $||\cdot||$) and the corresponding scalar product by $(\cdot,\cdot)(resp.((\cdot,\cdot)))$. Assume that the injection of V into H is continuous. The antidual of V is denoted by V^*, and the norm of V^* by $||\cdot||^*$. Identifying H with its antidual we may consider that H is embedded in V^*. Hence we have $V \subset H \subset V^*$ densely and continuously.

We realize the operator $\mathcal{A}(x, D_x)$, $\mathcal{A}_\iota(x, D_x)$, $\iota = 1,\ 2$, in Hilbert spaces by

$$A_0 v = -\mathcal{A}(x, D_x)v, \quad A_\iota v = -\mathcal{A}_\iota(x, D_x)v,\ \iota = 1,\ 2, \quad v \in V$$

in the distribution sense. The mixed problem (2.1.1) can be formulated abstractly as

$$
\begin{cases}
\frac{d}{dt}u(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds \\
\qquad\qquad +F(t, u(t-h), \int_0^t k(t,s,u(s-h))ds) + f(t), \quad 0 \le t \le T \\
u(0) = g^0, \quad u(s) = g^1(s), \quad -h \le s \le 0.
\end{cases}
\tag{SLE}
$$

Let $b(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$
\mathrm{Re}\ b(v,v) \ge c_0||v||^2 - c_1|v|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{2.2.1}
$$

Let A_0 be the operator associated with the sesquilinear form $-b(\cdot,\cdot)$:

$$
(A_0v_1, v_2) = -b(v_1, v_2), \quad v_1,\ v_2 \in V.
$$

A_0 is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A_0 to

$$
D(A_0) = \{v \in V; A_0v \in H\}
$$

is also denoted by A_0. Then A_0 generates an analytic semigroup in both of H and V^*(see [7]).

The operators A_1 and A_2 are bounded linear operators from V to V^* such that their restrictions to $D(A_0)$ are bounded linear operators from $D(A_0)$ equipped with the graph norm of A_0 to H. The function $a(\cdot)$ is assumed to be real valued and belongs to $L^2(-h, 0)$.

First, we consider some basic results on the following linear functional differential initial value problem:

$$\begin{cases} \frac{d}{dt}u(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^0 a(s)A_2u(t+s)ds + f(t), \\ u(0) = g^0, \quad u(s) = g^1(s) \quad -h \leq s \leq 0. \end{cases} \tag{LE}$$

By assumption there exists a positive constant M_0 such that

$$|v| \leq M_0||v||. \tag{2.2.2}$$

Then, for any $f \in H$ we have

$$||f||_* \leq M_0|f|. \tag{2.2.3}$$

It follows from (2.2.1) that for $u \in V$

$$\mathrm{Re}\,((c_1 - A_0)v, v) \geq c_0||v||^2.$$

Hence there exists a constant C_0 such that

$$||v|| \leq C_0||v||_{D(A_0)}^{1/2}|v|^{1/2} \tag{2.2.4}$$

for every $v \in D(A_0)$, where

$$||v||_{D(A_0)} = (|A_0v|^2 + |v|^2)^{1/2}$$

is the graph norm of $D(A_0)$.

Now, we introduce some basic notations. If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X whose p-th powers of norms are integrable and $W^{m,p}(0,T;X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0,T;X)$.

By virtue of Theorem 3.3 of [2] we have the following result on the corresponding linear equation of (LE).

Proposition 2.2.1. *Suppose that the assumptions stated above are satisfied. Then the following properties hold:*

1) Let $X = (D(A_0), H)_{1/2,2}$ *where* $(D(A_0), H)_{1/2,2}$ *is the real interpolation space between* $D(A_0)$ *and* H *(see [[8]; Section 1.3.3]). For* $(g^0, g^1) \in X \times L^2(-h, 0; D(A_0))$ *and* $f \in L^2(0, T; H)$, $T > 0$, *there exists a unique solution* u *of (LE) belonging to*

$$\mathcal{W}_0(T) \equiv L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; X)$$

and satisfying

$$||u||_{\mathcal{W}_0(T)} \leq C_1(||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + ||f||_{L^2(0,T;H)}), \tag{2.2.5}$$

where C_1 *is a constant depending on* T.

2) Let $(g^0, g^1) \in H \times L^2(-h, 0; V)$ *and* $f \in L^2(0, T; V^*)$, $T > 0$. *Then there exists a unique solution* u *of (LE) in case* $G(\cdot, u) \equiv 0$ *belonging to*

$$\mathcal{W}_1(T) \equiv L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||u||_{\mathcal{W}_1(T)} \leq C_1(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}), \tag{2.2.6}$$

where C_1 *is a constant depending on* T.

Given $u \in L^2(0, T; V)$ we extend it to the space $L^2(-h, T; V)$ by setting $u(s) = g^1(s)$ for $s \in (-h, 0)$.

We assume the following hypotheses on the nonlinear mappings F, k in (SLE):

(A1) $F : [0,T] \times L^2(0,T;V) \times H \to H$ is a nonlinear mapping such that for $\phi \in L^2(0,T;V)$ and $x \in H$, $F(t,\phi,x)$ is strongly measurable on $[0,T]$ and there exist positive constants L_0, L_1, L_2 and L_3 such that

$$|F(t,\phi_1,x_1) - F(t,\phi_2,x_2)| \le L_1||\phi_1 - \phi_2|| + L_2|x_1 - x_2|, \quad t \in [0,T].$$

(A2) Let $\Delta_T = \{(s,t) : 0 \le s \le t \le T\}$. Then $k : \Delta_T \times L^2(0,T;V) \to H$ is a nonlinear mapping such that for $x \in H$, $k(t,s,x)$ is strongly measurable on Δ_T and there exists positive constant L_3 such that

$$|k(t,s,x_1) - k(t,s,x_2)| \le L_3||x_1 - x_2||, \quad (s,t) \in \Delta_T.$$

(A3) $|F(t,0,0)| \le L_0, \quad |k(t,s,0)| \le L_0.$

Remark 2.2.1. *The above operator F is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [9].*

For $u \in L^2(-h,T;V)$, $T > 0$ we set

$$G(t,u) = F(t,u(t-h), \int_0^t k(t,s,u(s-h))ds).$$

Lemma 2.2.1. *Let $u \in L^2(-h,T;V)$ $T > 0$. Then $G(\cdot,u) \in L^2(0,T;H)$ and*

$$||G(\cdot,u)||_{L^2(0,T;H)} \le L_0\sqrt{T} + (L_1 + L_2L_3T/\sqrt{2})||u||_{L^2(-h,T-h;V)}. \qquad (2.2.7)$$

Moreover if $u_1,\ u_2 \in L^2(-h,T;V)$, then

$$||G(\cdot,u_1) - G(\cdot,u_2)||_{L^2(0,T;H)} \le (L_1 + L_2L_3T/\sqrt{2})||u_1 - u_2||_{L^2(-h,T-h;V)}. \qquad (2.2.8)$$

Proof. For $u \in L^2(-h, T; V)$, since

$$\int_0^T |\int_0^t k(t,s,u(s-h))ds|^2 dt \le L_3^2 \int_0^T (\int_0^t ||u(s-h)||ds)^2 dt$$
$$\le L_3^2 \int_0^T t \int_0^t ||u(s-h)||^2 ds dt$$
$$\le L_3^2 \frac{T^2}{2} \int_0^T ||u(s-h)||^2 ds,$$

from (A1) and (A2), it is easily seen that

$$||G(\cdot,u)||_{L^2(0,T;H)} = \{\int_0^T |F(t, u(t-h), \int_0^t k(t,s,u(s-h))ds)|^2 dt\}^{1/2}$$
$$= \{\int_0^T |F(t, u(t-h), \int_0^t k(t,s,u(s-h))ds) - F(t,0,0) + F(t,0,0)|^2 dt\}^{1/2}$$
$$\le \{\int_0^T |F(t, u(t-h), \int_0^t k(t,s,u(s-h))ds) - F(t,0,0)|^2 dt\}^{1/2} + L_0\sqrt{T}$$
$$\le L_0\sqrt{T} + L_1||u||_{L^2(-h,T-h;V)} + L_2\{\int_0^T |\int_0^t k(t,s,u(s-h))ds|^2 dt\}^{1/2}.$$

The proof of (2.2.8) is similar. □

Now we are ready to give the following result on the local solvability of (SLE).

Theorem 2.2.1. *Suppose that the assumptions (A1), (A2) and (A3) are satisfied. Then for any* $(g^0, g^1) \in H \times L^2(-h, 0; V)$ *and* $f \in L^2(0, T; V^*)$, $T > 0$, *there exists a time* $T_0 > 0$ *such that the functional differential equation (SLE) admits a unique solution* u *in* $\mathcal{W}_1(T_0) \equiv L^2(-h, T_0; V) \cap W^{1,2}(0, T_0; V^*)$.

Proof. Let us fix $T_0 > 0$ so that

$$M := C_0C_1(L_1 + L_2L_3T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2} < 1, \tag{2.2.9}$$

where C_0 and C_1 are constants in (2.2.4) and (2.2.5) respectively. Let w be the solution of

$$\frac{d}{dt}w(t) = A_0w(t) + A_1w(t-h) \tag{2.2.10}$$

$$+ \int_{-h}^{0} a(s)A_2w(t+s)ds + G(t,v) + f(t),$$

$$w(0) = g^0, \quad w(s) = g^1(s), \qquad s \in [-h, 0). \tag{2.2.11}$$

We are going to show that $v \mapsto w$ is strictly contractive from $L^2(0,T_0;V)$ to itself if the condition (2.2.9) is satisfied. Let $w_1,\ w_2$ be the solutions of (2.2.10), (2.2.11) with v replaced by $v_1,\ v_2 \in L^2(0,T_0;V)$, respectively. From (2.2.5) and (2.2.8) it follows that

$$||w_1 - w_2||_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)} \le C_1||G(\cdot,v_1) - G(\cdot,v_2)||_{L^2(0,T_0;H)}$$

$$\le C_1(L_1 + L_2L_3\frac{T_0}{\sqrt{2}})||v_1 - v_2||_{L^2(0,T_0;V)},$$

and hence in view of (2.2.4) we have

$$||w_1 - w_2||_{L^2(0,T_0;V)} \le C_0||w_1 - w_2||^{\frac{1}{2}}_{L^2(0,T_0;D(A_0))}||w_1 - w_2||^{\frac{1}{2}}_{L^2(0,T_0;H)} \tag{2.2.12}$$

$$\le C_0||w_1 - w_2||^{\frac{1}{2}}_{L^2(0,T_0;D(A_0))}(\frac{T_0}{\sqrt{2}})^{\frac{1}{2}}||w_1 - w_2||^{\frac{1}{2}}_{W^{1,2}(0,T_0;H)}$$

$$\le C_0(\frac{T_0}{\sqrt{2}})^{\frac{1}{2}}||w_1 - w_2||_{L^2(0,T_0;D(A_0))\cap W^{1,2}(0,T_0;H)}$$

$$\le C_0C_1(L_1 + L_2L_3\frac{T_0}{\sqrt{2}})(\frac{T_0}{\sqrt{2}})^{1/2}||v_1 - v_2||_{L^2(0,T_0;V)}.$$

Here we used the following inequality

$$\begin{aligned}||w_1 - w_2||_{L^2(0,T_0;H)} &= \{\int_0^{T_0} |w_1(t) - w_2(t)|^2 dt\}^{\frac{1}{2}} \\ &= \{\int_0^{T_0} |\int_0^t (\dot{w}_1(\tau) - \dot{w}_2(\tau))d\tau|^2 dt\}^{\frac{1}{2}} \\ &\le \{\int_0^{T_0} t\int_0^t |\dot{w}_1(\tau) - \dot{w}_2(\tau)|^2 d\tau dt\}^{\frac{1}{2}} \\ &\le \frac{T_0}{\sqrt{2}}||w_1 - w_2||_{W^{1,2}(0,T_0;H)}.\end{aligned}$$

So by virtue of (2.2.9) the contraction mapping principle gives that equation (SLE) has a unique solution in $[-h, T_0]$. □

2.3 Global existence and behavior of solution

In this section we give norm estimate of the solution of (SLE) and which is helpful to establish the global existence of solutions with the aid of norm estimations.

Theorem 2.3.1. *Suppose that the assumptions (A1), (A2) and (A3) are satisfied. Then for any* $(g^0, g^1) \in H \times L^2(-h, 0; V)$ *and* $f \in L^2(0, T; V^*)$, $T > 0$, *the solution* u *of (SLE) exists and is unique in* $\mathcal{W}_1(T) \equiv L^2(-h, T; V) \cap W^{1,2}(0, T; V^*)$, *and there exists a constant* C_2 *depending on* T *such that*

$$||u||_{\mathcal{W}_1(T)} \le C_2(1 + |g^0| + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T;V^*)}). \tag{2.3.1}$$

Proof Let $u(\cdot)$ be the solution of (SLE) in the interval $[-h, T_0]$ where T_0 is a constant in (2.2.9) and $w(\cdot)$ be the solution of the following equation

$$\frac{d}{dt}w(t) = A_0 w(t) + A_1 w(t-h) + \int_{-h}^{0} a(s) A_2 w(t+s) ds + f(t),$$

$$w(0) = g^0, \quad w(s) = g^1(s), \qquad -h \le s < 0.$$

Then in view of (2.2.5), (2.2.7)

$$||u - w||_{L^2(0,T_0;D(A_0)) \cap W^{1,2}(0,T_0;H)} \le C_1 ||G(\cdot, u)||_{L^2(0,T_0;H)}$$

$$\le C_1\{L_0\sqrt{T}_0 + (L_1 + L_2 L_3 T_0/\sqrt{2})(||u||_{L^2(0,T_0;V)}$$

$$+ ||g^1||_{L^2(-h,0;V)})\}.$$

$$\le C_1\{L_0\sqrt{T}_0 + (L_1 + L_2 L_3 T_0/\sqrt{2})(||u - w||_{L^2(0,T_0;V)}$$

$$+ ||w||_{L^2(0,T_0;V)} + ||g^1||_{L^2(-h,0;V)})\}.$$

Thus, arguing as in the proof of (2.2.12)

$$||u - w||_{L^2(0,T_0;V)} \le C_0 (\frac{T_0}{\sqrt{2}})^{\frac{1}{2}} ||u - w||_{L^2(0,T_0;D(A_0)) \cap W^{1,2}(0,T_0;H)}$$

$$\le C_0 (\frac{T_0}{\sqrt{2}})^{\frac{1}{2}} C_1\{L_0\sqrt{T}_0 + (L_1 + L_2 L_3 T_0/\sqrt{2})(||u - w||_{L^2(0,T_0;V)}$$

$$+ ||w||_{L^2(0,T_0;V)} + ||g^1||_{L^2(-h,0;V)})\}.$$

For brevity, set

$$M := C_0 C_1 (L_1 + L_2 L_3 T_0/\sqrt{2})(T_0/\sqrt{2})^{1/2}$$

in the sense of (2.2.9). Therefore, we have

$$||u - w||_{L^2(0,T_0;V)} \le \frac{C_0 C_1 L_0 \sqrt{T_0}(T_0/\sqrt{2})^{1/2} + M(||w||_{L^2(0,T_0;V)} + ||g^1||_{L^2(-h,0;V)})}{1 - M}$$

and hence, with the aid of 2) of Proposition 2.2.1

$$||u||_{L^2(0,T_0;V)} \le \frac{C_0C_1L_0\sqrt{T_0}(T_0/\sqrt{2})^{1/2}}{1-M} + \frac{||w||_{L^2(0,T_0;V)} + ||g^1||_{L^2(-h,0;V)}}{1-M} \tag{2.3.2}$$

$$\le \frac{C_0C_1L_0\sqrt{T_0}(T_0/\sqrt{2})^{1/2}}{1-M}$$

$$+ \frac{1}{1-M}\{C_1(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T_0;V^*)}) + ||g^1||_{L^2(-h,0;V)}\}.$$

On the other hand using (2.2.6), (2.2.3), (2.2.7) we get

$$||u||_{\mathcal{W}_1(T_0)\equiv} \le C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||G(\cdot,u) + f||_{L^2(0,T_0;V^*)}) \tag{2.3.3}$$

$$\le C(|g^0| + ||g^1||_{L^2(-h,0;V)} + M_0||G(\cdot,u)||_{L^2(0,T_0;H)} + ||f||_{L^2(0,T_0;V^*)})$$

$$\le C[|g^0| + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T_0;V^*)}$$

$$+ M_0\{L_0\sqrt{T}_0 + (L_1 + L_2L_3T_0/\sqrt{2})(||u||_{L^2(0,T_0;V)}$$

$$+ ||g^1||_{L^2(-h,0;V)})\}]$$

for some constant C. Combining (2.3.2), and (2.3.3) we obtain

$$||u||_{\mathcal{W}_1(T_0)} \le C(1 + |g^0| + ||g^1||_{L^2(-h,0;V)} + ||f||_{L^2(0,T_0;V^*)}) \tag{2.3.4}$$

for some constant C_2. Since the condition (2.2.9) is independent of the initial values, the solution of (SLE) can be extended to the interval $[-h, nT_0]$ for every natural number n. An estimate analogous to (2.3.4) holds for the solution in $[-h, nT_0]$, and hence for the initial value $(u(nT_0), u_{nT_0})$ in the interval $[nT_0, (n+1)T_0]$. □

