Thesis for the Degree of Doctor of Philosophy

# Soft Proximity Spaces

by

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# 부드러운 근접공간

Advisor : Prof. Jin Han Park

by

Oe Hyeon Kim



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## 요 약

본 논문은 soft 집합의 동치관계와 soft 위상공간을 특별히 다루고 soft 근접공간을 연구한 것으로 내용은 다음과 같이 요약된다.

첫째, soft 집합상의 동치관계를 소개하고, 일반 집합에서의 동치관계와 분할에 관한 결과들과 유사한 결과들을 얻는다. soft 집합 관계의 추이적 폐포에 대한 성질을 조사하고, 특히 주어진 soft 집합 $\langle F,A\rangle$의 동치관계의 반순서집합 $(\mathrm{ESSR}(\langle F,A\rangle),\subseteq)$이 최소요소와 최대요소를 가진 완전한 속임을 보인다.

둘째, soft 위상공간상의 특성을 보이기 위해 근방 및 폐포와 같은 기본개념을 소개하고, 보통 위상공간상에서의 결과들을 soft 위상공간으로 확장한다. soft 집합에 필터의 개념을 도입하여 기초적이지만 중요한 성질을 찾고, 특히 null soft 집합이 아닌 soft 집합상의 모든 soft 필터는 자신을 포함하는 극대 soft 필터들의 교집합임을 보인다. 또한, soft 위상공간상에서의 soft 필터의 수렴성과 adherence에 관하여 조사한다.

셋째, soft 집합상의 soft 근접을 정의하고 이와 관련된 성질을 제시한다. soft 위상공간에서와 유사하게 soft 근접공간의 연구에 대한 방안을 제공할 $\delta$-근방, soft proximally 연속, 그리고 soft cluster를 소개한다. 또한, 극대 soft 필터와 soft cluster의 관계를 이용해 soft 근접공간에서 얻을 수 있는 이론상의 중요한 결과들을 조사한다.

# Chapter 1

# Introduction

Most of problems in real life situation such as economics, engineering, environment, social sciences and medical sciences not always involve crisp data. So we cannot successfully use the traditional methods because of various types of uncertainties presented in these problems. Since Zadeh [65] introduced fuzzy sets in 1965, a lot of new theories treating imprecision and uncertainty have been introduced. Some of these theories are extensions of fuzzy set theory and the others try to handle imprecision and uncertainty in different ways. Kerre [30] has given a summary of the links that exist between fuzzy sets and other mathematical models such as flou sets [51], two-fold fuzzy sets [15] and $L$-fuzzy sets [21].

The theories such as probability theory [56], fuzzy set theory [65, 66], intuitionistic fuzzy set theory [4, 6], vague set theory [20] and rough set theory [47], which can be considered as mathematical tools for dealing with uncertainties, have their inherent difficulties (see [42]). The reason for these difficulties is possibly the inadequacy of parameterization tool of the theories. Molodtsov [42] introduced soft sets as a mathematical tool for dealing with uncertainties which is free from the above-mentioned difficulties. Since the soft set theory offers mathematical tool for dealing with uncertain, fuzzy and not clearly defined objects, it has a rich potential for applications to problems in real life situation. The concept and basic properties of soft set theory are presented in [38, 42]. He also showed how soft set theory is free from the parameterization inadequacy syndrome of fuzzy set theory, rough set theory, probability theory and game theory. However, several assertions presented by Maji et al. [38] are not true in general [2]. Based on the analysis of several operations on soft sets introduced in [38], Ali et al. [2] present some new algebraic operations for soft sets and prove that certain De Morgan's laws in soft set theory with respect to these new definitions. Maji et al. [37] used the soft sets into the decision making problems that are based on the concept of knowledge reduction in rough set theory. Chen et al. [13] presented a new definition of soft set parameterization reduction and compared this definition with related concept of knowledge reduction in rough set theory. Pei and Miao [48] showed that soft sets are a class of special information systems. Kong et al. [32] introduced the notion of normal parameter reduction of soft sets and its use to investigate the problem of sub-optimal choice and added a parameter set in soft sets. Zou and Xiao [69] discussed the soft data analysis approach. Xiao et al. [60] proposed the notion of exclusive disjunctive soft sets and studied some of its operations. The application of soft set theory in algebraic structures was introduced by Aktaş and Çağman [1]. They discussed the notion of soft groups and derived some basic properties and shows that soft groups extend the concept of fuzzy groups. Jun [25] and Jun and Park [26], respectively, investigated soft BCK/BCI-algebras and its application in ideal theory. Feng et al. [17] worked on soft semirings, soft ideals and idealistic soft semirings. Ali et al. [2] and Shabir and Ali [54] studied soft semigroups and soft ideals over a semigroup which characterize generalized fuzzy ideals and fuzzy ideals with thresholds of semigroup. Babitha and Sunil [9] attempted to open the theoretical aspects of soft sets by extending the notions of equivalence relations, composition of relations, partitions and functions to soft sets. Yang and Guo [64] introduced the notions of kernels and closures of soft set relation and soft set relation mappings and obtained some related properties. Shabir and Naz [55] applied the soft set theory in topological structures and introduced soft topological spaces. Çağman et al. [12] introduced a topology on a soft set, so-called "soft topology", and its related properties. They then presented the foundations of the theory of soft topological spaces. This is the starting point for soft mathematical concepts and structures that are based on soft set-theoretic operations.

In this thesis, we attempt to conduct a further study of equivalence soft set relations and to broad the theoretical aspects of soft topological spaces, and introduce the concept of soft proximity and investigate its properties. We briefly summarize the contents of the each chapter as follows.

In Chapter 2, we firstly review basic notions about soft sets. We discuss and study the equivalence soft set relations and give soft analogues of many results concerning ordinary equivalence relations and partitions, and then present the concept of transitive closure of soft set relation with related results. We prove that the poset $(\mathrm{ESSR}(\langle F, A\rangle), \wedge_e, \vee_e)$ of the equivalence soft set relations on a given soft set $\langle F, A\rangle$ is complete lattice with the least element and greatest element.

In Chapter 3, we focus our attention on soft topological spaces and give soft analogues of many results concerning neighborhoods and closures in ordinary topological space. Further, we present the concept of soft filters and gives that every soft filter on non-null soft set is the intersection of the family of ultra soft filters which include it. The adherence and convergence of soft filters in a soft topology with related results are also discussed.

In Chapter 4, we define the soft proximity on a soft set, and present its related properties. The concepts of $\delta$-neighborhood, soft proximally continuity and soft cluster are discussed. They furnish approaches to the study of soft proximity spaces. We show that ultra soft filters and soft clusters are closely related, and used this relationship to drive several important results in the theory of soft proximity spaces.

# Chapter 2

# Some properties of equivalence soft set relations

The soft set theory is a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Babitha and Sunil [Computers and Mathematics with Applications 60 (7) (2010) 1840-1849] introduced the notion of soft set relations as a soft subset of the Cartesian product of soft sets and discussed many related concepts such as equivalence soft set relations, partitions and functions. In this chapter, we further study the equivalence soft set relations and obtain soft analogues of many results concerning ordinary equivalence relations and partitions. Furthermore, we introduce and discuss the transitive closure of soft set relation and prove that the poset of the equivalence soft set relations on a given soft set is complete lattice with the least element and greatest element.

## 2.1 Preliminaries

In this section, we recall some basic notions in soft set theory. Let $U$ be an initial universe of objects and $E$ the set of parameters in relation to objects in $U$. Parameters are often attributes, characteristics or properties of objects. Let

$\mathcal{P}(U)$ denote the power set of $U$ and $A, B \subseteq E$.

**Definition 2.1.1.** [42] A pair $\langle F, A\rangle$ is called a soft set over $U$, where $F$ is a function given by

$$F : A \to \mathcal{P}(U). \tag{2.1}$$

In other words, a soft set over $U$ is a parameterized family of subsets of the universe $U$. For any parameter $x \in A$, $F(x)$ may be considered the set of $x$-approximate elements of the soft set $\langle F, A\rangle$. Note that the set of all soft sets over $U$ will denoted by $\mathcal{SS}(U)$.

**Definition 2.1.2.** Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two soft set in $\mathcal{SS}(U)$. Then

(1) $\langle F, A\rangle$ is called a soft subset [38] of $\langle G, B\rangle$, denoted by $\langle F, A\rangle \widetilde{\subseteq} \langle G, B\rangle$, if $A \subseteq B$ and $F(x) \subseteq G(x)$ for all $x \in A$;

(2) $\langle F, A\rangle$ is called a soft superset [38] of $\langle G, B\rangle$, denoted by $\langle F, A\rangle \widetilde{\supseteq} \langle G, B\rangle$, if $\langle G, B\rangle$ is a soft subset of $\langle F, A\rangle$;

(3) $\langle F, A\rangle$ is called soft equal [38] to $\langle G, B\rangle$, denoted by $\langle F, A\rangle = \langle G, B\rangle$, if $\langle F, A\rangle \widetilde{\subseteq} \langle G, B\rangle$ and $\langle F, A\rangle \widetilde{\supseteq} \langle G, B\rangle$;

(4) $\langle F, A\rangle$ is called a relative null soft set [2] (with respect to the parameter set $A$), denoted by $\Phi_A$, if $F(x) = \emptyset$ for all $x \in A$;

(5) $\langle F, A\rangle$ is called a relative whole soft set [2] (with respect to the parameter set $A$), denoted by $U_A$, if $F(x) = U$ for all $x \in A$;

(6) the complement [2] of $\langle F, A\rangle$, denoted by $\langle F, A\rangle^c$, is defined by $\langle F, A\rangle^c = \langle F^c, A\rangle$, where $F^c : A \to \mathcal{P}(U)$ is a function given by $F^c(x) = U \setminus F(x)$ for all $x \in A$.

The relative whole soft set with respect to the set of parameters $E$ is called the absolute soft set over $U$ and simply denoted by $U_E$. In a similar way, the relative null soft set with respect to $E$ is called the null soft set over $U$ and is denoted by $\Phi_E$.

Clearly, ${U_A}^c = \Phi_A$, ${\Phi_A}^c = U_A$, and $\Phi_A \widetilde{\subseteq} \langle F, A\rangle \widetilde{\subseteq} U_A \widetilde{\subseteq} U_E$ for all soft set $\langle F, A\rangle$ over $U$ [3].

**Definition 2.1.3.** Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two soft set in $\mathcal{SS}(U)$. Then

(1) the union [38] of $\langle F, A\rangle$ and $\langle G, B\rangle$ is the soft set $\langle H, C\rangle$, where $C = A\cup B$ and for all $x \in C$,

$$H(x) = \begin{cases} F(x), & \text{if } x \in A \setminus B, \\ G(x), & \text{if } x \in B \setminus A, \\ F(x) \cup G(x), & \text{if } x \in A \cap B, \end{cases} \tag{2.2}$$

and is written as $\langle F, A\rangle \widetilde{\cup} \langle G, B\rangle = \langle H, C\rangle$;

(2) the intersection [2] of $\langle F, A\rangle$ and $\langle G, B\rangle$ is the soft set $\langle H, C\rangle$, where $C = A \cup B$ and for all $x \in C$,

$$H(x) = \begin{cases} F(x), & \text{if } x \in A \setminus B, \\ G(x), & \text{if } x \in B \setminus A, \\ F(x) \cap G(x), & \text{if } x \in A \cap B, \end{cases} \tag{2.3}$$

and is written as $\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle = \langle H, C\rangle$.

The following shows that the basic properties of operations on soft sets such as union, intersection and De Morgan's laws for union, intersection and complement.

**Proposition 2.1.4.** For two soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over $U$, the following are true:

(1) $\langle F, A\rangle \widetilde{\cap} \langle F, A\rangle = \langle F, A\rangle$ [38], $\langle F, A\rangle \widetilde{\cup} \langle F, A\rangle = \langle F, A\rangle$ [2].

(2) $\langle F, A\rangle \widetilde{\cap} \Phi_A = \Phi_A$ [52], $\langle F, A\rangle \widetilde{\cup} \Phi_A = \langle F, A\rangle$ [48].

(3) $\langle F, A\rangle \widetilde{\cap} U_A = \langle F, A\rangle$ [52], $\langle F, A\rangle \widetilde{\cup} U_A = U_A$ [48].

(4) $\langle F, A\rangle \widetilde{\cup} \langle F, A\rangle^c = U_A$ [3].

(5) $(\langle F, A\rangle \widetilde{\cap} \langle G, B\rangle)^c = \langle F, A\rangle^c \widetilde{\cup} \langle G, B\rangle^c$ [2].

(6) $(\langle F, A\rangle \widetilde{\cup} \langle G, B\rangle)^c = \langle F, A\rangle^c \widetilde{\cap} \langle G, B\rangle^c$ [2].

## 2.2 Equivalence soft set relations

**Definition 2.2.1.** [9] Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two soft sets over a universe $U$. Then the Cartesian product of $\langle F, A\rangle$ and $\langle G, B\rangle$ is defined as $\langle F, A\rangle \times \langle G, B\rangle = \langle H, A \times B\rangle$, where

$H : A \times B \to \mathcal{P}(U \times U)$ and $H(a,b) = F(a) \times G(b)$ for all $(a,b) \in A \times B$, i.e., $H(a,b) = \{(h_i, h_j) : h_i \in F(a) \text{ and } h_j \in G(b)\}$.

**Example 2.2.2.** [9] Consider the soft set $\langle F, A\rangle$ which describes the "peoples having different jobs" and the soft set $\langle G, B\rangle$ which describes the "peoples qualified in various courses" in a social gathering. Suppose that $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}\}$ denotes the set of peoples in a social gathering, $A = \{$ chartered account, doctor, engineer, teacher$\}$ and $B = \{$ B.Sc., B.Tech., MBBS, M.Sc.$\}$. Let $F$(chartered account) $= \{h_1, h_2\}$, $F$(doctor) $= \{h_4, h_5\}$, $F$(engineer) $= \{h_7, h_9\}$, $F$(teacher) $= \{h_3, h_4, h_7\}$, $G$(B.Sc.) $= \{h_1, h_6, h_8, h_{10}\}$, $G$(B.Tech.) $= \{h_3, h_6, h_7, h_9\}$, $G$(MBBS) $= \{h_3, h_4, h_5, h_8\}$ and $G$(M.Sc.) $= \{h_3, h_8\}$.
Now $\langle F, A\rangle \times \langle G, B\rangle = \langle H, A \times B\rangle$ where a typical element will look like

$$\begin{aligned} H(\text{doctor}, \text{MBBS}) &= \{h_4, h_5\} \times \{h_3, h_4, h_5, h_8\} \\ &= \{(h_4, h_3), (h_4, h_4), (h_4, h_5), (h_4, h_8), (h_5, h_3), \\ &\quad (h_5, h_4), (h_5, h_5), (h_5, h_8)\}. \end{aligned}$$

**Definition 2.2.3.** [9] Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two soft sets over a universe $U$. Then a soft set relation from $\langle F, A\rangle$ to $\langle G, B\rangle$ is a soft subset of $\langle F, A\rangle \times \langle G, B\rangle$.

In other words, a soft set relation from $\langle F, A\rangle$ to $\langle G, B\rangle$ is the form $\langle H_1, S\rangle$, where $S \subseteq A \times B$ and $H_1(a,b) = H(a,b)$ for all $(a,b) \in S$, where $\langle H, A \times B\rangle = \langle F, A\rangle \times \langle G, B\rangle$ as defined in Definition 2.2.1. Any soft subset of $\langle F, A\rangle \times \langle F, A\rangle$ is called a soft set relation on $\langle F, A\rangle$.

In an equivalent way, we can define the soft set relation $\mathcal{R}$ on $\langle F, A\rangle$ in the parameterized form as follows: If $\langle F, A\rangle = \{F(a), F(b), \ldots\}$, then

$$F(a)\mathcal{R}F(b) \Leftrightarrow F(a) \times F(b) \in \mathcal{R}. \tag{2.4}$$

**Definition 2.2.4.** Let $\mathcal{R}$ and $\mathcal{S}$ be soft set relations on $\langle F, A\rangle$. Then

(1) the inverse of the relation $\mathcal{R}$ is the soft set relation on $\langle F, A\rangle$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{F(b) \times F(a) : F(a) \times F(b) \in \mathcal{R}\}$ [9];

(2) the union of two soft set relations $\mathcal{R}$ and $\mathcal{S}$ on $\langle F, A\rangle$, denoted by $\mathcal{R} \cup \mathcal{S}$, is defied by $\mathcal{R} \cup \mathcal{S} = \{F(a) \times F(b) : F(a) \times F(b) \in \mathcal{R} \text{ or } F(a) \times F(b) \in \mathcal{S}\}$ [64];

(3) the intersection of two soft set relations $\mathcal{R}$ and $\mathcal{S}$ on $\langle F, A\rangle$, denoted by $\mathcal{R}\cap\mathcal{S}$, is defied by $\mathcal{R}\cap\mathcal{S} = \{F(a)\times F(b) : F(a)\times F(b) \in \mathcal{R}$ and $F(a)\times F(b) \in \mathcal{S}\}$ [64];

(4) $\mathcal{R} \subseteq \mathcal{S}$ if for any $a, b \in A$, $F(a) \times F(b) \in \mathcal{R} \Rightarrow F(a) \times F(b) \in \mathcal{S}$ [64].

**Example 2.2.5.** Consider a soft set $\langle F, A\rangle$ over $U$ where $U = \{h_1, h_2, h_3, h_4\}$, $A = \{m_1, m_2\}$, $F(m_1) = \{h_1, h_2\}$ and $F(m_2) = \{h_2, h_3, h_4\}$. Two soft set relations $\mathcal{R}$ and $\mathcal{S}$ on $\langle F, A\rangle$ are given by

$$\mathcal{R} = \{F(m_1) \times F(m_1), F(m_2) \times F(m_1)\},$$
$$\mathcal{S} = \{F(m_1) \times F(m_1), F(m_2) \times F(m_2)\}.$$

Then the union and intersection of $\mathcal{R}$ and $\mathcal{S}$ are

$$\mathcal{R} \cup \mathcal{S} = \{F(m_1) \times F(m_1), F(m_2) \times F(m_1), F(m_2) \times F(m_2)\},$$
$$\mathcal{R} \cap \mathcal{S} = \{F(m_1) \times F(m_1)\}.$$

Consider another soft set relation $\mathcal{Q}$ on $\langle F, A\rangle$ is given by

$$\mathcal{Q} = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1)\}.$$

Then $\mathcal{R} \subseteq \mathcal{Q}$ but $\mathcal{S} \nsubseteq \mathcal{Q}$.

**Definition 2.2.6.** [9] Let $\mathcal{R}$ be a soft set relation from $\langle F, A\rangle$ to $\langle G, B\rangle$ and $\mathcal{S}$ be a soft set relation from $\langle G, B\rangle$ to $\langle H, C\rangle$. Then the composition of $\mathcal{R}$ and $\mathcal{S}$, denoted by $\mathcal{S} \circ \mathcal{R}$, is soft set relation from $\langle F, A\rangle$ to $\langle H, C\rangle$ defined as follows: If $F(a) \in \langle F, A\rangle$ and $H(c) \in \langle H, C\rangle$, then

$$F(a) \times H(c) \in \mathcal{S} \circ \mathcal{R}$$
$$\Leftrightarrow F(a) \times G(b) \in \mathcal{R} \text{ and } G(b) \times H(c) \in \mathcal{S} \text{ for some } G(b) \in \langle G, B\rangle.$$

**Definition 2.2.7.** [9] Let $\langle F, A\rangle$ be a soft set over $U$. The identity soft set relation $I_{FA}$ on $\langle F, A\rangle$ is defined as follows: $F(a) \times F(b) \in I_{FA} \Leftrightarrow a = b$. That is, $I_{FA} = \{F(a) \times F(a) : F(a) \in \langle F, A\rangle\}$.

Clearly, $I_{FA}^{-1} = I_{FA}$ and $I_{FA} \circ I_{FA} = I_{FA}$.

**Proposition 2.2.8.** Let $\mathcal{R}, \mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{S}, \mathcal{S}_1$ and $\mathcal{S}_2$ be soft set relations on $\langle F, A\rangle$. Then

(1) $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$.

(2) If $\mathcal{R}_1 \subseteq \mathcal{S}_1$ and $\mathcal{R}_2 \subseteq \mathcal{S}_2$, then $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{S}_1 \circ \mathcal{S}_2$.

(3) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$, $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$.

(4) If $\mathcal{R}_1 \subseteq \mathcal{R}_2$, then $\mathcal{R}_1^{-1} \subseteq \mathcal{R}_2^{-1}$ [64].

(5) $(\mathcal{R}^{-1})^{-1} = \mathcal{R}$ [64], $(\mathcal{R}_1 \circ \mathcal{R}_2)^{-1} = \mathcal{R}_2^{-1} \circ \mathcal{R}_1^{-1}$ [9].

(6) $(\mathcal{R}_1 \cup \mathcal{R}_2)^{-1} = \mathcal{R}_1^{-1} \cup \mathcal{R}_2^{-1}$, $(\mathcal{R}_1 \cap \mathcal{R}_2)^{-1} = \mathcal{R}_1^{-1} \cap \mathcal{R}_2^{-1}$ [64].

(7) $\mathcal{R} \subseteq \mathcal{R} \cup \mathcal{S}$, $\mathcal{S} \subseteq \mathcal{R} \cup \mathcal{S}$ [64].

(8) $\mathcal{R} \cap \mathcal{S} \subseteq \mathcal{R}$, $\mathcal{R} \cap \mathcal{S} \subseteq \mathcal{S}$ [64].

**Proof** Obviously, (1) and (2) hold. We only show (3).

(3) By Definitions 2.2.4(2) and 2.2.6,

$$
\begin{aligned}
& F(a) \times F(b) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) \\
& \quad \Leftrightarrow F(a) \times F(c) \in \mathcal{R}_1 \text{ and } F(c) \times F(b) \in \mathcal{R}_2 \cup \mathcal{R}_3 \text{ for some } F(c) \in \langle F, A\rangle \\
& \quad \Leftrightarrow F(a) \times F(b) \in \mathcal{R}_1 \circ \mathcal{R}_2 \text{ or } F(a) \times F(b) \in \mathcal{R}_1 \circ \mathcal{R}_3 \\
& \quad \Leftrightarrow F(a) \times F(b) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3).
\end{aligned}
$$

Hence $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$. The proof of $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$ is similar. □

**Example 2.2.9.** Consider a soft set $\langle F, A\rangle$ over $U$ where $U = \{h_1, h_2, h_3, h_4\}$, $A = \{m_1, m_2, m_3\}$, $F(m_1) = \{h_1, h_2\}$, $F(m_2) = \{h_2, h_4\}$ and $F(m_3) = \{h_1, h_3, h_4\}$. Three soft set relations $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ on $\langle F, A\rangle$ are given by

$$
\begin{aligned}
\mathcal{R}_1 &= \{F(m_1) \times F(m_1), F(m_2) \times F(m_1)\}, \\
\mathcal{R}_2 &= \{F(m_1) \times F(m_2), F(m_2) \times F(m_2)\}, \\
\mathcal{R}_3 &= \{F(m_1) \times F(m_3), F(m_3) \times F(m_1)\}.
\end{aligned}
$$

Then $\mathcal{R}_2 \cup \mathcal{R}_3$, $\mathcal{R}_1 \circ \mathcal{R}_2$ and $\mathcal{R}_1 \circ \mathcal{R}_3$ are given by

$$\mathcal{R}_2 \cup \mathcal{R}_3 = \{F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_2), F(m_3) \times F(m_1)\},$$
$$\mathcal{R}_1 \circ \mathcal{R}_2 = \{F(m_1) \times F(m_1), F(m_2) \times F(m_1)\},\ \mathcal{R}_1 \circ \mathcal{R}_3 = \{F(m_3) \times F(m_1)\},$$

and thus

$$\begin{aligned}\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) &= \{F(m_1) \times F(m_1), F(m_2) \times F(m_1), F(m_3) \times F(m_1)\} \\ &= (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3).\end{aligned}$$

Now, we redefine the notions of reflexiveness, symmetry and transitivity of soft set relation and effectively use to prove their properties.

**Definition 2.2.10.** Let $\mathcal{R}$ be a soft set relation on $\langle F, A\rangle$. Then $\mathcal{R}$ is said to be

(1) reflexive if $I_{FA} \subseteq \mathcal{R}$;

(2) symmetric if $\mathcal{R}^{-1} = \mathcal{R}$;

(3) transitive if $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$;

(4) equivalence soft set relation if it is reflexive, symmetric and transitive.

**Theorem 2.2.11.** Let $\mathcal{R}$ and $\mathcal{S}$ be two soft set relations on $\langle F, A\rangle$.

(1) $\mathcal{R}$ is equivalence if and only if $\mathcal{R}^{-1}$ is equivalence.

(2) If $\mathcal{R}$ and $\mathcal{S}$ are equivalence, then so are $\mathcal{R} \circ \mathcal{R}$ and $\mathcal{R} \cap \mathcal{S}$.

(3) If $\mathcal{R}$ is equivalence, $\mathcal{R} \circ \mathcal{R} = \mathcal{R}$.

(4) If $\mathcal{R}$ and $\mathcal{S}$ are equivalence, then $\mathcal{R} \cup \mathcal{S}$ is equivalence if and only if $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{R} \cup \mathcal{S}$ and $\mathcal{S} \circ \mathcal{R} \subseteq \mathcal{R} \cup \mathcal{S}$.

**Proof** (1) Since $I_{FA} \subseteq \mathcal{R} \Leftrightarrow I_{FA} \subseteq \mathcal{R}^{-1}$, $\mathcal{R}$ is reflexive $\Leftrightarrow \mathcal{R}^{-1}$ is reflexive.

By Proposition 2.2.8(5), $\mathcal{R}$ is symmetric $\Leftrightarrow \mathcal{R}^{-1} = \mathcal{R} = (\mathcal{R}^{-1})^{-1} \Leftrightarrow \mathcal{R}^{-1}$ is symmetric.

By Proposition 2.2.8(4), $\mathcal{R}$ is transitive $\Leftrightarrow \mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R} \Leftrightarrow \mathcal{R}^{-1} \circ \mathcal{R}^{-1} \subseteq \mathcal{R}^{-1}$ $\Leftrightarrow \mathcal{R}^{-1}$ is transitive.

(2) First, we show that $\mathcal{R} \circ \mathcal{R}$ is equivalence soft set relation. Since $\mathcal{R}$ is reflexive, $I_{FA} \subseteq \mathcal{R}$ and hence by Proposition 2.2.8(2), $I_{FA} = I_{FA} \circ I_{FA} \subseteq \mathcal{R} \circ \mathcal{R}$,

i.e., $\mathcal{R}\circ\mathcal{R}$ is reflexive. Since $\mathcal{R}$ is symmetric, $\mathcal{R}^{-1}=\mathcal{R}$ and hence by Proposition 2.2.8(6), $(\mathcal{R}\circ\mathcal{R})^{-1}=\mathcal{R}^{-1}\circ\mathcal{R}^{-1}=\mathcal{R}\circ\mathcal{R}$, i.e., $\mathcal{R}\circ\mathcal{R}$ is symmetric. Since $\mathcal{R}$ is transitive, $\mathcal{R}\circ\mathcal{R}\subseteq\mathcal{R}$ and hence by Proposition 2.2.8(2), $(\mathcal{R}\circ\mathcal{R})\circ(\mathcal{R}\circ\mathcal{R})\subseteq\mathcal{R}\circ\mathcal{R}$, i.e., $\mathcal{R}\circ\mathcal{R}$ is transitive.

Next, we show that $\mathcal{R}\cap\mathcal{S}$ is equivalence soft set relation. Since $\mathcal{R}$ and $\mathcal{S}$ is reflexive, $I_{FA}\subseteq\mathcal{R}$ and $I_{FA}\subseteq\mathcal{S}$, and hence $I_{FA}\subseteq\mathcal{R}\cap\mathcal{S}$, i.e., $\mathcal{R}\cap\mathcal{S}$ is reflexive. Since $\mathcal{R}$ and $\mathcal{S}$ are symmetric, $\mathcal{R}^{-1}=\mathcal{R}$ and $\mathcal{S}^{-1}=\mathcal{S}$ and hence by Proposition 2.2.8(6), $(\mathcal{R}\cap\mathcal{S})^{-1}=\mathcal{R}^{-1}\cap\mathcal{S}^{-1}=\mathcal{R}\cap\mathcal{S}$, i.e., $\mathcal{R}\cap\mathcal{S}$ is symmetric. Since $\mathcal{R}$ and $\mathcal{S}$ are transitive, $\mathcal{R}\circ\mathcal{R}\subseteq\mathcal{R}$ and $\mathcal{S}\circ\mathcal{S}\subseteq\mathcal{S}$ and hence by Proposition 2.2.8(2), $(\mathcal{R}\cap\mathcal{S})\circ(\mathcal{R}\cap\mathcal{S})=(\mathcal{R}\cap\mathcal{S}\circ\mathcal{R})\cap(\mathcal{R}\cap\mathcal{S}\circ\mathcal{S})\subseteq(\mathcal{R}\circ\mathcal{R})\cap(\mathcal{S}\circ\mathcal{S})\subseteq\mathcal{R}\cap\mathcal{S}$, i.e., $\mathcal{R}\cap\mathcal{S}$ is transitive.

(3) Since $\mathcal{R}$ is transitive, $\mathcal{R}\circ\mathcal{R}\subseteq\mathcal{R}$. So we show that $\mathcal{R}\subseteq\mathcal{R}\circ\mathcal{R}$. Let $F(a)\times F(b)\in\mathcal{R}$. Since $\mathcal{R}$ is reflexive, $F(b)\times F(b)\in\mathcal{R}$ and thus $F(a)\times F(b)\in\mathcal{R}\circ\mathcal{R}$, i.e., $\mathcal{R}\subseteq\mathcal{R}\circ\mathcal{R}$. Hence $\mathcal{R}=\mathcal{R}\circ\mathcal{R}$.

(4) Suppose that $\mathcal{R}\cup\mathcal{S}$ is equivalence. Then

$$\begin{aligned}&F(a)\times F(b)\in\mathcal{R}\circ\mathcal{S}\\&\quad\Leftrightarrow F(a)\times F(c)\in\mathcal{S}\text{ and }F(c)\times F(b)\in\mathcal{R}\text{ for some }c\in A\\&\quad\Rightarrow F(a)\times F(c)\in\mathcal{S}\cup\mathcal{R}\text{ and }F(c)\times F(b)\in\mathcal{S}\cup\mathcal{R}\\&\quad\Rightarrow F(a)\times F(b)\in\mathcal{S}\cup\mathcal{R},\quad\text{by equivalence of }\mathcal{S}\cup\mathcal{R}.\end{aligned}$$

Hence $\mathcal{R}\circ\mathcal{S}\subseteq\mathcal{R}\cup\mathcal{S}$. Similarly, we have $\mathcal{S}\circ\mathcal{R}\subseteq\mathcal{R}\cup\mathcal{S}$.

Conversely, suppose that $\mathcal{R}\circ\mathcal{S}\subseteq\mathcal{R}\cup\mathcal{S}$ and $\mathcal{S}\circ\mathcal{R}\subseteq\mathcal{R}\cup\mathcal{S}$. Since $\mathcal{R}$ and $\mathcal{S}$ are reflexive, by the hypothesis, $I_{FA}=I_{FA}\circ I_{FA}\subseteq\mathcal{R}\circ\mathcal{S}\subseteq\mathcal{R}\cup\mathcal{S}$, i.e., $\mathcal{R}\cup\mathcal{S}$ is reflexive. Since $\mathcal{R}$ and $\mathcal{S}$ are symmetric, by Proposition 2.2.8(6), $(\mathcal{R}\cup\mathcal{S})^{-1}=\mathcal{R}^{-1}\cup\mathcal{S}^{-1}=\mathcal{R}\cup\mathcal{S}$, i.e., $\mathcal{R}\cup\mathcal{S}$ is symmetric. Since $\mathcal{R}$ and $\mathcal{S}$ are transitive, by Proposition 2.2.8(3), (7) and the hypothesis, $(\mathcal{R}\cup\mathcal{S})\circ(\mathcal{R}\cup\mathcal{S})=[(\mathcal{R}\circ\mathcal{R})\cup(\mathcal{S}\circ\mathcal{R})]\cup[(\mathcal{R}\circ\mathcal{S})\cup(\mathcal{S}\circ\mathcal{S})]\subseteq[\mathcal{R}\cup(\mathcal{R}\cup\mathcal{S})\cup\mathcal{S}]=\mathcal{R}\cup\mathcal{S}$, i.e., $\mathcal{R}\cup\mathcal{S}$ is transitive. Hence $\mathcal{R}\cup\mathcal{S}$ is equivalence. □

**Proposition 2.2.12.** Let $\mathcal{R}$ and $\mathcal{S}$ be soft set relations on $\langle F,A\rangle$.

(1) If $\mathcal{R}$ is reflexive and $\mathcal{S}$ is reflexive and transitive, then $\mathcal{R} \subseteq \mathcal{S}$ if and only if $\mathcal{R} \circ \mathcal{S} = \mathcal{S}$.

(2) If $\mathcal{R}$ and $\mathcal{S}$ are reflexive, then so is $\mathcal{R} \circ \mathcal{S}$.

**Proof** (1) Suppose that $\mathcal{R} \subseteq \mathcal{S}$. Since $\mathcal{R}$ is reflexive, $I_{FA} \subseteq \mathcal{R}$ and then $\mathcal{S} = I_{FA} \circ \mathcal{S} \subseteq \mathcal{R} \circ \mathcal{S}$. On the other hand, since $\mathcal{S}$ is transitive, by Proposition 2.2.8(2), $\mathcal{R} \circ \mathcal{S} \subseteq \mathcal{S} \circ \mathcal{S} \subseteq \mathcal{S}$. Hence $\mathcal{R} \circ \mathcal{S} = \mathcal{S}$.

Conversely, suppose that $\mathcal{R} \circ \mathcal{S} = \mathcal{S}$. Since $\mathcal{S}$ is reflexive, by Proposition 2.2.8(2), $\mathcal{R} = \mathcal{R} \circ I_{FA} \subseteq \mathcal{R} \circ \mathcal{S} = \mathcal{S}$. Hence $\mathcal{R} \subseteq \mathcal{S}$.

(2) Since $\mathcal{R}$ and $\mathcal{S}$ are reflexive, by Proposition 2.2.8(2), $I_{FA} = I_{FA} \circ I_{FA} \subseteq \mathcal{R} \circ \mathcal{S}$. Hence $\mathcal{R} \circ \mathcal{S}$ is reflexive. □

**Theorem 2.2.13.** Let $\mathcal{R}$ and $\mathcal{S}$ be equivalence soft set relations on $\langle F, A \rangle$. Then $\mathcal{R} \circ \mathcal{S}$ is equivalence if and only if $\mathcal{R} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}$.

**Proof** Suppose that $\mathcal{R} \circ \mathcal{S}$ is equivalence. Since $\mathcal{R}$ and $\mathcal{S}$ are symmetric, $\mathcal{R}^{-1} = \mathcal{R}$ and $\mathcal{S}^{-1} = \mathcal{S}$. Since $\mathcal{R} \circ \mathcal{S}$ is symmetric, by Proposition 2.2.8(5), $\mathcal{R} \circ \mathcal{S} = (\mathcal{R} \circ \mathcal{S})^{-1} = \mathcal{S}^{-1} \circ \mathcal{R}^{-1} = \mathcal{S} \circ \mathcal{R}$.

Conversely, suppose that $\mathcal{R} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}$. Then, by Proposition 2.2.12(2), $\mathcal{R} \circ \mathcal{S}$ is reflexive. Since $\mathcal{R}$ and $\mathcal{S}$ is symmetric, by the hypothesis, $(\mathcal{R} \circ \mathcal{S})^{-1} = \mathcal{S}^{-1} \circ \mathcal{R}^{-1} = \mathcal{S} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{S}$, i.e., $\mathcal{R} \circ \mathcal{S}$ is symmetric. On the other hand, since $\mathcal{R}$ and $\mathcal{S}$ are transitive, by Proposition 2.2.8(2) and the hypothesis, $(\mathcal{R} \circ \mathcal{S}) \circ (\mathcal{R} \circ \mathcal{S}) = (\mathcal{R} \circ \mathcal{R}) \circ (\mathcal{S} \circ \mathcal{S}) \subseteq \mathcal{R} \circ \mathcal{S}$, i.e., $\mathcal{R} \circ \mathcal{S}$ is transitive. Hence $\mathcal{R} \circ \mathcal{S}$ is equivalence soft set relation. □

**Remark 2.2.14.** Let $\{\mathcal{R}_\gamma : \gamma \in \Gamma\}$ be a family of equivalence soft set relations on $\langle F, A \rangle$. Then, clearly, $\cap_{\gamma \in \Gamma} \mathcal{R}_\gamma$ is equivalence soft set relation on $\langle F, A \rangle$. But, in general, $\cup_{\gamma \in \Gamma} \mathcal{R}_\gamma$ need not be equivalence soft set relation on $\langle F, A \rangle$.

**Example 2.2.15.** Let $\langle F, A \rangle$ be a soft set over $U$ where $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$, $A = \{m_1, m_2, m_3\}$ and $F(m_1) = \{h_1, h_2, h_5, h_6\}$, $F(m_2) = \{h_3, h_4, h_7,$

$h_8\}$, $F(m_3) = \{h_2, h_4, h_9\}$. Consider equivalence soft set relations $\mathcal{R}$ and $\mathcal{S}$ on $\langle F, A\rangle$ given by

$$\begin{aligned}\mathcal{R} = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_2) \times F(m_1), F(m_2) \times F(m_2),\\ &F(m_3) \times F(m_3)\};\\ \mathcal{S} = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_3), F(m_2) \times F(m_2), F(m_3) \times F(m_1),\\ &F(m_3) \times F(m_3)\}.\end{aligned}$$

Then $\mathcal{R} \cup \mathcal{S}$ is soft set relation on $\langle F, A\rangle$ given by

$$\begin{aligned}\mathcal{R} \cup \mathcal{S} = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_1),\\ &F(m_2) \times F(m_2), F(m_3) \times F(m_1), F(m_3) \times F(m_3)\}.\end{aligned}$$

Since $F(m_2)\times F(m_1), F(m_1)\times F(m_3) \in \mathcal{R}\cup\mathcal{S}$, $F(m_2)\times F(m_3) \in (\mathcal{R}\cup\mathcal{S})\circ(\mathcal{R}\cup\mathcal{S})$ but $F(m_2) \times F(m_3) \notin \mathcal{R} \cup \mathcal{S}$, i.e., $(\mathcal{R} \cup \mathcal{S}) \circ (\mathcal{R} \cup \mathcal{S}) \not\subseteq (\mathcal{R} \cup \mathcal{S})$. This shows that $\mathcal{R} \cup \mathcal{S}$ is not transitive. Hence $\mathcal{R} \cup \mathcal{S}$ is not equivalence soft set relation.

**Definition 2.2.16.** [9] Let $\mathcal{R}$ be an equivalence soft set relation on $\langle F, A\rangle$ and $a \in A$. Then equivalence class of $F(a)$, denoted by $F(a)/\mathcal{R}$, is defined as $F(a)/\mathcal{R} = \{F(b) : F(a) \times F(b) \in \mathcal{R}\}$. The set of $\{F(a)/\mathcal{R} : a \in A\}$ is called the quotient soft set of $\langle F, A\rangle$ and denoted by $\langle F, A\rangle/\mathcal{R}$.

**Theorem 2.2.17.** Let $\mathcal{R}$ be an equivalence soft set relation on $\langle F, A\rangle$ and $a, b \in A$. Then

(1) Every $F(a)/\mathcal{R}$ is a non null soft subset of $\langle F, A\rangle$.

(2) $F(a)/\mathcal{R} = F(b)/\mathcal{R}$ if and only if $F(a) \times F(b) \in \mathcal{R}$ if and only if $F(a)/\mathcal{R}\widetilde{\cap}F(b)/\mathcal{R} \neq \Phi_A$.

**Proof** (1) Since $\mathcal{R}$ is reflexive, $F(a) \times F(a) \in \mathcal{R}$ for any $a \in A$ and hence by Definition 2.2.16, $F(a) \in F(a)/\mathcal{R}$. Hence $F(a)/\mathcal{R}$ is non null soft subset of $\langle F, A\rangle$.

(2) By Lemma 4.5 of [9], $F(a)/\mathcal{R} = F(b)/\mathcal{R}$ if and only if $F(a) \times F(b) \in \mathcal{R}$. We only prove that $F(a) \times F(b) \in \mathcal{R}$ if and only if $F(a)/\mathcal{R}\widetilde{\cap}F(b)/\mathcal{R} \neq \Phi_A$. Since

$\mathcal{R}$ is equivalence soft set relation on $\langle F, A\rangle$,

$$\begin{aligned} F(a)/\mathcal{R}\widetilde{\cap}F(b)/\mathcal{R} \neq \Phi_A & \\ & \Leftrightarrow\ F(c) \in F(a)/\mathcal{R}\widetilde{\cap}F(b)/\mathcal{R} \text{ for some } F(c) \in F(a)/\mathcal{R} \\ & \Leftrightarrow F(a) \times F(c) \in \mathcal{R} \text{ and } F(c) \times F(b) \in \mathcal{R} \\ & \Leftrightarrow F(a) \times F(b) \in \mathcal{R}. \end{aligned}$$

Hence $F(a) \times F(b) \in \mathcal{R}$ if and only if $F(a)/\mathcal{R}\widetilde{\cap}F(b)/\mathcal{R} \neq \Phi_A$. □

**Definition 2.2.18.** [9] A collection $\mathcal{P} = \{\langle F_i, A_i\rangle : i \in I\}$ of nonempty soft subsets of soft set $\langle F, A\rangle$ is called a partition of $\langle F, A\rangle$ if

(1) $\widetilde{\cup}_{i\in I}\langle F_i, A_i\rangle = \langle F, A\rangle$;

(2) $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

**Definition 2.2.19.** [9] Let $\mathcal{P} = \{\langle F_i, A_i\rangle\}$ be a partition of $\langle F, A\rangle$. We define a soft set relation $\langle F, A\rangle/\mathcal{P}$ on $\langle F, A\rangle$ by $F(a) \times F(b) \in \langle F, A\rangle/\mathcal{P}$ if and only if there exists $\langle F_i, A_i\rangle \in \mathcal{P}$ such that $F(a), F(b) \in \langle F_i, A_i\rangle$.

Babitha and Sunil [9] proved that an equivalence soft set relation on soft set gives rise to a partition of soft set, and each partition of soft set gives rise to an equivalence soft set relation as follows.

**Theorem 2.2.20.** [9] Let $\mathcal{R}$ be an equivalence soft set relation on $\langle F, A\rangle$ and $\mathcal{P}$ be a partition of $\langle F, A\rangle$. Then

(1) $\langle F, A\rangle/\mathcal{R}$ is a partition of $\langle F, A\rangle$.

(2) $\langle F, A\rangle/\mathcal{P}$ is an equivalence soft set relation on $\langle F, A\rangle$.

The following gives the intimate connection between equivalence soft set relations and partitions.

**Theorem 2.2.21.** Let $\mathcal{R}$ be an equivalence soft set relation on $\langle F, A\rangle$ and $\mathcal{P} = \{\langle F_i, A_i\rangle\}$ be a partition of $\langle F, A\rangle$. Then

(1) $\langle F, A\rangle/\mathcal{P} = \widetilde{\cup}_i\langle F_i, A_i\rangle \times \langle F_i, A_i\rangle$.

(2) $\langle F, A\rangle/(\langle F, A\rangle/\mathcal{P}) = \mathcal{P}$.

(3) $\langle F, A\rangle/(\langle F, A\rangle/\mathcal{R}) = \mathcal{R}$.

**Proof** (1) By Definition 2.2.19, we have

$$\begin{aligned} & F(a) \times F(b) \in \langle F, A\rangle/\mathcal{P} \\ & \quad \Leftrightarrow F(a) \in \langle F_i, A_i\rangle \text{ and } F(b) \in \langle F_i, A_i\rangle \text{ for some } \langle F_i, A_i\rangle \in \mathcal{P} \\ & \quad \Leftrightarrow F(a) \times F(b) \in \langle F_i, A_i\rangle \times \langle F_i, A_i\rangle \text{ for some } \langle F_i, A_i\rangle \in \mathcal{P} \\ & \quad \Leftrightarrow F(a) \times F(b) \in \widetilde{\cup}_i \langle F_i, A_i\rangle \times \langle F_i, A_i\rangle. \end{aligned}$$

Hence $\langle F, A\rangle/\mathcal{P} = \widetilde{\cup}_i \langle F_i, A_i\rangle \times \langle F_i, A_i\rangle$.

(2) Let $F(a)/(\langle F, A\rangle/\mathcal{P}) \in \langle F, A\rangle/(\langle F, A\rangle/\mathcal{P})$. Since $F(a) \in F(a)/(\langle F, A\rangle/\mathcal{P})$ and $\mathcal{P}$ is partition, there exists unique $\langle F_i, A_i\rangle \in \mathcal{P}$ such that $F(a) \in \langle F_i, A_i\rangle$. By Definition 2.2.19, we have $\langle F_i, A_i\rangle = F(a)/(\langle F, A\rangle/\mathcal{P})$. Hence $F(a)/(\langle F, A\rangle/\mathcal{P}) \in \mathcal{P}$. On the other hand, let $\langle F_i, A_i\rangle \in \mathcal{P}$. Since $\langle F_i, A_i\rangle$ is non null soft set, there exists a $F(a) \in \langle F, A\rangle$ such that $F(a) \in \langle F_i, A_i\rangle$. By our previous argument, $F(a)/(\langle F, A\rangle/\mathcal{P}) = \langle F_i, A_i\rangle$. Hence $\langle F_i, A_i\rangle \in \langle F, A\rangle/(\langle F, A\rangle/\mathcal{P})$. Therefore, we have $\langle F, A\rangle/(\langle F, A\rangle/\mathcal{P}) = \mathcal{P}$.

(3) By (1), Definition 2.2.19 and Theorem 2.2.20, we have

$$\begin{aligned} & F(a) \times F(b) \in \langle F, A\rangle/(\langle F, A\rangle/\mathcal{R}) \\ & \quad \Leftrightarrow F(a) \times F(b) \in \langle F_i, A_i\rangle \times \langle F_i, A_i\rangle \text{ for some } \langle F_i, A_i\rangle \in \langle F, A\rangle/\mathcal{R} \\ & \quad \Leftrightarrow F(a) \times F(b) \in F(c)/\mathcal{R} \times F(c)/\mathcal{R} \text{ for some } F(c) \in \langle F, A\rangle \\ & \quad \Leftrightarrow F(a) \times F(b) \in \mathcal{R}. \end{aligned}$$

Hence $\langle F, A\rangle/(\langle F, A\rangle/\mathcal{R}) = \mathcal{R}$. □

Babitha and Sunil [9] introduced the induced soft set relation from the relation on set of parameters as follows.

**Definition 2.2.22.** [9] Let $\langle F, A\rangle$ be a soft set defined on the universal set $U$ and $\Re$ be a relation defined on $A$, i.e., $\Re \subseteq A \times A$. Then the induced soft set relation $\Re_A$ on $\langle F, A\rangle$ is defined as follows:

$$F(a) \times F(b) \in \Re_A \Leftrightarrow (a, b) \in \Re. \tag{2.5}$$

**Theorem 2.2.23.** Let $\langle F, A\rangle$ be a soft set defined on $U$ and $\Re$ be a relation defined on $A$. Then $\Re$ is equivalence relation if and only if the induced relation $\Re_A$ is equivalence soft set relation.

**Proof** By Definitions 2.2.10 and 2.2.22, we have

(a) $\Re$ is reflexive $\Leftrightarrow \Delta_A \subseteq \Re \Leftrightarrow I_{FA} \subseteq \Re_A \Leftrightarrow \Re_A$ is reflexive. Here $\Delta_A = \{(a,a) : a \in A\}$ is diagonal relation on $A$.

(b) $\Re$ is symmetric $\Leftrightarrow \Re^{-1} = \Re \Leftrightarrow \Re_A^{-1} = \Re_A \Leftrightarrow \Re_A$ is symmetric.

(c) $\Re$ is transitive $\Leftrightarrow \Re \circ \Re \subseteq \Re \Leftrightarrow \Re_A \circ \Re_A \subseteq \Re_A \Leftrightarrow \Re_A$ transitive.

From (a), (b) and (c), $\Re$ is equivalence relation if and only if $\Re_A$ is equivalence soft set relation. □

Let $\langle F, A\rangle$ be a soft set on universal set $U$, $\Re$ be a equivalence relation on $A$ and $f : A \to A$ be a function. Then we say that $f$ is compatible with $\Re$ if and only if for all $a, b \in A$,

$$(a, b) \in \Re \;\Rightarrow\; (f(a), f(b)) \in \Re. \tag{2.6}$$

**Theorem 2.2.24.** Let $\langle F, A\rangle$ be a soft set on $U$, $f : A \to A$ be a function, and $\Re$ be a equivalence relation on $A$. If $f$ is compatible with $\Re$, then there exists a unique function $g : \langle F, A\rangle/\Re_A \to \langle F, A\rangle/\Re_A$ such that

$$g(F(a)/\Re_A) = F(f(a))/\Re_A \text{ for all } F(a) \in \langle F, A\rangle. \tag{2.7}$$

**Proof** Suppose that $f$ is compatible with $\Re$. Let $g = \{(F(a)/\Re_A, F(f(a))/\Re_A) : F(a) \in \langle F, A\rangle\}$. We shall prove that $g$ is a function. Clearly, the domain of $g$ is $\langle F, A\rangle/\Re_A$. Let $(F(a)/\Re_A, F(f(a))/\Re_A)$ and $(F(b)/\Re_A, F(f(b))/\Re_A)$ be elements in $g$. Then we have

$$\begin{aligned} F(a)/\Re_A = F(b)/\Re_A &\Rightarrow F(a) \times F(b) \in \Re_A, \text{ by Theorem 2.2.17} \\ &\Rightarrow (a, b) \in \Re, \text{ by Definition 2.2.22} \\ &\Rightarrow (f(a), f(b)) \in \Re, \text{ by compatibility} \\ &\Rightarrow F(f(a))/\Re_A = F(f(b))/\Re_A, \text{ by Theorem 2.2.17,} \end{aligned}$$

and hence $g$ is a function. Finally (2.7) holds because $(F(a)/\Re_A, F(f(a))/\Re_A) \in g$. The uniqueness can be easily checked. $\square$

**Example 2.2.25.** Let $\langle F, A\rangle$ be a soft set on universal set $U$, $A$ and $B$ be two sets of parameters and $f : A \to B$ be a function. Define the relation $\Re$ on $A$ by, for points in $A$,

$$(a, b) \in \Re \ \Leftrightarrow f(a) = f(b).$$

Then, clearly, $\Re$ is equivalence relation on $A$ and thus there is a unique one-to-one function $\bar{f} : A/\Re \to B$ such that $f = \bar{f} \circ \varphi$ (where $\varphi : A \to A/\Re$ is the natural map).