Theorem 2.3.2. *Suppose that the assumptions (A1), (A2) and (A3) are satisfied. If* $(g^0, g^1) \in X \times L^2(-h, 0; D(A_0))$ *and* $f \in L^2(0, T; H)$*, then* $u \in \mathcal{W}_0(T) \equiv L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H)$*, and the mapping* $(g^0, g^1, f) \mapsto u \in \mathcal{W}_0(T)$ *is continuous.*

Proof It is easy to show that if $(g^0, g^1) \in X \times L^2(-h, 0; D(A_0))$ and $f \in L^2(0, T; H)$, then from Proposition 2.2.1 it follows that u belongs to $\mathcal{W}_0(T)$. Let $(g_i^0, g_i^1, f_i) \in X \times L^2(-h, 0; D(A_0)) \times L^2(0, T; H)$, and u_i be the solution of (SLE) with (g_i^0, g_i^1, f_i) in place of (g^0, g^1, f) for $i = 1,\ 2$. Then in view of Proposition 2.2.1 and Lemma 2.2.1 we have

$$
\begin{aligned}
||u_1 - u_2||_{\mathcal{W}_0(T))} &\le C_1\{||g_1^0 - g_2^0||_X \qquad\qquad (2.3.5)\\
&+ ||g_1^1 - g_2^1||_{L^2(-h,0:D(A_0))} + ||G(\cdot, u_1) - G(\cdot, u_2)||_{L^2(0,T;H)}\\
&+ ||f_1 - f_2||_{L^2(0,T;H)}\}\\
&\le C_1\{||g_1^0 - g_2^0||_X + ||g_1^1 - g_2^1||_{L^2(-h,0:D(A_0))} + ||f_1 - f_2||_{L^2(0,T;H)}\\
&+ (L_1 + L_2 L_3 T/\sqrt{2})(||u_1 - u_2||_{L^2(0,T:V)} + ||g_1^1 - g_2^1||_{L^2(-h,0;V)})\}.
\end{aligned}
$$

Since

$$u_1(t) - u_2(t) = g_1^0 - g_2^0 + \int_0^t (\dot{u}_1(s) - \dot{u}_2(s))ds,$$

we get

$$||u_1 - u_2||_{L^2(0,T;H)} \le \sqrt{T}|g_0^1 - g_2^0| + \frac{T}{\sqrt{2}}||u_1 - u_2||_{W^{1,2}(0,T;H)}.$$

Hence, arguing as in (2.2.12) we get

$$||u_1 - u_2||_{L^2(0,T;V)} \le C_0||u_1 - u_2||^{1/2}_{L^2(0,T;D(A_0))}||u_1 - u_2||^{1/2}_{L^2(0,T;H)} \tag{2.3.6}$$

$$\le C_0||u_1 - u_2||^{1/2}_{L^2(0,T;D(A_0))}$$

$$\times \{T^{1/4}|g_1^0 - g_2^0|^{1/2} + (\frac{T}{\sqrt{2}})^{1/2}||u_1 - u_2||^{1/2}_{W^{1,2}(0,T;H)}\}$$

$$\le C_0T^{1/4}|g_1^0 - g_2^0|^{1/2}||u_1 - u_2||^{1/2}_{L^2(0,T;D(A_0))} + C_0(\frac{T}{\sqrt{2}})^{1/2}||u_1 - u_2||_{\mathcal{W}_0(T)}$$

$$\le 2^{-7/4}C_0|g_1^0 - g_2^0| + 2C_0(\frac{T}{\sqrt{2}})^{1/2}||u_1 - u_2||_{\mathcal{W}_0(T)}.$$

Combining (2.3.5) and (2.3.6) we obtain

$$||u_1 - u_2||_{\mathcal{W}_0(T)} \le C_1\{||g_1^0 - g_2^0||_X + ||g_1^1 - g_2^1||_{L^2(-h,0:D(A_0))} \tag{2.3.7}$$

$$+ ||f_1 - f_2||_{L^2(0,T;H)} + (L_1 + L_2L_3T/\sqrt{2})||g_1^1 - g_2^1||_{L(-h,0;V))}\}$$

$$+ 2^{-7/4}C_0C_1(L_1 + L_2L_3T/\sqrt{2})|g_1^0 - g_2^0| + 2C_0C_1(\frac{T}{\sqrt{2}})^{1/2}$$

$$\times (L_1 + L_2L_3T/\sqrt{2})||u_1 - u_2||_{\mathcal{W}_0(T)}.$$

Suppose that $(g_n^0, g_n^1, f_n) \to (g^0, g^1, f)$ in $X\times L^2(-h, 0; D(A_0))\times L^2(0, T; H)$, and let u_n and u be the solutions (SLE) with (g_n^0, g_n^1, f_n) and (g^0, g^1, f) respectively. Let $0 < T_1 \le T$ be such that

$$2C_0C_1(T_1/\sqrt{2})^{1/2}(L_1 + L_2L_3T_1/\sqrt{2}) < 1.$$

Then by virtue of (2.3.7) with T replaced by T_1 we see that $u_n \to u$ in $\mathcal{W}_0(T_1)$. This implies that $(u_n(T_1), (u_n)_{T_1}) \mapsto (u(T_1), u_{T_1})$ in $X\times L^2(-h, 0; D(A_0))$.

Hence the same argument shows that $u_n \to u$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A_0)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $u_n \to u$ in $\mathcal{W}_0(T)$. □

Theorem 2.3.3. *For $f \in L^2(0,T;H)$ let u_f be the solution of equation (SLE). Let us assume that the embedding $D(A_0) \subset V$ is compact. Then the mapping $f \mapsto u_f$ is compact from $L^2(0,T;H)$ to $L^2(0,T;V)$.*

Proof If $f \in L^2(0,T;H)$, then in view of Theorem 2.3.1

$$||u_f||_{\mathcal{W}_1(T)} \le C_2(1 + |g^0| + ||g^1||_{L^2(-h,0;V)} + M_0||f||_{L^2(0,T;H)}). \tag{2.3.8}$$

Since $u_f \in L^2(0,T;V)$, $G(\cdot, u_f) \in L^2(0,T;H)$. Consequently $u_f \in L^2(0,T;D(A_0) \cap W^{1,2}(0,T^*;V)$ and with aid of Proposition 2.2.1, Lemma 2.2.1, and (2.3.8),

$$\begin{aligned} &||u_f||_{L^2(0,T;D(A_0)\cap W^{1,2}(0,T;V)} \qquad (2.3.9)\\ &\le C_1(||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + ||G(\cdot, u_f) + f||_{L^2(0,T;H)})\\ &\le C_1\{||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + L_0\sqrt{T}\\ &\quad + (L_1 + L_2L_3T/\sqrt{2})||u||_{L^2(-h,T-h;V)} + ||f||_{L^2(0,T;H)}\}\\ &\le C_1[||g^0||_X + ||g^1||_{L^2(-h,0;D(A_0))} + L_0\sqrt{T}\\ &\quad + (L_1 + L_2L_3T/\sqrt{2})\{||g^1||_{L^2(-h,0;V)} + C_2(1 + M_0||f||_{L^2(0,T;H)})\}\\ &\quad + ||f||_{L^2(0,T;H)}]. \end{aligned}$$

Hence if f is bounded in $L^2(0,T;H)$, then so is u_f in $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)$. Since $D(A_0)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A_0) \cap W^{1,2}(0,T;V) \subset L^2(0,T;V)$$

is compact in view of Theorem 2 of J. P. Aubin [1]. □

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Chapter 3
Semilinear nonlocal differential equations with delay terms

3.1 Introduction

In this paper we deal with the nonlocal initial value problem governed by retarded semilinear parabolic type equation in a Hilbert space as follows.

$$\begin{cases} \frac{d}{dt}x(t) = A_0 x(t) + \int_{-h}^{0} a(s)A_1 x(t+s)ds \\ \quad +f(t, x(t), x(b_1(t)), ..., x(b_m(t))) + k(t), \ t \geq 0, \\ x(0) = g^0 - \phi(x), \quad x(s) = g^1(s) - e^s\phi(x), \quad -h \leq s < 0, \end{cases} \tag{NRE}$$

Let H and V be complex Hilbert spaces such that the embedding $V \subset H$ is continuous. Let A_0 be the operator associated with a bounded sesquilinear form defined in $V \times V$ satisfying Gårding inequality. Then A_0 generates an analytic semigroup $S(t)$ in both H and V^* and so the equation (NRE) may be considered as an equation in both H and V^*. The operator A_1 is bounded linear from V to V^* such that its restriction to $D(A_0)$ is bounded linear operator from $D(A_0)$ to H. The function $a(\cdot)$ is assumed to be a real valued and Hölder continuous in the interval $[-h, 0]$, and f, ϕ, $b_i(i = 1, \cdots, m)$ are given functions satisfying some assumptions.

In view of the maximal regularity result by Di Blasio, Kunisch and Sinestrari [2] the retarded functional differential equation of parabolic type

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds + k(t), \\ x(0) = g^0, \quad x(s) = g^1(s), \quad -h \le s < 0 \end{cases} \tag{RE}$$

has unique solution x in the class $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)$ (or see [10, 6] in case the class $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$). There are many papers which studied for the existence of solutions of nonlocal abstract initial value problems without delay(see the bibliographies of [11, 12]. Results about the existence of mild and classical solutions of nonlocal Cauchy problem for a semilinear functional differential evolution equation was obtained by Byszewski and Akca [12].

In recent year, Obukhovski and Zecca [13] discussed the controllability for system governed by semilinear differential inclusions in a Banach space with noncompact semigroup and Xue [14, 15] studied Semilinear nonlocal problems without the assumptions of compactness in Banach spaces. Zhu et el. [16] concerned with impulsive differential equations with nonlocal condition in general Banach spaces.

In this paper, we extend these results to the equation (NRE) with unbounded principal operators and delay term. Let $W(\cdot)$ be the fundamental solution of the linear equation associated with (RE) which is defined to be the operator valued function satisfying

$$\frac{d}{dt}W(t) = A_0W(t) + \int_{-h}^{0} a(s)A_1W(t+s)ds,$$

$$W(0) = I, \quad W(s) = 0, \quad s \in [-h, 0).$$

The fundamental solution enables us to solve the equation (NRE). For the basis of our arguments, we construct the fundamental solution in the sense of Nakagiri [17] to (RE) and establish the Hölder continuity results concerning the fundamental solution $W(t)$ of the equation (RE) and obtain the regularity and existence of solutions of (NRE) by applying Schauder's fixed point theorem. According to Tanabe [18, Theorem 1], we will also prove the uniqueness of solutions of the equation (NRE).

3.2 Semilinear equation and its fundamental solution

The inner product and norm in H are denoted by $(\cdot, \cdot)$ and $|\cdot|$. V is another Hilbert space densely and continuously embedded in H. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. For brevity we may regard that

$$||u||_* \le |u| \le ||u||, \quad u \in V. \tag{3.2.1}$$

Let $B(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } B(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{3.2.2}$$

Let A_0 be the operator associated with the sesquilinear form $-B(\cdot, \cdot)$:

$$(A_0u, v) = -B(u, v), \quad u,\ v \in V.$$

It follows from (3.2.2) that for every $u \in V$

$$\text{Re}\,((c_1 - A_0)u, u) \ge c_0||u||^2.$$

Then A_0 is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V; A_0 u \in H\}$$

is also denoted by A_0. Then A_0 generates an analytic semigroup $S(t) = e^{tA_0}$ in both H and V^* as in Theorem 3.6.1 of [19]. Hence we may assume that $0 \in \rho(A_0)$ according to the Lax-Milgram theorem where $\rho(A_0)$ denotes the resolvent set of A_0. Moreover, there exists a constant C_0 such that

$$||u|| \le C_0 ||u||_{D(A_0)}^{1/2} |u|^{1/2}, \tag{3.2.3}$$

for every $u \in D(A_0)$, where

$$||u||_{D(A_0)} = (|A_0 u|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A_0)$.

For the sake of simplicity we assume that $S(t)$ is uniformly bounded. Then

$$|S(t)| \le M_0,\ |A_0 S(t)| \le M_0/t,\ |A_0^2 S(t)| \le M_0/t^2,\ t > 0 \tag{3.2.4}$$

for some constant M_0(e.g., [15]). We also assume that $a(\cdot)$ is Hölder continuous of order ρ:

$$|a(\cdot)| \le H_0, \quad |a(s) - a(\tau)| \le H_1 (s - \tau)^\rho \tag{3.2.5}$$

for some constants H_0, H_1.

Lemma 3.2.1. *For* $0 < s < t$ *and* $0 < \alpha < 1$

$$|S(t) - S(s)| \le \frac{M_0}{\alpha}(\frac{t-s}{s})^\alpha, \tag{3.2.6}$$

$$|A_0S(t) - A_0S(s)| \le M_0(t-s)^\alpha s^{-\alpha-1}. \tag{3.2.7}$$

Proof. From (3.2.4) for $0 < s < t$

$$|S(t) - S(s)| = |\int_s^t A_0S(\tau)d\tau| \le M_0 \log\frac{t}{s}. \tag{3.2.8}$$

It is easily seen that for any $t > 0$ and $0 < \alpha < 1$

$$\log(1+t) = \int_1^{1+t}\frac{1}{s}ds \le \int_1^{1+t}\frac{1}{s^{1-\alpha}}ds \tag{3.2.9}$$

$$= \frac{1}{\alpha}\{(1+t)^\alpha - 1\} \le t^\alpha/\alpha.$$

Combining (3.2.9) with (3.2.8) we get (3.2.6). For $0 < s < t$

$$|A_0S(t) - A_0S(s)| = |\int_s^t A_0^2S(\tau)d\tau| \le M_0(t-s)/ts. \tag{3.2.10}$$

Noting that $(t-s)/t \le ((t-s)/s)^\alpha$ for $0 < \alpha < 1$, we obtain (3.2.7) from (3.2.10). $\square$

First, we introduce the following linear retarded functional differential equation:

$$\frac{d}{dt}x(t) = A_0x(t) + \int_{-h}^0 a(s)A_1x(t+s)ds + k(t).$$

Let $W(\cdot)$ be the fundamental solution of the above linear equation in the sense of Nakagiri [17], which is the operator valued function satisfying

$$\begin{cases} \frac{d}{dt}W(t) = A_0W(t) + \int_{-h}^{0} a(s)A_1W(t+s)ds, \\ W(0) = I, \quad W(s) = 0, \quad s \in [-h, 0). \end{cases}$$

According to Duhamel's principle, the problem mentioned above is transformed to the following integral equation:

$$\begin{cases} W(t) = S(t) + \int_0^t S(t-s)\int_{-h}^{0} a(\tau)A_1W(s+\tau)d\tau ds, \quad t > 0, \\ W(0) = I, \qquad W(s) = 0, \quad -h \le s < 0. \end{cases} \tag{3.2.11}$$

where $S(\cdot)$ is the semigroup generated by A_0. Then

$$\begin{cases} x(t) = W(t)(g^0 - \phi(x)) + \int_{-h}^{0} U_t(s)(g^1(s) - e^s\phi(x))ds \\ \qquad + \int_0^t W(t-s)\{f(s, x(s), x(b_1(s)), ..., x(b_m(s))) + k(s)\}ds, \\ U_t(s) = \int_{-h}^{s} W(t-s+\sigma)a(\sigma)A_1 d\sigma. \end{cases} \tag{3.2.12}$$

Recalling the formulation of mild solutions, we know that the mild solution of (RE) is also represented by

$$x(t) = \begin{cases} S(t)(g^0 - \phi(x)) + \int_0^t S(t-s)\{\int_{-h}^{0} a(\tau)A_1x(s+\tau)d\tau \\ +f(s, x(s), x(b_1(s)), ..., x(b_m(s))) + k(s)\}ds, \ 0 \le t \\ g^1(s) - e^s\phi(x), \quad -h \le s < 0. \end{cases}$$

According to H. Tanabe [18] we set

$$V(t) = \begin{cases} A_0(W(t) - S(t)), \quad t \in (0, h] \\ A_0W(t), \quad t \in (nh, (n+1)h], \ n = 1, 2, ... \end{cases} \tag{3.2.13}$$

For $0 < t \leq h$

$$W(t) = S(t) + A_0^{-1} V(t)$$

and from (3.2.11) we have

$$W(t) = S(t) + \int_0^t \int_\tau^t S(t-s) a(\tau - s) ds A_1 W(\tau) d\tau.$$

Hence,

$$V(t) = V_0(t) + \int_0^t A_0 \int_\tau^t S(t-s) a(\tau - s) ds A_1 A_0^{-1} V(\tau) d\tau$$

where

$$V_0(t) = \int_0^t A_0 \int_\tau^t S(t-s) a(\tau - s) ds A_1 S(\tau) d\tau.$$

For $nh \leq t \leq (n+1)h (n = 1, 2, \ldots)$ the fundamental solution $W(t)$ is represented by

$$\begin{aligned} W(t) =& S(t) + \int_0^{t-h} \int_\tau^{\tau+h} S(t-s) a(\tau - s) ds A_1 W(\tau) d\tau \\ &+ \int_{t-h}^{nh} \int_\tau^t S(t-s) a(\tau - s) ds A_1 W(\tau) d\tau \\ &+ \int_{nh}^t \int_\tau^t S(t-s) a(\tau - s) ds A_1 W(\tau) d\tau. \end{aligned}$$

The integral equation to be satisfied by (3.2.13) is

$$V(t) = V_0(t) + \int_{nh}^t A_0 \int_\tau^t S(t-s) a(\tau - s) ds A_1 A_0^{-1} V(\tau) d\tau$$

where

$$V_0(t) = A_0S(t) + \int_0^{t-h} A_0 \int_\tau^{\tau+h} S(t-s)a(\tau-s)dsA_1W(\tau)d\tau$$

$$+ \int_{t-h}^{nh} A_0 \int_0^t S(t-s)a(\tau-s)dsA_1W(\tau)d\tau.$$

Thus, the integral equation (3.2.13) can be solved by successive approximation and $V(t)$ is uniformly bounded in $[nh,(n+1)h]$:

$$\sup_{nh\le t\le (n+1)h} |V(t)| < \infty, \quad n = 0, 1, 2, ...$$

It is not difficult to show that for $n > 1$

$V_0(nh+0) \ne V_0(nh-0)$, $W(nh+0) = W(nh-0)$ and $V(nh+0) = V(nh-0)$.