By Definition 2.2.22 and Theorem 2.2.23, the induced soft set relation $\Re_A$ is equivalence soft set relation on $\langle F, A\rangle$. Then there is a function $\bar{F} : A/\Re \to \langle F, A\rangle/\Re_A$ such that $\bar{F}(a/\Re) = F(a)/\Re_A$ for all $a/\Re \in A/\Re$. In fact, let $\bar{F} = \{(a/\Re, F(a)/\Re_A) : a/\Re \in A/\Re\}$. Consider pairs $(a/\Re, F(a)/\Re_A)$ and $(b/\Re, F(b)/\Re_A)$ in $\bar{F}$. Then the calculation

$$\begin{aligned} a/\Re = b/\Re &\Rightarrow (a, b) \in \Re, \text{ by equivalence} \\ &\Rightarrow F(a) \times F(b) \in \Re_A, \text{ by Definition 2.2.22} \\ &\Rightarrow F(a)/\Re_A = F(b)/\Re_A, \text{ by Theorem 2.2.17} \end{aligned}$$

shows that $\bar{F}$ is a function. Hence there is unique a one-to-one function $f^* : \langle F, A\rangle/\Re_A \to B$ such that $\bar{f} = f^* \circ \bar{F}$. In fact, let $f^* = \{(F(a)/\Re_A, f(a)) : F(a)/\Re_A \in \langle F, A\rangle/\Re_A\}$. Consider the pairs $(F(a)/\Re_A, f(a))$ and $(F(a)/\Re_A, f(a))$ in $f^*$. The calculation

$$\begin{aligned} F(a)/\Re_A = F(b)/\Re_A &\Rightarrow F(a) \times F(b) \in \Re_A, \text{ by Theorem 2.2.17} \\ &\Rightarrow a/\Re = b/\Re, \text{ by definition of } \bar{F} \\ &\Rightarrow f(a) = f(b), \text{ by definition of } \bar{f} \end{aligned}$$

shows that $f^*$ is a function. The uniqueness and one-to-one of $f^*$ can be easily checked. Finally, by the definitions of $\bar{f}$ and $\bar{F}$, for any $a/\Re \in A/\Re$,

$$(f^* \circ \bar{F})(a/\Re) = f^*(\bar{F}(a/\Re)) = f^*(F(a)/\Re_A) = f(a) = \bar{f}(a/\Re)$$

and hence $\bar{f} = f^* \circ \bar{F}$.

## 2.3 Transitive closure of soft set relation

For a soft set relation $\mathcal{R}$ on $\langle F, A\rangle$, there exists at least one transitive soft set relation containing $\mathcal{R}$, namely the trivial one $\langle F, A\rangle \times \langle F, A\rangle$. Furthermore, the intersection of any family of transitive soft set relations is again transitive. Thus we need the smallest transitive soft set relation containing the soft set relation $\mathcal{R}$. Now we define the transitive closure of soft set relation as follows.

**Definition 2.3.1.** Let $\mathcal{R}$ be a soft set relation on $\langle F, A\rangle$. Then the transitive closure of $\mathcal{R}$, denoted by $\hat{\mathcal{R}}$, is the soft set relation on $\langle F, A\rangle$ defined as follows:

$$\hat{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}^2 \cup \mathcal{R}^3 \cup \cdots \cup \mathcal{R}^n \cup \cdots \tag{2.8}$$

where $\mathcal{R}^1 = \mathcal{R}$ and $\mathcal{R}^n = \overbrace{\mathcal{R} \circ \mathcal{R} \circ \cdots \circ \mathcal{R}}^{n}$, $n \geq 2$.

**Remark 2.3.2.** By Definition 2.3.1, $\mathcal{R} \subseteq \hat{\mathcal{R}}$. Since every element of $\hat{\mathcal{R}}$ is in one of the $\mathcal{R}^i$, $\hat{\mathcal{R}}$ must be transitive by the following reasoning: if $F(a) \times F(b) \in \mathcal{R}^j$ and $F(b) \times F(c) \in \mathcal{R}^k$, then from composition's associativity, $F(a) \times F(c) \in \mathcal{R}^{j+k}$ (and thus in $\hat{\mathcal{R}}$) due to the definition of $\mathcal{R}^i$. Let $\mathcal{S}$ be any transitive soft set relation on $\langle F, A\rangle$ containing $\mathcal{R}$. Since $\mathcal{S}$ is transitive, whenever $\mathcal{R}^i \subseteq \mathcal{S}$, $\mathcal{R}^{i+1} \subseteq \mathcal{S}$ according to the construction of $\mathcal{R}^i$. Then, by induction, $\mathcal{S}$ contains every $\mathcal{R}^i$ and thus also $\hat{\mathcal{R}}$. Therefore, the transitive closure $\hat{\mathcal{R}}$ is the smallest transitive soft set relation containing $\mathcal{R}$.

**Proposition 2.3.3.** Let $\mathcal{R}$ be a soft set relation on $\langle F, A\rangle$. Then

(1) $\hat{\mathcal{R}}$ is transitive.

(2) If there exists $n \in \mathbf{N}$ such that $\mathcal{R}^{n+1} = \mathcal{R}^n$, then $\hat{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}^2 \cup \cdots \cup \mathcal{R}^n$.

**Proof** It follows from Definition 2.3.1 and Remark 2.3.2. □

**Example 2.3.4.** Let $\langle F, A\rangle$ be a soft set over $U$ where $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$, $A = \{m_1, m_2, m_3\}$ and $F(m_1) = \{h_1, h_2, h_5, h_6\}$, $F(m_2) = \{h_3, h_4, h_7, h_8\}$, $F(m_3) = \{h_2, h_4, h_9\}$. Consider soft set relations $\mathcal{R}$ on $\langle F, A\rangle$ given by

$$\mathcal{R} = \{F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_3) \times F(m_1), F(m_2) \times F(m_3)\}.$$

Then

$$\begin{aligned}
\mathcal{R}^2 = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_1),\\
&F(m_3) \times F(m_1), F(m_3) \times F(m_2)\},\\
\mathcal{R}^3 = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_1),\\
&F(m_2) \times F(m_2), F(m_3) \times F(m_1), F(m_3) \times F(m_2), F(m_3) \times F(m_3)\},\\
\mathcal{R}^4 = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_1),\\
&F(m_2) \times F(m_2), F(m_2) \times F(m_3), F(m_3) \times F(m_1), F(m_3) \times F(m_2),\\
&F(m_3) \times F(m_3)\},\\
\mathcal{R}^5 = \{&F(m_1) \times F(m_1), F(m_1) \times F(m_2), F(m_1) \times F(m_3), F(m_2) \times F(m_1),\\
&F(m_2) \times F(m_2), F(m_2) \times F(m_3), F(m_3) \times F(m_1), F(m_3) \times F(m_2),\\
&F(m_3) \times F(m_3)\}.
\end{aligned}$$

Thus the transitive closure of $\mathcal{R}$ is $\hat{\mathcal{R}} = \mathcal{R} \cup \mathcal{R}^2 \cup \mathcal{R}^3 \cup \mathcal{R}^4$.

**Proposition 2.3.5.** Let $\mathcal{R}$ and $\mathcal{S}$ be two soft set relations on $\langle F, A\rangle$. Then

(1) $\mathcal{R}$ is symmetric, then so is $\hat{\mathcal{R}}$.

(2) If $\mathcal{R} \subseteq \mathcal{S}$, then $\hat{\mathcal{R}} \subseteq \hat{\mathcal{S}}$.

(3) If $\mathcal{R}$ and $\mathcal{S}$ are equivalence soft set relations and $\mathcal{R} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}$, then $(\mathcal{R} \circ \mathcal{S})\hat{} = \mathcal{R} \circ \mathcal{S}$.

**Proof** (1) and (2) are proved from Proposition 2.2.8 and Definitions 2.2.10 and 2.3.1.

(3) By Theorem 2.2.13 and hypothesis, since $\mathcal{R} \circ \mathcal{S}$ is equivalence soft set relation, $(\mathcal{R} \circ \mathcal{S})^n \subset \mathcal{R} \circ \mathcal{S}$ for any $n \geq 1$. Hence $(\mathcal{R} \circ \mathcal{S})\hat{} = \mathcal{R} \circ \mathcal{S}$. □

Let $\mathcal{R}$ be a soft set relation on $\langle F, A\rangle$ and $\{\mathcal{R}_\gamma : \gamma \in \Gamma\}$ be family of equivalence soft set relations on $\langle F, A\rangle$ such that $\mathcal{R} \subseteq \mathcal{R}_\gamma$ for each $\gamma \in \Gamma$. Then clearly $\cap_{\gamma\in\Gamma}\mathcal{R}_\gamma$ is the smallest equivalence soft set relation such that $\mathcal{R} \subseteq \cap_{\gamma\in\Gamma}\mathcal{R}_\gamma$ and denoted by $\mathcal{R}^e$.

**Theorem 2.3.6.** If $\mathcal{R}$ is a soft set relation on $\langle F, A\rangle$, then $\mathcal{R}^e = (\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA})\hat{}$.

**Proof** Let $\mathcal{S} = (\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA})\hat{}$. Then clearly $\mathcal{R} \subseteq \mathcal{S}$. By Proposition 2.3.3(1), $\mathcal{S}$ is transitive. Since $I_{FA} \subseteq (\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA})$, $\mathcal{S}$ is reflexive. By Proposition 2.2.8(6), $(\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA})^{-1} = \mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA}$, i.e., $\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA}$ is symmetric. By Proposition 2.3.5(1), $\mathcal{S}$ is symmetric. Thus $\mathcal{S}$ is equivalence soft set relation such that $\mathcal{R} \subseteq \mathcal{S}$. Now let $\mathcal{K}$ be equivalence soft set relation on $\langle F, A\rangle$ such that $\mathcal{R} \subseteq \mathcal{K}$. Since $\mathcal{K}$ is equivalence, by Proposition 2.2.8(4) and Definition 2.2.10, $I_{FA} \subseteq \mathcal{K}$ and $\mathcal{R}^{-1} \subseteq \mathcal{R}^{-1} \subseteq \mathcal{K}^{-1} = \mathcal{K}$. Then, by Proposition 2.2.8(2), $(\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA})^n \subseteq \mathcal{K}^n = \mathcal{K}$ for any $n \geq 1$. Thus $\mathcal{S} \subseteq \mathcal{K}$. This show that $\mathcal{R}^e = (\mathcal{R}\cup\mathcal{R}^{-1}\cup I_{FA})\hat{}$. □

**Theorem 2.3.7.** Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence soft set relations on $\langle F, A\rangle$. Then $(\mathcal{R}\cup\mathcal{S})\hat{}$ is equivalence soft set relation.

**Proof** By Proposition 2.3.3(1), $(\mathcal{R}\cup\mathcal{S})\hat{}$ is transitive. Since $\mathcal{R}$ and $\mathcal{S}$ is symmetric, $(\mathcal{R}\cup\mathcal{S})^{-1} = \mathcal{R}^{-1}\cup\mathcal{S}^{-1} = \mathcal{R}\cup\mathcal{S}$, i.e., $\mathcal{R}\cup\mathcal{S}$ is symmetric. Then by Proposition 2.3.5(1), $(\mathcal{R}\cup\mathcal{S})\hat{}$ is symmetric. Since $\mathcal{R}$ and $\mathcal{S}$ is reflexive, $I_{FA} = I_{FA}\cup I_{FA} \subseteq \mathcal{R}\cup\mathcal{S}$. Thus $I_{FA} \subseteq (\mathcal{R}\cup\mathcal{S})\hat{}$, i.e., $(\mathcal{R}\cup\mathcal{S})\hat{}$ is reflexive. Hence $(\mathcal{R}\cup\mathcal{S})\hat{}$ is equivalence soft set relation. □

**Theorem 2.3.8.** Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence soft set relations on $\langle F, A\rangle$. If $\mathcal{R}\circ\mathcal{S}$ is an equivalence soft set relation on $\langle F, A\rangle$, then $\mathcal{R}\circ\mathcal{S}$ is the least upper bound for $\{\mathcal{R}, \mathcal{S}\}$ with respect to $\subseteq$.

**Proof** Since $\mathcal{S}$ is reflexive, by Proposition 2.2.8(2), $\mathcal{R} = \mathcal{R}\circ I_{FA} \subseteq \mathcal{R}\circ\mathcal{S}$. By the similar argument, $\mathcal{S} \subseteq \mathcal{R}\circ\mathcal{S}$. So, $\mathcal{R}\circ\mathcal{S}$ is upper bound for $\{\mathcal{R}, \mathcal{S}\}$ with respect to $\subseteq$. Now let $\mathcal{K}$ be any equivalence soft set relation on $\langle F, A\rangle$ such that $\mathcal{R} \subseteq \mathcal{K}$ and $\mathcal{S} \subseteq \mathcal{K}$. Since $\mathcal{K}$ is transitive, by Proposition 2.2.8(2), $\mathcal{R}\circ\mathcal{S} \subseteq \mathcal{K}\circ\mathcal{K} \subseteq \mathcal{K}$. Hence $\mathcal{R}\circ\mathcal{S}$ is least upper bound for $\{\mathcal{R}, \mathcal{S}\}$ with respect to $\subseteq$. □

**Theorem 2.3.9.** Let $\mathcal{R}$ and $\mathcal{S}$ be two equivalence soft set relations on $\langle F, A\rangle$ such that $\mathcal{R}\circ\mathcal{S} = \mathcal{S}\circ\mathcal{R}$. Then $(\mathcal{R}\cup\mathcal{S})^e = (\mathcal{R}\cup\mathcal{S})\hat{} = (\mathcal{R}\circ\mathcal{S})\hat{} = \mathcal{R}\circ\mathcal{S}$.

**Proof** Clearly, $\mathcal{R}\cup\mathcal{S}$ is a soft set relation. Since $\mathcal{R}$ and $\mathcal{S}$ are equivalence soft set relation, by Theorem 2.3.6, Proposition 2.2.8(6) and Definition 2.2.10,

$$(\mathcal{R}\cup\mathcal{S})^e = ((\mathcal{R}\cup\mathcal{S})\cup(\mathcal{R}\cup\mathcal{S})^{-1}\cup I_{FA})\hat{} = (\mathcal{R}\cup\mathcal{S})\hat{}.$$

By Proposition 2.2.8(2),(3),(7) and the hypothesis, we have $\mathcal{R}\circ\mathcal{S}\subseteq(\mathcal{R}\cup\mathcal{S})\circ(\mathcal{R}\cup\mathcal{S}) = \mathcal{R}\cup\mathcal{S}$ and thus, by Proposition 2.3.5(2), $(\mathcal{R}\circ\mathcal{S})\hat{}\subseteq(\mathcal{R}\cup\mathcal{S})\hat{}$. On the other hand, by Theorems 2.2.13 and 2.3.8, $\mathcal{R}\circ\mathcal{S}$ is equivalence soft set relation and $\mathcal{R}\circ\mathcal{S}$ is the least upper bound for $\{\mathcal{R},\mathcal{S}\}$ with respect to $\subseteq$. Since $\mathcal{R}\subseteq\mathcal{R}\circ\mathcal{S}$ and and $\mathcal{S}\subseteq\mathcal{R}\circ\mathcal{S}$, by Proposition 2.3.5(2), $(\mathcal{R}\cup\mathcal{S})\hat{}\subseteq(\mathcal{R}\circ\mathcal{S})\hat{}$. Hence $(\mathcal{R}\circ\mathcal{S})\hat{} = (\mathcal{R}\cup\mathcal{S})\hat{}$.

Therefore, by Proposition 2.3.5(3), $(\mathcal{R}\cup\mathcal{S})^e = (\mathcal{R}\cup\mathcal{S})\hat{} = (\mathcal{R}\circ\mathcal{S})\hat{} = \mathcal{R}\circ\mathcal{S}$. □

**Example 2.3.10.** Let $\langle F, A\rangle$ be a soft set over $U$ where $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9\}$, $A = \{m_1, m_2, m_3, m_4, m_5\}$ and $F(m_1) = \{h_1, h_2, h_5, h_6\}$, $F(m_2) = \{h_3, h_4, h_7, h_8\}$, $F(m_3) = \{h_2, h_4, h_9\}$, $F(m_4) = \{h_1, h_6, h_8\}$, $F(m_5) = \{h_2, h_4, h_5\}$. If $\mathcal{R}$ and $\mathcal{S}$ soft set relations on $\langle F, A\rangle$ defined by

$$\begin{aligned}\mathcal{R} = \{&F(m_1)\times F(m_1), F(m_2)\times F(m_2), F(m_2)\times F(m_4), F(m_3)\times F(m_2),\\ &F(m_3)\times F(m_3), F(m_4)\times F(m_2), F(m_4)\times F(m_4)\}\\ \mathcal{S} = \{&F(m_2)\times F(m_2), F(m_2)\times F(m_3), F(m_3)\times F(m_3), F(m_3)\times F(m_4),\\ &F(m_4)\times F(m_3), F(m_4)\times F(m_4), F(m_5)\times F(m_5)\}.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{R}\circ\mathcal{S} &= \mathcal{S}\circ\mathcal{R}\\ &= \{F(m_2)\times F(m_2), F(m_2)\times F(m_3), F(m_2)\times F(m_4), F(m_3)\times F(m_2),\\ &\quad F(m_3)\times F(m_3), F(m_3)\times F(m_4), F(m_4)\times F(m_2), F(m_4)\times F(m_3),\\ &\quad F(m_4)\times F(m_4)\},\end{aligned}$$

$$\begin{aligned}\mathcal{R} \cup \mathcal{S} = \{&F(m_1) \times F(m_1), F(m_2) \times F(m_2), F(m_2) \times F(m_3), F(m_2) \times F(m_4),\\ &F(m_3) \times F(m_2), F(m_3) \times F(m_3), F(m_3) \times F(m_4), F(m_4) \times F(m_2),\\ &F(m_4) \times F(m_3), F(m_4) \times F(m_4), F(m_5) \times F(m_5)\},\end{aligned}$$

and so $(\mathcal{R} \cup \mathcal{S})\hat{} = (\mathcal{R} \cup \mathcal{S})^e = \mathcal{R} \cup \mathcal{S}$ and $(\mathcal{R} \circ \mathcal{S})\hat{} = \mathcal{R} \circ \mathcal{S}$ but $\mathcal{R} \cup \mathcal{S} \neq \mathcal{R} \circ \mathcal{S}$ because $\mathcal{R}$ and $\mathcal{S}$ is not equivalence soft set relations on $\langle F, A\rangle$.

If $\mathcal{R}$ is equivalence soft set relation on $\langle F, A\rangle$ given in Example 2.2.15 and $\mathcal{S} = I_{FA}$ is identity soft set relation on $\langle F, A\rangle$, then $\mathcal{R} \cup \mathcal{S} = \mathcal{R} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{R}$ and thus $(\mathcal{R} \cup \mathcal{S})^e = (\mathcal{R} \cup \mathcal{S})\hat{} = (\mathcal{R} \circ \mathcal{S})\hat{} = \mathcal{R} \circ \mathcal{S}$.

Let ESSR($\langle F, A\rangle$) be a set of all equivalence soft set relations on $\langle F, A\rangle$. Then (ESSR($\langle F, A\rangle$), $\subseteq$) is a poset. Moreover, for any $\mathcal{R}, \mathcal{S} \in$ ESSR($\langle F, A\rangle$), $\mathcal{R} \cap \mathcal{S}$ is the greatest lower bound for $\{\mathcal{R}, \mathcal{S}\}$ with respect to $\subseteq$.

Now we define two binary operation $\wedge_e$ and $\vee_e$ on ESSR($\langle F, A\rangle$) as follows: for any $\mathcal{R}, \mathcal{S} \in$ ESSR($\langle F, A\rangle$),

$$\mathcal{R} \wedge_e \mathcal{S} = \mathcal{R} \cap \mathcal{S} \ \text{ and } \ \mathcal{R} \vee_e \mathcal{S} = (\mathcal{R} \cup \mathcal{S})^e.$$

Then we obtain the following result from Remark 2.2.14 and Theorems 2.3.6 and 2.3.9.

**Theorem 2.3.11.** (ESSR($\langle F, A\rangle$), $\wedge_e, \vee_e$) is a complete lattice with the least element $I_{FA}$ and the greatest element $\langle F, A\rangle \times \langle F, A\rangle$.

## 2.4 Conclusions

Soft set theory is an effective method for solving problems of uncertainty. Babitha and Sunil [9] extended the concepts of relation and functions in soft set theory. In this chapter, we further study the equivalence soft set relations and obtain soft analogues of many results concerning ordinary equivalence relations and partitions. The transitive closure of soft set relation is discussed and some basic properties are proved. There exists compact connections between soft sets and information systems and so one can apply the results deducted from the studies on soft set relations to solve these connections. Thus, one can get more affirmative solution in decision making problems in real life situations.

# Chapter 3

# Filterness on soft topological spaces

The soft topology on a soft set and its related properties is presented by Çağman et al. [Computers and Mathematics with Applications 62 (1) (2011) 351]. In this chapter, we attempt to broad the theoretical aspects of soft topological spaces and so give soft analogues of many results concerning neighborhoods and closures in ordinary topological spaces. The notions of soft filters, ultra soft filters and bases of a soft filter are introduced and their basic properties are investigated. The adherence and convergence of soft filters in soft topological spaces with related results is also discussed.

## 3.1 Preliminaries

In the previous chapter, we present the basic definitions and results of soft set theory which may be found earlier studies [9, 38, 42]. For illustration, Molodtsov [42] considered several examples. The following example shows that every Zadeh's fuzzy $A$ may be considered a special case of the soft set.

**Example 3.1.1.** [42] Let $A$ be a fuzzy set and $\mu_A$ be the membership function of the fuzzy set $A$. That is $\mu_A$ is a function of $U$ onto $[0, 1]$. Consider the family of $\alpha$-level sets for the function $\mu_A$:

$$F(\alpha) = \{x \in U : \mu_A(x) \geq \alpha\}, \quad \alpha \in [0, 1].$$

If we know the family $F$, we can find the functions $\mu_A(x)$ by means of the following formula:

$$\mu_A(x) = \sup_{\substack{\alpha \in [0,1] \\ x \in F(\alpha)}} \alpha.$$

Thus, every Zadeh's fuzzy set $A$ may be considered the soft set $\langle F, [0, 1]\rangle$.

Now, in order to show the set theoretic relation in soft topological spaces, the distributive law with respect to the union and intersection operations holds. But the distributive law with respect to $\widetilde{\cup}$ and $\widetilde{\cap}$ does not hold. Thus, Maji et al. [38] defined the operation $\Cap$ (restricted intersection) which establish the distributive law about the union $\widetilde{\cup}$.

**Definition 3.1.2.** Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two soft set in $\mathcal{SS}(U)$. Then the intersection [38] of $\langle F, A\rangle$ and $\langle G, B\rangle$ is the soft set $\langle H, C\rangle$, where $C = A \cap B$ and for all $x \in C$, $H(x) = F(x) \cap G(x)$, and is written as $\langle F, A\rangle \Cap \langle G, B\rangle = \langle H, C\rangle$.

Maji et al. [38] proposed several operations on soft sets, and some basic properties of these operations are revealed. However, several assertions presented by Maji et al. are not true in general [2]. In order to efficiently discuss, we consider only soft sets $\langle F, A\rangle$ over a universe $U$ in which all parameter set $A$ are same. We denote the family of these soft sets by $\mathcal{SS}(U)_A$. In fact, for the family $\mathcal{SS}(U)_A$, Ali et al. [2] investigated some properties for algebraic structures on $\mathcal{SS}(U)_A$ and Shabir and Naz [55] introduced the notion of soft topology on $U$. Zorlutuna et al. [68] presented basic properties and operations induced by the family $\mathcal{SS}(U)_A$.

**Example 3.1.3.** Let $U = \{h_1, h_2, h_3\}$, $A = \{e_1, e_2\}$ and $U_A = \{\langle e_1, \{h_1, h_2\}\rangle, \langle e_2, \{h_2, h_3\}\rangle\}$. Then

$$\langle F_1, A\rangle = \{\langle e_1, \{h_1\}\rangle\}, \langle F_2, A\rangle = \{\langle e_1, \{h_2\}\rangle\}, \langle F_3, A\rangle = \{\langle e_1, \{h_1, h_2\}\rangle\},$$

$$\langle F_4, A\rangle = \{\langle e_2, \{h_2\}\rangle\}, \langle F_5, A\rangle = \{\langle e_2, \{h_3\}\rangle\}, \langle F_6, A\rangle = \{\langle e_2, \{h_2, h_3\}\rangle\},$$
$$\langle F_7, A\rangle = \{\langle e_1, \{h_1\}\rangle, \langle e_2, \{h_2\}\rangle\}, \langle F_8, A\rangle = \{\langle e_1, \{h_1\}\rangle, \langle e_2, \{h_3\}\rangle\},$$
$$\langle F_9, A\rangle = \{\langle e_1, \{h_1\}\rangle, \langle e_2, \{h_2, h_3\}\rangle\}, \langle F_{10}, A\rangle = \{\langle e_1, \{h_2\}\rangle, \langle e_2, \{h_2\}\rangle\},$$
$$\langle F_{11}, A\rangle = \{\langle e_1, \{h_2\}\rangle, \langle e_2, \{h_3\}\rangle\}, \langle F_{12}, A\rangle = \{\langle e_1, \{h_2\}\rangle, \langle e_2, \{h_2, h_3\}\rangle\},$$
$$\langle F_{13}, A\rangle = \{\langle e_1, \{h_1, h_2\}\rangle, \langle e_2, \{h_2\}\rangle\}, \langle F_{14}, A\rangle = \{\langle e_1, \{h_1, h_2\}\rangle, \langle e_2, \{h_3\}\rangle\},$$
$$\langle F_{15}, A\rangle = U_A, \langle F_{16}, A\rangle = \Phi_A$$

are all soft subsets of $U_A$ and so $\mathcal{SS}(U)_A = \{\langle F_i, A\rangle : i = 1, \ldots, 16\}$.

**Proposition 3.1.4.** Let $\langle F, A\rangle$, $\langle G, A\rangle$, $\langle H, A\rangle$ and $\langle K, A\rangle$ be soft sets in $\mathcal{SS}(U)_A$ and $\{\langle F_i, A\rangle\}_{i\in I}$ be a subfamily of $\mathcal{SS}(U)_A$. Then:

(1) $U_A^c = \Phi_A$ and $\Phi_A^c = U_A$ [38].

(2) $\langle F, A\rangle \Cap \Phi_A = \Phi_A$ [3].

(3) $\langle F, A\rangle \Cap U_A = \langle F, A\rangle$ [3].

(4) $\langle F, A\rangle \Cap \langle G, A\rangle = \Phi_A$ iff $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle^c$.

(5) $\langle F, A\rangle \Cap \langle G, A\rangle = \langle F, A\rangle$ iff $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle$ iff $\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle = \langle G, A\rangle$ [68].

(6) If $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle$ and $\langle H, A\rangle \widetilde{\subseteq} \langle K, A\rangle$, then $\langle F, A\rangle \Cap \langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle \Cap \langle K, A\rangle$ [68].

(7) $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle$ iff $\langle G, A\rangle^c \widetilde{\subseteq} \langle F, A\rangle^c$ [68].

(8) $[\widetilde{\cup}_{i\in I}\langle F_i, A\rangle]^c = \Cap_{i\in I}\langle F_i, A\rangle^c$, $[\Cap_{i\in I}(F_i, A)]^c = \widetilde{\cup}_{i\in I}\langle F_i, A\rangle^c$ [68].

(9) $\langle F, A\rangle \Cap \langle F, A\rangle = \langle F, A\rangle$ [48].

(10) $(\langle F, A\rangle \Cap \langle G, B\rangle)^c = \langle F, A\rangle^c \widetilde{\cup} \langle G, B\rangle^c$, $(\langle F, A\rangle \widetilde{\cup} \langle G, B\rangle)^c = \langle F, A\rangle^c \Cap \langle G, B\rangle^c$ [55].

(11) $(\langle F, A\rangle \Cap \langle G, B\rangle) \Cap \langle H, C\rangle = \langle F, A\rangle \Cap (\langle G, B\rangle \Cap (\gamma, C))$, $(\langle F, A\rangle \widetilde{\cup} \langle G, B\rangle) \widetilde{\cup} \langle H, C\rangle = \langle F, A\rangle \widetilde{\cup} (\langle G, B\rangle \widetilde{\cup} \langle H, C\rangle)$ [48].

(12) $\langle F, A\rangle \widetilde{\cup} (\langle G, B\rangle \Cap \langle H, C\rangle) = (\langle F, A\rangle \widetilde{\cup} \langle G, B\rangle) \Cap (\langle F, A\rangle \widetilde{\cup} \langle H, C\rangle)$, $\langle F, A\rangle \Cap (\langle G, B\rangle \widetilde{\cup} \langle H, C\rangle) = (\langle F, A\rangle \Cap \langle G, B\rangle) \widetilde{\cup} (\langle F, A\rangle \Cap \langle H, C\rangle)$ [48].

**Proof** (4) Necessity follows from [68]. To prove sufficiency, suppose $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle^c$. Let $x \in A$. Then $F(x) \subseteq G^c(x) = U \setminus G(x)$ and so $F(x) \cap G(x) = \emptyset$. Hence $\langle F, A\rangle \Cap \langle G, A\rangle = \Phi_A$. □

**Definition 3.1.5.** The soft set $\langle F, A\rangle \in \mathcal{SS}(U)_A$ is called a soft point in $U_A$, denoted by $x_F$, if for the element $x \in A$, $F(x) \neq \emptyset$ and $F(y) = \emptyset$ for all $y \in A\backslash\{x\}$.

**Definition 3.1.6.** The soft point $x_F$ is said to be in the soft set $\langle G, A\rangle$, denoted by $x_F \widetilde{\in} \langle G, A\rangle$, if for the element $x \in A$ and $F(x) \subset G(x)$.

**Remark 3.1.7.** If $x_F \widetilde{\in} \langle G, A\rangle \in \mathcal{SS}(U)_A$, then $x_F \widetilde{\notin} \langle G, A\rangle^c$. However, the converse is not true in general. In fact, let $A = \{e_1, e_2, e_3\}$ be a parameter set and $U = \{h_1, h_2, h_3, h_4\}$ be a universe. Let $e_{2_F} = \langle e_2, \{h_1, h_2, h_3\}\rangle$ and $\langle G, A\rangle = \{\langle e_1, \{h_1, h_4\}\rangle, \langle e_2, \{h_1, h_3\}\rangle\}$. Then $e_{2_F} \widetilde{\notin} \langle G, A\rangle$ and $e_{2_F} \widetilde{\notin} \langle G, A\rangle^c = \{\langle e_1, \{h_, h_3\}\rangle, \langle e_2, \{h_2, h_4\}\rangle, \langle e_3, U\rangle\}$.

## 3.2 Soft topology on soft sets

In this section, we present some results concerning neighborhoods in soft topological spaces.

**Definition 3.2.1.** [55] Let $\tau$ be a collection of soft sets over a universe $U$ with a fixed set $A$ of parameters, then $\tau \subset \mathcal{SS}(U)_A$ is called a soft topology on $U$ with a fixed set $A$ if it satisfies the conditions:

(T1) $\Phi_A \in \tau$, $U_A \in \tau$;
(T2) if $\langle G_i, A\rangle \in \tau$, $i \in I$, then $\widetilde{\bigcup}_{i\in I} \langle G_i, A\rangle \in \tau$;
(T3) if $\langle G_i, A\rangle \in \tau$, $i \in I$, where $I$ is finite set, then $\Cap_{i\in I} \langle G_i, A\rangle \in \tau$.

The triplet $(U, \tau, A)$ is called a soft topological space over $U$. The elements of $\tau$ are called the soft open sets in $U$ and the complements of soft open sets is called soft closed sets in $U$.

$\{\Phi_A, U_A\}$ and $\mathcal{SS}(U)_A$ are two examples for soft topology on $X$ and shall call indiscrete soft topology and discrete soft topology respectively as called in point-set topology. Moreover, $\mathcal{SS}(U)$ is a soft topology on $\widetilde{U}$.

For two soft topologies $\tau$ and $\tau'$ on $U_A$, $\tau$ is said to be finer than $\tau'$ and $\tau'$ coarser than $\tau$ if $\tau' \subseteq \tau$; thus $\tau$ is finer than $\tau'$ if and only if every $\tau'$-soft open subset of $U_A$ is $\tau$-open.

**Definition 3.2.2.** [68] A soft set $\langle G, A\rangle$ in a soft topological space $(U, \tau, A)$ is called a neighborhood of the soft point $x_F \widetilde{\in} U_A$ if there exists a soft open set $\langle H, A\rangle$ such that $x_F \widetilde{\in} \langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$.

The family of all neighborhoods of $x_F$ is called the neighborhood system [55] of $x_F$ up to soft topology $\tau$ and is denoted by $\mathcal{N}_\tau(x_F)$ (or simply by $\mathcal{N}(x_F)$). By a neighborhood base of $x_F$ we mean a collection $\mathcal{BN}_\tau(x_F)$ (or simply by $\mathcal{BN}(x_F)$) of neighborhood of $x_F$ such that for every neighborhood $\langle G, A\rangle$ of $x_F$, there exists a soft set $\langle H, A\rangle$ in $\mathcal{BN}_\tau(x_F)$ such that $\langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$.

**Example 3.2.3.** Let us consider the soft subsets of $U_A$ given in Example 3.1.3. Let $\tau_1 = \{\Phi_A, U_A, \langle F_2, A\rangle, \langle F_{11}, A\rangle, \langle F_{13}, A\rangle\}$ and $\tau_2 = \{\Phi_A, U_A, \langle F_3, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle\}$ be two soft topologies on $U_A$. Consider a soft point $e_{1_F} = \langle e_1, \{h_2\}\rangle$ in $U_A$. Then

$$\mathcal{N}_{\tau_1}(e_{1_F}) = \{\langle F_2, A\rangle, \langle F_3, A\rangle, \langle F_{10}, A\rangle, \langle F_{11}, A\rangle, \langle F_{12}, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle, U_A\},$$
$$\mathcal{N}_{\tau_2}(e_{1_F}) = \{\langle F_3, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle, U_A\}$$

are the neighborhood system of $e_{1_F}$ with respect to $\tau_1$ and $\tau_2$, respectively.

**Theorem 3.2.4.** Let $(U, \tau, A)$ be a soft topological space. A soft set $\langle G, A\rangle$ in $\mathcal{SS}(U)_A$ is soft open if and only if it is a neighborhood of each of its soft points.

**Proof** Let $\langle G, A\rangle$ be a soft open set and $x_F \widetilde{\in} \langle G, A\rangle$. Since $\langle G, A\rangle$ is a soft open set containing $x_F$ and included in $\langle G, A\rangle$, it follows that $\langle G, A\rangle$ is neighborhood of $x_F$. Conversely, suppose that $\langle G, A\rangle$ is neighborhood of each of its soft points. Then for each soft point $x_F$ of $\langle G, A\rangle$ there is a soft open set $\langle G, A\rangle_{x_F}$ such that $x_F \widetilde{\in} \langle G, A\rangle_{x_F}$ and $\langle G, A\rangle_{x_F} \widetilde{\subseteq} \langle G, A\rangle$. Then $\langle G, A\rangle = \widetilde{\cup}_{x_F \widetilde{\in} \langle G, A\rangle} \langle G, A\rangle_{x_F}$ and hence is soft open. □

When two soft topologies are given, a criterion in terms of the neighborhoods for determining whether one soft topology is finer than another is the following:

**Theorem 3.2.5.** Let $\tau$ and $\tau'$ be soft topologies on $U_A$. Then $\tau$ is finer than $\tau'$ if and only if for every soft point $x_F$ in $U_A$, every $\tau'$-neighborhood of $x_F$ is a $\tau$-neighborhood of $x_F$.

**Proof** Suppose $\tau$ is finer than $\tau'$. Let $x_F$ be a soft point in $U_A$ and $\langle G, A\rangle$ be a $\tau$-neighborhood of $x_F$. Then there is a $\tau'$-soft open set $\langle H, A\rangle$ such that $x_F \widetilde{\in} \langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$. Since $\tau' \subseteq \tau$, $\langle H, A\rangle$ is $\tau$-soft open set. Hence $\langle G, A\rangle$ is a $\tau$-neighborhood of $x_F$.

Conversely, suppose that for every soft point $x_F$ in $U_A$, every $\tau'$-neighbor-hood is a $\tau$-neighborhood. Let $\langle H, A\rangle$ be a $\tau'$-soft open set. Then by Theorem 3.2.4, $\langle H, A\rangle$ is a $\tau'$-neighborhood of each of its soft points and hence a $\tau$-neighborhood of its soft points. Thus $\langle H, A\rangle$ is $\tau$-soft open set. So $\tau' \subseteq \tau$, i.e., $\tau$ is finer than $\tau'$. □

**Lemma 3.2.6.** [68] Let $(U, \tau, A)$ be a soft topological space, $x_F$ be a soft point in $U_A$ and $\mathcal{N}_\tau(x_F)$ be the set of neighborhoods of $x_F$. Then :

(1) If $\langle G, A\rangle \in \mathcal{N}_\tau(x_F)$, then $x_F \widetilde{\in} \langle G, A\rangle$.

(2) If $\langle G, A\rangle \in \mathcal{N}_\tau(x_F)$ and $\langle G, A\rangle \widetilde{\subseteq} \langle H, A\rangle$, then $\langle H, A\rangle \in \mathcal{N}_\tau(x_F)$.

(3) If $\langle G, A\rangle, \langle H, A\rangle \in \mathcal{N}_\tau(x_F)$, then $\langle G, A\rangle \Cap \langle H, A\rangle \in \mathcal{N}_\tau(x_F)$.

(4) If $\langle G, A\rangle \in \mathcal{N}_\tau(x_F)$, then there is a $\langle H, A\rangle \in \mathcal{N}_\tau(x_F)$ such that $\langle G, A\rangle \in \mathcal{N}_\tau(x'_F)$ for every $x'_F \widetilde{\in} \langle H, A\rangle$.

The following theorem shows that a soft topology may be defined on a soft set $U_A$ by prescribing for each soft point its neighborhoods with respect to the soft topology. Referring to Theorem 3.2.4, we see that the open soft sets in the soft topology must be those which belong to the proposed neighborhood collections for each of their soft points.

**Theorem 3.2.7.** Let $U_A$ be a soft set and $(\mathcal{N}(x_F))_{x_F \widetilde{\in} U_A}$ be a family of non-empty sets of soft subsets in $\mathcal{SS}(U)_A$ such that

(1) for each soft point $x_F$ in $U_A$, every soft set in $\mathcal{SS}(U)_A$ which includes a soft set in $\mathcal{N}(x_F)$ belongs to $\mathcal{N}(x_F)$;

(2) for each soft point $x_F$ in $U_A$, the intersection of each finite family of soft sets in $\mathcal{N}(x_F)$ belongs to $\mathcal{N}(x_F)$;

(3) for each soft point $x_F$ in $U_A$, the soft point $x_F$ is in every soft set in $\mathcal{N}(x_F)$;

(4) for each soft point $x_F$ in $U_A$ and each soft set $\langle F, A\rangle$ in $\mathcal{N}(x_F)$ there exists a soft set $\langle G, A\rangle$ in $\mathcal{N}(x_F)$ such that $\langle F, A\rangle$ belongs to $\mathcal{N}(y_F)$ for every soft point $y_F \widetilde{\in} \langle G, A\rangle$.

Then there exists a unique soft topology $\tau$ on $U_A$ such that for each soft point $x_F$ in $U_A$, the set $\mathcal{N}(x_F)$ is the set of $\tau$-neighborhoods of $x_F$.

**Proof** Let $\tau = \{\langle G, A\rangle \in \mathcal{SS}(U)_A : \langle G, A\rangle \in \mathcal{N}(x_F)$ for all soft points $x_F$ of $U_A\}$. Then

(T1) $U_A$ and $\Phi_A$ belong to $\tau$.

(T2) Let $\{\langle G_i, A\rangle : i \in I\}$ be a family of soft sets in $\tau$ and let $x_F \widetilde{\in} \widetilde{\bigcup}_{i\in I}\langle G_i, A\rangle$. Then there is an index $i_0$ in $I$ such that $x_F \widetilde{\in} \langle G_{i_0}, A\rangle$. Since $\langle G_{i_0}, A\rangle$ is in $\tau$, we have $\langle G_{i_0}, A\rangle \in \mathcal{N}(x_F)$. Since $\widetilde{\bigcup}_{i\in I}\langle G_i, A\rangle \widetilde{\supseteq} \langle G_{i_0}, A\rangle$, it follows that $\widetilde{\bigcup}_{i\in I}\langle G_i, A\rangle \in \mathcal{N}(x_F)$. Hence $\widetilde{\bigcup}_{i\in I}\langle G_i, A\rangle \in \tau$.

(T3) Let $\{\langle G_i, A\rangle : i \in J\}$ be a finite family of soft sets in $\tau$ and let $x_F \widetilde{\in} \Cap_{i\in J} \langle G_i, A\rangle$. Then for each $i \in J$, $\langle G_i, A\rangle \in \mathcal{N}(x_F)$. It follows that for each $x_F \widetilde{\in} \Cap_{i\in J} \langle G_i, A\rangle$, we have $\Cap_{i\in J}\langle G_i, A\rangle \in \mathcal{N}(x_F)$. Hence $\Cap_{i\in J}\langle G_i, A\rangle \in \tau$.

So $\tau$ is a soft topology on $U_A$.

Let $x_F$ be a soft point in $U_A$ and $\langle G, A\rangle$ be any $\tau$-soft neighborhood of $x_F$. Then there is $\langle H, A\rangle \in \tau$ such that $x_F \widetilde{\in} \langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$. Since $\langle H, A\rangle \in \mathcal{N}(x_F)$ (because $\langle H, A\rangle \in \tau$), it follows that $\langle G, A\rangle \in \mathcal{N}(x_F)$.

Conversely, let $x_F$ be a soft point in $U_A$ and $\langle G, A\rangle$ be any soft set in $\mathcal{N}(x_F)$. Let $\langle H, A\rangle = \{y_F \widetilde{\in} U_A : \langle G, A\rangle \in \mathcal{N}(y_F)\}$. Clearly, $x_F \widetilde{\in} \langle H, A\rangle$. Next, $\langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$; for if $y_F \widetilde{\in} \langle H, A\rangle$ we have $\langle G, A\rangle \in \mathcal{N}(y_F)$ and hence $y_F \widetilde{\in} \langle G, A\rangle$. Finally, $\langle H, A\rangle \in \tau$. To see this, let $z_F \widetilde{\in} \langle H, A\rangle$. Since $\langle G, A\rangle \in \mathcal{N}(z_F)$ there is a soft set $\langle K, A\rangle$ in $\mathcal{N}(z_F)$ such that $\langle G, A\rangle \in \mathcal{N}(w_F)$ for all soft points $w_F$ of $\langle K, A\rangle$. Since $\langle G, A\rangle \in \mathcal{N}(w_F)$ for all soft points $w_F$ of $\langle K, A\rangle$, it follows that $\langle K, A\rangle \widetilde{\subseteq} \langle H, A\rangle$. Since $\langle K, A\rangle \in \mathcal{N}(z_F)$ it follows that $\langle H, A\rangle \in \mathcal{N}(z_F)$. Thus $\langle H, A\rangle \in \tau$ and so $\langle G, A\rangle$ is a $\tau$-neighborhood of $x_F$.

The uniqueness of $\tau$ is clear. □

**Definition 3.2.8.** [55, 68] Let $(U, \tau, A)$ be a soft topological space and $\langle F, A\rangle$ be a soft set in $\mathcal{SS}(U)_A$. Then

(1) The soft closure of $\langle F, A\rangle$ is the soft set
$\overline{\langle F, A\rangle} = \Cap\{\langle G, A\rangle \in \mathcal{SS}(U)_A : \langle G, A\rangle$ is soft closed and $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle\}$;
(2)The soft interior $\langle F, A\rangle^o$ is the soft set
$\langle F, A\rangle^o = \widetilde{\cup}\{\langle G, A\rangle \in \mathcal{SS}(U)_A : \langle G, A\rangle$ is soft open and $\langle G, A\rangle \widetilde{\subseteq} \langle F, A\rangle\}$.

Clearly $\langle F, A\rangle^o$ is the largest soft open subset included in $\langle F, A\rangle$ and $\overline{\langle F, A\rangle}$ is the smallest soft closed subset which includes $\langle F, A\rangle$.

**Lemma 3.2.9.** [68] Let $(U, \tau, A)$ be a soft topological space and $\langle F, A\rangle$ be a soft set in $\mathcal{SS}(U)_A$. Then $\overline{\langle F, A\rangle^c} = (\langle F, A\rangle^o)^c$.

The following shows that a soft topology may be defined on a soft set $U_A$ by prescribing for each soft subset its closure in the soft topology.

**Theorem 3.2.10.** Let $U_A$ be a soft set and $f : \mathcal{SS}(U)_A \to \mathcal{SS}(U)_A$ be a mapping such that
(1) for every soft subset $\langle G, A\rangle$ of $U_A$ we have $f(\langle G, A\rangle) \widetilde{\supseteq} \langle G, A\rangle$;
(2) for every soft subset $\langle G, A\rangle$ of $U_A$ we have $f(f(\langle G, A\rangle)) = f(\langle G, A\rangle)$;
(3) for all soft subsets $\langle G, A\rangle$ and $\langle H, A\rangle$ of $U_A$ we have $f(\langle G, A\rangle \widetilde{\cup} \langle H, A\rangle) = f(\langle G, A\rangle) \widetilde{\cup} f(\langle H, A\rangle)$;
(4) $f(\Phi_A) = \Phi_A$.
Then $\tau_f = \{\langle G, A\rangle \in \mathcal{SS}(U)_A : f(\langle G, A\rangle^c) = \langle G, A\rangle^c\}$ is a soft topology on $U_A$ such that $\overline{\langle G, A\rangle} = f(\langle G, A\rangle)$ for every soft subset $\langle G, A\rangle$ of $U_A$.

**Proof** It follows from condition (3) that if $\langle G, A\rangle$ and $\langle H, A\rangle$ are soft subsets of $U_A$ such that $\langle G, A\rangle \widetilde{\subseteq} \langle H, A\rangle$ then $f(\langle G, A\rangle) \widetilde{\subseteq} f(\langle H, A\rangle)$.
(T1) Since $f(\Phi_A^c) = f(U_A) = U_A = \Phi_A^c$ and $f(U_A^c) = f(\Phi_A) = \Phi_A = U_A^c$, we have $\Phi_A, U_A \in \tau_f$.
(T2) Let $\{\langle G_i, A\rangle : i \in I\}$ be a family of soft sets in $\tau_f$. Then we have $f((\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c) \widetilde{\supseteq} (\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c$. On the other hand, we have, for all $j \in I$,

$$f((\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c) = f(\Cap_{i\in I}\langle G_i, A\rangle^c) \widetilde{\subseteq} f(\langle G_j, A\rangle^c) = \langle G_j, A\rangle^c.$$

Hence $f((\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c) \widetilde{\subseteq} \Cap_{i\in I}\langle G_i, A\rangle^c = (\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c$. Thus $f((\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c) = (\widetilde{\cup}_{i\in I}\langle G_i, A\rangle)^c$ and so $\widetilde{\cup}_{i\in I}\langle G_i, A\rangle \in \tau_f$.

(T3) Let $\{\langle G_i, A\rangle : i \in J\}$ be a finite family of soft sets in $\tau_f$. Then, by condition (3), we have $f((\widetilde{\bigcap}_{i\in J}\langle G_i, A\rangle)^c) = f(\widetilde{\bigcup}_{i\in J}\langle G_i, A\rangle^c) = \widetilde{\bigcup}_{i\in J} f(\langle G_i, A\rangle^c) = \widetilde{\bigcup}_{i\in J}\langle G_i, A\rangle^c = (\widetilde{\bigcap}_{i\in J}\langle G_i, A\rangle)^c$. So we have $\widetilde{\bigcap}_{i\in J}\langle G_i, A\rangle \in \tau_f$.

Thus $\tau_f$ is a soft topology on $U_A$.

Let $\langle K, A\rangle$ be any soft set in $\mathcal{SS}(U)_A$. Then $\langle K, A\rangle$ is $\tau_f$-closed $\Longleftrightarrow \langle K, A\rangle^c \in \tau_f \Longleftrightarrow f((\langle K, A\rangle^c)^c) = (\langle K, A\rangle^c)^c \Longleftrightarrow f(\langle K, A\rangle) = \langle K, A\rangle$.

Now let $\langle G, A\rangle$ be any soft set in $\mathcal{SS}(U)_A$. Since $f(\langle G, A\rangle)\widetilde{\supseteq}\langle G, A\rangle$ and $f(f(\langle G, A\rangle)) = f(\langle G, A\rangle)$, we see that $f(\langle G, A\rangle)$ is a soft closed subset which includes $\langle G, A\rangle$. So $f(\langle G, A\rangle)\widetilde{\supseteq}\overline{\langle G, A\rangle}$. On the other hand, since $\overline{\langle G, A\rangle}$ is $\tau_f$-closed and $\overline{\langle G, A\rangle}\widetilde{\supseteq}\langle G, A\rangle$, we deduce that $\overline{\langle G, A\rangle} = f(\overline{\langle G, A\rangle})\widetilde{\supseteq} f(\langle G, A\rangle)$. Thus $\overline{\langle G, A\rangle} = f(\langle G, A\rangle)$. □

**Definition 3.2.11.** Let $(U, \tau, A)$ be a soft topological space, $x_F$ be a soft point and $\langle G, A\rangle$ be the soft sets in $\mathcal{SS}(U)_A$. Then $x_F$ is said to be an adherent soft point of $\langle G, A\rangle$ if every neighborhood of $x_F$ meets $\langle G, A\rangle$, i.e., has non-null soft set intersection with $\langle G, A\rangle$.

**Lemma 3.2.12.** Let $(U, \tau, A)$ be a soft topological space, $x_F$ be a soft point and $\langle G, A\rangle$ be the soft sets in $\mathcal{SS}(U)_A$. Then $x_F\widetilde{\in}\overline{\langle G, A\rangle}$ if and only if $x_F$ is an adherent soft set of $\langle G, A\rangle$.

**Proof** Let $x_F\widetilde{\in}\overline{\langle G, A\rangle}$. If $x_F$ is not an adherent soft set of $\langle G, A\rangle$, there is a neighborhood $\langle H, A\rangle$ of $x_F$ such that $\langle H, A\rangle \widetilde{\cap} \langle G, A\rangle = \Phi_A$. Then there is a soft open set $\langle K, A\rangle$ such that $x_F\widetilde{\in}\langle K, A\rangle\widetilde{\subseteq}\langle H, A\rangle$ and of course $\langle K, A\rangle\widetilde{\cap}\langle G, A\rangle = \Phi_A$, so that $\langle K, A\rangle^c\widetilde{\supseteq}\langle G, A\rangle$. Since $\langle K, A\rangle^c$ is a soft closed set which includes $\langle G, A\rangle$, we have $\langle F, A\rangle^c\widetilde{\supseteq}\overline{\langle G, A\rangle}$ and so $x_F \widetilde{\notin}\overline{\langle G, A\rangle}$, which is a contradiction.