Lemma 3.2.2. *There exists a constant $C'_n > 0$ such that*

$$|\int_{nh}^t a(\tau-s)A_1W(\tau)d\tau| \le C'_n \tag{3.2.14}$$

for $n = 0, 1, 2,\ ...,\ t \in [nh,(n+1)h]$ and $t \le s \le t+h$.

Proof. For $t \in (0,h]$(i.e.,$n = 0$), from (3.2.13) it follows

$$\int_0^t a(\tau-s)A_1W(\tau)d\tau = \int_0^t a(\tau-s)A_1A_0^{-1}(A_0S(\tau)+V(\tau))d\tau$$

$$= \int_0^t (a(\tau-s)-a(s))A_1A_0^{-1}A_0S(\tau)d\tau + a(s)A_1A_0^{-1}(S(t)-I)$$

$$+ \int_0^t a(\tau-s)A_1A_0^{-1}V(\tau)d\tau.$$

Noting that

$$|\int_0^t (a(\tau - s) - a(s))A_1 A_0^{-1} A_0 S(\tau)d\tau| \le M_0 H_1 |A_1 A_0^{-1}| \int_0^t \tau^{\rho-1} d\tau,$$

we have

$$\begin{aligned} |\int_0^t a(\tau - s)A_1 W(\tau)d\tau| \le & |A_1 A_0^{-1}|\{\rho^{-1} h^\rho M_0 H_1 + H_0(M_0 + 1) \\ & + hH_0(\sup_{0\le t\le h} |V(t)|)\}. \end{aligned}$$

Thus the assertion (3.2.14) holds in $[0, h]$. For $t \in [nh, (n+1)h],\ n \ge 1$,

$$\int_{nh}^t a(\tau - s)A_1 W(\tau)d\tau = \int_{nh}^t a(\tau - s)A_1 A_0^{-1} V(\tau)d\tau.$$

The term of the right of the above equality is estimated as

$$|\int_{nh}^t a(\tau - s)A_1 A_0^{-1} V(\tau)d\tau| \le hH_0 |A_1 A_0^{-1}|(\sup_{nh\le t\le (n+1)h} |V(t)|).$$

Hence, we get the assertion (3.2.14). □

Proposition 3.2.1. *The fundamental solution* $W(t)$ *of (RE) exists uniquely. For* $0 < t < t' \le nh$, $n > 1$, *there exists a constant* $C_n > 0$ *such that*

$$|W(t') - W(t)| \le C_n (t' - t)^\alpha, \quad 0 < \alpha < 1. \tag{3.2.15}$$

Proof. The existence and uniqueness of the fundamental solution $W(t)$ of (RE) is due to Tanabe [18]. With the aid of suitable changes of variables, from (3.2.11) we obtain

$$W(t) = \begin{cases} S(t) + \int_0^t S(t-s) \int_0^s a(\tau - s) A_1 W(\tau) d\tau ds, \ 0 < t \le h, \\ S(t) + \int_0^t S(t-s) \int_{s-h}^s a(\tau - s) A_1 W(\tau) d\tau ds, \ h < t. \end{cases}$$

For $0 < t \le h$, since

$$\begin{aligned} |W(t') - W(t)| \le & |S(t') - S(t)| \\ & + |\int_0^t (S(t'-s) - S(t-s)) \int_0^s a(\tau - s) A_1 W(\tau) d\tau ds| \\ & + |\int_t^{t'} S(t'-s) \int_0^s a(\tau - s) A_1 W(\tau) d\tau ds|, \end{aligned}$$

from Lemmas 3.2.1, 3.2.2 it follows that

$$|W(t') - W(t)| \le \text{const.}(\frac{t'-t}{t})^\alpha \le C_n (t'-t)^\alpha, \quad 0 < \alpha < 1.$$

For $h < t$, we get (3.2.15) by the similar way. □

Considering as an equation in V^* we also obtain the same norm estimates of (3.2.4)-(3.2.7), (3.2.15) in the space V^*. By virtue of Theorem 3.3 of [10], [11] we have the following regularity results on the retarded linear equation (RE).

Proposition 3.2.2. *1) Let $F := (D(A_0), H)_{\frac{1}{2},2}$ where $(D(A_0), H)_{1/2,2}$ denote the real interpolation space between $D(A_0)$ and H. For $(g^0, g^1) \in F \times$*

$L^2(-h, 0; D(A_0))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (RE) belonging to

$$L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; F)$$

and satisfying

$$||x||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} \leq C_T(||g^0||_F + ||g^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}), \tag{3.2.16}$$

where C_T is a constant depending on T.

2) Let $(g^0, g^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (RE) belonging to

$$L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_T(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}). \tag{3.2.17}$$

3.3 Existence and uniqueness of solutions

In this section we investigate the regularity for solutions of the equation (NRE) with the operator A_0 associated with the sesquilinear form $-B(\cdot, \cdot)$ satisfying Gårding's inequality

$$\text{Re } B(u, u) \geq c_0||u||^2, \quad c_0 > 0.$$

Hence, we have $0 \in \rho(A_0)$ In what follows this paper, we assume that embedding $D(A_0) \hookrightarrow V$ is compact. Then $A_0^{-1} : H \to D(A_0) \hookrightarrow V \hookrightarrow H$ is compact. It is equivalent that the semigroup $S(t)$ is completely continuous [20, Corollary 3.4], and hence $W(t)$ defined as (3.2.11) is completely continuous(for more information of the fundamental solution refers [21, Proposition 3.1] or [22, Lemma 2.4]). For brebity we assume that

$$||W(t)|| \leq M_1, \quad t > 0 \tag{3.3.1}$$

for the sake of simplicity.

Let $T > 0$ be fixed and $X = C([0, T]; H)$. Put

$$H_r = \{z \in H : |z| \leq r\} \text{ and } X_r = \{x \in X : ||x||_X \leq r\}$$

for some $r > 0$.

Let $k \in L^2(0, T; H)$ and let $f : [0, T] \times H^{m+1} \to H$, $\phi : X \to H$, $b_i : [0, T] \to [0, T] (i = 1, ..., m)$ satisfying the following assumptions:

Assumption (A). (i) $f \in C([0, T] \times H^{m+1}; H)$, $\phi \in C(X; H)$ and $b_i \in C([0, T]; \mathbb{R}^+)(i = 1, ..., m)$. Moreover, there are $L_i > 0 (i = 1, 2)$ such that

$$|f(s, z_0, z_1, ..., z_m)| \leq L_1 \ \text{ for } \ s \in [0, T], \ z_i \in H_r(i = 1, ..., m).$$

(ii) ϕ is completely continuous such that

$$|\phi(x)| \leq L_2 \ \text{ for } \ x \in X_r.$$

Lemma 3.3.1. *Let $h \in L^2(0, T; H)$. Then for any $t > 0$, the operators P_t and Q_t defined by from $L^2(0, T; H)$ into H defined by*

$$P_t h = \int_0^t S(t - s) h(s) ds, \quad \text{and } Q_t h = \int_0^t W(t - s) h(s) ds$$

are completely continuous.

Proof. We define the ϵ-approximation $P_t^\epsilon : L^2(0,T;H) \to H$ of P_t for $\epsilon \in (0,t]$ by

$$P_t^\epsilon h = S(\epsilon)\int_0^{t-\epsilon} S(t-\epsilon-s)h(s)ds.$$

Since $S(t)$ is completely continuous, so is P_t^ϵ. The complete continuity of P_t follows from

$$|(P_t^\epsilon - P_t)h| \le \sqrt{\epsilon}M||h||_{L^2(0,T;H)}.$$

The ϵ-approximation $Q_t^\epsilon : L^2(0,T;H) \to H$ of Q_t is defined by

$$Q_t^\epsilon h = \int_0^{t-\epsilon} W(t-s)h(s)ds$$

Noting that

$$W(t+t') = S(t')W(t) + \int_0^{t'} S(t'-\sigma)\int_{-h}^0 a(\tau)A_1 x(\sigma+t+\tau)d\tau d\sigma, \quad 0 < t, t' \le T,$$

we have

$$\begin{aligned} Q_t^\epsilon h =& S(\epsilon)\int_0^{t-\epsilon} W(t-\epsilon-s)h(s)ds \\ &+ \int_0^{t-\epsilon}\int_0^{\epsilon} S(\epsilon-\sigma)\int_0^{-h'} a(\tau)A_1 x(\sigma+t-\epsilon-s+\tau)d\tau d\sigma ds. \end{aligned}$$

By using a similar procedure to the case of P_t, we obtain that Q_t is completely continuous from the complete continuity of $W(t)$ and Q_t^ϵ. □

Theorem 3.3.1. *1) Let $(g^0, g^1) \in H \times L^2(-h, 0; D(A_0))$ and $k \in L^2(0,T;H)$. Assume that f, ϕ and $b_i (i = 1, ..., m)$ satisfy Asuumption (A). Then there exists a mild solution x of (NRE) belonging to $C([0,T];H)$. Furthermore, if $g^0 - \phi(x) \in F = (D(A_0), H)_{\frac{1}{2},2}$ then a solution x of (NRE) belongs to*

$$L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset C([0,T];F)$$

and satisfying

$$\begin{aligned} ||x||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} \le C_T'(1 + ||g^0||_{(D(A_0),H)_{\frac{1}{2},2}} \\ + ||g^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}), \end{aligned}$$

where C_T' is a constant depending on T.

Proof. Let

$$\begin{aligned} r =& M_1(|g^0| + L_2) + hM_1H_0\|A_1A_0^{-1}\|(L_2h + \sqrt{h}\|g^1\|_{L^2(-h,0:D(A_0))}) \\ &+ M_1L_1T + M_1\sqrt{T}||k||_{L^2(0,T;H)}. \end{aligned} \tag{3.3.2}$$

Define a mapping $\mathcal{F}$ on X_r by the fomular

$$\begin{aligned} (\mathcal{F}x)(t) =& W(t)(g^0 - \phi(x)) + \int_{-h}^{0} U_t(s)(g^1(s) - e^s\phi(x))ds \\ &+ \int_0^t W(t-s)\{f(s, x(s), x(b_1(s)), ..., x(b_m(s))) + k(s)\}ds. \end{aligned}$$

In view of (3.3.1) and Assumption (A),

$$\begin{aligned} |(\mathcal{F}x)(t)| \le& M_1(|g^0| + L_2) + hM_1H_0\|A_1A_0^{-1}\|(L_2h + \sqrt{h}\|g^1\|_{L^2(-h,0:D(A_0))}) \\ &+ M_1L_1T + M_1\sqrt{T}||k||_{L^2(0,T;H)}, \end{aligned}$$

then $\mathcal{F}(X_r) \subset X_r \subset C([0,T];H)$. Observe that $0 < t < t' \leq T$, from (3.3.1), Assumption (A) and Proposition 3.2.1 we have

$$|x(t') - x(t)| \leq |(W(t') - W(t))(g^0 - \phi(x))| \tag{3.3.3}$$

$$+ \int_{-h}^{0}\int_{-h}^{s} |(W(t'-s+\sigma) - W(t-s+\sigma))a(\sigma)A_1(g^1(s) - e^s\phi(x))|d\sigma ds$$

$$+ \int_{0}^{t} |W(t'-s) - W(t-s)||f(s,x(s),x(b_1(s)),...,x(b_m(s))) + k(s)|ds$$

$$+ \int_{t}^{t'} |W(t'-s)||f(s,x(s),x(b_1(s)),...,x(b_m(s))) + k(s)|ds$$

$$\leq C_n(t^{'} - t)^{\alpha}(|g^0 - \phi(x)| + TL_1 + \sqrt{T}||k||_{L^2(0,T;H)})$$

$$+ C_n\int_{-h}^{0}\int_{-h}^{s}(t'-t)^{\alpha}H_0\|A_1A_0^{-1}\|_{B(H,H)}\|g^1(s) - e^s\phi(x))\|_{D(A_0)}d\sigma ds$$

$$+ M_1L_1(t'-t) + \int_{t}^{t'}|k(s)|ds$$

$$\leq C_n(t^{'} - t)^{\alpha}(|g^0 - \phi(x)| + TL_1 + \sqrt{T}||k||_{L^2(0,T;H)})$$

$$+ C_nH_0\|A_1A_0^{-1}\|_{B(H,H)}(t'-t)^{\alpha}\int_{-h}^{0}\|g^1(s) - e^s\phi(x))\|_{D(A_0)}ds$$

$$+ M_1L_1(t'-t) + (t'-t)^{1/2}||k||_{L^2(0,T;H)}$$

$$\leq \text{const.}(t'-t)^{\kappa}(1 + |g^0| + ||g^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)})(0 < \kappa \leq \frac{1}{2}).$$

Hence, $\mathcal{F}(X_r)$ is a uniformly equicontinuous family of functions. Furthermore, from (3.2.17) in Proposition 3.2.2 and Assumption(A) it follows that

$$
\begin{aligned}
|(\mathcal{F}x)(t)| &\le ||\mathcal{F}x||_{C([0,T];H)} \\
&\le C_T(|g^0-\phi(x)| + ||g^1 - e^{\cdot}\phi(x)||_{L^2(-h,0;V)} \\
&\quad ||f(\cdot,x(\cdot),x(b_1(\cdot)),...,x(b_m(\cdot)) + k||_{L^2(0,T;V^*)}) \\
&\le \text{const.}(1+|g^0| + ||g^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}).
\end{aligned}
$$

Thus, $\mathcal{F}(X_r)$ is equibounded.

From Lemma 3.3.1 it follows that the set $V(t) = \{(\mathcal{F}x)(t) : x \in X_r\}$ is relatively compact in H. By (ii) of Assumption (A), $V(0)$ is obviously is relatively compact. The proof or the continuity of $\mathcal{F}$ is routine, and may be omitted. Therefore, applying Schauder's fixed point theorem it holds $\mathcal{F}$ has a fixed point in X_r and hence, any fixed point of $\mathcal{F}$ is a mild solution of (NRE).

Assume that $g^0 - \phi(x) \in F = (D(A_0), H)_{\frac{1}{2},2}$. Then in virtue of Proposition 3.2.2 there exists a solution x of (NRE) belonging to

$$
L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset C([0,T];F)
$$

and satisfying

$$
\begin{aligned}
&||x||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} \le C_1'(||g^0-\phi(x)||_{(D(A_0),H)_{\frac{1}{2},2}} \\
&\quad + ||g^1 - e^{\cdot}\phi(x)||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}) \\
&\quad \le C_2'(1 + ||g^0||_{(D(A_0),H)_{\frac{1}{2},2}} + ||g^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}).
\end{aligned}
$$

□

Theorem 3.3.2. *Suppose that the functions f, ϕ and $b_i (i = 1, ..., m)$ satisfy Assumption (A) and g^1 is a Hölder continuous function in $[-h, 0]$ with values in $D(A_0)$ and k is a Hölder continuous function in $[0, T]$ with values in H. Assume, additionally, that*

(i) there exists a constant $L_3 > 0$ such that

$$|f(s, z_0, z_1, ..., z_m) - f(\tilde{s}, \tilde{z}_0, \tilde{z}_1, ..., \tilde{z}_m)| \leq L_3\big(|s - \tilde{s}| + \sum_{i=0}^{m} \|z_i - \tilde{z}_i\|\big)$$

$$\text{for} \quad s, \tilde{s} \in I, \quad z_i, \tilde{z}_i \in H_r (i = 0, 1, ..., m),$$

where r is the constant in (3.3.2),

(ii) x is a solution of problem (NRE) and there is a constant $\mathcal{H} > 0$ such that

$$|x(b_i(s)) - x(b_i(\tilde{s}))| \leq \mathcal{H}|x(s) - x(\tilde{s})| \qquad \text{for} \quad s, \tilde{s} \in I.$$