Conversely, suppose $x_F \widetilde{\notin}\overline{\langle G, A\rangle}$. Then there is a soft closed subset $\langle L, A\rangle$ such that $\langle L, A\rangle\widetilde{\supseteq}\langle G, A\rangle$ and $x_F \widetilde{\notin}\langle L, A\rangle$. Then $\langle L, A\rangle^c$ is a soft open set including $x_F$ and hence $\langle L, A\rangle^c$ is a neighborhood of $x_F$ and $\langle L, A\rangle^c \widetilde{\cap} \langle G, A\rangle = \Phi_A$. So $x_F$ is not adherent soft set of $\langle G, A\rangle$. □

**Theorem 3.2.13.** Let $(U, \tau, A)$ be a soft topological space. If $\langle G, A\rangle$ and $\langle H, A\rangle$ are soft sets in $\mathcal{SS}(U)_A$ such that $U_A = \langle G, A\rangle\widetilde{\cup}\langle H, A\rangle$, then $U_A = \overline{\langle G, A\rangle}\widetilde{\cup}\langle H, A\rangle^o$.

**Proof** Suppose $U_A = \langle G, A\rangle \widetilde{\cup} \langle H, A\rangle$. Let $x_F$ be any soft point in $\mathcal{SS}(U)_A$ not included in $\overline{\langle G, A\rangle}$. Then $x_F \widetilde{\notin} \langle G, A\rangle$ and hence $x_F \widetilde{\in} \langle H, A\rangle$. By Lemma 3.2.12, there is a neighborhood $\langle K, A\rangle$ of $x_F$ such that $\langle K, A\rangle \Cap \langle G, A\rangle = \Phi_A$ and hence $\langle K, A\rangle \widetilde{\subseteq} \langle H, A\rangle$. Thus $\langle H, A\rangle$ is a neighborhood of $x_F$, i.e., $x_F \widetilde{\in} \langle H, A\rangle^o$. So $U_A = \overline{\langle G, A\rangle} \widetilde{\cup} \langle H, A\rangle^o$. □

## 3.3 Soft filter

In this section, we present the notion of soft filters and obtain some results of soft filter on soft topological spaces.

**Definition 3.3.1.** A soft filter on $U_A$ is a non-empty subfamily $\mathcal{F}$ of $\mathcal{SS}(U)_A$ having the following properties:

(F1) Every soft subset of $\mathcal{SS}(U)_A$ which includes a soft set in $\mathcal{F}$ belongs to $\mathcal{F}$;

(F2) The intersection of each finite family of soft sets in $\mathcal{F}$ belongs to $\mathcal{F}$;

(F3) All the soft sets in $\mathcal{F}$ are not null soft set.

Let $\mathcal{F}$ be a soft filter on $U_A$. A collection $\mathcal{B}$ of soft subsets of $\mathcal{SS}(U)_A$ is called a base for the soft filter $\mathcal{F}$ if (1) $\mathcal{B} \subseteq \mathcal{F}$ and (2) for every soft set $\langle F, A\rangle$ in $\mathcal{F}$, there is a soft set $\langle G, A\rangle$ in $\mathcal{B}$ such that $\langle G, A\rangle \widetilde{\subseteq} \langle F, A\rangle$; we say also that $\mathcal{B}$ generates $\mathcal{F}$.

Observe that the family $\mathcal{N}(x_F)$ of all neighborhoods of a soft point $x_F$ in a soft topological space $(U, \tau, A)$ is always a soft filter on $U_A$. Note also that if $\mathcal{F}$ is a soft filter, then $(\mathcal{F}, \widetilde{\subseteq})$ is a direct set.

**Remark 3.3.2.** Let $(U, \tau, A)$ be a soft topological space and $x_F$ be any soft point of $U_A$. Then, by Lemma 3.2.6, the set of all neighborhood of $x_F$ is a soft filter on $U_A$. We call this the neighborhood soft filter of $x_F$ and denoted by $\mathcal{V}_\tau(x_F)$ (simply $\mathcal{V}(x_F)$). Furthermore, every neighborhood base of $x_F$ is a base for this soft filter.

**Example 3.3.3.** Let us consider the soft subsets of $U_A$ given in Example 3.1.3. Then $\mathcal{F}_1 = \{\langle F_1, A\rangle, \langle F_3, A\rangle, \langle F_7, A\rangle, \langle F_8, A\rangle, \langle F_9, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle, U_A\}$, $\mathcal{F}_2 = \{\langle F_3, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle, U_A\}$ and $\mathcal{F}_3 = \{\langle F_6, A\rangle, \langle F_9, A\rangle, \langle F_{12}, A\rangle, U_A\}$ are soft filters on $U_A$. And $\mathcal{B}_1 = \{\langle F_1, A\rangle, \langle F_3, A\rangle, \langle F_7, A\rangle, \langle F_8, A\rangle\}$, $\mathcal{B}_2 = \{\langle F_3, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle\}$ and $\mathcal{B}_3 = \{\langle F_6, A\rangle, \langle F_9, A\rangle, \langle F_{12}, A\rangle\}$ are bases for the soft filters $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$, respectively.

**Theorem 3.3.4.** Let $\mathcal{F}$ and $\mathcal{G}$ be soft filters on $U_A$. Then a soft set $\langle H, A\rangle$ in $\mathcal{SS}(U)_A$ belongs to both $\mathcal{F}$ and $\mathcal{G}$ if and only if there are soft sets $\langle F, A\rangle \in \mathcal{F}$ and $\langle G, A\rangle \in \mathcal{G}$ such that $\langle H, A\rangle = \langle F, A\rangle \widetilde{\cup} \langle G, A\rangle$.

**Proof** Suppose $\langle H, A\rangle \in \mathcal{F} \cap \mathcal{G}$. Then $\langle H, A\rangle = \langle H, A\rangle \widetilde{\cup} \langle H, A\rangle$, $\langle H, A\rangle \in \mathcal{F}$ and $\langle H, A\rangle \in \mathcal{G}$. Conversely, suppose $\langle H, A\rangle = \langle F, A\rangle \widetilde{\cup} \langle G, A\rangle$ where $\langle F, A\rangle \in \mathcal{F}$ and $\langle G, A\rangle \in \mathcal{G}$. Then $\langle H, A\rangle \widetilde{\supseteq} \langle F, A\rangle$, so $\langle H, A\rangle \in \mathcal{F}$ and $\langle H, A\rangle \widetilde{\supseteq} \langle G, A\rangle$, so $\langle H, A\rangle \in \mathcal{G}$. □

**Theorem 3.3.5.** Let $\mathcal{B}$ be a collection of soft sets in $\mathcal{SS}(U)_A$. Then $\mathcal{B}$ is a base for a soft filter on $U_A$ if and only if (1) the finite intersection of members of $\mathcal{B}$ includes a member of $\mathcal{B}$ and (2) $\mathcal{B}$ is non-empty and $\Phi_A$ does not belong to $\mathcal{B}$.

**Proof** Suppose that $\mathcal{B}$ is a base for a soft filter $\mathcal{F}$ on $U_A$. Let $\{\langle G_i, A\rangle : i = 1, \ldots, n\}$ be a finite family of soft sets in $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{F}$, it follows that $\Cap_{i=1}^{n}(\langle G_i, A\rangle) \in \mathcal{F}$ and so $\Cap_{i=1}^{n}(\langle G_i, A\rangle)$ includes a soft set in $\mathcal{B}$. Since $\mathcal{F}$ is non-empty and every soft set in $\mathcal{F}$ includes a soft set in $\mathcal{B}$, it follows that $\mathcal{B}$ is non-empty. Since $\Phi_A \notin \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{F}$, we have $\Phi_A \notin \mathcal{B}$.

Conversely, suppose the conditions are satisfied. Let $\mathcal{F} = \{\langle F, A\rangle \in \mathcal{SS}(U)_A : \langle F, A\rangle$ includes a soft set in $\mathcal{B}\}$. Then $\mathcal{F}$ is a soft filter on $U_A$ with base $\mathcal{B}$. □

Note that a non-empty family $\mathcal{B}$ of soft subsets of $\mathcal{SS}(U)_A$ is called a soft filter base on $U_A$ provided $\mathcal{B}$ does not contain the null soft set and provided the intersection of any two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$. A family $\mathcal{S}$ is called a subbase of a soft filter iff it is nonvoid and the intersection of any finite number of elements of $\mathcal{S}$ is not the null soft set.

**Theorem 3.3.6.** If $\mathcal{S}$ is a subbase of a soft filter on $U_A$, then the family $\mathcal{B}(\mathcal{S})$ consisting of all finite intersections of elements of $\mathcal{S}$ is a soft filter base. If $\mathcal{B}$ is a soft filter base, then the family $\mathcal{F}(\mathcal{B})$, consisting of all soft sets $\langle F, A\rangle \in \mathcal{SS}(U)_A$ such that $\langle F, A\rangle \widetilde{\supseteq} \langle G, A\rangle$ for some $\langle G, A\rangle \in \mathcal{B}$, is a soft filter. $\mathcal{B}(\mathcal{S})$ and $\mathcal{F}(\mathcal{B})$ are uniquely determined by $\mathcal{S}$ and $\mathcal{B}$, respectively.

**Proof** Clearly, the family $\mathcal{B}(\mathcal{S})$ satisfies the requirements of a soft filter base. That $\mathcal{F}(\mathcal{B})$ is a soft filter if $\mathcal{B}$ is a soft filter base is also easily shown. We have $\Phi_A \notin \mathcal{F}$, since $\Phi_A \notin \mathcal{B}$. If $\langle F, A\rangle \in \mathcal{F}(\mathcal{B}$ and $\langle H, A\rangle \widetilde{\supseteq} \langle F, A\rangle$, then $\langle G, A\rangle \widetilde{\subseteq} \langle F, A\rangle \widetilde{\subseteq} \langle H, A\rangle$ for some $\langle G, A\rangle \in \mathcal{B}$, and hence $\langle H, A\rangle \in \mathcal{F}(\mathcal{B})$. Finally, let $\langle F_1, A\rangle$ and $\langle F_2, A\rangle$ be in $\mathcal{F}(\mathcal{B})$. Then there exist $\langle G_1, A\rangle$ and $\langle G_2, A\rangle$ in $\mathcal{B}$ such that $\langle F_1, A\rangle \widetilde{\supseteq} \langle G_1, A\rangle$ and $\langle F_2, A\rangle \widetilde{\supseteq} \langle G_2, A\rangle$. It follows that $\langle F_1, A\rangle \Cap \langle F_2, A\rangle \widetilde{\supseteq} \langle G_1, A\rangle \Cap \langle G_2, A\rangle$. Since $\mathcal{B}$ is a soft filter base, there exists $\langle G, A\rangle \in \mathcal{B}$ such that $\langle G, A\rangle \widetilde{\subseteq} \langle G_1, A\rangle \Cap \langle G_2, A\rangle$. Hence $\langle F_1, A\rangle \Cap \langle F_2, A\rangle \in \mathcal{F}(\mathcal{B})$. That $\mathcal{B}(\mathcal{S})$ and $\mathcal{F}(\mathcal{B})$ are uniquely determined by $\mathcal{S}$ and $\mathcal{B}$, respectively, is an immediate consequence of their definitions. □

Let $\mathbf{A}$ be a collection of soft subsets of $U_A$; let $\mathbf{A}'$ be the collection of intersections of all finite families of soft sets in $\mathbf{A}$. If $\mathbf{A}'$ does not contain the null soft set $\Phi$, then it satisfies the conditions of Theorem 3.3.5 and hence is a base for a soft filter $\mathcal{F}$ on $U_A$. We call $\mathcal{F}$ the soft filter generated by $\mathbf{A}$.

**Theorem 3.3.7.** Let $\mathcal{F}$ and $\mathcal{G}$ be soft filters on $U_A$. Suppose that for every pair of soft subsets $\langle F, A\rangle, \langle G, A\rangle$ of $U_A$ in $\mathcal{F} \cup \mathcal{G}$, we have $\langle F, A\rangle \Cap \langle G, A\rangle \neq \Phi_A$. Then the soft filter generated by $\mathcal{F} \cup \mathcal{G}$ consists of all soft sets of the form $\langle H, A\rangle \Cap \langle K, A\rangle$ where $\langle H, A\rangle \in \mathcal{F}$ and $\langle K, A\rangle \in \mathcal{G}$.

**Proof** Let $\mathcal{H}$ be the soft filter generated by $\mathcal{F} \cup \mathcal{G}$. Let $\mathcal{S}$ be the set of intersections of all finite families of soft sets from $\mathcal{F} \cup \mathcal{G}$. Let $\langle F, A\rangle \in \mathcal{H}$. Then $\langle F, A\rangle$ includes a soft set in $\mathcal{S}$. Every soft set in $\mathcal{S}$ has the form $\langle H, A\rangle \Cap \langle K, A\rangle$ where $\langle H, A\rangle \in \mathcal{F}$ and $\langle K, A\rangle \in \mathcal{G}$. If $\langle F, A\rangle \widetilde{\supseteq} \langle H, A\rangle \Cap \langle K, A\rangle$ where $\langle H, A\rangle \in \mathcal{F}$ and $\langle K, A\rangle \in \mathcal{G}$, then it follows that we have

$$\langle F, A\rangle = \langle F, A\rangle \widetilde{\cup} (\langle H, A\rangle \Cap \langle K, A\rangle) = (\langle F, A\rangle \widetilde{\cup} \langle H, A\rangle) \Cap (\langle F, A\rangle \widetilde{\cup} \langle K, A\rangle).$$

Since $\langle F, A\rangle\widetilde{\cup}\langle H, A\rangle\widetilde{\supseteq}\langle H, A\rangle$ and $\langle F, A\rangle\widetilde{\cup}\langle K, A\rangle\widetilde{\supseteq}\langle K, A\rangle$, we have $\langle F, A\rangle\widetilde{\cup}\langle H, A\rangle \in \mathcal{F}$ and $\langle F, A\rangle\widetilde{\cup}\langle K, A\rangle \in \mathcal{G}$. So $\langle F, A\rangle \in \mathcal{S}$. Thus $\mathcal{H} = \mathcal{S}$, as required. □

**Theorem 3.3.8.** The set of all soft filters on a non-null soft set $U_A$ is inductively ordered by inclusion.

**Proof** Let $\mathbf{F} = \{\mathcal{F} : \mathcal{F}$ is a soft filter on $U_A\}$ be totally ordered by inclusion $\subseteq$. Let $\mathbf{A}$ be the union of $\mathbf{F}$. Let $\{\langle F_i, A\rangle : i \in I\}$ be a finite family of soft sets in $\mathbf{A}$. For each $i \in I$, there is a soft filter $\mathcal{F}_i$ in $\mathbf{F}$ such that $\langle F_i, A\rangle \in \mathcal{F}_i$. Since $\mathbf{F}$ is $\subseteq$-totally ordered, there is an index $j$ in $I$ such that $\langle F_i, A\rangle \in \mathcal{F}_j$ for all $i \in I$. Hence $\Cap_{i\in I}\langle F_i, A\rangle \neq \Phi_A$. By Theorem 3.3.5, $\mathbf{A}$ generates a soft filter $\mathcal{F}$ on $U_A$ which is clearly the $\subseteq$-supremum of $\mathbf{F}$. □

It follows from Theorem 3.3.8 by the application of Zorn's Lemma that the collection of soft filters on a non-null soft set $U_A$ has $\subseteq$-maximal elements: these maximal soft filters are called ultra soft filters. It is also easy to show that for every soft filter $\mathcal{F}$ on a soft set $U_A$ there is an ultra soft filter on $U_A$ which includes $\mathcal{F}$.

**Theorem 3.3.9.** Let $\mathcal{F}$ be an ultra soft filter on a soft set $U_A$. If $\langle F, A\rangle$ and $\langle G, A\rangle$ are soft sets in $\mathcal{SS}(U)_A$ such that $\langle F, A\rangle\widetilde{\cup}\langle G, A\rangle \in \mathcal{F}$, then either $\langle F, A\rangle \in \mathcal{F}$ or $\langle G, A\rangle \in \mathcal{F}$.

**Proof** Suppose $\langle F, A\rangle \notin \mathcal{F}$ and $\langle G, A\rangle \notin \mathcal{F}$. Let $\mathcal{F}' = \{\langle H, A\rangle \in \mathcal{SS}(U)_A : \langle F, A\rangle\widetilde{\cup}\langle H, A\rangle \in \mathcal{F}\}$. Then

(F1) Let $\langle H, A\rangle \in \mathcal{F}'$ and $\langle K, A\rangle \in \mathcal{SS}(U)_A$ with $\langle H, A\rangle\widetilde{\subseteq}\langle K, A\rangle$. Since $\langle F, A\rangle\widetilde{\cup}\langle H, A\rangle \in \mathcal{F}$ and $\langle F, A\rangle\widetilde{\cup}\langle H, A\rangle\widetilde{\subseteq}\langle F, A\rangle\widetilde{\cup}\langle K, A\rangle$, we have $\langle F, A\rangle\widetilde{\cup}\langle K, A\rangle \in \mathcal{F}$. So $\langle K, A\rangle \in \mathcal{F}'$.

(F2) Let $\{\langle H_i, A\rangle : i \in I\}$ be a finite family of soft sets in $\mathcal{F}'$. Since $\mathcal{F}$ is a soft filter, we have $\langle F, A\rangle\widetilde{\cup}(\Cap_{i\in I}\langle H_i, A\rangle) = \Cap_{i\in I}(\langle F, A\rangle\widetilde{\cup}\langle H_i, A\rangle) \in \mathcal{F}$. So $\Cap_{i\in I}\langle H_i, A\rangle \in \mathcal{F}'$.

(F3) Since $\langle F, A\rangle \notin \mathcal{F}$, we have $\Phi_A \notin \mathcal{F}'$.

Thus $\mathcal{F}'$ is a soft filter on $U_A$. Clearly, $\mathcal{F}' \supseteq \mathcal{F}$ and $\langle G, A\rangle \in \mathcal{F}'$ although $\langle G, A\rangle \notin \mathcal{F}$. So $\mathcal{F}' \supset \mathcal{F}$, which contradicts the fact that $\mathcal{F}$ is an ultra soft filter. $\square$

**Theorem 3.3.10.** Let $U_A$ be a non-null soft set and $\mathbf{A}$ be a collection of soft sets in $\mathcal{SS}(U)_A$ which generates a soft filter $\mathcal{F}$ on $U_A$. If for every soft set $\langle F, A\rangle \in \mathcal{SS}(U)_A$ we have either $\langle F, A\rangle \in \mathbf{A}$ or $\langle F, A\rangle^c \in \mathbf{A}$, then $\mathbf{A}$ is an ultra soft filter on $U_A$.

**Proof** Let $\mathcal{F}$ be the soft filter generated by $\mathbf{A}$ and $\mathcal{F}'$ be any ultra soft filter which includes $\mathcal{F}$. Then clearly $\mathcal{F}' \supseteq \mathbf{A}$. Let $\langle F, A\rangle$ be any soft set in $\mathcal{F}'$. Then $\langle F, A\rangle^c \notin \mathbf{A}$, for if $\langle F, A\rangle^c \in \mathbf{A}$ then $\langle F, A\rangle^c \in \mathcal{F}'$ and $\langle F, A\rangle \Cap \langle F, A\rangle^c = \Phi_A \in \mathcal{F}'$. This is a contradiction since $\mathcal{F}'$ is a soft filter. Hence $\langle F, A\rangle \in \mathbf{A}$ and so $\mathcal{F}' \subseteq \mathbf{A}$. So $\mathbf{A} = \mathcal{F}'$, an ultra soft filter. $\square$

**Theorem 3.3.11.** Let $\langle F, A\rangle$ be a soft set in $\mathcal{SS}(U)_A$ and $\mathcal{F}$ be a soft filter on $U_A$. Let $\mathcal{F}_{\langle F,A\rangle} = \{\langle F, A\rangle \Cap \langle G, A\rangle : \langle G, A\rangle \in \mathcal{F}\}$. Then:

(1) $\mathcal{F}_{\langle F,A\rangle}$ is a soft filter on $\langle F, A\rangle$ if and only if all these soft sets are non-null soft sets.

(2) If $\mathcal{F}$ is an ultra soft filter on $U_A$, then $\mathcal{F}_{\langle F,A\rangle}$ is an ultra soft filter on $\langle F, A\rangle$ if and only if $\langle F, A\rangle \in \mathcal{F}$.

**Proof** (1) Suppose $\mathcal{F}_{\langle F,A\rangle}$ is a soft filter on $\langle F, A\rangle$. Then all the soft sets in $\mathcal{F}_{\langle F,A\rangle}$ are non-null soft sets. Conversely, suppose all soft sets in $\mathcal{F}_{\langle F,A\rangle}$ are non-null soft sets.

(F1) Let $\langle G, A\rangle \Cap \langle F, A\rangle \in \mathcal{F}_{\langle F,A\rangle}$ and $\langle H, A\rangle$ be a soft subset of $\langle F, A\rangle$ such that $\langle H, A\rangle \widetilde{\supseteq} \langle G, A\rangle \Cap \langle F, A\rangle$. Then we have

$$\begin{aligned}\langle H, A\rangle &= \langle H, A\rangle \widetilde{\cup} (\langle G, A\rangle \Cap \langle F, A\rangle) = (\langle H, A\rangle \widetilde{\cup} \langle G, A\rangle) \Cap (\langle H, A\rangle \widetilde{\cup} \langle F, A\rangle) \\ &= (\langle H, A\rangle \widetilde{\cup} \langle G, A\rangle) \Cap \langle F, A\rangle \in \mathcal{F}_{\langle F,A\rangle}\end{aligned}$$

since $\langle H, A\rangle \widetilde{\cup} \langle G, A\rangle \in \mathcal{F}$ (because $\langle G, A\rangle \in \mathcal{F}$ and $\langle H, A\rangle \widetilde{\cup} \langle G, A\rangle \widetilde{\supseteq} \langle G, A\rangle$).

(F2) Let $\{\langle G_i, A\rangle \Cap \langle F, A\rangle : i \in I\}$ be a finite family of soft sets in $\mathcal{F}_{\langle F,A\rangle}$. Then we have

$$\Cap_{i\in I}(\langle G_i, A\rangle \Cap \langle F, A\rangle) = (\Cap_{i\in I}\langle G_i, A\rangle) \Cap \langle F, A\rangle \in \mathcal{F}_{\langle F,A\rangle}$$

since $\Cap_{i\in I}\langle G_i, A\rangle \in \mathcal{F}$.

(F3) By hypothesis all the soft sets in $\mathcal{F}_{\langle F,A\rangle}$ are non-null soft sets.

Hence $\mathcal{F}_{\langle F,A\rangle}$ is a soft filter on $\langle F, A\rangle$.

(2) Suppose $\mathcal{F}$ is an ultra soft filter. If $\langle F, A\rangle \in \mathcal{F}$, then $\langle G, A\rangle \Cap \langle F, A\rangle \neq \Phi_A$ for all $\langle G, A\rangle \in \mathcal{F}$. By (1), $\mathcal{F}_{\langle F,A\rangle}$ is a soft filter on $\langle F, A\rangle$. If $\mathcal{F}_{\langle F,A\rangle}$ is not an ultra soft filter on $\langle F, A\rangle$, there is a soft filter $\mathcal{F}'$ on $\langle F, A\rangle$ properly including $\mathcal{F}_{\langle F,A\rangle}$. Let $\langle F', A\rangle$ be a soft subset of $\langle F, A\rangle$ which belongs to $\mathcal{F}'$ but not to $\mathcal{F}_{\langle F,A\rangle}$. Then $\mathcal{F} \cup \{\langle F', A\rangle\}$ is a soft filter on $U_A$ which properly includes $\mathcal{F}$. This is impossible. So $\mathcal{F}_{\langle F,A\rangle}$ is an ultra soft filter.

Conversely, suppose $\mathcal{F}$ is an ultra soft filter and $\mathcal{F}_{\langle F,A\rangle}$ is an ultra soft filter on $\langle F, A\rangle$. If $\langle F, A\rangle \notin \mathcal{F}_{\langle F,A\rangle}$ then $\mathcal{F} \cup \{\langle F, A\rangle\}$ generates a soft filter which properly includes $\mathcal{F}$. This is impossible since $\mathcal{F}$ is an ultra soft filter. So $\langle F, A\rangle \in \mathcal{F}$. □

**Theorem 3.3.12.** Every soft filter $\mathcal{F}$ on non-null soft set $U_A$ is the intersection of the family of ultra soft filters which include $\mathcal{F}$.

**Proof** Let $\langle F, A\rangle \in \mathcal{SS}(U)_A$ be a soft set which does not belong to $\mathcal{F}$. Then for each soft set $\langle G, A\rangle$ in $\mathcal{F}$ we cannot have $\langle G, A\rangle \widetilde{\subseteq} \langle F, A\rangle$ and hence we must have $\langle G, A\rangle \Cap \langle F, A\rangle^c \neq \Phi_A$. So $\mathcal{F} \cup \{\langle F, A\rangle^c\}$ generates a soft filter on $U_A$, which is included in some ultra soft filter $\mathcal{F}_{\langle F,A\rangle}$. Since $\langle F, A\rangle^c \in \mathcal{F}_{\langle F,A\rangle}$ we must have $\langle F, A\rangle \notin \mathcal{F}_{\langle F,A\rangle}$. Thus $\langle F, A\rangle$ does not belong to the intersection of the set of all ultra soft filters which include $\mathcal{F}$. Hence this intersection is just the soft filter $\mathcal{F}$ itself. □

Now, let $(U, \tau, A)$ be a soft topological space and $\mathcal{F}$ be a soft filter on $U_A$. A soft point $x_F$ in $\mathcal{SS}(U)_A$ is said to be a limit or a limit soft point of the soft filter $\mathcal{F}$ and $\mathcal{F}$ is said to converge to $x_F$ or to be convergent to $U_A$ if the neighborhood soft filter $\mathcal{V}(x_F)$ of $x_F$ is included in the soft filter $\mathcal{F}$. If $\mathcal{B}$ is a base for a soft filter on $U_A$ then $x_F$ is a limit of $\mathcal{B}$ and $\mathcal{B}$ converges to $x_F$ if the soft filter generated by $\mathcal{B}$ converges to $x_F$.

**Example 3.3.13.** Let $\tau_1$ and $\tau_2$ be two soft topologies on $U_A$ given in Example 3.2.3. Consider a soft filter $\mathcal{F} = \{\langle F_1, A\rangle, \langle F_3, A\rangle, \langle F_7, A\rangle, \langle F_8, A\rangle, \langle F_9, A\rangle, \langle F_{13}, A\rangle, \langle F_{14}, A\rangle, U_A\}$ on $U_A$. Then $e_{1_F} = \langle e_1, \{h_2\}\rangle$ is a limit of $\mathcal{F}$ for $\tau_2$ but not a limit of $\mathcal{F}$ for $\tau_1$, i.e., $\mathcal{F}$ is convergent to $e_{1_F}$ for $\tau_2$ but not convergent to $U_A$ for $\tau_1$.

In classical (point-set) topology, when topologies are given, it is useful to have a criterion in terms of the filters for determining whether one topology is finer than another. One such criterion for soft topological spaces is the following:

**Theorem 3.3.14.** Let $\tau$ and $\tau'$ be soft topologies on a soft set $U_A$. Then $\tau$ is finer than $\tau'$ if and only if every soft filter $\mathcal{F}$ on $U_A$ which converges to $x_F$ for the soft topology $\tau$ also converges to $x_F$ for the soft topology $\tau'$.

**Proof** Suppose $\tau$ is finer than $\tau'$. Let $\mathcal{F}$ be a soft filter which is $\tau$-convergent to $x_F$. Then $\mathcal{F} \supseteq \mathcal{V}_\tau(x_F)$, the $\tau$-neighborhood soft filter of $x_F$. Since $\tau$ is finer than $\tau'$, every $\tau'$-neighborhood of $x_F$ is a $\tau$-neighborhood. So $\mathcal{F} \supseteq \mathcal{V}_{\tau'}(x_F)$, the $\tau'$-neighborhood soft filter of $x_F$, and hence $\mathcal{F}$ is $\tau'$-convergent to $x_F$.

Conversely, suppose that every soft filter on $U_A$ which is $\tau$-convergent to $x_F$ is also $\tau'$-convergent to $x_F$. Let $\langle G', A\rangle$ be any $\tau'$-soft open set and $x_{F'}$ be any soft point of $\langle G', A\rangle$. Then $\langle G', A\rangle \in \mathcal{V}_{\tau'}(x_{F'})$. Since $\mathcal{V}_\tau(x_{F'})$ is $\tau$-convergent to $x_{F'}$, it follows from our hypothesis that it is $\tau'$-convergent to $x_{F'}$. Thus $\mathcal{V}_\tau(x_{F'}) \supseteq \mathcal{V}_{\tau'}(x_{F'})$ and in particular $\langle G', A\rangle \in \mathcal{V}_\tau(x_{F'})$. Thus $\langle G', A\rangle$ is a $\tau$-neighborhood of each of its soft points and hence by Theorem 3.2.4, $\langle G', A\rangle$ is $\tau$-open. So $\tau' \subseteq \tau$, i.e., $\tau$ is finer than $\tau'$. □

Again let $(U, \tau, A)$ be a soft topological space and $\mathcal{F}$ be a soft filter on $U_A$. A soft point $x_F$ in $U_A$ is said to be an adherent soft point of $\mathcal{F}$ if $x_F$ is an adherent soft point of every soft set in $\mathcal{F}$. The adherence of $\mathcal{F}$, $\mathrm{Adh}(\mathcal{F})$, is the set of all adherent soft points of $\mathcal{F}$; so $\mathrm{Adh}(\mathcal{F}) = \Cap_{\langle F,A\rangle\in\mathcal{F}} \overline{\langle F, A\rangle}$. If $\mathcal{B}$ is a base for a soft filter on $U_A$, then $x_F$ is an adherent soft point of $\mathcal{B}$ if it is an adherent soft point of the soft filter generated by $\mathcal{B}$. The adherence of $\mathcal{B}$, $\mathrm{Adh}(\mathcal{B})$, is the set of its adherent soft points.

**Theorem 3.3.15.** Let $(U, \tau, A)$ be a soft topological space and $\mathcal{B}$ be a base for a soft filter on $U_A$. Then $\mathrm{Adh}(\mathcal{B}) = \Cap_{\langle F,A\rangle \in \mathcal{B}} \overline{\langle F, A\rangle}$.

**Proof** Let $\mathcal{F}$ be the soft filter which $\mathcal{B}$ is a base. Then, according to the definition of the adherence of a soft filter base,

$$\mathrm{Adh}(\mathcal{B}) = \mathrm{Adh}(\mathcal{F}) = \Cap_{\langle F,A\rangle \in \mathcal{F}} \overline{\langle F, A\rangle} \,\widetilde{\subseteq}\, \Cap_{\langle F,A\rangle \in \mathcal{B}} \overline{\langle F, A\rangle}.$$

Let $\langle G, A\rangle$ be any soft set in $\mathcal{F}$. Then there is a soft set $\langle H, A\rangle$ in $\mathcal{B}$ such that $\langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$ and so $\overline{\langle G, A\rangle} \,\widetilde{\supseteq}\, \overline{\langle H, A\rangle} \,\widetilde{\supseteq}\, \Cap_{\langle F,A\rangle \in \mathcal{B}} \overline{\langle F, A\rangle}$. Thus we have $\Cap_{\langle F,A\rangle \in \mathcal{F}} \overline{\langle F, A\rangle} \,\widetilde{\supseteq}\, \Cap_{\langle F,A\rangle \in \mathcal{B}} \overline{\langle F, A\rangle}$. Hence $\Cap_{\langle F,A\rangle \in \mathcal{F}} \overline{\langle F, A\rangle} = \Cap_{\langle F,A\rangle \in \mathcal{B}} \overline{\langle F, A\rangle}$. □

**Theorem 3.3.16.** Let $(U, \tau, A)$ be a soft topological space and $\langle G, A\rangle$ be a soft set in $\mathcal{SS}(U)_A$. Then a soft point $x_F$ in $U_A$ is adherent to $\langle G, A\rangle$ if and only if there is a soft filter $\mathcal{F}$ on $U_A$ such that $\langle G, A\rangle \in \mathcal{F}$ and $\mathcal{F}$ converges to $x_F$.

**Proof** Suppose $x_F$ is adherent to $\langle G, A\rangle$. Then every neighborhood $\langle H, A\rangle$ of $x_F$ meets $\langle G, A\rangle$, i.e., $\langle H, A\rangle \Cap \langle G, A\rangle \neq \Phi_A$. Thus $\mathcal{V}_\tau(x_F) \cup \langle G, A\rangle$, where $\mathcal{V}_\tau(x_F)$ is the neighborhood soft filter of $x_F$, generates a soft filter which contains $\langle G, A\rangle$ and is convergent to $x_F$.

Conversely, suppose there is a soft filter $\mathcal{F}$ such that $\langle G, A\rangle \in \mathcal{F}$ and $\mathcal{F}$ is convergent to $x_F$. Let $\langle H, A\rangle$ be any neighborhood of $x_F$. Then $\langle H, A\rangle \in \mathcal{F}$, and since $\langle G, A\rangle \in \mathcal{F}$ it follows that $\langle G, A\rangle \Cap \langle H, A\rangle \neq \Phi_A$. So $x_F$ is adherent to $\langle G, A\rangle$. □

**Theorem 3.3.17.** Let $(U, \tau, A)$ be a soft topological space and $\mathcal{B}$ be a base for a soft filter on $U_A$. Let $x_F$ be a soft point in $U_A$ and $\mathcal{N}$ be a neighborhood base of $x_F$. Then:

(1) $x_F$ is a limit soft point of $\mathcal{B}$ if and only if every soft set in $\mathcal{N}$ includes a soft set in $\mathcal{B}$.

(2) $x_F$ is an adherent soft point of $\mathcal{B}$ if and only if every soft set in $\mathcal{N}$ meets every soft set in $\mathcal{B}$.

**Proof** (1) Suppose $x_F$ is a limit soft point of $\mathcal{B}$. Thus the soft filter $\mathcal{F}$ generated by $\mathcal{B}$ converges to $x_F$. Let $\langle G, A\rangle$ be any soft set in $\mathcal{N}$. Then $\langle G, A\rangle \in \mathcal{F}$. Hence $\langle G, A\rangle$ includes a soft set in $\mathcal{B}$.

Conversely, suppose every soft set in $\mathcal{N}$ includes a soft set in $\mathcal{B}$. Let $\langle G, A\rangle$ be any neighborhood of $x_F$. Then $\langle G, A\rangle$ includes a soft set in $\mathcal{N}$ and hence a soft set in $\mathcal{B}$. So $\langle G, A\rangle$ belongs to the soft filter $\mathcal{F}$ generated by $\mathcal{B}$. Hence $\mathcal{F}$ (and so $\mathcal{B}$) converges to $x_F$.

(2) Suppose $x_F$ is an adherent soft point of $\mathcal{B}$. Then $x_F$ is adherent to every soft set in the soft filter generated by $\mathcal{B}$. So every neighborhood of $x_F$ meets every soft set in that soft filter. Since every soft set in $\mathcal{N}$ is a neighborhood of $x_F$ and every soft set in $\mathcal{B}$ belongs to the soft filter, it follows that every soft set in $\mathcal{N}$ meets every soft set in $\mathcal{B}$.

Conversely, suppose every soft set in $\mathcal{N}$ meets every soft set in $\mathcal{B}$. Let $\langle G, A\rangle$ be any neighborhood of $x_F$ and $\langle H, A\rangle$ be any soft set in the filter generated by $\mathcal{B}$. Then $\langle G, A\rangle$ includes a soft set $\langle K, A\rangle$ in $\mathcal{N}$ and $\langle H, A\rangle$ includes a soft set $\langle L, A\rangle$ in $\mathcal{B}$. Since $\langle K, A\rangle \Cap \langle L, A\rangle \neq \Phi_A$, it follows that $\langle G, A\rangle \Cap \langle H, A\rangle \neq \Phi_A$. So $x_F$ is adherent to $\mathcal{B}$. □

The following corollary is simple consequence.

**Corollary 3.3.18.** Let $(U, \tau, A)$ be a soft topological space and $\mathcal{F}$ be a soft filter on $U_A$. Then:

(1) A soft set $x_F$ is adherent to a soft filter $\mathcal{F}$ if and only if there is a soft filter $\mathcal{F}'$ which includes $\mathcal{F}$ and converges to $x_F$.

(2) Every limit soft point of a soft filter $\mathcal{F}$ is adherent to $\mathcal{F}$.

(3) Every adherent soft point of an ultra soft filter $\mathcal{F}$ is a limit soft point of $\mathcal{F}$.

**Proof** (1) Suppose $x_F$ is adherent to $\mathcal{F}$. Then every soft set in $\mathcal{V}_\tau(x_F)$, the neighborhood filter of $x_F$, meets every soft set in $\mathcal{F}$. Hence $\mathcal{F} \cup \mathcal{V}_\tau(x_F)$ generates a soft filter $\mathcal{F}'$ on $U_A$. Clearly, $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F}'$ converges to $x_F$.

Conversely, suppose $\mathcal{F} \subseteq \mathcal{F}'$ where $\mathcal{F}'$ is a soft filter which converges to $x_F$. Then every neighborhood of $x_F$ belongs to $\mathcal{F}'$. Since every soft set in $\mathcal{F}$ also belongs to $\mathcal{F}'$, it follows that every soft set in $\mathcal{F}$ meets every neighborhood of $x_F$. So $x_F$ is adherent to $\mathcal{F}$.

(2) If $x_F$ is a limit soft point of a soft filter $\mathcal{F}$ then $\mathcal{V}_\tau(x_F) \subseteq \mathcal{F}$, where $\mathcal{V}_\tau(x_F)$ is neighborhood filter of $x_F$, and so every neighborhood of $x_F$ meets every soft set in $\mathcal{F}$. So $x_F$ is adherent to $\mathcal{F}$.

(3) If $x_F$ is adherent to an ultra soft filter $\mathcal{F}$ then a soft filter $\mathcal{F}'$ such that $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{F}'$ converges to $x_F$. But, since $\mathcal{F}$ is an ultra soft filter, $\mathcal{F}' = \mathcal{F}$. So $\mathcal{F}$ converges to $x_F$. □

## 3.4 Soft separation axioms and soft continuity

**Definition 3.4.1.** Let $(U, \tau, A)$ be a soft topological space. Then

(1) $(U, \tau, A)$ is said to be $T_1$ if for every pair of distinct soft points $x_F, y_G$ of $U_A$, there exists a neighborhood of each which does not contain the other;

(2) $(U, \tau, A)$ is said to be $T_2$ if for every pair of distinct soft points $x_F, y_G$ of $U_A$, there exist disjoint neighborhoods of $x_F$ and $y_G$.

**Theorem 3.4.2.** For a soft topological space $(U, \tau, A)$, the following are equivalent:

(1) $(U, \tau, A)$ is $T_1$;

(2) For every soft point $x_F$ of $U_A$, the soft set $\{x_F\}$ is soft closed;

(3) For every soft point $x_F$ of $U_A$, the intersection of the neighborhood soft filter of $x_F$ is $\{x_F\}$.

**Proof** (1)⇒(2): Let $x_F$ be any soft point of $U_A$. We claim that $\overline{\{x_F\}} = \{x_F\}$. If $y_G$ is any soft point of $U_A$ distinct from $x_F$, there is a soft open set $\langle G, A\rangle$ containing $y_G$ which does not contain $x_F$. Thus $y_G \widetilde{\notin} \overline{\{x_F\}}$. It follows that $\overline{\{x_F\}} = \{x_F\}$ and so $\{x_F\}$ is soft closed.

(2)⇒(3): Suppose that for every soft point $y_G$ of $U_A$, the soft set $\{y_G\}$ is soft closed. Let $x_F$ be any soft point of $U_A$ and $y_G$ be any soft point of $\widetilde{\cup}\mathcal{N}(x_F)$, where $\mathcal{N}(x_F)$ is the neighborhood soft filter of $x_F$. Then every neighborhood of

$x_F$ contains $y_G$ and so meets $\{y_G\}$. Thus $x_F \widetilde{\in} \overline{\{y_G\}} = \{y_G\}$. So $x_F = y_G$. Hence $\Cap \mathcal{N}(x_F) = \{x_F\}$.

(3)⇒(4): Suppose that for every soft point $y_G$ of $U_A$, we have $\Cap \mathcal{N}(y_G) = \{y_G\}$. Let $x_F$ and $y_G$ be distinct soft points of $U_A$. Since $y_G \widetilde{\notin} \{x_F\} = \Cap \mathcal{N}(x_F)$, there is a neighborhood of $x_F$ which does not contain $y_G$. Similarly, there is a neighborhood of $y_G$ which does not contain $x_F$. Thus $(U, \tau, A)$ is $T_1$. □

**Theorem 3.4.3.** For a soft topological space $(U, \tau, A)$, the following are equivalent:

(1) $(U, \tau, A)$ is $T_2$;

(2) For every soft point $x_F$ of $U_A$, the intersection of the family of soft closed neighborhoods of $x_F$ is $\{x_F\}$;

(3) If a soft filter $\mathcal{F}$ on $U_A$ converges to a soft point $x_F$, then $x_F$ is the only adherent soft point of $\mathcal{F}$;

(4) A soft filter $\mathcal{F}$ on $U_A$ can have at most one limit soft point.

**Proof** (1)⇒(2): Let $x_F$ be any soft point of $U_A$ and $y_G$ be any soft point distinct from $x_F$. By (a), there exist disjoint neighborhoods $\langle F, A\rangle$ of $x_F$ and $\langle G, A\rangle$ of $y_G$. Since $\langle F, A\rangle \Cap \langle G, A\rangle = \Phi_A$, it follows that $y_G \widetilde{\notin} \overline{\langle F, A\rangle}$, which is a soft closed neighborhood of $x_F$. Thus $y_G$ is not in every soft closed neighborhood of $x_F$. So the intersection of the family of soft closed neighborhoods of $x_F$ is $\{x_F\}$.

(2)⇒(3): Let $\mathcal{F}$ be a soft filter on $U_A$ which converges to $x_F$, i.e., $\mathcal{F} \supseteq \mathcal{N}(x_F)$, the neighborhood soft filter of $x_F$. If $y_G$ is adherent to $\mathcal{F}$, then $y_G$ belongs to the soft closure of neighborhood of $x_F$ and hence to every soft closed neighborhood of $x_F$. Thus $y_G = x_F$, i.e., $x_F$ is the only adherent soft point of $\mathcal{F}$.

(3)⇒(4): Let $\mathcal{F}$ be a soft filter on $U_A$. If $x_F$ and $y_G$ are limit soft points of $\mathcal{F}$, they are also adherent soft points and so, by (3), $x_F = y_G$.

(4)⇒(1): If $(U, \tau, A)$ is not $T_2$, then there is a pair of distinct soft points $x_F$ and $y_G$ such that every neighborhood of $x_F$ meets every neighborhood of $y_G$. Thus $\mathcal{N}(x_F) \cup \mathcal{N}(y_G)$ generates a soft filter $\mathcal{F}$ which converges to both $x_F$ and $y_G$. This is a contradiction. Hence $(U, \tau, A)$ is $T_2$. □

Now, we review the following notions to introduce the soft continuous.

Let $\mathcal{SS}(U)_A$ and $\mathcal{SS}(V)_B$ be families of soft sets. Let $u : U \to V$ and $p : A \to B$ be functions. Then a function $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ is defined as

(1) Let $\langle F, A\rangle \in \mathcal{SS}(U)_A$. The image of $\langle F, A\rangle$ under $f_{pu}$ [31], written as $f_{pu}(\langle F, A\rangle) = \langle f_{pu}(F), p(A)\rangle$, is a soft set in $\mathcal{SS}(V)_B$ such that

$$f_{pu}(F)(y) = \begin{cases} \cup_{x\in p^{-1}(y)\cap A} u(F(x)), & p^{-1}(y) \cap A \neq \emptyset, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $y \in B$.

(2) Let $\langle G, B\rangle \in \mathcal{SS}(V)_B$. The inverse image of $\langle G, B\rangle$ under $f_{pu}$ [31], written as $f^{-1}_{pu}(\langle G, B\rangle) = \langle f^{-1}_{pu}(G), p^{-1}(B)\rangle$, is a soft set in $\mathcal{SS}(U)_A$ such that

$$f^{-1}_{pu}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $x \in A$.

The function $f_{pu}$ is called surjective [68] iff $p$ and $u$ are surjective and is called injective [68] iff $p$ and $u$ are injective.

**Proposition 3.4.4.** [31, 68] Let $\mathcal{SS}(U)_A$ and $\mathcal{SS}(V)_B$ be families of soft sets. For a function $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$, the following are true:

(1) $f_{pu}(\Phi_A) = \Phi_B$, $f_{pu}(U_A)\widetilde{\subseteq}V_B$.

(2) $f_{pu}(\langle F, A\rangle\widetilde{\cup}\langle G, A\rangle) = f_{pu}(\langle F, A\rangle)\widetilde{\cup}f_{pu}(\langle G, A\rangle)$, where $\langle F, A\rangle, \langle G, A\rangle \in \mathcal{SS}$ $(U)_A$.

(3) If $\langle F, A\rangle\widetilde{\subseteq}\langle G, A\rangle \in \mathcal{SS}(U)_A$, then $f_{pu}(\langle F, A\rangle)\widetilde{\subseteq}f_{pu}(\langle G, A\rangle)$.

(4) If $\langle H, B\rangle\widetilde{\subseteq}\langle K, B\rangle \in \mathcal{SS}(V)_B$, then $f^{-1}_{pu}(\langle H, B\rangle)\widetilde{\subseteq}f^{-1}_{pu}(\langle K, B\rangle)$.

(5) $f^{-1}_{pu}(\langle H, B\rangle^c) = (f^{-1}_{pu}(\langle H, B\rangle))^c$ for any $\langle H, B\rangle \in \mathcal{SS}(V)_B$.

(6) $f_{pu}(f^{-1}_{pu}(\langle H, B\rangle))\widetilde{\subseteq}\langle H, B\rangle$ for any $\langle H, B\rangle \in \mathcal{SS}(V)_B$. If $f_{pu}$ is surjective, the equality holds.

(7) $\langle F, A\rangle\widetilde{\subseteq}f^{-1}_{pu}(f_{pu}(\langle F, A\rangle))$ for any $\langle F, A\rangle \in \mathcal{SS}(U)_A$. If $f_{pu}$ is injective, the equality holds.

Let $(U, \tau, A)$ and $(V, \tau^*, B)$ be soft topological spaces and $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ be a function, where $u : U \to V$ and $p : A \to B$ are two functions. Then $f_{pu}$ is soft continuous [68] if $f_{pu}^{-1}(\langle H, B\rangle) \in \tau$ for each $\langle H, B\rangle \in \tau^*$.

**Theorem 3.4.5.** Let $(U, \tau, A)$ and $(V, \tau^*, B)$ be soft topological spaces. For a function $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$, the following are equivalent:

(1) $f_{pu}$ is soft $pu$-continuous.

(2) For each soft closed set $\langle H, B\rangle$ of $V_B$, $f_{pu}^{-1}(\langle H, B\rangle)$ is soft closed of $U_A$.

(3) For each soft set $\langle H, B\rangle \in \mathcal{SS}(V)_B$, $\overline{f_{pu}^{-1}(\langle H, B\rangle)} \widetilde{\subseteq} f_{pu}^{-1}(\overline{\langle H, B\rangle})$.

(4) For each soft set $\langle F, A\rangle \in \mathcal{SS}(U)_A$, $f_{pu}(\overline{\langle F, A\rangle}) \widetilde{\subseteq} \overline{f_{pu}(\langle F, A\rangle)}$.

**Proof** (1)⇔(2) is proved in [68].

(2)⇒(3) Let $\langle H, B\rangle \in \mathcal{SS}(V)_B$. Then $\overline{\langle H, B\rangle}$ is soft closed set of $V_B$ and by (2) $f_{pu}^{-1}(\overline{\langle H, B\rangle})$ is soft closed set of $U_A$ containing $f_{pu}^{-1}(\langle H, B\rangle)$. Hence, by Theorem 1(3) and (5) of [55], $\overline{f_{pu}^{-1}(\langle H, B\rangle)} \widetilde{\subseteq} \overline{f_{pu}^{-1}(\overline{\langle H, B\rangle})} = f_{pu}^{-1}(\overline{\langle H, B\rangle})$.

(3)⇒(2) Let $\langle H, B\rangle$ be a soft closed set of $V_B$. Then by Theorem 1 of [55], $\overline{\langle H, B\rangle} = \langle H, B\rangle$ and so by (3), $\overline{f_{pu}^{-1}(\langle H, B\rangle)} \widetilde{\subseteq} f_{pu}^{-1}(\overline{\langle H, B\rangle}) = f^{-1}(\langle H, B\rangle)$. Hence $f^{-1}(\langle H, B\rangle)$ is soft closed of $U_A$.

(3)⇔(4) follows from Proposition 3.4.4. □

## 3.5 Conclusions

Many scholars have grafted the soft set theory onto some areas of mathematics, in particular algebra structures. Topology is a major area of mathematics. In this chapter, we have attempted to conduct a further study of soft topology along the work of Çağman et al. [12]. First, we have presented soft analogues of many results concerning neighborhoods and closures in ordinary topological spaces. Next, we have defined the soft filter on a soft set and have presented its related properties. Its convergence and adherence in the soft topology have been discussed. Consequently, we have used the soft filter as an effective tool for studying on soft topological structures. To extend our work, further research could be done to study the interconnection between soft filters and soft separation axioms.

# Chapter 4

# Soft proximity spaces

In this chapter, we define the soft proximity on a soft set, and present its related properties. The concepts of $\delta$-neighborhood, soft proximally continuity and soft cluster are discussed. They furnish approaches to the study of sot proximity spaces.

## 4.1 Soft proximity

**Definition 4.1.1.** A soft proximity on $U_A$ is a binary relation $\delta$ on $\mathcal{SS}(U)_A$ satisfying the following properties: for any $\langle F, A\rangle, \langle G, A\rangle, \langle H, A\rangle \in \mathcal{SS}(U)_A$,

(SP1) $\langle F, A\rangle \delta \langle G, A\rangle$ implies $\langle G, A\rangle \delta \langle F, A\rangle$;

(SP2) $(\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)\delta\langle H, A\rangle$ if and only if $\langle F, A\rangle\delta\langle H, A\rangle$ or $\langle G, A\rangle\delta\langle H, A\rangle$;

(SP3) $\langle F, A\rangle\delta\langle G, A\rangle$ implies $\langle F, A\rangle \neq \Phi_A$ and $\langle G, A\rangle \neq \Phi_A$;

(SP4) $\langle F, A\rangle \not\delta \langle G, A\rangle$ implies that there exists a soft set $\langle H, A\rangle \in \mathcal{SS}(U)_A$ such that $\langle F, A\rangle \not\delta \langle H, A\rangle$ and $\langle H, A\rangle^c \not\delta \langle G, A\rangle$;

(SP5) $\langle F, A\rangle \Cap \langle G, A\rangle \neq \Phi_A$ implies $\langle F, A\rangle\delta\langle G, A\rangle$.