Then x represented as (3.2.12) is the unique solution of (NRE) satisfying the initial condition

$$x(s) = g^1(s) - e^s \phi(x), \quad s \in [-h, 0].$$

Proof. Put

$$G(s) = g^1(s) - e^s \phi(s), \ s \in [-h, 0],$$

$$K(t) = f(t, x(t), x(b_1(t)), ..., x(b_m(t))) + k(t), \ t \in [0, T].$$

Then in virtue of Theorem 2 of [18] it is sufficient to prove that G and K are Lipschitz continuous in $[-h, 0]$ and $[0, T]$, respectively. Since g^1 is a Hölder continuous function in $[-h, 0]$ with values in $D(A_0)$ and

$$|e^{s'}\phi(x) - e^{s}\phi(x)| = |\int_0^1 \frac{d}{d\sigma}(e^{s'\sigma}e^{s(1-\sigma)})\phi(x)d\sigma|$$

$$\leq \int_0^1 |e^{s'\sigma}e^{s(1-\sigma)}(s'-s)\phi(x)|d\sigma$$

$$\leq (s'-s)e^{s'}||\phi(x)||_{D(A_0)}$$

it holds that G is Hölder continuous. Furthermore, since

$$|K(t') - K(t)| \leq |k(t') - k(t)|$$

$$+ |f(t', x(t'), x(b_1(t')), ..., x(b_m(t'))) - f(t, x(t), x(b_1(t)), ..., x(b_m(t)))|$$

$$\leq |k(t') - k(t)| + L_3(|t'-t| + \sum_{i=1}^{m} |x(b_i(t')) - x(b_i(t))|)$$

$$\leq |k(t') - k(t)| + L_3(|t'-t| + m\mathcal{H}|x(t') - x(t)|),$$

from (3.3.3) and the Hölder continuity of k it follows that K is Hölder continuous in $[0, T]$. □

3.4 Example

Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$

$$B(u,v) = \int_0^{\pi} \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A_i = d^2/dx^2 (i = 0, 1) \quad \text{with} \quad D(A_i) = \{y \in H^2(0, \pi) : y(0) = y(\pi) = 0\}.$$

We consider the following nonlinear term:

$$f(s, z_0, z_1, ..., z_m) = l(s) + \frac{\gamma \sum_{i=1}^m z_i}{1 + |\sum_{i=1}^m z_i|}, \quad \gamma \in \mathbb{R}$$

where

$$|l(s) - l(\tilde{s})| \leq \sigma |s - \tilde{s}|, \quad l(0) = 0,$$

which comes out in a feedback control system for a diffusion and reaction process in a enzyme membrane. Then

$$\begin{aligned} |f(s, z_0, z_1, ..., z_m)| &\leq \sup_{0 \leq t \leq T} |l(t)| + |\gamma|, \\ |f(s, z_0, z_1, ..., z_m) - f(\tilde{s}, \tilde{z}_0, \tilde{z}_1, ..., \tilde{z}_m)| &\leq |l(s) - l(\tilde{s})| \\ &+ \frac{|\gamma|(1 + 2|\sum_{i=1}^m \tilde{z}_i|)(\sum_{i=1}^m z_i - \sum_{i=1}^m \tilde{z}_i)}{(1 + |\sum_{i=1}^m z_i|)(1 + |\sum_{i=1}^m \tilde{z}_i|)} \\ &\leq \sigma |s - \tilde{s}| + 2|\gamma| \sum_{i=1}^m |z_i - \tilde{z}_i|. \end{aligned}$$

Let $t_1, ..., t_p$ be given real numbers such that $0 < t_1 < ... < t_p < T$. Then we can give ϕ by the formula

$$\phi(x) = \sum_{i=1}^p d_i x(t_i) \quad x \in C([0, T]; L^2(0, \pi))$$

where $d_i(i = 1, ..., p)$ are given constants. Let the solution x be represented by the following retarded semilinear parabolic type equation:

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds \\ \ +f(t, x(t), x(b_1(t)), ..., x(b_m(t))) + k(t), \ t \geq 0, \\ x(0) = g^0 - \sum_{i=1}^{p} d_i x(t_i), \quad x(s) = g^1(s) - e^s\phi(x), \quad -h \leq s < 0, \end{cases}$$

where the forcing term k belongs to $L^2(0, T; V^*)$, $b_i(t) = t(i = 1, ..., m)$. Then the nonlinear term f, ϕ and $b_i(i = 1, ..., m)$ satisfy the conditions of Theorems 3.3.1, 3.3.2.

Chapter 4
Regularity for nonlinear variational inequalities of hyperbolic type

4.1 Introduction

The subject of this paper is to consider the initial value problem of the following nonlinear variational inequalities of second order in Hilbert spaces;

$$\begin{cases} (u''(t)+Au(t),u(t)-z)+\phi(u(t))-\ \phi(z) \\ \quad \le (f(t,u(t))+k(t),u(t)-z),\ \text{a.e.},\ \forall z\in V \\ u(0)=u^0, \quad u'(0)=u^1. \end{cases} \tag{NVE}$$

Let H and V be two complex Hilbert spaces. Assume that V is dense subspace in H and the injection of V into H is continuous. Let A be a continuous linear operator from V into V^* which is assumed to satisfy Gårding's inequality, and let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. The nonlinear term $f(\cdot, u)$, which is a locally Lipschitz continuous operator with respect to u from V to H, is a semilinear version of the quasilinear one considered in [23, 24, 25], and a forcing term $k \in L^2(0,T;V^*)$. By the definition of the subdifferential operator $\partial\phi$, the problem (NVE) is represented by the following nonlinear functional differential problem:

$$\begin{cases} u''(t)+Au(t)+\partial\phi(u(t)) \ni f(t,u(t))+k(t),\ 0<t, \\ u(0)=u^0, \quad u(s)=u^1. \end{cases} \tag{NDE}$$

The background of these variational problems are physics, especially in solid mechanics, where nonconvex and multi-valued constitutive laws lead to differential inclusions. We refer to [26, 27, 28, 29, 30] to see the applications of differential inclusions. There are extensive literatures on parabolic variational inequalities of first order and the Stefan problems(see Babue [31, 32] and the book by Duvaut and Lions [33]. But the papers treating the variational inequalities of second order with nonlinear inhomogeneous terms are not many.

In this paper we are primarily interested in the regular problem for the variational inequalities of second order with nonlinear inhomogeneous terms for that arise as direct consequences of the general theory developed previously, and we consider to put in perspective those models of initial value problems which can be formulated as nonlinear differential equations of variational inequalities. The approach used here is similar to that developed in Yosida [34] in which more general hyperbolic equations are also treated. When the nonlinear mapping k is a locally Lipschitz continuous from $\mathbb{R} \times V$ into H, we will obtain that the most part of the regularity for parabolic variational inequalities of first order can also applicable to (NDE) with nonlinear perturbations(see [31-36]).

Section 2 gives some basic properties on the principal operator A to consider a representation formula of solutions for the general hyperbolic semilinear equations in case $\phi \equiv 0$ [31-35, 37]. In section 3, we will introduce single valued smoothing systems corresponding to nonlinear variational inequalities (NDE) by using approximate function $\phi_\epsilon(x) = \inf\{||x-y||_*^2/2\epsilon + \phi(y) : y \in H\}$(see [31, 32]).

Section 4 deals with the wellposedness for solutions of (NDE) by converting the problem into the contraction mapping principle with more general conditions on the nonlinear terms and without conditions of the compactness of the principal operators, and obtain the norm estimate of a solution of the above nonlinear equation on $C([0,T];V)\cap C^1((0,T];H)\cap C^2((0,T];V^*)$ by using the results of its corresponding the hyperbolic semilinear part in case $\phi\equiv 0$ as seen in [35].

4.2 Parabolic variational inequalities

Let H be a complex Hilbert space with inner product $(\ ,\ )$ and norm $|\cdot|$. Let V be embedded in H as a dense subspace with inner product and norm by $((\ ,\ ))$ and $||\cdot||$, respectively. By considering $H=H^*$. We may write $V\subset H\subset V^*$ where H^* and V^* denote the dual spaces of H and V, respectively. For $l\in V^*$ we denoted (l,v) by the value $l(v)$ of l at $v\in V$. The norm of l as element of V^* is given by

$$||l||_*=\sup_{v\in V}\frac{|(l,v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for the brevity, we may regard that

$$||u||_*\le |u|\le ||u||,\quad \forall u\in V.$$

Definition 4.2.1. *Let X and Y be complex Banach spaces. An operator S from X to Y is called antilinear if $S(u+v)=S(u)+S(v)$ and $S(\lambda u)=\bar{\lambda}S(u)$ for $u,v\in X$ and for $\lambda\in\mathbb{C}$*

Let $a(u,v)$ be a quadratic form defined on $V \times V$ which is linear in u and antilinear in v.

We make the following assumptions: i) $a(u,v)$ is bounded, i.e. $\exists c_0 > 0$ such that

$$|a(u,v)| \le c_0 ||u|| \cdot ||v||; \qquad (4.2.1)$$

ii) $a(u,v)$ is symmetric, i.e.

$$a(u,v) = \overline{a(v,u)};$$

iii) $a(u,v)$ satisfies the Gårding's inequality, i.e.

$$\operatorname{Re} a(u,u) \ge \delta ||u||^2, \quad \delta > 0. \qquad (4.2.2)$$

Let A be the operator such that $(Au,v) = a(u,v)$ for any $u,v \in V$. Then, as seen in Theorem 2.2.3 of [19], the operator A is positive definite and self-adjoint, $D(A^{1/2}) = V$, and

$$a(u,v) = (A^{1/2}u, A^{1/2}v), \quad u,\ v \in V.$$

It is also known that the operator A is a bounded linear from V to V^*. The realization of A in H which is the restriction of A to $D(A) = \{v \in V : Av \in H\}$ is denoted by A_H, which is structured as a Hilbert space with the norm $||v||_{D(A)} = |A_H v|$. Then the operators A_H and A generate analytic semigroups in both of H and V^*, respectively. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one which continuous injection.

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers whose norms are integrable and $W^{m,p}(0,T;X)$ is the set of all functions f whose derivatives $D^\alpha f$ up to degree m in the distribution sense belong to $L^p(0,T;X)$. $C^m([0,T];X)$ is the set of all m-times continuously differentiable functions from $[0,T]$ into X. Let X and Y be complex Banach spaces. Denote by $B(X,Y)$(resp. $\overline{B}(X,Y)$) the set of all bounded linear(resp. anti-linear) operators from X and Y. Let $B(X) = B(X,X)$.

We consider the initial value problem of the following variational inequality

$$\begin{cases} (u''(t) + Au(t), u(t) - z) + \phi(u(t)) - \phi(z) \\ \quad \le (f(t,u(t)) + k(t), u(t) - z), \text{ a.e., } \forall z \in V \\ u(0) = u^0, \quad u(s) = u^1. \end{cases} \tag{NVE}$$

Definition 4.2.2. *A function $u : [0,T] \to H$ is called a solution of equation (NVE) on $[0,T]$ if*

i) $u \in C([0,T];V) \cap C^1((0,T];H) \cap C^2((0,T];V^*)$,

ii) u satisfies (NVE) on $[0,T]$.

Let us introduce a new norm in V^* as follows. For $g, k \in V^*$, putting

$$(g,k)_{-1} = a(A^{-1}g, A^{-1}k) = (AA^{-1}g, A^{-1}k) = (g, A^{-1}k),$$

in virtue of the condition of a $(g,k)_{-1}$, it satisfies the inner product properties and its norm is given by

$$||g||_{-1} = a(A^{-1}g, A^{-1}g)^{1/2}.$$

Lemma 4.2.1. *The norm* $||g||_{-1}$ *is equivalent to* $||\cdot||_*$ *, i.e, we have*

$$\frac{\delta}{\sqrt{c_0}}||g||_{-1} \le ||g||_* \le \frac{c_0}{\sqrt{\delta}}||g||_{-1}.$$

The proof follows immediately from Definition 4.2.2.

If we set $X = (V \times H)^T$ with inner product and norm given by

$$\left\langle \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right\rangle = ((u_0, v_0)) + (u_1, v_1) \quad \text{and} \quad \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_X = \{||u_0||^2 + |u_1|^2\}^{1/2},$$

respectively. Noting that $a(u,v)$ is inner product in V and $a(u,u)^{1/2}$ is equivalent to the norm $||u||$, we can also rewrite inner product and norm as

$$\left\langle \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right\rangle = a(u_0, v_0) + (u_1, v_1) \quad \text{and} \quad \left\| \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right\|_X = \{a(u_0, u_0) + |u_1|^2\}^{1/2},$$

respectively.

Putting $\widetilde{X} = (H \times V^*)^T$, for every $\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \in \widetilde{X}$, we define an inner product and norm by

$$\left(\begin{pmatrix} g_0 \\ g_1 \end{pmatrix}, \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} \right)_{\widetilde{X}} = (g_0, k_0) + (g_1, k_1)_{-1} \quad \text{and} \quad \left| \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \right|_{\widetilde{X}} = \left(|g_0|^2 + ||g_1||_{-1}^2 \right)^{1/2},$$

respectively. Let $\mathcal{A}_X$ be the operator defined by

$$D(\mathcal{A}_X) = (D(A_H) \times V)^T, \quad \mathcal{A}_X \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_H & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ -A_H u_0 \end{pmatrix} \in X.$$

In virtue of Lax-Milgram theorem we can also define $\mathcal{A}$ as

$$D(\mathcal{A}) = (V \times H)^T = X, \quad \mathcal{A} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} = \begin{pmatrix} g_1 \\ -A g_0 \end{pmatrix} \in \widetilde{X}.$$

Theorem 4.2.1. *The linear operators $\mathcal{A}_X$ and $\mathcal{A}$ mentioned above are the infinitesimal generators of C_0-groups of unitary operators in X and $\widetilde{X}$, respectively.*

Since the proof is easy, it is omitted.

Lemma 4.2.2. *Let the linear operator $\mathcal{A}$ is the infinitesimal generators of C_0-group of unitary operator in $\widetilde{X}$ as in Theorem 4.2.1. Then*

$$\min\{\delta,1\}(||u_0(t)||^2+|u_1(t)|^2)^{\frac{1}{2}} \leq \left|\mathcal{A}\begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix}\right|_{\widetilde{X}} \leq \max\{c_0,1\}(||u_0(t)||^2+|u_1(t)|^2)^{\frac{1}{2}}. \tag{4.2.3}$$

Proof. From (4.2.1), (4.2.2) it follows that

$$\begin{aligned} \left|\mathcal{A}\begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix}\right|_{\widetilde{X}} = \left\|\begin{pmatrix} u_1(t) \\ -Au_0(t) \end{pmatrix}\right\|_{\widetilde{X}} &\geq (\delta||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}} \\ &\geq \min\{\delta,1\}(||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} \left|\mathcal{A}\begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix}\right|_{\widetilde{X}} = \left\|\begin{pmatrix} u_1(t) \\ -Au_0(t) \end{pmatrix}\right\|_{\widetilde{X}} &= (|u_1(t)|^2 + ||Au_0(t)||_*^2)^{\frac{1}{2}} \\ &\leq \max\{c_0,1\}(||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}}, \end{aligned}$$

hence, we readily get (4.2.3) □

4.3 Smoothing system corresponding to variational inequalities

For every $\epsilon > 0$, define

$$\phi_\epsilon(x) = \inf\{||x-y||_*^2/2\epsilon + \phi(y) : y \in H\}.$$

Then the function ϕ_ϵ is Fréchet differentiable on H and its Frećhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1} where $\partial\phi_\epsilon = \epsilon^{-1}(I-(I+\epsilon\partial\phi)^{-1})$ as is seen in Corollary 2.2 of Chapter II of [31]. It is also well known results that $\lim_{\epsilon\to 0}\phi_\epsilon = \phi$ and $\lim_{\epsilon\to 0}\partial\phi_\epsilon(x) = (\partial\phi)^0(x)$ for every $x \in D(\partial\phi)$ where $(\partial\phi)^0$ is the minimal segment of $\partial\phi$. .

Now, we introduce the smoothing system corresponding to (NDE) as follows.

$$\begin{cases} u''(t) + Au(t) + \partial\phi_\epsilon(u(t)) = f(t,u(t)) + k(t), & 0 < t \le T, \\ u(0) = u^0, \quad u(s) = u^1. \end{cases} \tag{SDE}$$

Now, we assume the hypothesis that $V \subset D(\partial\phi)$ and $(\partial\phi)^0$ is uniformly bounded, i.e.,

(A) $\qquad |(\partial\phi)^0 x| \le M_0, \quad x \in H.$

We will need the following hypotheses on the nonlinear term;

Assumption (F). Let $f : [0,T]\times V \to H$ $(T > 0)$ be a nonlinear mapping such that $t \mapsto f(t,\cdot)$ is continuous on $[0,T]$ and $u \mapsto f(\cdot,u)$ is locally Lipschitz continuous on V: there exists constant $L : \mathbb{R}^+ \to \mathbb{R}$ such that $L(r_1) \le L(r_2)$ if $r_1 \le r_2$ and

$$|f(\cdot,u)| \le L(r), \quad |f(\cdot,u) - f(\cdot,v)| \le L(r)|u-v|$$

holds for $||u|| < r$ and $||v|| < r$.