The pair $(U_A, \delta)$ is called a soft proximity space. The phrase $\langle F, A\rangle\delta\langle G, A\rangle$ is read '$\langle F, A\rangle$ is near $\langle G, A\rangle$' and '$\langle F, A\rangle$ is not near $\langle G, A\rangle$' is denoted by $\langle F, A\rangle \not\delta \langle G, A\rangle$.

**Example 4.1.2.** Just as discrete and indiscrete soft topologies can be defined on any soft set, we have discrete and indiscrete soft proximities. If we define

$\langle F, A\rangle \delta_1 \langle G, A\rangle$ iff $\langle F, A\rangle \Cap \langle G, A\rangle \neq \Phi_A$, then $\delta_1$ is the discrete soft proximity on $U_A$. On the other hand, if $\langle F, A\rangle \delta_2 \langle G, A\rangle$ for every pair of non-null soft sets $\langle F, A\rangle$ and $\langle G, A\rangle$ in $\mathcal{SS}(U)_A$, then we obtain the indiscrete soft proximity on $U_A$.

Properties of the following lemma, which follows directly from definition, are useful in several proofs.

**Lemma 4.1.3.** Let $(U_A, \delta)$ be a soft proximity space and $\langle F, A\rangle, \langle G, A\rangle \in \mathcal{SS}(U)_A$. Then the following properties hold:

(1) If $\langle F, A\rangle \delta \langle G, A\rangle$, $\langle F, A\rangle \widetilde{\subseteq} \langle F_1, A\rangle \in \mathcal{SS}(U)_A$ and $\langle G, A\rangle \widetilde{\subseteq} \langle G_1, A\rangle \in \mathcal{SS}(U)_A$, then $\langle F_1, A\rangle \delta \langle G_1, A\rangle$. Hence $U_A \delta \langle F, A\rangle$ for every $\langle F, A\rangle \neq \Phi_A$.

(2) If there exists a soft point $x_F \widetilde{\in} U_A$ such that $\langle F, A\rangle \delta x_F$ and $x_F \delta \langle G, A\rangle$, then $\langle F, A\rangle \delta \langle G, A\rangle$.

**Proof** (1) Let $\langle F_1, A\rangle = \langle F, A\rangle \widetilde{\cup} \langle H, A\rangle$ for some $\langle H, A\rangle \in \mathcal{SS}(U)_A$. Then $\langle F_1, A\rangle \delta \langle H, A\rangle$ by (SP2). Applying (SP2) once more, we obtain $\langle F_1, A\rangle \delta \langle G_1, A\rangle$.

(2) If $\langle F, A\rangle \not\delta \langle G, A\rangle$, by (SP4) there exists a soft set $\langle H, A\rangle$ such that $\langle F, A\rangle \not\delta \langle H, A\rangle$ and $\langle H, A\rangle^c \not\delta \langle G, A\rangle$. But since either $x_F \widetilde{\in} \langle H, A\rangle$ or $x_F \widetilde{\in} \langle H, A\rangle^c$, by (1) we have either $\langle F, A\rangle \delta \langle H, A\rangle$ or $\langle H, A\rangle^c \delta \langle G, A\rangle$, a contradiction. □

**Theorem 4.1.4.** If a soft set $\langle F, A\rangle$ of a soft proximity space $(U_A, \delta)$ is defined to be soft closed iff $x_F \delta \langle F, A\rangle$ implies $x_F \widetilde{\in} \langle F, A\rangle$, then the collection of soft complements of all soft closed sets so defined yields a soft topology $\tau = \tau(\delta)$ on $U_A$.

**Proof** Obviously $\Phi_A$ and $U_A$ are soft closed sets. Let $\{\langle F_i, A\rangle : i \in I\}$ be an arbitrary collection of soft closed sets. If $x_F \delta \Cap_{i \in I} \langle F_i, A\rangle$ then by Lemma 4.1.3, $x_F \delta \langle F_i, A\rangle$ for each $i \in I$, and so $x_F \widetilde{\in} \langle F_i, A\rangle$ for each $i \in I$ since $\langle F_i, A\rangle$ is soft closed. Thus $x_F \widetilde{\in} \Cap_{i \in I} \langle F_i, A\rangle$, which means $\Cap_{i \in I} \langle F_i, A\rangle$ is soft closed. Finally, if $\langle F_1, A\rangle$ and $\langle F_2, A\rangle$ are soft closed and $x_F \delta(\langle F_1, A\rangle \widetilde{\cup} \langle F_2, A\rangle)$ then by (SP2), either $x_F \delta \langle F_1, A\rangle$ or $x_F \delta \langle F_2, A\rangle$. But since $\langle F_1, A\rangle$ and $\langle F_2, A\rangle$ are soft closed, implying that $x_F \widetilde{\in} \langle F_1, A\rangle$ or $x_F \widetilde{\in} \langle F_2, A\rangle$, i.e., $x_F \widetilde{\in} (\langle F_1, A\rangle \widetilde{\cup} \langle F_2, A\rangle)$. Thus $\langle F_1, A\rangle \widetilde{\cup} \langle F_2, A\rangle$ is soft closed. □

**Theorem 4.1.5.** Let $(U_A, \delta)$ be a soft proximity space and $\tau = \tau(\delta)$. Then the $\tau$-soft closure $\overline{\langle F, A\rangle}$ of a soft set $\langle F, A\rangle$ in $\mathcal{SS}(U)_A$ is given by

$$\overline{\langle F, A\rangle} = \{x_F : x_F \delta \langle F, A\rangle\}.$$

**Proof** If $\overline{\langle F, A\rangle}$ denotes the intersection of all soft closed sets containing $\langle F, A\rangle$ and $\langle F, A\rangle^\delta = \{x_F : x_F\delta\langle F, A\rangle\}$, then we must show that $\overline{\langle F, A\rangle} = \langle F, A\rangle^\delta$. If $x_F \widetilde{\in} \langle F, A\rangle^\delta$ then $x_F\delta\langle F, A\rangle$. By Lemma 4.1.3, this implies $x_F\delta\overline{\langle F, A\rangle}$ and, since $\overline{\langle F, A\rangle}$ is soft closed, $x_F\widetilde{\in}\overline{\langle F, A\rangle}$. Thus $\overline{\langle F, A\rangle}\widetilde{\subseteq}\langle F, A\rangle^\delta$. To prove the reverse inclusion it suffices to prove that $\langle F, A\rangle^\delta$ is soft closed, i.e., $x_F\delta\langle F, A\rangle^\delta$ implies $x_F\widetilde{\in}\langle F, A\rangle^\delta$. Assuming $x_F \widetilde{\notin} \langle F, A\rangle^\delta$, then $x_F \not\delta \langle F, A\rangle$ so that, by (SP4), there is a soft set $\langle G, A\rangle \in \mathcal{SS}(U)_A$ such that $x_F \not\delta\langle G, A\rangle$ and $\langle G, A\rangle^c \not\delta \langle F, A\rangle$. Thus $y_F \not\delta\langle F, A\rangle$ any soft point $y_F\widetilde{\in}\langle G, A\rangle^c$, i.e., $\langle F, A\rangle^\delta\widetilde{\subseteq}\langle G, A\rangle$, which together with $x_F \not\delta\langle G, A\rangle$ implies that $x_F \not\delta\langle F, A\rangle^\delta$. □

**Corollary 4.1.6.** If $(U_A, \delta)$ is a soft proximity space and $\langle G, A\rangle$ is a soft set in $\mathcal{SS}(U)_A$, then $\langle G, A\rangle \in \tau(\delta)$ iff $x_F \not\delta\langle G, A\rangle^c$ for every $x_F\widetilde{\in}\langle G, A\rangle$.

**Proof** $\langle G, A\rangle \in \tau(\delta)$ iff $\langle G, A\rangle^c$ is $\tau(\delta)$-closed iff (by [55]) $\overline{\langle G, A\rangle^c} = \langle G, A\rangle^c$ iff $x_F \not\delta\langle G, A\rangle^c$ for every $x_F\widetilde{\in}\langle G, A\rangle$. □

**Corollary 4.1.7.** If $(U_A, \delta)$ is a soft proximity space and $\langle F, A\rangle, \langle G, F\rangle \in \mathcal{SS}(U)_A$, then $\langle F, A\rangle \not\delta\langle G, A\rangle$ implies

(1) $\overline{\langle G, A\rangle}\widetilde{\subseteq}\langle F, A\rangle^c$ and

(2) $\langle G, A\rangle\widetilde{\subseteq}(\langle F, A\rangle^c)^o$,

where the soft closure and soft interior are taken with respect to $\tau(\delta)$.

**Proof** (1) follows from directly from Lemma 3.3.3. To prove (2), we use the identity: $(\langle F, A\rangle^c)^o = (\overline{\langle F, A\rangle})^c$ [68]. Then $x_F \widetilde{\notin}(\langle F, A\rangle^c)^o$ implies $x_F\widetilde{\in}\overline{\langle F, A\rangle}$, so that $x_F\delta\langle F, A\rangle$ and hence $x_F \widetilde{\notin}(G, A)$. □

**Remark 4.1.8.** (1) Theorem 4.1.5 is true if we omit the axiom (SP1) and add the following condition:

$$\langle F, A\rangle\delta[\langle G, A\rangle\widetilde{\cup}\langle H, A\rangle] \text{ iff } \langle F, A\rangle\delta\langle G, A\rangle \text{ or } \langle F, A\rangle\delta\langle H, A\rangle. \tag{4.1}$$

(2) An alternative method of introducing the same soft topology on a soft proximity space $(U_A, \delta)$ would be define for each soft set $\langle F, A\rangle$ in $\mathcal{SS}(U)_A$,

$$\langle F, A\rangle^{\delta} = \{x_F : x_F \delta (F, A)\} \tag{4.2}$$

and show that $^{\delta}$ is Kuratowski closure operator as follows:

(a) By (SP3), $x_F \not\delta \Phi_A$ implies $\Phi_A = (\Phi_A)^{\delta}$.

(b) By (SP5), $x_F \widetilde{\in} \langle F, A\rangle$ implies $x_F \delta \langle F, A\rangle$, so that $\langle F, A\rangle \widetilde{\subseteq} \langle F, A\rangle^{\delta}$.

(c) By (SP2), then $x_F \widetilde{\in} (\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)^{\delta}$ iff $x_F \delta (\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)$ iff $x_F \delta \langle F, A\rangle$ or $x_F \delta \langle G, A\rangle$ iff $x_F \widetilde{\in} \langle F, A\rangle^{\delta}$ or $x_F \widetilde{\in} \langle G, A\rangle^{\delta}$ iff $x_F \widetilde{\in} (\langle F, A\rangle^{\delta} \widetilde{\cup} \langle G, A\rangle^{\delta})$. So $(\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)^{\delta} = (\langle F, A\rangle^{\delta} \widetilde{\cup} \langle G, A\rangle^{\delta})$.

(d) To prove $(\langle F, A\rangle^{\delta})^{\delta} \widetilde{\subseteq} \langle F, A\rangle^{\delta}$, suppose $x_F \widetilde{\notin} \langle F, A\rangle$, i.e., $x_F \not\delta \langle F, A\rangle^{\delta}$. Then by (SP4), there exists an soft set $\langle H, A\rangle$ such that $x_F \not\delta \langle H, A\rangle$ and $\langle H, A\rangle^c \not\delta \langle F, A\rangle$. Now $\langle F, A\rangle^{\delta} \widetilde{\subseteq} \langle H, A\rangle$ and $x_F \not\delta \langle H, A\rangle$, so that $x_F \not\delta \langle F, A\rangle^{\delta}$ and $x_F \widetilde{\notin} (\langle F, A\rangle^{\delta})^{\delta}$.

**Lemma 4.1.9.** For soft subsets $\langle F, A\rangle$ and $\langle G, A\rangle$ of a soft proximity space $(U_A, \delta)$,

$$\langle F, A\rangle \delta \langle G, A\rangle \text{ iff } \overline{\langle F, A\rangle} \delta \overline{\langle G, A\rangle},$$

where the soft closure is taken with respect to $\tau(\delta)$.

**Proof** Necessity follows from Lemma 4.1.3. To prove sufficiency, suppose $\langle F, A\rangle \not\delta \langle G, A\rangle$. Then by (SP4), there exists a soft set $\langle H, A\rangle$ in $\mathcal{SS}(U)_A$ such that $\langle F, A\rangle \not\delta \langle H, A\rangle$ and $\langle H, A\rangle^c \not\delta \langle G, A\rangle$. By Corollary 4.1.7, $\overline{\langle G, A\rangle} \widetilde{\subseteq} \langle H, A\rangle$ and by Lemma 4.1.3, $\langle F, A\rangle \not\delta \langle H, A\rangle$ implies $\langle F, A\rangle \not\delta \overline{\langle G, A\rangle}$. It then follows from (SP1) that $\overline{\langle F, A\rangle} \not\delta \overline{\langle G, A\rangle}$. □

Just as the class of soft topologies on a given soft set can be partially ordered by soft inclusion, one can impose a partial order on the class of soft proximities defined on a soft set in the following manner:

**Definition 4.1.10.** If $\delta_1$ and $\delta_2$ are two soft proximities on a soft set $U_A$, we define

$$\delta_1 > \delta_2 \text{ iff } \langle F, A\rangle \delta_1 \langle G, A\rangle \text{ implies } \langle F, A\rangle \delta_2 \langle G, A\rangle. \tag{4.3}$$

The above is expressed by saying that $\delta_1$ is finer than $\delta_2$, or $\delta_2$ is coarser than $\delta_1$.

The following theorem shows that a finer soft proximity structure induces a finer soft topology:

**Theorem 4.1.11.** Let $\delta_1$ and $\delta_2$ be two soft proximities defined on a soft set $U_A$. Then $\delta_1 < \delta_2$ implies $\tau(\delta_1) \subseteq \tau(\delta_2)$.

**Proof** Suppose $\langle F, A\rangle \in \tau(\delta_1)$. Then by Corollary 4.1.6, $x_F \not\delta_1 \langle F, A\rangle^c$ for each $x_F \widetilde{\in} \langle F, A\rangle$. Moreover, since $\delta_1 < \delta_2$, $x_F \not\delta_2 \langle F, A\rangle^c$ for each $x_F \widetilde{\in} \langle F, A\rangle$. Thus $\langle F, A\rangle \in \tau(\delta_2)$, from which we conclude that $\tau(\delta_1) \subseteq \tau(\delta_2)$. □

Given a soft topological space $(U, \tau, A)$, a soft set $\langle G, A\rangle$ is said to be a neighborhood [68] of a soft set $\langle F, A\rangle$ iff there exists a $\langle H, A\rangle \in \tau$ such that $\langle F, A\rangle \widetilde{\subseteq} \langle H, A\rangle \widetilde{\subseteq} \langle G, A\rangle$. An analogous concept, that of a $\delta$-neighborhood, can be introduced in a soft proximity space and furnishes an alternative approach to the study of sot proximity spaces.

**Definition 4.1.12.** A soft subset $\langle G, A\rangle$ of a soft proximity space $(U_A, \delta)$ is a $\delta$-neighborhood of a soft set $\langle F, A\rangle$ (in symbols $\langle F, A\rangle \ll \langle G, A\rangle$) iff $\langle F, A\rangle \not\delta \langle G, A\rangle^c$.

The second part of the following lemma (which is a strengthened form of Corollary 4.1.7) justifies the term '$\delta$-neighborhood'.

**Lemma 4.1.13.** Let $(U_A, \delta)$ be a soft proximity space and $\overline{\langle F, A\rangle}$ and $\langle F, A\rangle^o$ denote, respectively, the soft closure and soft interior of $\langle F, A\rangle$ in $\tau(\delta)$. Then

(1) $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\overline{\langle F, A\rangle} \ll \langle G, A\rangle$, and

(2) $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\langle F, A\rangle \ll \langle G, A\rangle^o$.

Therefore $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle^o$, showing that a $\delta$-neighborhood is a soft topological neighborhood.

**Proof** (1) Using Lemma 4.1.9, $\langle F, A\rangle \not\delta \langle G, A\rangle^c$ implies $\overline{\langle F, A\rangle} \not\delta \langle G, A\rangle^c$, i.e., $\overline{\langle F, A\rangle} \ll \langle G, A\rangle$.

(2) Using Lemma 4.1.9, $\langle F, A\rangle \not\delta \langle G, A\rangle^c$ implies $\langle F, A\rangle \not\delta \overline{\langle G, A\rangle^c}$. By Lemma 3.2.9, equivalently, $\langle F, A\rangle \not\delta (\langle G, A\rangle^o)^c$, i.e., $\langle F, A\rangle \ll \langle G, A\rangle^o$. □

**Lemma 4.1.14.** Let $(U_A, \delta)$ be a soft proximity space. Then the axiom (SP4) is equivalent to

$$\langle F, A\rangle \not\delta \langle G, A\rangle \text{ implies there exist soft sets } \langle \mathrm{H, A}\rangle \text{ and } \langle \mathrm{K, A}\rangle \text{ in } \mathcal{SS}(\mathrm{U})_{\mathrm{A}}$$
$$\text{such that } \langle \mathrm{F, A}\rangle \ll \langle \mathrm{H, A}\rangle, \langle \mathrm{G, A}\rangle \ll \langle \mathrm{K, A}\rangle \text{ and } \langle \mathrm{H, A}\rangle \not\delta \langle \mathrm{K, A}\rangle. \quad (4.4)$$

**Proof** To prove (4.4) implies (SP4), we note that if $\langle H, A\rangle \not\delta \langle K, A\rangle$ then $\langle H, A\rangle \tilde{\subseteq} \langle K, A\rangle^c$ by (SP5) and Proposition 3.1.4(4). Setting $\langle S, A\rangle = \langle H, A\rangle^c$, we have $\langle F, A\rangle \not\delta \langle S, A\rangle$ and $\langle S, A\rangle^c \not\delta \langle G, A\rangle$. On the other hand, suppose (SP4) holds. Then $\langle F, A\rangle \not\delta \langle G, A\rangle$ implies there is a soft set $\langle K, A\rangle$ such that $\langle F, A\rangle \not\delta \langle K, A\rangle$ and $\langle K, A\rangle^c \not\delta \langle G, A\rangle$. Moreover, there exists a soft set $\langle H, A\rangle$ such that $\langle F, A\rangle \not\delta \langle H, A\rangle^c$ and $\langle H, A\rangle \not\delta \langle K, A\rangle$. Thus we have $\langle F, A\rangle \ll \langle H, A\rangle$ and $\langle G, A\rangle \ll \langle K, A\rangle$. $\square$

**Theorem 4.1.15.** Given a soft proximity space $(U_A, \delta)$, the relation $\ll$ satisfies the following properties:

(1) $U_A \ll U_A$.

(2) $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\langle F, A\rangle \tilde{\subseteq} \langle G, A\rangle$. The converse holds if $(U_A, \delta)$ is discrete.

(3) $\langle F, A\rangle \tilde{\subseteq} \langle G, A\rangle \ll \langle H, A\rangle \tilde{\subseteq} \langle K, A\rangle$ implies $\langle F, A\rangle \ll \langle K, A\rangle$.

(4) $\langle F, A\rangle \ll \langle G_i, A\rangle$ for $i = 1, 2, \ldots, n$ iff $\langle F, A\rangle \ll \tilde{\cap}_{i=1}^{n} \langle G_i, A\rangle$.

(5) $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\langle G, A\rangle^c \ll \langle F, A\rangle^c$.

(6) $\langle F, A\rangle \ll \langle G, A\rangle$ implies there is a soft set $\langle H, A\rangle$ such that $\langle F, A\rangle \ll \langle H, A\rangle \ll \langle G, A\rangle$.

**Proof** (1) Since $U_A \not\delta \Phi_A$ by (SP3), $U_A \ll U_A$.

(2) If $\langle F, A\rangle \not\delta \langle G, A\rangle^c$ then $\langle F, A\rangle \Cap \langle G, A\rangle^c = \Phi_A$, implying $\langle F, A\rangle \tilde{\subseteq} \langle G, A\rangle$. The second part is easy consequence of the definition of discrete soft proximity given in Example 4.1.2.

(3) If $\langle F, A\rangle \not\ll \langle K, A\rangle$, then $\langle F, A\rangle \delta \langle K, A\rangle^c$. This implies that $\langle G, A\rangle \delta \langle H, A\rangle^c$ or $\langle G, A\rangle \not\ll \langle H, A\rangle$, a contradiction.

(4) It suffices to consider $n = 2$. $\langle F, A\rangle \ll \langle G_1, A\rangle$ and $\langle F, A\rangle \ll \langle G_2, A\rangle$ iff $\langle F, A\rangle \not\delta \langle G_1, A\rangle^c$ and $\langle F, A\rangle \not\delta \langle G_2, A\rangle^c$ iff (by (SP2)) $\langle F, A\rangle \not\delta [\langle G_1, A\rangle^c \tilde{\cup} \langle G_2, A\rangle^c]$ iff $\langle F, A\rangle \not\delta\ [\langle G_1, A\rangle \Cap \langle G_2, A\rangle]^c$ iff $\langle F, A\rangle \ll (\langle G_1, A\rangle \Cap \langle G_2, A\rangle)$.

(5) $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\langle F, A\rangle \not\delta \langle G, A\rangle^c$. By (SP1), $\langle G, A\rangle^c \not\delta \langle F, A\rangle$, i.e., $\langle G, A\rangle^c \ll \langle F, A\rangle^c$.

(6) $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\langle F, A\rangle \not\delta \langle G, A\rangle^c$. By (SP4), there exists a soft set $\langle H, A\rangle^c$ such that $\langle F, A\rangle \not\delta \langle H, A\rangle^c$ and $\langle H, A\rangle \not\delta \langle G, A\rangle^c$; that is, $\langle F, A\rangle \ll \langle H, A\rangle \ll \langle G, A\rangle$. □

**Corollary 4.1.16.** Let $(U_A, \delta)$ be a soft proximity space. Then $\langle F_i, A\rangle \ll \langle G_i, A\rangle$ for $i = 1, 2, \ldots, n$ implies

$$\Cap_{i=1}^{n} \langle F_i, A\rangle \ll \Cap_{i=1}^{n} \langle G_i, A\rangle \text{ and } \widetilde{\bigcup}_{\mathrm{i=1}}^{\mathrm{n}} \langle \mathrm{F_i, A}\rangle \ll \widetilde{\bigcup}_{\mathrm{i=1}}^{\mathrm{n}} \langle \mathrm{G_i, A}\rangle.$$

Note that (SP4) is equivalent to Theorem 4.1.15(6). The following is a converse of Theorem 4.1.15.

**Theorem 4.1.17.** If $\ll$ is a binary relation on $\mathcal{SS}(U)_A$ satisfying the conditions (1)-(6) of Theorem 4.1.15 and $\delta$ is defined by

$$\langle F, A\rangle \not\delta \langle G, A\rangle \text{ iff } \langle F, A\rangle \ll \langle G, A\rangle^c,$$

then $\delta$ is a soft proximity on $U_A$. Thus $\langle G, A\rangle$ is $\delta$-neighborhood of $\langle F, A\rangle$ iff $\langle F, A\rangle \ll \langle G, A\rangle$.

**Proof** (SP1) $\langle F, A\rangle \not\delta \langle G, A\rangle$ implies $\langle F, A\rangle \ll \langle G, A\rangle^c$. By Theorem 4.1.15(5), $\langle G, A\rangle \ll \langle F, A\rangle^c$, and so $\langle G, A\rangle \not\delta \langle F, A\rangle$.

(SP2) $(\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle) \not\delta \langle H, A\rangle$ implies $(\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle) \ll \langle H, A\rangle^c$. By Theorem 4.1.15(3), $\langle F, A\rangle \ll \langle H, A\rangle^c$ and $\langle G, A\rangle \ll \langle H, A\rangle^c$; that is, $\langle F, A\rangle \not\delta \langle H, A\rangle$ and $\langle G, A\rangle \not\delta \langle H, A\rangle$. Conversely, if $(\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)\delta\langle H, A\rangle$ then by (SP1), $\langle H, A\rangle\delta$ $(\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)$. Hence $\langle H, A\rangle \not\ll (\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle)^c$, or $\langle H, A\rangle \not\ll (\langle F, A\rangle^c \Cap \langle G, A\rangle^c)$. Thus by Theorem 4.1.15(4), $\langle H, A\rangle \not\ll \langle F, A\rangle^c$ or $\langle H, A\rangle \not\ll \langle G, A\rangle^c$. Hence $\langle H, A\rangle\delta\langle F, A\rangle$ or $\langle H, A\rangle\delta\langle G, A\rangle$ and it follows, since $\delta$ is symmetric, that $\langle F, A\rangle\delta\langle H, A\rangle$ or $\langle G, A\rangle\delta\langle H, A\rangle$.

(SP3) is a direct consequence of Theorem 4.1.15(1).

(SP4) Suppose $\langle F, A\rangle \not\delta \langle G, A\rangle$, i.e., $\langle F, A\rangle \ll \langle G, A\rangle^c$. Then Theorem 4.1.15(6) assures the existence of a soft set $\langle H, A\rangle$ such that $\langle F, A\rangle \ll \langle H, A\rangle^c \ll \langle G, A\rangle^c$. Thus there is a soft set $\langle H, A\rangle$ such that $\langle F, A\rangle \not\delta \langle H, A\rangle$ and $\langle H, A\rangle^c \not\delta \langle G, A\rangle$.