Let $\mathbf{x}(t) = \binom{u_0(t)}{u_1(t)}$, $\partial\Phi_\epsilon(\mathbf{x}(t)) = \binom{0}{\partial\phi_\epsilon(u_0(t))}$, let $F(\mathbf{x}(t)) = \binom{0}{f(\cdot,u_0(t))}$ and $K(t) = \binom{0}{k(t)}$ Then problem (SDE) are equivalent to

$$\begin{cases} \mathbf{x}'(t) + \partial\Phi_\epsilon(\mathbf{x}(t)) = \mathcal{A}\mathbf{x}(t) + F(\mathbf{x}(t)) + K(t) \\ \mathbf{x}(0) = \binom{u^0}{u^1}. \end{cases} \tag{4.3.1}$$

Lemma 4.3.1. *Let u_ϵ and u_λ be the solutions of (SDE) with constans ϵ and λ, respectively. Then there exists a constant C independent of ϵ and λ such that*

$$||u_\epsilon - u_\lambda||_{C([0,T];V)\cap C^1((0,T];H)} \leq C(\epsilon + \lambda), \quad 0 < T. \tag{4.3.2}$$

Proof. For given ϵ, $\lambda > 0$, let $\mathbf{x}_\epsilon = \binom{u_\epsilon}{u_\epsilon{}'}$ and $\mathbf{x}_\lambda = \binom{u_\lambda}{u_\lambda{}'}$ be the solutions of (4.3.1) corresponding to ϵ and λ such that $||u_\epsilon||_{C([0,T];V)} \leq r$ and $||u_\lambda||_{C([0,T];V)} \leq r$, respectively. Then from the equation (4.3.1) we have

$$\mathbf{x}_\epsilon^{'}(t) - \mathbf{x}_\lambda^{'}(t) + \mathcal{A}(\mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)) + \partial\Phi_\epsilon(\mathbf{x}_\epsilon(t)) - \partial\phi_\lambda(\mathbf{x}_\lambda(t)) = F(\mathbf{x}_\epsilon(t)) - F(\mathbf{x}_\lambda(t)),$$

and hence, from (4.2.2) and multiplying by $\mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)$, it follows that

$$\frac{1}{2}\frac{d}{dt}||\mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)||_X^2 + (\mathcal{A}(\mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)), \mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)) \tag{4.3.3}$$

$$+ (\partial\Phi_\epsilon(\mathbf{x}_\epsilon(t)) - \partial\phi_\lambda(\mathbf{x}_\lambda(t)), \mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)) \leq (F(\mathbf{x}_\epsilon(t)) - F(\mathbf{x}_\lambda(t), \mathbf{x}_\epsilon(t) - \mathbf{x}_\lambda(t)).$$

For every $\binom{u_0}{u_1} \in D(\mathcal{A}) = V \times H$, since

$$\left| \left(\mathcal{A}\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right) \right| = |(u_1, u_0) - (u_0, u_1)| = |-2\,\mathrm{Im}(u_0, u_1)| \leq ||\binom{u_0}{u_1}||_X^2,$$

we have

$$|(\mathcal{A}(\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t)),\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t))|\le ||\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t))||_X^2.$$

Then by (F), we have

$$\begin{aligned}(F(\mathbf{x}_\epsilon(t))-F(\mathbf{x}_\lambda(t)),\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t)) &\le |F(\mathbf{x}_\epsilon(t))-F(\mathbf{x}_\lambda(t))|\cdot|\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t)|\\ &\le L(r)|\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t)|^2.\end{aligned}$$

Integrating (4.3.3) over $[0,T]$ we have

$$\begin{aligned}\frac{1}{2}||\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t)||^2 \le &\int_0^T(\partial\Phi_\epsilon(\mathbf{x}_\epsilon(t))-\partial\Phi_\lambda(\mathbf{x}_\lambda(t)),\lambda\partial\Phi_\lambda(\mathbf{x}_\lambda(t))-\epsilon\partial\Phi_\epsilon(\mathbf{x}_\epsilon(t)))dt\\ &+(L(r)+1)\int_0^T||\mathbf{x}_\epsilon(t)-\mathbf{x}_\lambda(t)||_X^2dt.\end{aligned}$$

Here, we used that

$$\partial\Phi_\epsilon(\mathbf{x}_\epsilon(t))=\epsilon^{-1}(\mathbf{x}_\epsilon(t)-(I+\epsilon\partial\Phi)^{-1}\mathbf{x}_\epsilon(t)).$$

Since $|\partial\Phi_\epsilon(\mathbf{x})|\le|(\partial\Phi)^0\mathbf{x}|$ for every $\mathbf{x}\in D(\partial\Phi)$, it follows from (A) and using Gronwall's inequality that

$$||\mathbf{x}_\epsilon-\mathbf{x}_\lambda||_{C([0,T];X)}\le C(\epsilon+\lambda),\quad 0<T,$$

hence, (4.3.2) follows. □

Theorem 4.3.1. *Let the assumptions (F) and (A) be satisfied. Then* $u=\lim_{\epsilon\to 0}u_\epsilon$ *in* $C([0,T];H)\cap C^1((0,T];V^*)$ *is a solution of the equation (NDE) where* u_ϵ *is the solution of (SDE) .*

Proof. In virtue of Lemma 4.3.1, there exists $\mathbf{x}(\cdot) = \left(\begin{smallmatrix} u \\ u' \end{smallmatrix}\right) \in C([0,T];X)$ such that

$$\mathbf{x}_\epsilon(\cdot) \to \mathbf{x}(\cdot) \quad \text{in} \quad C([0,T];X).$$

From (F) it follows that

$$F(\mathbf{x}_\epsilon) \to F(\mathbf{x}), \quad \text{strongly in } C([0,T];X) \tag{4.3.4}$$

and

$$\mathcal{A}\mathbf{x}_\epsilon \to \mathcal{A}\mathbf{x}, \quad \text{strongly in } C((0,T];\widetilde{X}). \tag{4.3.5}$$

Since $\partial\phi_\epsilon(\mathbf{x}_\epsilon)$ are uniformly bounded by assumption (A), from (4.3.4), (4.3.5) we have that

$$\frac{d}{dt}\mathbf{x}(t)_\epsilon \to \frac{d}{dt}\mathbf{x}(t), \quad \text{weakly in } C((0,T];\widetilde{X}),$$

therefore

$$\partial\phi_\epsilon(\mathbf{x}_\epsilon) \to F(\mathbf{x}) + K - \mathbf{x}' + \mathcal{A}\mathbf{x}, \quad \text{weakly in } C([0,T];\widetilde{X}),$$

Note that $\partial\Phi_\epsilon(\mathbf{x}_\epsilon) = \partial\Phi((I+\epsilon\partial\Phi)^{-1}\mathbf{x}_\epsilon)$. Since $(I+\epsilon\partial\Phi)^{-1}\mathbf{x}_\epsilon \to \mathbf{x}$ strongly and $\partial\Phi$ is demiclosed, we have that

$$F(\mathbf{x}) + K - \mathbf{x}' + \mathcal{A}\mathbf{x} \in \partial\Phi(\mathbf{x}) \text{ in } C([0,T];\widetilde{X}).$$

Thus we have proved that $u(t)$ satisfies on $C([0,T];H) \cap C^1((0,T];V^*)$ the equation (NDE). □

4.4 Variational inequalities with nonlinear perturbations

In virtue of Theorem 4.2.1, if $\{\mathcal{U}(t)\}$ is a C_0-group generated by $\mathcal{A}$ then, for a solution of (SDE) in the wide sense, we are going to find a solution of the integral equation

$$\mathbf{x}(t) = \mathcal{U}(t)\mathbf{x}(0) + \int_0^t \mathcal{U}(t-s)\{\partial\Phi_\epsilon(\mathbf{x}(s)) + F(\mathbf{x}(s)) + K(s)\}ds. \quad (4.4.1)$$

For the sake of simplicity, we assume

$$M_1 = \sup_{0\le t\le T} ||\mathcal{U}(t)||. \quad (4.4.2)$$

The following lemma is from Theorems 6.1.1 and 6.1.5 in [19].

Lemma 4.4.1. *Let us assume the assumption (F). Then for every* $u^0 \in V, u^1 \in H$, *a given* $T > 0$ *and* $h \in C([0,T]; V^*)$. *The equation*

$$\mathbf{x}(t) = \mathcal{U}(t)\mathbf{x}(0) + \int_0^t \mathcal{U}(t-s)\{F(\mathbf{x}(s)) + K(s)\}ds. \quad (4.4.3)$$

has a unique local solution on interval $[0, T_0]$ *for* $0 < T_0 \le T$

Now, we consider the global existence of a solution of (4.4.1).

Theorem 4.4.1. *Let us assume the assumption (F). Then for every* $u^0 \in V, u^1 \in H$ *and* $k \in C([0,T]; V^*)$, *the equation (NDE) has a unique solution on* $[0,T]$ *for a given* $T > 0$.

Proof. First we prove that the equation (4.4.1) has a unique local solution. For a given $\mathbf{x} \in C([0,T];\widetilde{X})$, let y be the solution of

$$\mathbf{y}(t) = \mathcal{U}(t)\mathbf{x}(0) + \int_0^t \mathcal{U}(t-s)\{\partial\Phi_\epsilon(\mathbf{x}(s)) + F(\mathbf{y}(s)) + K(s)\}ds. \quad (4.4.4)$$

Since the Frećhet differential $\partial\phi_\epsilon$ is Lipschitz continuous on H with Lipschitz constant ϵ^{-1}, by Lemma 4.4.1, the equation (4.4.4) has a unique local solution on interval $[0,T_0]$ for $0 < T_0 \leq T$. Let B_r be the ball of radius r centered at zero of $C([0,T_0];\widetilde{X})$, i.e., $B_r = \{v \in C([0,T_0];\widetilde{X}) : ||v|| \leq r\}$. Let us fix $T_1 > 0$ satisfying

$$T_1 \equiv \min\{T_0, \epsilon^{-1}M_1T_0\} < 1 - M_1L(r)T_0\} \quad (4.4.5)$$

where $L(r)$ and M_1 are given by (F) and (4.4.2), respectively. We are going to show that the mapping defined by $\mathbf{x} \mapsto \mathbf{y}$ maps is strictly contractive from the ball B_r into itself if the condition (4.4.4) is satisfied. Let $\mathbf{y}$,$\hat{\mathbf{y}}$ be solution (4.4.4) corresponding to $\mathbf{x}$, $\hat{\mathbf{x}}$ in $[0,T_1]$, respectively. Then from assumption (F), (4.4.2) and

$$\mathbf{y}(t) - \hat{\mathbf{y}}(t) = \int_0^t \mathcal{U}(t-s)\{\partial\Phi_\epsilon(\mathbf{x}(s)) - \partial\Phi_\epsilon(\hat{\mathbf{x}}(s))$$
$$+ \int_0^t \mathcal{U}(t-s)\{F(\mathbf{y}(s)) - F(\hat{\mathbf{y}}(s))\}ds,$$

we have

$$|\mathbf{y}(t) - \hat{\mathbf{y}}(t)|_{\widetilde{X}} \leq \epsilon^{-1}M_1t|\mathbf{x}(t) - \hat{\mathbf{x}}(t)|_{\widetilde{X}} + M_1L(r)t|\mathbf{y}(t) - \hat{\mathbf{y}}(t)|_{\widetilde{X}}$$

So by virtue of (4.4.5), the mapping defined by $\mathbf{x} \mapsto \mathbf{y}$ maps is strictly contractive from B_r into itself. Therefore, the contraction mapping principle gives that the equation (4.4.4) has a unique solution in $[0, T_1]$. Since A is an isomorphism from V onto $\widetilde{V}^*$, we note that the solution of (SDE) belongs to $C^2([0, T_1]; V^*)$.

Now, we give a norm estimation of the solution of (SDE) and establish the global existence of solutions with the aid of norm estimations. So, it is enough to show that if u is solution in $0 \le t \le T_1$, then $u(t)$ is bounded in $0 \le t \le T_1$, i.e., there exists a constant $C > 0$ such that

$$\left| \mathbf{x}(t) = \begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix} \right|_{C([0,T_0];\widetilde{X})} \le C, \quad 0 \le t \le T_1.$$

Therefore, from (4.4.1) and (4.2.3) we obtain that

$$\min\{\delta, 1\}(||u_0(t)||^2 + |u_1(t)|^2)^{\frac{1}{2}} \le \left| \mathcal{A} \begin{pmatrix} u_0(t) \\ u_1(t) \end{pmatrix} \right|_{\widetilde{X}}$$

$$\le \left| \mathcal{A}\mathcal{U}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right|_{\widetilde{X}} + \left| \mathcal{A} \int_0^t \mathcal{U}(t-s) \left\{ \begin{pmatrix} 0 \\ \partial\phi_\epsilon(u_0(s)) \end{pmatrix} + \begin{pmatrix} 0 \\ f(s, u_0(s)) \end{pmatrix} \right\} ds \right|_{\widetilde{X}}.$$

Here, we can calculate from (4.2.3) that

$$\left| \mathcal{A}\mathcal{U}(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right|_{\widetilde{X}} = \left| \mathcal{A}\mathcal{U}(t)\mathcal{A}^{-1}\mathcal{A} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right|_{\widetilde{X}} \le c_1 \left| \mathcal{A} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right|_{\widetilde{X}} \le c_1 \max\{c_0, 1\}(||u_0|| + |u_1|)$$

where $c_1 = |\mathcal{A}\mathcal{U}(t)\mathcal{A}^{-1}|_{B(\widetilde{X})}$,

$$\left| \mathcal{A} \int_0^t \mathcal{U}(t-s) \begin{pmatrix} 0 \\ \partial\phi_\epsilon(u_0(s)) \end{pmatrix} ds \right|_{\widetilde{X}} \le c_0 M_0 M_1 \int_0^t ||u_0(s)|| ds,$$

and

$$\left|\mathcal{A}\int_0^t \mathcal{U}(t-s)\begin{pmatrix}0\\ f(s,u_0(s))\end{pmatrix}ds\right|_{\widetilde{X}} \le \left|\int_0^t \mathcal{U}(t-s)\mathcal{A}\left(\begin{pmatrix}0\\ f(s,u_0(s))\end{pmatrix}-\begin{pmatrix}0\\ f(s,0)\end{pmatrix}\right)ds\right|_{\widetilde{X}}$$

$$+\left|\int_0^t \mathcal{U}(t-s)\mathcal{A}\begin{pmatrix}0\\ f(s,0)\end{pmatrix})ds\right|_{\widetilde{X}}$$

$$\le c_0L(r)Mt + c_0L(r)M\int_0^t ||u_0(s)||ds \le c_0L(r)M\{t+\int_0^t (||u_0(s)||^2+|u(s)|^2)^{1/2}ds\}.$$

Combining inequalities mentioned above and (4.2.3) it follows from Gronwall's inequality that there exists a constant c_1 such that

$$(||u_0(t)||^2+|u_1(t)|^2)^{1/2} \le c_1(1+||u^0||+|u^1|). \tag{4.4.6}$$

By the calculation similar to those in the proof of mentioned above, a solution $\mathbf{y}=\binom{v_0}{v_1}$ of

$$\begin{pmatrix}v_0(t)\\ v_1(t)\end{pmatrix} = \mathcal{U}(t-T_1)\begin{pmatrix}u_0(T_1)\\ u_1(T_1)\end{pmatrix}+\int_{T_0}^t \mathcal{U}(t-s)\mathcal{U}(t-s)\{\partial\Phi_\epsilon(\mathbf{y}(s))+F(\mathbf{y}(s))+K(s)\}ds$$

exists in some interval $[T_1, T_2]$ with the initial value

$$\hat{\mathbf{x}}(T_1) = \mathcal{U}(T_1)\hat{\mathbf{x}}(0)+\int_0^{T_1}\mathcal{U}(T_1-s)\{\partial\Phi_\epsilon(\hat{\mathbf{x}}(s))+F(\hat{\mathbf{x}}(s))+K(s)\}ds.$$

By letting $\hat{\mathbf{x}}(t)=\mathbf{x}(t)$ for $0\le t\le T_1$ and $\hat{\mathbf{x}}(t)=\mathbf{y}(t)$ for $T_1\le t<T_2$, it is easy to see that $\hat{\mathbf{x}}$ is a solution in $0\le t\le T_2$. Let $\hat{\mathbf{x}}$ be a bounded solution of (4.4.1): $||\hat{\mathbf{x}}||_{C([0,T_0];\widetilde{X})} < C'$. Then, since $||\binom{0}{f(t,u_0(t))}||_X \le L(C')$ for $T_1\le t<T_2$ by Assumption (F), it satisfies the variational inequality

(4.4.6) on $[T_1, T_2]$. Hence, $\hat{\mathbf{x}}$ can be extended to the interval $[0, T_2]$ as a solution and u_0 is the desired solution. So the equation (SDE) has a unique solution on $[0, T]$ for given $T > 0$. The results for (NDE) follows now directly from Theorem 4.3.1. □

Example. Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$

$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx.$$

Define the operator A by

$$(Au, v) = a(u,v), \quad \forall v,\ u \in V.$$

Then we know

$$A = \partial^2/\partial x^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

For any $u \in D(A)$, we let

$$f(t, u(t,x)) = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(s, \nabla u(s,x))ds.$$

Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function. Then we treat (NDE) as the initial value problem for the abstract second order equations.

We assume the following:

Assumption (F1). The partial derivatives $\sigma_i(s,\xi)$, $\partial/\partial t\, \sigma_i(s,\xi)$ and $\partial/\partial \xi_j\, \sigma_i(s,\xi)$ exist and continuous for $i = 1, 2,\ j = 1, 2, \cdots, n$, and $\sigma_i(s,\xi)$

satisfies an uniform Lipschitz condition with respect to ξ, that is, there exists a constant $L > 0$ such that

$$|\sigma_i(s,\xi) - \sigma_i(s,\widehat{\xi})| \le L|\xi - \widehat{\xi}|$$

where $|\cdot|$ denotes the norm of $L^2(\Omega)$.

Lemma 4.4.2. *If Assumption (F1) is satisfied, then the mapping* $t \mapsto f(t,\cdot)$ *is continuously differentiable on* $[0,T]$ *and* $u \mapsto f(\cdot,u)$ *is Lipschitz continuous on* V.

Proof. Put

$$g(s,u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(s, \nabla u).$$

Then we have $g(s,u) \in H^{-1}(\Omega)$. For each $w \in H_0^1(\Omega)$, we satisfy the following that

$$(g(s,u), w) = -\sum_{i=1}^{n} (\sigma_i(s,\nabla u), \frac{\partial}{\partial x_i} w).$$

The nonlinear term is given by

$$f(t,u) = \int_0^t g(s,u)ds.$$

For any $w \in H_0^1(\Omega)$, if u and $\hat{u}$ belong to $H_0^1(\Omega)$, by Assumption (F1) we obtain

$$|(f(t,u) - f(t,\hat{u})), w| \le LT||u - \hat{u}||\,||w||.$$

□

Now in virtue of Lemma 4.4.1, we can apply the results of Theorem 4.3.2 as follows.

Theorem 4.4.2. *Let Assumption (F1) be satisfied. Assume that* $k \in C([0,T]; H^{-1}(\Omega)) \cap W^{1,2}(0,T; H^{-1}(\Omega))(T > 0)$ *and* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. *Then the solution* u *of (NDE) exists and is unique in*

$$u \in \widetilde{W}_T \cap C([0,T]; H_0^1(\Omega)) \cap C^1([0,T); L^2(\Omega)), \quad T > 0$$

where

$$\widetilde{W}_T = L^2(0,T; H_0^1(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap W^{2,2}(0,T; H^{-1}(\Omega)).$$

Furthermore, the following energy inequality holds: there exists a constant C_T *depending on* T *such that*

$$||u||_{\widetilde{W}_T} \leq C_T(1 + ||u_0|| + |u_1| + ||k||_{W^{1,2}(0,T;H^{-1}(\Omega))}).$$

Chapter 5
Approximate controllability of linear retarded systems in Hilbert spaces

5.1 Introduction

The object of this paper is to construct some results on the control problems for the following retarded functional differential equation of parabolic type in a Hilbert space H:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \mathcal{A}_0(x,D_x)u(x,t) + \mathcal{A}_1(x,D_x)u(x,t-h) \\ \quad + \int_{-h}^{0} a(s)\mathcal{A}_2(x,D_x)u(x,t+s)ds + (B_0w(t))(x), \quad (x,t) \in \Omega \times (0,T] \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t \in (0,T], \\ u(x,0) = g^0(x), \ u(x,s) = g^1(x,s), \quad x \in \Omega, \ s \in [-h,0). \end{cases} \tag{5.1.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and h is some positive number. $\mathcal{A}_\iota(x,D_x)$, $\iota = 0,\ 1,\ 2$, are second order linear differential operators with smooth coefficients in $\overline{\Omega}$, and $\mathcal{A}_0(x,D_x)$ is elliptic. We note that in order to guarantee the existence of fundamental solution of system (5.1.1), we must need the assumption that $a(\cdot)$ is Hölder continuous as seen in [7]. Let U be a Banach space of control variables and the controller B_0 be a bounded linear operator from U to $L^2(\Omega)$. Let

$$H = L^2(\Omega), \quad V = W_0^{1,2}(\Omega), \quad \text{and } \mathrm{V}^* = \mathrm{W}^{-1,2}(\Omega).$$

We realize the operators $\mathcal{A}_\iota(x, D_x)(\iota = 0,\ 1,\ 2)$ in the space V by

$$D(A_\iota) = V, \quad \text{and} \quad A_\iota u = \mathcal{A}_\iota(x, D_x)u \quad \forall u \in V$$

in the distribution sense. Then it is well know result that $0 \in \rho(A_0)$(the resolvent set of A_0) and A generates an analytic semigroup in both H and V^*(see [19, Theorem 3.6.1]) and so the equation (5.1.1) may be considered as an equation in H as well as in V^*. Thus, we formulated the problem (5.1.1) as

$$\begin{cases} u'(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds + B_0w(t), \quad t > 0, \\ u(0) = g^0, \quad u(s) = g^1(s) \quad s \in [-h, 0). \end{cases} \tag{5.1.2}$$

Many authors have discussed the structural properties for retarded systems(see [2, 6, 21, 42, 43, 45, 49]). Further, in the case of infinite dimensional spaces, we refer to [44, 52] and references therein. Recently, Approximate controllability for semilinear control systems can be founded in [40, 41, 50], and for stochastic systems in [54, 55] with a range condition of the control action operator. In Di Blasio et al. [2], they have developed an excellent state space theory for retarded system in the product space $F \times L^2(-h, 0; D(A_0))(h > 0)$, where $F = D_{A_0}(1/2, 2)$ is the Lions real interpolation space between $D(A_0)$ and H. Since it enables us to express the solution with the aid of the solution semigroup (cf. [2, 21]), it is convenient to consider the original equation in the space $Z \equiv H \times L^2(-h, 0; V)$.

Now, we introduce the solution semigroup $S(t)$ for the system (5.1.2) defined by

$$S(t)g = (u(t; g, 0), u_t(\cdot; g, 0)),$$

where $g = (g^0, g^1) \in Z \equiv H \times L^2(-h, 0; V)$, $u(t; g, 0)$ is the solution of (5.1.2) with $B_0 = 0$ and $u_t(\cdot; g, 0)$ is the function $u_t(s; g, 0) = u(t + s; g, 0)$ defined in $[-h, 0]$. With the aid of the solution semigroup, we can define the approximate controllability and observability in Z without using the fundamental solution. We define the set of attainability by

$$R = \Big\{ \int_0^t S(t-\tau)Bw(t)d\tau : w \in L^2(0, t; U), \quad t > 0 \Big\},$$

where $Bw = (B_0 w, 0)$. Let $v(t; \phi)$ be a solution the following adjoint system of (5.1.2):

$$\begin{cases} v'(t) = A_0^* v(t) + A_1^* v(t-h) + \int_{-h}^0 a(s) A_2^* u(t+s)ds, \\ v(0) = \phi^0, \quad v(s) = \phi^1(s), \quad s \in [-h, 0), \end{cases} \tag{5.1.3}$$

where A_ι^*, $\iota = 0, 1, 2$, are adjoint operators of A_ι, repectively, and $\phi = (\phi^0, \phi^1) \in Z$. We say that the system (5.1.2) is approximately controllable if R is dense in Z and the adjoint system (5.1.3) is observability if $\phi \in Z$, $B_0^* v(t; \phi) \equiv 0$ implies $\phi = 0$.

When X is a reflexive Banach space, Nakagiri and Yamamoto [45] developed the controllability of (5.1.2) in the product space $X \times L^p(-h, 0; X)(p > 1)$ with bounded principal operators under the condition of the completeness of the infinitesimal generators A_0.

In this paper, assuming that $a(\cdot)$ has only to belong to $L^2(-h, 0)$ with unbounded principal operators, we obtain a number of criteria for various controllability and observability for (5.1.2) and (5.1.3) in Hilbert spaces, respectively.

The structural operator $F : Z \longrightarrow Z^* \equiv H \times L^2(-h, 0; V^*)$ is defined by

$$Fg = (g^0, A_1 g^1(-h-s) + \int_{-h}^{0} a(\tau) A_2 g^1(\tau - s) d\tau).$$

In section 2, we will show that if F is an isomorphism, then the approximate controllability of (5.1.2) is equivalent to the observability of (5.1.3), Further, since we can not define the attainability set using solution semigroup $S(t)$ in the space V^*, we will prove that the system (5.1.3) is observable if $\phi \in Z$, $B_0^* v(t; \phi) \equiv 0$ almost everywhere implies $\phi = 0$ except using solution semigroup.

In section 3, when $A_1 = \gamma A_0$, γ is a real constant, $A_2 = A_0$, we deal with the spectrum of the infinitesimal generator A of $S(t)$. Moreover, we study the problem of completeness of generalized eigenspaces of A. We also prove that the condition of the completeness of between A_0 and the infinitesimal generator of the solution semigroup is the necessary and sufficient property.

Finally, when the control space U is a finite dimensional space, a necessary and sufficient for the controllability of (5.1.2) is given as the so called Rank Condition. The rank condition of linear equations without delay terms(in case $A_1 = A_2 \equiv 0$) is given in [21, 39, 46, 47]. In order to obtain the approximate controllability of (5.1.2), we no longer require the condition of the compactness of the infinitesimal generator of solution semigroup, but we need the compactness of the embedding $V \subset H$.

5.2 Controllability and observability

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X,Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X,X)$ is simply written as $\mathcal{L}(X)$.

Let V and H be complex Hilbert spaces forming a Gelfand triple $V \subset H \subset V^*$ with pivot space H. The notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^* as usual, respectively. For the sake of simplicity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad u \in V.$$

The duality pairing $(\cdot,\cdot)$ between V^* and V is the extension by continuity of inner product in H.

From now on, we consider the control system with initial values of the following form:

$$\begin{cases} x'(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + B_0u(t), \\ x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h,0). \end{cases} \tag{5.2.1}$$

Equations of the type (5.2.1) were investigated in the state space $D_{A_0}(1/2,2)\times L^2(-h,0;D(A_0))(h>0)$ by Di Blasio et el. [2]. See also the bibliography of this paper. If an operator A_0 is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{1/2,2} = H,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(see [38, 8]). Using the maximal regularity for more general retarded parabolic system, we can follow the argument of [2] term by term to deduce the following results as seen in [6].

Proposition 5.2.1. *Let $T > 0$, $g = (g^0, g^1) \in H \times L^2(-h, 0; V)$, and $u \in L^2(0, T; U)$. Then there exists a unique solution x of equation (5.2.1) such that*

$$x \in L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

Moreover, there exists a constant C such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;U)}).$$

Let

$$Z \equiv H \times L^2(-h, 0; V).$$

be the state space of the equation (5.2.1). Z is a product Hilbert space with the norm

$$||g|| = (|g^0|^2 + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

The adjoint space Z^* of Z is identified with the product space $H \times L^2(-h, 0; V^*)$ via the duality pairing

$$(g, f)_Z = (g^0, f^0) + \int_{-h}^{0} ((g^1(s), f^1(s)))ds, \quad g = (g^0, g^1) \in Z, \ f = (f^0, f^1) \in Z^*,$$

where $(\cdot, \cdot)$ denote the inner product on H and $((\cdot, \cdot))$ the duality pairing between V and V^*. Let $g \in Z$ and $x(t; g, u)$ be a solution of (5.2.1) associated with control u at time t. The segment x_t be given by $x_t(s; g, u) = x(t+s; g, u)$, $s \in [-h, 0)$. Thus, we can define the solution semigroup for the system (5.2.1) as follows [2, Theorem 4.1]:

$$S(t) = (x(t; g, 0), x_t(\cdot; g, 0)),$$

where $g = (g^0, g^1) \in Z$. Then, we have the following proposition which can be shown just as in theorem 4.2 of [2].

Proposition 5.2.2. *(i) The operator $S(t)$ is a C_0-semigroup on Z.*

(ii) The infinitesimal generator A of $S(t)$ is characterized by

$$D(A) = \{g = (g^0, g^1) : \ g^0 \in D(A_0), \ g^1 \in W^{1,2}(-h, 0; V),$$

$$g^1(0) = g^0, \ A_0 g^0 + \int_{-h}^{0} a(s) A_1 g^1(s) ds \in H\},$$

$$Ag = (A_0 g^0 + \int_{-h}^{0} a(s) A_1 g^1(s) ds, g^1).$$

Let A be the infinitesimal generator of $S(t)$ as in Proposition 5.2.2. Then the equation (5.2.1) can be transformed into an abstract equation in Z as follows.

$$\begin{cases} z'(t) = Az(t) + Bu(t), \\ z(0) = g \end{cases} \tag{5.2.2}$$

where $z(t) = (x(t; g, f, u), x_t(\cdot; g, f, u)) \in Z$ and $g = (g^0, g^1) \in Z$. The control operator B defined by $Bu = (B_0 u, 0)$. The mild solution of initial value problem (5.2.2) is the following form:

$$z(t; g, u) = S(t)g + \int_0^t S(t-s)Bu(s)\}ds.$$

We introduce the transposed problem of (5.2.1):

$$\begin{cases} y'(t) = A_0^* y(t) + A_1^* y(t-h) + \int_{-h}^0 a(s)A_2^* y(t+s)ds, \quad t \in (0, T], \\ y(0) = \phi^0, \quad y(s) = \phi^1(s), \quad s \in [-h, 0). \end{cases} \tag{5.2.3}$$

We can also define the solution semigroup $S_T(t)$ of (5.2.3) by

$$S_T(t)\phi = (y(t; \phi), y_t(\cdot, \phi))$$

for $\phi = (\phi^0, \phi^1) \in Z$, where $y(t; \phi)$ is the solution of (5.2.3). Let A_{T} be the infinitesimal generator of $S_{\mathrm{T}}(t)$ associated with the system (5.2.3). Then the equation (5.2.3) can also be transformed into the following equation:

$$\begin{cases} \hat{z}'(t) = A_{\mathrm{T}} \hat{z}(t), \\ \hat{z}(0) = \phi, \end{cases} \tag{5.2.4}$$

where $\hat{z}(t) = (y(t;\phi), (y_t(\cdot;\phi)) \in Z$ and $\phi = (\phi^0, \phi^1) \in Z$. Let Π_0 be the projection of Z onto H, i.e., $\Pi_0(g^0, g^1) = g^0$ for $(g^0, g^1) \in Z$.

The structural operator F is defined by

$$Fg = ([Fg]^0, [Fg]^1),$$

$$[Fg]^0 = g^0,$$

$$[Fg]^1(s) = A_1 g^1(-h-s) + \int_{-h}^{0} a(\tau) A_2 g^1(\tau - s) d\tau$$

for $g = (g^0, g^1) \in Z$. It is easy to see that for any $\phi = (\phi^0, \phi^1) \in Z$

$$[F^*\phi]^0 = \phi^0,$$

$$[F^*\phi]^1(s) = A_1^*\phi^1(-h-s) + \int_{-h}^{0} a(\tau) A_2^* \phi^1(\tau - s) d\tau.$$

As in [6, 21] we have that $F \in \mathcal{L}(Z, Z^*)$ and

$$FS(t) = S_T^*(t)F^*, \quad F^*S_T(t) = S^*(t)F^*. \tag{5.2.5}$$

We denote the set of attainability by

$$R = \{\int_0^t S(t-s)Bw(s)ds : w \in L^2(0,T;U), \quad t \geq 0\}.$$

Definition 5.2.1. *(1) The system (5.2.1) is approximately controllable if* $\overline{R} = Z$*, where* $\overline{R}$ *is the closure of* R *in* Z*.*

(2) The system (5.2.3) is observable if $\phi = (\phi^0, \phi^1) \in Z$*,* $B_0^*\Pi_0[S_T(t)\phi] = 0$ *a.e. implies* $\phi = 0$*.*

Here we note that $\Pi_0\big[\int_0^t S(t-s)Bw(s)ds\big] = x(t,0,w)$. This means that the approximate controllability of system (5.2.2) implies the approximate controllability of system (5.2.1).

Theorem 5.2.1. *Let the structural operator F be an isomorphism. Then the system (5.2.1) is approximately controllable if and only if The system (5.2.3) is observable .*

Proof. Using the duality theorem, we obtain

$$
\begin{aligned}
&\{\int_0^t S(t-s)Bw(s)ds : w \in L^2(0,T;U),\ t \geq 0\}^{\perp} \\
= &\ \{f \in Z^* : B^*S^*(t)f = 0,\ t > 0\}.
\end{aligned}
$$

Thus, the system (5.2.2) is approximately controllable iff $B^*S^*(t)f = 0 (t > 0)$ for any $f \in Z^*$. Since F^* is isomorphism, there exists $\phi \in Z$ such that $F^*\phi = f$. From (5.2.5) it follows that

$$S^*(t)f = F^*S_T(t)\phi = F^*S_T(t)\phi.$$

Noting that $B^*F^*(\phi^0, \phi^1) = B_0^*\phi^0$, we have

$$B^*S^*(t)f = B^*S^*(t)F^*\phi = B^*F^*S_{\mathrm{T}}(t)\phi = B_0^*\Pi_0[S_{\mathrm{T}}(t)\phi].$$

Consequently, the approximately controllability od (5.2.1) is equivalent to the fact that for any $\phi \in Z_\lambda^{\mathrm{T}}$, $B_0^*\Pi_0[S_{\mathrm{T}}(t)\phi] = 0$ a.e. implies $\phi = 0$, or the observability of (5.2.3). $\square$

Remark 5.2.1. *Let* $A_1 : V \to V^*$ *be an isomorphism. Then for* $f = (f^0, f^1) \in Z^*$*, the element* $g \in Z$ *satisfying*

$$\begin{cases} g^0 = f^0 \\ g^1(-h-s) + \int_{-h}^0 a(\tau)A_1^{-1}A_2 g^1(\tau - s)d\tau = A_1^{-1}f^1(s) \end{cases}$$

is the unique solution of $Fg = f$*. The above integral equation is of Volterra type, and so it can be solved by successive approximation. Therefore,* $F : Z \to Z^*$ *is an isomorphism.*

For $\lambda \in \mathbb{C}$ we define a densely defined closed linear operator by

$$\Delta(\lambda) = \lambda - A_0 - e^{-\lambda h}A_1 - \int_{-h}^0 e^{\lambda s}A_2 ds,$$

$$\Delta_T(\lambda) = \lambda - A_0^* - e^{-\lambda h}A_1^* - \int_{-h}^0 e^{\lambda s}A_2^* ds.$$

Noting that if $\lambda \in \rho(A_0)$

$$\Delta(\lambda) = \big\{I - \big(e^{-\lambda h}A_1 + \int_{-h}^0 e^{\lambda s}A_2 ds\big)(\lambda - A_0)^{-1}\big\}(\lambda - A_0)$$

Lemma 5.2.1. $(\lambda - A)f = \phi$ *if and only if*

$$\Delta(\lambda) = \phi^0 + \int_{-h}^0 e^{-\lambda(h+\tau)}A_1\phi^1(\tau)d\tau + \int_{-h}^0 a(s)\int_s^0 e^{s-\tau}A_2\phi^1(\tau)d\tau ds,$$

$$f^1(s) = e^{\lambda s}f^0 + \int_s^0 e^{s-\tau}\phi^1(\tau)d\tau.$$

Let $\lambda \in \sigma(A)$ be an isolated point of $\sigma(A)$ and P_λ be the spectral projection associated with λ:

$$P_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu - A)^{-1} d\mu,$$

where Γ_λ is a small circle centered at λ such that it surrounds no point of $\sigma(A)$ except λ. And we know that $\overline{\lambda} \in \sigma(A_T)$ and the spectral projection is given by

$$P_{\overline{\lambda}}^T = \frac{1}{2\pi i}\int_{\Gamma_{\overline{\lambda}}} (\mu - A^T)^{-1} d\mu.$$

It is well known that λ is an eigenvalue of A and the generalized eigenspace corresponding to λ is given by

$$Z_\lambda = P_\lambda Z = \{P_\lambda u : u \in Z\} \ (\text{or } Z_{\overline{\lambda}}^T = P_{\overline{\lambda}}^T Z).$$

Moreover, we set

$$Z_{\overline{\lambda}}^* = \mathrm{Im}(P_\lambda)^*.$$

It is also well known that λ is a pole of $(\lambda - A)^{-1}$ whose order we denote by k_λ and $\dim Z_{\lambda_j} < \infty$. Let us set

$$Q_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\lambda - \lambda)(\lambda - A)^{-1} d\lambda.$$

Then we remark that

$$Q_{\lambda_j}^i = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_j}} (\lambda - \lambda_j)^i (\lambda - A)^{-1} d\lambda.$$

It is also well known that $Q_{\lambda_j}^{k_{\lambda_j}} = 0$ (nilpotent) and $(A-\lambda)P_{\lambda_j} = Q_{\lambda_j}$(cf. [48, 53]). The following set subset of $\sigma(A)$ are especially of use:

$$\sigma_p(A) = \text{the point spectrum of } A$$

$$\sigma_d(A) = \{\lambda \in \sigma(A) : \lambda \text{ is isolated and } \dim(Z_\lambda) < \infty\}.$$

We know that $\lambda \in \sigma_d(A)$ if and only if $\overline{\lambda} \in \sigma_d(A_T)$.

Lemma 5.2.2. *Let $\lambda \in \sigma_p(A)$. Then*

1) $\mathrm{Ker}\,(\lambda I - A) = Z_\lambda \cap \mathrm{Ker}\, G_\lambda$

2)

$$Ker\,(\lambda - A)^k = \Big\{\big(\phi_0^0, e^{\lambda s}\sum_{i=0}^{k-1}(-s)^i\phi_i^0/i!\big) : \sum_{i=j-1}^{k-1}(-1)^{i-j}\Delta^{(i-j+1)}(\lambda)\phi_i^0/(i-j+1)! = 0,\ j = 1,\cdots,k\Big\}.$$

3) $\lambda \in \rho(A) = \rho(A_T^*)$,

$$F(\lambda - A)^{-1} = (\lambda - A_T^*)^{-1}F.$$

In particular, if $\lambda \in \sigma_p(A)$ then

$$FP_\lambda = (P_\lambda^T)^*F.$$

The proof of 1) is from Suzuki and Yamamoto [47, Appendix I], and 2)and (3) from Nakagiri [21, Proposition 7.2] and [21, Theorem 6.1], respectively.

Definition 5.2.2. *The system of generalized eigenspaces of A is complete if*

$$\mathrm{Cl}(\mathrm{span}\{Z_\lambda : \lambda \in \sigma_p(A)\}) = Z,$$

where Cl *denotes the closure in Z.*

Lemma 5.2.3. *Let $\lambda \in \sigma_p(A)$. Then*

1) Let the system of generalized eigenspaces of A_T be complete and F be one to one. Then $P_\lambda g = 0$ implies g=0.

2) Let the system of generalized eigenspaces of A be complete and F^* be one to one. Then $P_\lambda^T f = 0$ implies f=0.

Proof. For any $\lambda \in \sigma_p(A)$, if $P_\lambda f = 0$ then $FP_\lambda f = 0$. Thus, from (5.2.3) or 3) of Lemma 5.2.2 it follows that

$$(FP_\lambda f, g) = ((P_\lambda^T)^* Ff, g) = (Ff, P_\lambda^T g) = 0. \tag{5.2.6}$$

Since he system of generalized eigenspaces of A_T be complete, (5.2.6) implies $f = 0$. The proof of 2) is similar. □

Theorem 5.2.2. *Assume that the system of generalized eigenspaces of A_T be complete and F be one to one. Then the system (5.2.3) is observable if and only if*

$$\mathrm{Ker}\, B_0^* \cap \mathrm{Ker}\, \Delta_T(\lambda) = \{0\}, \quad \forall \lambda \in \sigma_p(A_T).$$

Proof. Let $B_0^*\Pi_0[S_T(t)\phi] = 0$ a.e. for $\phi = (\phi^0, \phi^1) \in Z$. Since $S_T(t)$ is C_0-semigroup, there exist $M \geq 1$ and $\beta \in \mathbb{R}$ such that

$$||S_T(t)||_{\mathcal{L}(Z)} \leq Me^{\beta t}.$$

For $\mathrm{Re}\,\mu > \beta$, we have

$$(\mu - A_T)^{-1}\phi = \int_0^\infty e^{-\mu t} S_T(t)\phi dt,$$

and

$$B_0^*\Pi_0[(\mu - A_T)^{-1}\phi] = \int_0^\infty e^{-\mu t} B_0^*\Pi_0[S_T(t)\phi]dt.$$

This implies

$$B_0^*\Pi_0[(Q_\lambda^T)^j\phi] = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu - \lambda)^j B_0^*\Pi_0[(\mu - A_T)^{-1}\phi]d\mu = 0 \qquad (5.2.7)$$

for any $j = 0, ...k_\lambda - 1$ and $\lambda \in \sigma_p(A_T)$, In what follows we follow the method of [45] and [47]. Put

$$\phi_1 = (Q_\lambda^T)^{k_\lambda - 1}\phi,$$

then $\phi_1 \in KerQ_\lambda^T$, so that $\phi_1 \in Ker(\lambda - A_T)$ by 1) of Lemma 5.2.2. As is seen in [21, Proposition 7.2], there exists $\phi_1^0 \in Ker\Delta_T(\lambda)$ such that $\phi_1 = (\phi_1^0, e^{\lambda s}\phi_1^0)$. It follows from (5.2.7) that $B_0^*\phi_1^0 = B_0^*[(Q_\lambda^T)^{k_\lambda - 1}\phi]^0 = 0$. From the hypothesis we have $\phi_1^0 = 0$, hence $\phi_1 = 0$. Put $\phi_2 = (Q_\lambda^T)^{k_\lambda - 2}\phi$, then $\phi_2 \in Ker(\lambda - A_T)$. Hence by the same way we obtain that $\phi_2 = 0$. Continuing this procedure, we have $P_\lambda^T\phi = 0$. Therefore, from Lemma 5.2.3 it follows that $\phi = 0$.

(Necessity). Suppose that $\phi^0 \in Ker\, B_0^* \cap Ker\, \Delta_T(\lambda)$ for some $\lambda \in \sigma_p(A_T)$. Then

$$\phi = (\phi^0, e^{\lambda s}\phi^0) \in Ker(\lambda - A_T) \quad \text{and} \quad B_0^*\phi^0 = 0.$$

It implies that

$$S_T(t)\phi = e^{\lambda t}\phi \quad \text{and} \quad B_0^*\Pi_0[S_T(t)\phi] = B_0^*(e^{\lambda t}\phi^0) = e^{\lambda t}B_0^*\phi^0 = 0.$$

By the hypothesis we obtain that $\phi = 0$, and hence $\phi^0 = 0$. □

5.3 Spectral properties in case $A_1 = \gamma A_0$, $A_2 = A_0$

In this section we investigate the spectral properties of the infinitesimal generator A of $S(t)$ in the special case where $A_1 = \gamma A_0$ with some constant γ, $A_2 = A_0$ and the embedding $V \subset H$ is compact. Thus, in what follows we consider the equation

$$\begin{cases} x'(t) = A_0x(t) + \gamma A_0 x(t-h) + \int_{-h}^0 a(s)A_0x(t+s)ds + B_0u(t), \\ x(0) = g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{5.3.1}$$

According to Riesz-Schauder theorem A_0 has discrete spectrum

$$\sigma(A_0) = \{\mu_j : j = 1, \cdots\}$$

which has no point of accumulation except possibly $\lambda = \infty$.

For $\lambda \in \mathbb{C}$ we have

$$\Delta(\lambda) = 1 - m(\lambda)A_0$$

where

$$m(\lambda) = 1 + \gamma e^{-\lambda h} + \int_{-h}^{0} e^{\lambda s} a(s) ds.$$

It is clear that $m(0) \neq 0$ is an entire function and

$$m(\lambda) \to 1 \text{ as } \mathrm{Re} \to \infty. \tag{5.3.2}$$

Lemma 5.3.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be analytic on a neighborhood of z_0 and z_0 be a zero of f multiplicity $k \geq 1$. Then there exists a neighborhood V at zero and analytic function $\phi : V \to D(f)$ such that $f(\phi(w)) = w^k$, where $D(f)$ denotes the domain of f.*

Proof. There exists an analytic function g on neighborhood at z_0 such that $f(z) = (z - z_0)^k g(z)$, where $g(z_0) \neq 0$. Since $g(z) \neq 0$ on neighborhood at z_0 there exists a analytic function h such that $g(z) = h(z)^k$. Thus $(z - z_0)h(z)|_{z=z_0} = 0$ and

$$\frac{d}{dz}((z - z_0)h(z))|_{z=z_0} = (h(z) + (z - z_0)\frac{d}{dz}h(z))|_{z=z_0} = h(z_0) \neq 0.$$

Hence, from inverse mapping theorem it follows that there exist a neighborhood U at z_0 and a neighborhood V at zero such that the mapping $z \mapsto (z - z_0)h(z)$ is a homeomorphism from U onto V. If we denote by $\phi(w)$ the inverse of such mapping, then the function ϕ is analytic on V, $\phi(0) = z_0$ and $(\phi(w) - z_0)h(\phi(w)) = w$ for any $w \in V$. Therefore, it holds that

$$f(\phi(w)) = (\phi(w) - z_0)^k g(\phi(w)) = (\phi(w) - z_0)^k (h(\phi(w)))^k = w^k.$$

□

Theorem 5.3.1. *(i) Let $\rho(A)$ be the resolvent set of the infinitesimal generator A of $S(t)$. Then*

$$\rho(A) = \{\lambda : m(\lambda) \neq 0, \ \ \frac{\lambda}{m(\lambda)} \in \rho(A_0)\}$$

$$= \{\lambda : \Delta(\lambda) \ \ is \ \ \ isomorphism \ \ from \ V \ \ \ onto \ V^*\}$$

(ii) Let $\sigma(A)$ be the spectrum of A. Then

$$\sigma(A) = \sigma_e(A) \cup \sigma_p(A),$$

where $\sigma_e(A) = \{\lambda : m(\lambda) = 0\}$ and $\sigma_p(A) = \{\lambda : m(\lambda) \neq 0, \ \lambda/m(\lambda) \in \sigma(A_0)\}$. Each nonzero point of $\sigma_e(A)$ is not an eigenvalue of A but a cluster point of $\sigma(A)$. $\sigma_p(A)$ consists only of discrete eigenvalues.

(iii) Suppose $m(0) = 0$. Then 0 is an eigenvalue of A with infinity multiplicity. 0 is an isolated point of $\sigma(A)$ if it is a simple zero of $m(\lambda)$ and a cluster point of $\sigma(A)$ if it is a multiple zero of $m(\lambda)$.

Proof. (i) If $m(\lambda) \neq 0$ and $\lambda/m(\lambda) \in \rho(A_0)$, then for all $\phi \in X$, there exists $f = (f^0, f^1) \in D(A)$ such that Lemma 5.2.1 holds. Hence $R(\lambda - A) = X$ where $R(A)$ denotes the range of A. Let $(\lambda - A) = f = 0$. Then from Lemma 5.2.1 it follows that $\Delta(\lambda)f^1(0) = 0$. Therefore $f^1(0) = 0$ and hence $f^1(s) = 0$. We have proved that $\lambda \in \rho(A)$.

Conversely, if $m(\lambda) = 0$, then since $\Delta(\lambda) = \lambda I_v$, $\Delta(\lambda)$ is not onto H. If $m(\lambda) \neq 0$ and $\lambda/m(\lambda) \in \sigma(A_0)$, then the mapping $\Delta(\lambda) = m(\lambda)(\lambda/m(\lambda) -$

A_0) is not onto. Let $\phi = (\phi^0, 0)$ where $\phi^0 \in H \backslash \mathrm{Im}\Delta(\lambda)$. Then there does not exist $f^1(0)$ such that Lemma 5.2.1 holds.

(ii) Let $\lambda_0 \neq 0$ be a zero of $m(\lambda)/\lambda$ of multiplicity $k \geq 1$. From Lemma 5.3.1, it follows that there exists an analytic function ϕ on a neighborhood V at zero such that for any $\mu \in V$,

$$m(\phi(\mu))/\phi(\mu) = \mu^k, \quad \phi(0) = \lambda_0.$$

Let us denote by λ_j a k-th root of $1/\mu_j$, then λ_j converges to zero as j tends to infinity. Infact, $\sigma(A_0) = \{\mu_j : j = 1, 2, \cdots\}$ has no point of cluster point except for infinity point. If j is sufficiently large then $\lambda_j \in V$ and $\phi(\lambda_j)/m(\phi(\lambda_j)) = \mu_j \in \sigma(A_0)$. Hence, it holds that $\phi(\lambda_j) \in \sigma(A)$ and $\phi(\lambda_j)$ tends to $\phi(0) = \lambda_0$ as j tends to infinity. We have proved nonzero point of $\sigma_e(A)$ is a cluster point of $\sigma(A)$.

Next, suppose $m(\lambda_0) \neq 0$, $\lambda_0/m(\lambda_0) \in \sigma(A_0)$. If there exists a sequence $\{\lambda_j\}$ such that $\lambda_j/m(\lambda_j) \in \sigma(A_0)$. Since $\sigma(A_0)$ consists only of isolated points, we have $\lambda_j/m(\lambda_j) = \lambda_0/m(\lambda_0)$ for sufficiently large j. In view of the theorem of identity we have $m(\lambda) = \lambda_0\lambda/m(\lambda_0)$ which is contradictory to (5.3.2).

(iii) If $m(0) = 0$, then for all $v \in V$, $f = (f^0, f^1)$ defined by $f^0 = v$ and $f^1(s) \equiv v \ s \in [-h, 0]$ belongs to the eigenspace corresponding to zero of A with infinity multiplicity. The others of this assertion is obtained by similar way of (ii). □

Here, we note again that A_0 has discrete spectrum

$$\sigma(A_0) = \{\mu_j : j = 1, \cdots\}.$$

Defining the spectral operator p_n associated with A_0 by

$$p_n^T = \frac{1}{2\pi i}\int_{|\mu-\mu_n|=\epsilon_n} (\mu - A_0)^{-1} d\mu,$$

where the circle surrounds no point of $\sigma(A_0)$ except μ_n. Putting

$$H_n = p_n H = \{p_n u : u \in H\},$$

we have that $p_n^2 = p_n$, $H_n \subset V$ and $\dim H_n < \infty$. Hence, it follows that

$$p_n V = \{p_n u : u \in V\} = H_n.$$

Lemma 5.3.2. *Let $g = (g^0, g^1)$ belongs to $H_n \times L^2(-h, 0; H_n)$. Then for the solution x of (5.3.1) we have $p_n x(t) = x(t)$*

Proof. If we compose p_n on both sides of (5.3.1). then $p_n x(t)$ is also a solution of (5.3.1). From uniqueness of the solution the result follows. □

Put $A_{0n} = A_0|H_n$. For any $g \in H_n \times L^2(-h, 0; H_n)$ the solution $u(t)$ of (5.3.1) is the solution satisfied the following

$$\begin{cases} x'(t) = A_{0n}x(t) + \gamma A_{0n}x(t-h) + \int_{-h}^{0} a(s)A_{0n}u(t+s)ds, \\ x(0) = g^0, x(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{5.3.3}$$

If we denotes the solution semigroup of equation (5.3.3) with A_{0n} in place of A_0 by $S_n(t) = \exp(tA_n)$, then we have that

$$S_n(t) = S(t)|_{H_n \times L^2(-h,0;H_n)},$$

$$A_n = A|_{D(A_n)},$$

$$D(A_n) = \{(g^0, g^1); g^1 \in W^{1,2}(-h, 0; H_n), g^0 = g^1(0)\}.$$

Let $\lambda_{ni}/m(\lambda_{ni}) = \mu_n$, $n = 1, 2, ...,$, then

$$P_{ni} = \frac{1}{2\pi i}\int_{|\lambda-\lambda_{ni}|=\epsilon_{ni}} (\lambda - A)^{-1} d\lambda.$$

Set Z_{ni} =ImP_{ni}.

Lemma 5.3.3. *$\phi \in Z_{ni}$ if and only if there exists an integer k such that $(\lambda_{ni} - A_n)^k\phi = 0$.*

Proof. If $(\lambda_{ni} - A)^k\phi = 0$ where $\phi = (\phi^0, \phi^1)$, then from $\Delta(\lambda_{ni})^k\phi^0 = 0$ and $\Delta(\lambda_{ni})^k\phi^1(s) \equiv 0$ it follows that

$$(\mu_n - A_0)^k\phi^0 = 0, \ (\mu_n - A_0)^k\phi^1(s) \equiv 0.$$

Hence, since $\phi^0 = p_n\phi^0 \in H_n$ and $\phi^1(s) = p_n\phi^1(s) \in H_n$ we have $(\lambda_{ni} - A_n)^k\phi = 0$. In view of the Lemma 5.3.2 $(\lambda_{ni} - A_n)^k\phi = 0$ implies $(\lambda_{ni} - A)^k\phi = 0$. Thus Lemma is proved.

Lemma 5.3.4. *The adjoint operator of p_n is represented by*

$$p_n^* = \frac{1}{2\pi i}\int_{|\mu-\overline{\mu_n}|=\epsilon_n} (\mu - A_0^*)^{-1} d\mu$$

Proof. If $\mu \in \rho(A_0)$, then p_n is a bounded linear operator from V^* into V because $(\mu - A_0)^{-1}$ is an isomorphism from V^* onto V. For any $\phi^0, \psi \in V^*$, from $(\phi^0, (\overline{\mu} - A_0^*)^{-1}\psi^0) = ((\mu - A_0)^{-1}\phi^0, \psi^0)$, we have

$$
\begin{aligned}
(p_n^*\phi^0, \psi^0) &= \frac{1}{2\pi i}\int_{|\mu-\overline{\mu_n}|=\epsilon} ((\mu - A_0^*)^{-1}\phi^0, \psi^0)d\mu \\
&= \frac{1}{2\pi i}\int_{|\mu-\overline{\mu_n}|=\epsilon} (\phi^0, (\overline{\mu} - A_0^*)^{-1}\psi^0 d\mu) \\
&= (\phi^0, \frac{1}{2\pi i}\int_{|\mu-\mu_n|=\epsilon} (\mu - A_0^*)^{-1}\psi^0 d\mu) \\
&= (\phi^0, p_n\psi^0).
\end{aligned}
$$

Thus, the lemma is proved. □

Theorem 5.3.2. *Suppose that $m(0) \neq 0,\ \gamma \neq 0$. Then the system of generalizes eigenspaces of A_n is complete in $H_n \times L^2(-h, 0; H_n)$, and so is the system of generalized eigenspaces of A in Z.*

Proof. From the corresponding result of Manitius([41, Theorems 5.1 and 5.4(ii)]) in the case a finite dimensional space, the system of generalized eigenspaces of A_n is complete in $H_n \times L^2(-h, 0; H_n)$. In view of Lemma 5.3.3 the system of generalized eigenspaces of A_n is $\cup_{n=1}^{\infty} Z_{ni}$(we remark that in the case of a finite dimensional case the complex number λ satisfied with $m(0) = 0$ belongs to the resolvent set). Suppose that $(f, Z_{ni}) = 0$ for any n and any i, where $f = (f^0, f^1) \in H \times L^2(-h, 0; V^*)$. Then in view of Lemma

5.3.4 we have that for all $\phi = (\phi^0, \phi^1) \in Z_{ni}$

$$((p_n^* f^0, p_n^* f^1)(\phi^0, \phi^1))_Z = (p_n^* f^0, \phi^0) + \int_{-h}^{0} ((p_n^* f^1(s), \phi^1(s)))ds$$

$$= (f^0, p_n \phi^0) + \int_{-h}^{0} ((f^1(s), p_n \phi^1(s)))ds$$

$$= (f^0, \phi^0) + \int_{-h}^{0} ((f^1(s), \phi^1(s)))ds$$

$$= ((f^0, f^1)(\phi^0, \phi^1))_Z = 0.$$

Thus $((p_n^* f^0, p_n^* f^1), Z_{ni}) = 0$ for any $i = 1, 2, \ldots$. Hence the element $(p_n^* f^0, p_n^* f^1)$ is orthogonal to $H_n \times L^2(-h, 0; H_n)$, and hence $p_n^* f^0 = 0$ and $p_n^* f^1(s) \equiv 0$. Since n is arbitrary number we have that $f^0 = 0$ and $f_1 \equiv 0$. We have proved that the system of generalized eigenspaces of A which is the set $\cup_{n,i} Z_{ni}$ is complete in $Z = H \times L^2(-h, 0; V)$. □

Lemma 5.3.5. *The structural operator F defined by*

$$Fg = ([Fg]^0, [Fg]^1),$$

$$[Fg]^0 = g^0, \quad [Fg]^1(s) = \gamma A_0 g^1(-h - s) + \int_{-h}^{0} a(\tau) A_0 g^1(\tau - s) d\tau$$

for $g = (g^0, g^1) \in Z$ is isomorphism.

Proof. We have only to prove that for any $f \in L^2(-h, 0; V^*)$ there exists uniquely $g^1 \in L^2(-h, 0; V)$ such that

$$f(s) = \gamma A_0 g^1(-h - s) + \int_{-h}^{0} a(\tau) A_0 g^1(\tau - s) d\tau. \qquad (5.3.4)$$

For $0 < s < h$ we set $b(s) = a(s+h)\gamma^{-1}$. Then the second term of the right hand side of equation (5.3.4) is represented as

$$\int_0^{s+h} a(\tau - h)g^1(\tau - h - s)d\tau = \gamma \int_0^{s+h} b(\tau)g^1(\tau - h - s)d\tau.$$

Let $r(s)$ be the solution for the following equation:

$$r + b + r * b = 0, \tag{5.3.5}$$

where

$$(r * b)(s) = \int_0^s r(s-\tau)b(\tau)d\tau = \int_0^s r(\tau)b(s-\tau)d\tau.$$

Let

$$g^1(s) = (\gamma A_0)^{-1}\{f(-h-s) + \int_s^0 r(\tau - s)f(-\tau - h)d\tau\}.$$

Then $g^1 \in L^2(-h, 0; V)$ and from (5.3.5) it follows that

$$\begin{aligned} f(s) =& f(s) + \int_{-h-s}^0 r(\sigma + h + s)f(\sigma - h)d\sigma + \int_0^{s+h} b(\tau)f(s-\tau)d\tau \\ &+ \int_0^{s+h}\int_{\tau-h-s} r(\sigma - tau + h + s)f(-\sigma - h)d\sigma\tau. \end{aligned}$$

The third term of right hand side of the equation above is rewritten by

$$\int_{-h-s}^0 \int_0^{\sigma+h+s} b(\tau)r(\sigma + h + s - \tau)d\tau f(-\sigma - h)d\sigma,$$

and by (5.3.4) the proof of Lemma is complete. □

Thus, from Theorem 5.2.2, Theorem 5.3.2 and Lemma 5.3.5, we obtain the following.

Corollary 5.3.1. *The transposed system of (5.3.1) is observable if and only if*

$$\operatorname{Ker} B_0^* \cap \operatorname{Ker} \Delta_T(\lambda) = \{0\}, \quad \forall \lambda \in \sigma_p(A_T).$$

5.4 Rank condition

Now, we note that if we know that A_0 is self adjoint with a compact resolvent in virtue of compactness of the embedding $V \subset H$, the system of generalizes eigenspaces of A_0 is complete in H. In fact, since dim $\operatorname{Ker}(\lambda - A_0) = d_\lambda$, we suppose that $\{\phi^0_{\lambda i}; i = 1, ..., d_\lambda\}$ is a subset of Ker $\Delta(\lambda)$ and by Lemma 5.2.2,

$$\{\phi_{\lambda i} = (\phi^0_{\lambda i}, c^{\lambda s}\phi^0_{\lambda i}) : i = 1, ..., d_\lambda\} \subset \operatorname{Ker}(\lambda I - A), \quad \lambda \in \sigma_p(A).$$

Hence, $\{\phi_{\lambda i}\}_{\lambda \in \sigma_p(A), 1 \le \lambda \le d_\lambda}$ is a complete orthogonal system in Z and it holds

$$\mathrm{Cl(Span\{\Pi_0 Z_\lambda : \lambda \in \sigma_p(A)\}) \supset Cl(Span\{\phi^0_{\lambda i} : i = 1, ..., d_\lambda\} = X},$$

which means that the system of generalizes eigenspaces of A_0 is complete in H. Hence from Theorem we know that the system of generalizes eigenspaces of A is also complete in Z from Theorem 5.3.2. Thus, Combining Lemma 5.3.4 and Theorem 5.2.1, we obtain the following result.

Corollary 5.4.1. *The system (5.2.2) is approximately controllable if and only if the system (5.2.4) is observable .*

Next we consider the case where the control space U is a finite dimensional space $\mathbb{C}^N$. Then, the controller $B_0 : \mathbb{C}^N \longrightarrow L^1(\Omega)$ is expressed as

$$B_0 u = \sum_{i=1}^{N} u_i b_i^0, \quad \forall u = (u_1, \cdots, u_N) \in \mathbb{C}^N,$$

where $b_i^0(i = 1, ..., N)$, are some fixed elements of H. The adjoint operator $B_0^* : H \to \mathbb{C}^N$ of B_0 is given by

$$B_0^* w = ((w, b_1^0), ..., (w, b_N^0)), \quad w \in H.$$

We suppose that the basis $\{\varphi_{\lambda 1}, ..., \varphi_{\lambda m_\lambda}\}$ of $P_i^T Z$ is arranged so that $\{\varphi_{\lambda 1}, ..., \varphi_{\lambda m_\lambda}\}$ span $\mathrm{Ker}(\lambda - A_T)$ where $d_\lambda = \mathrm{dimKer}(\lambda - A_T)$. Then $\{\varphi_{\lambda i}^0; i = 1, ..., d_\lambda\}$ is a basis of $Ker\ \Delta_T(\lambda)$ and $\varphi_{\lambda i} = (\varphi_{\lambda i}^0, c^{\lambda s}\varphi_{\lambda i}^0)$ for $i = 1, ..., d_\lambda$. Since $\varphi_{ij}^0 \in L^\infty(\Omega)$, $(b_i^0, \varphi_{\lambda i}^0)$ are all meaningful. We assume

Rank Condition: For any $\lambda \in \sigma_p(A_T)$

$$\mathrm{rank} \begin{pmatrix} (b_1^0, \varphi_{\lambda 1}^0) & \cdots & (b_1^0, \varphi_{\lambda d_\lambda}^0) \\ \vdots & \ddots & \vdots \\ (b_N^0, \varphi_{\lambda 1}^0) & \cdots & (b_N^0, \varphi_{\lambda d_\lambda}^0) \end{pmatrix} = d_\lambda$$

Theorem 5.4.1. *If Rank Condition is satisfied, then the problem (5.2.3) is observable.*

Proof. Let $\varphi^0 \in Ker\ \Delta_T(\lambda)$ for some $\lambda \in \sigma_p(A_T)$. Then $\varphi = (\varphi^0, e^{\lambda s}\varphi^0) \in \mathrm{Ker}(\lambda - A_T)$ and $\varphi = \Sigma_{i=1}^{d_\lambda} c_i \varphi_{\lambda i}$ for $c_i \in \mathbf{C}^N$. Hence, by Rank Condition we obtain

$$B_0^*\varphi^0 = B_0^*(\sum_{i=1}^{d_\lambda} c_i\varphi_{\lambda i}^0)$$

$$= ((\sum_{i=1}^{d_\lambda} c_i\varphi_{\lambda i}^0, b_1^0), \cdots, (\sum_{i=1}^{d_\lambda} c_i\varphi_{\lambda i}^0, b_N^0))$$

$$= \begin{pmatrix} c_1 & \cdots & c_2 \end{pmatrix} \begin{pmatrix} (\varphi_{\lambda 1}^0, b_1^0) & \cdots & (\varphi_{\lambda 1}^0, b_N^0) \\ \vdots & \ddots & \vdots \\ (\varphi_{\lambda d_\lambda}^0, b_1^0) & \cdots & (\varphi_{\lambda d_\lambda}^0, b_N^0) \end{pmatrix}$$

$$= (0, \cdots, 0)$$

implies $c_1 = c_2 = \cdots = c_{d_\lambda} = 0$. Therefore, we have proved that $\mathrm{Ker}B_0^* \cap \mathrm{Ker}\ \Delta_T(\lambda) = \{0\}$ for $\lambda \in \sigma_p(A_T)$. The result follows from Theorem 5.2.2. □

Remark 5.4.1. *Let A_0 be the operator associated with a sesquilinear form $b(\cdot,\cdot)$ which is defined in $V \times V$ satisfying Gårding's inequality:*

$$\mathrm{Re}\ b(u, v) \geq c||u||^2 - c_1|u|^2, \quad c_0 > 0,\ c_1 \geq 0, \quad \forall u, v \in V.$$

We assume that $B_0 \in \mathcal{L}(U, V^*)$, where U is a Banach space and $A_1,\ A_2 \in \mathcal{L}(V, V^*)$. Suppose that the system of generalizes eigenspaces of A_0 is complete in H. Then The rank condition remains valid for this general case of the equation (5.3.1) with A_0 defined above in a Hilbert space.

Example

$$\frac{\partial u(x,t)}{\partial t} = a_1\frac{\partial^2 u(x,t)}{\partial x^2} + a_2 u(x, t-h) + \int_{-h}^0 a(s)u(x, t+s)ds + \sum_{i=1}^N u_i b_i^0(t) \tag{5.4.1}$$

for each $(x,t) \in \Omega \times (0,T]$ and $u_i \in \mathbb{C}(i=1,\cdots,N)$ with boundary and initial conditions

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = g^0(x),\ u(x,s) = g^1(x,s), \quad x \in [0,1],\ s \in [-h,0).$$

Here, $a_1 > 0$, $a \neq 0$, $b_i^0 \in L^2(0,1)(i=1,\cdots,N)$, and $a \in L^2(-h,0;H_0^1(0,1))$. Let

$$H = L^2(0,1),\ V = H_0^1(0,1),\ V^* = H^{-1}(0,1),$$

$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A_0 = \partial^2/\partial x^2 \quad \text{with} \quad D(A_0) = \{x \in H^2(0,\pi) : x(0) = x(1) = 0\}.$$

Let the controller $B_0 : \mathbb{C}^N \longrightarrow L^1(\Omega)$ be defined as

$$B_0 u = \sum_{i=1}^{N} u_i b_i^0, \quad \forall u = (u_1,\cdots,u_N) \in \mathbb{C}^N.$$

If we define the operators $A_1 = a_2 I$ and $A_2 = I$, then the system (5.4.1) can be written in the same form as of (5.1.2). The eigenvalue and the eigenfunction of A_0 are $\lambda_n = -a_1 n^2 \pi$ and $\phi_n(y) = \sqrt{2}\sin n\pi x$, respectively. It is well known that the spectrum $\sigma(A)$ of A defined in Proposition 5.2.2 is given by

$$\sigma(A) = \sigma_d(A) = \{\lambda \in \mathbb{C} : \Delta_n(\lambda) = 0 \quad \text{for some } n = 1,2,\cdots\},$$

where

$$\Delta_n(\lambda) = \lambda + a_1 n^2\pi^2 - a_2 e^{-\lambda h} - \int_{-h}^0 e^{\lambda s} a(s) ds$$

(see [44, 56]). Let $\{\lambda_{nj}\}_{j=1}^{\infty}$ be the set of root of $\Delta_n(\lambda) = 0$ and let k_{nj} be the multiplicity of λ_{nj} (in many cases $k_{nj} = 1$). The generalized eigenspace of A corresponding to $\lambda_{nj} \in \sigma(A)$ is given by

$$\text{span}\{e^{\lambda_{nj}s}\sin n\pi x, \cdots, s^{k_{nj}-1}e^{\lambda_{nj}s}\sin n\pi x\}.$$

Since $\{\sin n\pi x\}$ is complete in H, the system of generalized eigenspace of A_0 is complete. Hence, from Theorem 5.3.2 or [41, Theorem 5.4] it follows that the system of generalized eigenspace of A is complete in the product space in Z. Thus, according to Theorem 5.2.1 and 5.2.2, we can see that system (5.4.1) is approximately controllable if the rank condition is satisfied

5.5 Conclusion

This paper has established applicable conditions for the approximate controllability and observability of the adjoint system under assumptions that the system of generalized eigenspaces of the principal operator is complete and the structural operator of F defined as in Section 2 is isomorphism. With the aid of the structural operators of the adjoint system and spectral decomposition theory, we have obtained some general results of the approximate controllability of retarded systems or the observability of adjoint system without using the fundamental solution used methods commonly. We also investigated the condition of the completeness of the system of generalized eigenspaces of the principal operator. Moreover, it has been shown that when the control space is a finite dimensional space, a necessary and sufficient for the controllability of retarded systems is given as the so called rank condition, which is a generalization of the result for evolution systems without delay discussed in the previous results.

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