(SP5) If $\langle F, A\rangle \not\delta \langle G, A\rangle$, then $\langle F, A\rangle \ll \langle G, A\rangle^c$. From Theorem 4.1.15(2), we have $\langle F, A\rangle \widetilde{\subseteq} \langle G, A\rangle^c$, i.e., $\langle F, A\rangle \Cap \langle G, A\rangle = \Phi_A$.

Therefore $\delta$ is a soft proximity on $U_A$. Clearly, $\langle G, A\rangle$ is $\delta$-neighborhood of $\langle F, A\rangle$ iff $\langle F, A\rangle \not\delta \langle G, A\rangle^c$ iff $\langle F, A\rangle \ll ((\langle G, A\rangle^c)^c = \langle G, A\rangle$. □

**Theorem 4.1.18.** If $(U_A, \delta)$ is a soft proximity space and $\langle F, A\rangle \in \mathcal{SS}(U)_A$, then

$$\overline{\langle F, A\rangle} = \Cap_{\langle F,A\rangle \ll \langle G,A\rangle} \langle G, A\rangle.$$

**Proof** From Lemma 4.1.13(1) and Theorem 4.1.15(2), we conclude that $\langle F, A\rangle \ll \langle G, A\rangle$ implies $\overline{\langle F, A\rangle} \widetilde{\subseteq} \langle G, A\rangle$, and hence $\overline{\langle F, A\rangle} \widetilde{\subseteq} \Cap_{\langle F,A\rangle \ll \langle G,A\rangle} \langle G, A\rangle$. To show the reverse soft inclusion, suppose that $x_F \widetilde{\not\in} \overline{\langle F, A\rangle}$. Then $x_F \not\delta \overline{\langle F, A\rangle}$ and, by Lemma 4.3.14, $\overline{\langle F, A\rangle}$ has a $\delta$-neighborhood $\langle G, A\rangle_{x_F}$ not soft containing $x_F$. Thus $x_F \widetilde{\not\in} \Cap_{\langle F,A\rangle \ll \langle G,A\rangle} \langle G, A\rangle$. □

## 4.2 Soft proximally continuity

In the study of soft topological spaces, soft continuous functions play an important role. This analogue in the theory of soft proximity spaces is the concept of a soft proximally continuous mapping.

**Definition 4.2.1.** Let $(U_A, \delta_1)$ and $(V_B, \delta_2)$ be two soft proximity spaces. Let $u : U \to V$ and $p : A \to B$ be functions. A function $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ is said to be a soft proximally continuous mapping iff $\langle F, A\rangle \delta_1 \langle G, A\rangle$ implies $f_{pu}(\langle F, A\rangle) \delta_2 f_{pu}(\langle G, A\rangle)$.

Equivalently, $f_{pu}$ is a soft proximally continuous mapping iff $\langle H, B\rangle \not\delta_2 \langle K, B\rangle$ implies $f_{pu}^{-1}(\langle H, B\rangle) \not\delta_1 f_{pu}^{-1}(\langle K, B\rangle)$, or $\langle H, B\rangle \ll_2 \langle K, B\rangle$ implies $f_{pu}^{-1}(\langle H, B\rangle) \ll_1 f_{pu}^{-1}(\langle K, B\rangle)$.

It is easy to see that the composition of two soft proximally continuous mappings is a soft proximally continuous mapping.

**Theorem 4.2.2.** Let $(U_A, \delta_1)$ and $(V_B, \delta_2)$ be two soft proximity spaces and $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ be a function. If $f_{pu}$ is soft proximally continuous mapping, then $f_{pu}$ is soft $pu$-continuous with respect to $\tau(\delta_1)$ and $\tau(\delta_2)$.

**Proof** This result follows easily from Theorem 3.4.5 and the fact that $x_F \delta_1 \langle F, A\rangle$ implies $f_{pu}(x_F)\delta_2 f_{pu}(\langle F, A\rangle)$, i.e., $f_{pu}(\overline{\langle F, A\rangle})\widetilde{\subseteq}\overline{f_{pu}(\langle F, A\rangle)}$. □

It is natural to inquire as to when the converse of Theorem 4.4.2 is true.

**Theorem 4.2.3.** Let $(U_A, \delta_1)$ and $(V_B, \delta_2)$ be soft proximity spaces and $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ be a function. If $(U_A, \delta_1)$ is a soft proximity space satisfying the condition

$$\langle F, A\rangle \delta_1 \langle G, A\rangle \text{ iff } \overline{\langle F, A\rangle} \Cap \overline{\langle G, A\rangle} \neq \Phi_A,$$

then every soft *pu*-continuous mapping $f_{pu}$ is soft proximally continuous.

**Proof** If $\langle F, A\rangle$ and $\langle G, A\rangle$ are soft sets such that $\langle F, A\rangle\delta_1\langle G, A\rangle$, then $\overline{\langle F, A\rangle} \Cap \overline{\langle G, A\rangle} \neq \Phi_A$. But this implies that $f_{pu}(\overline{\langle F, A\rangle}) \Cap f_{pu}(\overline{(G, A)}) \neq \Phi_B$, i.e., $f_{pu}(\overline{\langle F, A\rangle})\delta_2 f_{pu}(\overline{\langle G, A\rangle})$. Since $f_{pu}$ is soft *pu*-continuous, it follows from Theorem 3.4.5 that $f_{pu}(\overline{\langle F, A\rangle})\widetilde{\subseteq}\overline{f_{pu}(\langle F, A\rangle)}$ and $f_{pu}(\overline{\langle G, A\rangle})\widetilde{\subseteq}\overline{f_{pu}(\langle G, A\rangle)}$, yielding $\overline{f_{pu}(\langle F, A\rangle)}\delta_2\overline{f_{pu}(\langle G, A\rangle)}$. From Lemma 4.1.9, it follows that $f_{pu}(\langle F, A\rangle)\delta_2 f_{pu}(\langle G, A\rangle)$, and we conclude that $f_{pu}$ is a soft proximally continuous mapping. □

**Theorem 4.2.4.** Let $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ be a function and $(V_B, \delta_2)$ be a soft proximity space. If we define a relation $\delta_0$ on $U_A$ by

$$\langle F, A\rangle \not\delta_0 \langle G, A\rangle \text{ iff there exists a } \langle H, B\rangle \in \mathcal{SS}(V)_B \text{ such that } f_{pu}(\langle F, A\rangle) \not\delta_2 \langle H, B\rangle^c \text{ and } f_{pu}^{-1}(\langle H, B\rangle)\widetilde{\subseteq}\langle G, A\rangle^c,$$

then $\delta_0$ is the coarsest soft proximity on $U_A$ such that $f_{pu}$ is soft proximally continuous mapping.

**Proof** We first verify that $\delta_0$ is a soft proximity on $U_A$.

(SP1) Suppose $\langle F, A\rangle \not\delta_0 \langle G, A\rangle$ and let $\langle K, B\rangle = (f_{pu}(\langle F, A\rangle))^c$. Since $f_{pu}(\langle G, A\rangle) \widetilde{\subseteq}\langle H.B\rangle^c$ and $f(\langle F, A\rangle) \not\delta_2 \langle H, B\rangle^c$, we have $f_{pu}(\langle G, A\rangle) \not\delta_2 \langle H, B\rangle^c$. Moreover, we have $f_{pu}^{-1}(\langle K, B\rangle) = (f_{pu}^{-1}(f_{pu}(\langle F, A\rangle)))^c\widetilde{\subseteq}\langle F, A\rangle^c$. Hence $\langle G, A\rangle \not\delta_0 \langle F, A\rangle$.

(SP2) $(\langle F, A\rangle\widetilde{\cup}\langle G, A\rangle) \not\delta_0 \langle H, A\rangle$ implies the existence of a $\langle K, B\rangle \in \mathcal{SS}(V)_B$ such that $[f_{pu}(\langle F, A\rangle)\widetilde{\cup} f_{pu}(\langle G, A\rangle)] \not\delta_2 \langle K, B\rangle^c$ and $f_{pu}^{-1}(\langle K, B\rangle)\widetilde{\subseteq}\langle H, A\rangle^c$, from which $\langle F,A\rangle \not\delta_0 \langle H,A\rangle$ and $\langle G,A\rangle \not\delta_0 \langle H,A\rangle$ follow. If $\langle F,A\rangle \not\delta_0 \langle H,A\rangle$ and $\langle G,A\rangle \not\delta_0 \langle H,A\rangle$, there exist $\langle K_1,B\rangle$ and $\langle K_2,B\rangle$ in $\mathcal{SS}(V)_B$ such that $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle K_1, B\rangle^c$, $f_{pu}(\langle G,A\rangle) \not\delta_2 \langle K_2,B\rangle^c$, $f_{pu}^{-1}(\langle K_1,B\rangle) \widetilde{\subseteq} \langle H,A\rangle^c$ and $f_{pu}^{-1}(\langle K_2,B\rangle) \widetilde{\subseteq} \langle H,A\rangle^c$. Hence $[f_{pu}(\langle F,A\rangle) \widetilde{\cup} f_{pu}(\langle G,A\rangle)] \not\delta_2 [\langle K_1,B\rangle \widetilde{\cup} \langle K_2,B\rangle]^c$ and $f_{pu}^{-1}(\langle K_1,B\rangle \widetilde{\cup} \langle K_2,B\rangle) \widetilde{\subseteq} \langle H,A\rangle^c$, i.e., $(\langle F,A\rangle \widetilde{\cup} \langle G,A\rangle) \not\delta_0 \langle H,A\rangle$.

(SP3) If $\langle F,A\rangle = \Phi_A$, then $f_{pu}(\langle F,A\rangle) \not\delta_2 V_B$ and $f_{pu}^{-1}(\Phi_B) \widetilde{\subseteq} \langle G,A\rangle^c$. Hence we have $\langle F,A\rangle \not\delta_0 \langle G,A\rangle$.

(SP4) If $\langle F,A\rangle \not\delta_0 \langle G,A\rangle$, then there exists a $\langle H,B\rangle$ such that $f_{pu}^{-1}(\langle H,B\rangle) \widetilde{\subseteq} \langle G, A\rangle^c$ and $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle H,B\rangle^c$. Since $\delta_2$ is soft proximity, the latter relation and (SP4) together assure the existence of a soft set $\langle K,B\rangle$ such that $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle K,B\rangle$ and $\langle K,B\rangle^c \not\delta_2 \langle H,B\rangle^c$. Let $\langle J,A\rangle = f_{pu}^{-1}(\langle K,B\rangle)$. Since $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle K, B\rangle$, $\langle F,A\rangle \not\delta_0 \langle J,A\rangle$. As $f_{pu}(\langle J,A\rangle^c) \widetilde{\subseteq} \langle K,B\rangle^c \not\delta_2 \langle H,B\rangle^c$ and $f_{pu}^{-1}(\langle H,B\rangle) \widetilde{\subseteq} \langle G,A\rangle^c$, we have $\langle J,A\rangle^c \not\delta_0 \langle G,A\rangle$.

(SP5) $\langle F,A\rangle \not\delta_0 \langle G,A\rangle$ implies there exists a $\langle H,B\rangle$ such that $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle H, B\rangle^c$ and $f_{pu}^{-1}(\langle H,B\rangle) \widetilde{\subseteq} \langle G,A\rangle^c$. Therefore $f_{pu}(\langle F,A\rangle) \Cap \langle H,B\rangle^c = \Phi_B$ and $f_{pu}^{-1}(f_{pu}(\langle F,A\rangle)) \Cap f_{pu}^{-1}(\langle H,B\rangle^c) = \Phi_A$. Since $\langle F,A\rangle \widetilde{\subseteq} f_{pu}^{-1}(f_{pu}(\langle F,A\rangle))$ and $\langle G,A\rangle \widetilde{\subseteq} f_{pu}^{-1}(\langle H,B\rangle^c)$, we have $\langle F,A\rangle \Cap \langle G,A\rangle = \Phi_A$.

In order to show that $f_{pu}$ is a soft proximally continuous mapping, suppose that $f(\langle F,A\rangle) \not\delta_2 f_{pu}(\langle G,A\rangle)$. Since $f_{pu}(\langle F,A\rangle) \ll (f_{pu}(\langle G,A\rangle))^c$, by Theorem 4.1.15(6), there exists a $\langle H,B\rangle$ such that $f_{pu}(\langle F,A\rangle) \ll \langle H,B\rangle \ll (f_{pu}(\langle G,A\rangle))^c$. Thus we have $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle H,B\rangle^c$ and $f_{pu}^{-1}(\langle H,B\rangle) \widetilde{\subseteq} (f_{pu}^{-1}(\langle G,A\rangle))^c \widetilde{\subseteq} \langle G,A\rangle^c$, $\langle F,A\rangle \not\delta_0 \langle G,A\rangle$.

It remains to show that if $\delta_1$ is any soft proximity on $U_A$ such that $f_{pu}$ is soft proximally continuous, then $\delta_1$ is finer than $\delta_0$. If $\langle F,A\rangle \not\delta_0 \langle G,A\rangle$, then there exists a $\langle H,B\rangle$ such that $f_{pu}(\langle F,A\rangle) \not\delta_2 \langle H,B\rangle^c$ and $f_{pu}^{-1}(\langle H,B\rangle) \widetilde{\subseteq} \langle G,A\rangle^c$. Since $f_{pu}$ is soft proximally continuous, $\langle F,A\rangle \not\delta_1 (f_{pu}^{-1}(\langle H,B\rangle))^c$, and $\langle G,A\rangle \widetilde{\subseteq} (f_{pu}^{-1}(\langle H,B\rangle))^c$ implies $\langle F,A\rangle \not\delta_1 \langle G,A\rangle$. Thus $\delta_1 > \delta_0$. □

**Corollary 4.2.5.** Let $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ be a function and $(V_B, \delta_2)$ be a soft proximity space. If $\delta_0$ is a soft proximity on $U_A$ given in Theorem 4.2.4, then $f_{pu}^{-1}[\tau(\delta_2)] \subseteq \tau(\delta)$.

**Proof** It follows from Theorems 4.2.2 and 4.2.3. □

## 4.3 Soft clusters and ultra soft filters

Ultra soft filters play an important role in soft topological spaces because such notions as soft convergence and soft compactness can be characterized in terms of ultra soft filters. In this section we consider their counterparts, namely soft clusters, in soft proximity spaces. We show that ultra soft filters and soft clusters are closely related, and used this relationship to drive several important results in the theory of soft proximity spaces.

**Definition 4.3.1.** A collection $\sigma$ of soft subsets of soft proximity space $(U_A, \delta)$ is called a soft cluster iff the following conditions are satisfied:

(C1) If $\langle F, A\rangle$ and $\langle G, A\rangle$ belong to $\sigma$, then $\langle F, A\rangle\delta\langle G, A\rangle$;

(C2) If $\langle F, A\rangle\delta\langle H, A\rangle$ for every $\langle H, A\rangle \in \sigma$, then $\langle F, A\rangle \in \sigma$;

(C3) If $(\langle F, A\rangle\widetilde{\cup}\langle G, A\rangle) \in \sigma$, then $\langle F, A\rangle \in \sigma$ or $\langle G, A\rangle \in \sigma$.

**Remark 4.3.2.** (1) For each $x_F\widetilde{\in}U_A$, the collection $\sigma_{x_F} = \{\langle F, A\rangle \in \mathcal{SS}(U)_A : \langle F, A\rangle\delta x_F\}$ is a soft cluster. We call such a soft cluster a soft point cluster and use this notation.

(2) If $\sigma$ is any soft cluster in $U_A$, then $U_A \in \sigma$ by (C2). Hence for each soft set $\langle F, A\rangle$ in $\mathcal{SS}(U)_A$, either $\langle F, A\rangle \in \sigma$ or $\langle F, A\rangle^c \in \sigma$. Recall that an ultra soft filter also has this property.

(3) If $\langle F, A\rangle \in \sigma$ and $\langle F, A\rangle\widetilde{\subseteq}\langle G, A\rangle$, then $\langle G, A\rangle \in \sigma$. This too is a property of an ultra soft filter.

(4) If $\sigma$ is any cluster in $U_A$, then $\langle F, A\rangle \in \sigma$ iff $\overline{\langle F, A\rangle} \in \sigma$. This follows from Lemma 4.1.9, (C2) and (3).

**Lemma 4.3.3.** If $\sigma_1$ and $\sigma_2$ are two soft cluster in proximity space $(U_A, \delta)$ such that $\sigma_1 \subseteq \sigma_2$, then $\sigma_1 = \sigma_2$.

**Proof** If $\langle F, A\rangle \in \sigma_2$, then $\langle F, A\rangle\delta_2\langle H, A\rangle$ for every $\langle H, A\rangle \in \sigma_2$. Since $\sigma_1 \subseteq \sigma_2$, $\langle F, A\rangle\delta\langle G, A\rangle$ for every $\langle G, A\rangle \in \sigma_1$, which shows that $\langle F, A\rangle \in \sigma_1$. Thus $\sigma_2 \subseteq \sigma_1$. □

The following lemma on ultra soft filters is useful in deriving the fundamental relationship between ultra soft filters and soft clusters.

**Lemma 4.3.4.** Let $\mathcal{P}$ be a collection of soft sets in $\mathcal{SS}(U)_A$ such that (1) $\Phi_A \notin \mathcal{P}$, and (2) $\langle F, A\rangle \widetilde{\cup} \langle G, A\rangle \in \mathcal{P}$ iff $\langle F, A\rangle \in \mathcal{P}$ or $\langle G, A\rangle \in \mathcal{P}$. If $\langle F_0, A\rangle \in \mathcal{P}$, then there exists an ultra soft filter $\mathcal{F}$ such that

(a) $\langle F_0, A\rangle \in \mathcal{F}$ and (b) $\mathcal{F} \subseteq \mathcal{P}$.

**Proof** By Zorn's lemma, there exists a maximal collection $\mathcal{F}$ of soft sets in $\mathcal{SS}(U)_A$ satisfying

(a) $\langle F_0, A\rangle \in \mathcal{F}$ and

(b)$'$ $\langle F_i, A\rangle \in \mathcal{F}$ for $i = 1, 2, \ldots, n$ implies $\Cap_{i=1}^{n} \langle F_i, A\rangle \in \mathcal{P}$.

Obviously $\Phi_A \notin \mathcal{F}$. If $\langle F, A\rangle$ and $\langle G, A\rangle$ belong to $\mathcal{F}$ then by (b)$'$, $\langle F, A\rangle \Cap \langle G, A\rangle \in \mathcal{P}$. Since $\mathcal{F}$ is maximal, we must have $\langle F, A\rangle \Cap \langle G, A\rangle \in \mathcal{F}$. If $\langle F, A\rangle \in \mathcal{F}$ and $\langle F, A\rangle \widetilde{\subseteq} \langle H, A\rangle$, then $\langle H, A\rangle \in \mathcal{P}$ and hence belongs to $\mathcal{F}$ since $\mathcal{F}$ is maximal. Having shown that $\mathcal{F}$ is soft filter, it remains to show that $\mathcal{F}$ is an ultra soft filter. Supposing the contrary, there would exist a soft set $\langle K, A\rangle$ in $\mathcal{SS}(U)_A$ such that neither $\langle K, A\rangle$ nor $\langle K, A\rangle^c$ belongs to $\mathcal{F}$. Hence there are soft sets $\langle F_1, A\rangle$ and $\langle F_2, A\rangle$ in $\mathcal{F}$ such that neither $\langle F_1, A\rangle \Cap \langle K, A\rangle$ nor $\langle F_2, A\rangle \Cap \langle K, A\rangle^c$ belongs to $\mathcal{P}$. If $\langle F, A\rangle = \langle F_1, A\rangle \Cap \langle F_2, A\rangle$, then $\langle F, A\rangle \in \mathcal{P}$ while neither $\langle F, A\rangle \Cap \langle K, A\rangle$ nor $\langle F, A\rangle \Cap \langle K, A\rangle^c$ belongs to $\mathcal{P}$, a contradiction. □

**Theorem 4.3.5.** A collection $\sigma$ of soft subsets of a soft proximity space $(U_A, \delta)$ is a soft cluster if and only if there exists an ultra soft filter $\mathcal{F}$ on $U_A$ such that

$$\sigma = \{\langle F, A\rangle \in \mathcal{SS}(U)_A : \langle F, A\rangle \delta \langle G, A\rangle \text{ for every } \langle G, A\rangle \in \mathcal{F}\}. \tag{4.5}$$

Moreover, given $\sigma$ and $\langle F_0, A\rangle \in \sigma$, there exists an ultra soft filter $\mathcal{F}$ satisfying (4.5) and which contains $\langle F_0, A\rangle$.

**Proof** Let $\mathcal{F}$ be an ultra soft filter on $U_A$ and let $\sigma$ be defined by (4.5). We shall first show that $\sigma$ is a soft cluster.

(C1) Suppose $\langle F, A\rangle$ and $\langle G, A\rangle$ belong to $\mathcal{F}$. For every soft set $\langle H, A\rangle$ in $\mathcal{SS}(U)_A$, either $\langle H, A\rangle$ or $\langle H, A\rangle^c$ is in $\mathcal{F}$. This means either (a) $\langle F, A\rangle \delta \langle H, A\rangle$

and $\langle G, A\rangle\delta\langle H, A\rangle$ or (b) $\langle F, A\rangle\delta\langle H, A\rangle^c$ and $\langle G, A\rangle\delta\langle H, A\rangle^c$. Hence for every soft set $\langle H, A\rangle$ in $\mathcal{SS}(U)_A$, either $\langle F, A\rangle\delta\langle H, C\rangle$ or $\langle H, A\rangle^c\delta\langle G, A\rangle$ which shows (by (SP4)) that $\langle F, A\rangle\delta\langle G, A\rangle$.

(C2) Suppose $\langle F, A\rangle\delta\langle H, A\rangle$ for every $\langle H, A\rangle \in \sigma$. Since $\mathcal{F} \subseteq \sigma$, $\langle F, A\rangle\delta\langle G, A\rangle$ for every $\langle G, A\rangle \in \mathcal{F}$, which shows that $\langle F, A\rangle \in \sigma$.

(C3) If neither $\langle F, A\rangle$ nor $\langle G, A\rangle$ belongs to $\sigma$, there exist $\langle F', A\rangle$ and $\langle G', A\rangle$ in $\mathcal{F}$ such that $\langle F, A\rangle \not\delta \langle F', A\rangle$ and $\langle G, A\rangle \not\delta \langle G', A\rangle$. Using Theorem 4.1.15(3), we obtain $\langle F, A\rangle \not\delta [\langle F', A\rangle \Cap \langle G', A\rangle]$ and $\langle G, A\rangle \not\delta [\langle F', A\rangle \Cap \langle G', A\rangle]$. Thus by (SP2), $[\langle F, A\rangle\widetilde{\cup}\langle G, A\rangle] \not\delta [\langle F', A\rangle \Cap \langle G', A\rangle]$. Since $[\langle F', A\rangle \Cap \langle G', A\rangle] \in \mathcal{F}$, it follows that $[\langle F, A\rangle\widetilde{\cup}\langle G, A\rangle] \notin \mathcal{F}$, as required.

Conversely, let $\sigma$ be a soft cluster and suppose $\langle F_0, A\rangle \in \mathcal{F}$. Taking $\mathcal{P} = \sigma$ in Lemma 4.3.4, we obtain an ultra soft filter $\mathcal{F} \subseteq \sigma$ such that $\langle F_0, A\rangle \in \mathcal{F}$. If $\sigma' = \{\langle F, A\rangle \in \mathcal{SS}(U)_A : \langle F, A\rangle\delta\langle G, A\rangle$ for every $\langle G, A\rangle \in \mathcal{F}\}$, then $\sigma \subseteq \sigma'$. Thus by Lemma 4.3.4, $\sigma = \sigma'$, and (4.5) is satisfied. □

**Corollary 4.3.6.** If $\mathcal{F}$ is an ultra soft filter such that $\mathcal{F} \subseteq \sigma$, then $\sigma$ is uniquely determined.

**Remark 4.3.7.** (1) If $\mathcal{F}$ and $\sigma$ are as in Theorem 4.3.5, we say that '$\mathcal{F}$ generates $\sigma$' or '$\sigma$ is determined by $\mathcal{F}$'.

(2) In Theorem 4.3.5, $\mathcal{F}$ need only be an ultra soft filterbase.

**Lemma 4.3.8.** If a soft cluster $\sigma$ in a soft proximity space $(U_A, \delta)$ is determined by an ultra soft filter $\mathcal{F}$, then $\sigma$ is a soft point cluster $\sigma_{x_F}$ if and only if $\mathcal{F}$ converges to $x_F$.

**Proof** $\sigma = \sigma_{x_F}$ iff $\{x_F\} \in \sigma$ iff $x_F\delta\langle F, A\rangle$ for every $\langle F, A\rangle \in \mathcal{F}$ iff $x_F$ is a cluster soft point of $\mathcal{F}$ iff (Corollary 3.3.18 in soft filter) $\mathcal{F}$ converges to $x_F$. □

A soft topological space is compact if and only if every ultra soft filter in the space converges to a soft point. The following analogue of this result follows directly from the above lemma.

**Theorem 4.3.9.** A soft proximity space is compact if and only if every soft cluster in the space is soft point cluster.

If $\langle F, A\rangle \Cap \langle G, A\rangle \neq \Phi_A$, then there exists an ultra soft filter $\mathcal{F}$ which contains both $\langle F, A\rangle$ and $\langle G, A\rangle$. A similar result holds for soft clusters in soft proximity spaces:

**Theorem 4.3.10.** If $\langle F, A\rangle\delta\langle G, A\rangle$, then there exists a soft cluster $\sigma$ in a soft proximity space $(U_A, \delta)$ such that $\langle F, A\rangle$ and $\langle G, A\rangle$ belong to $\sigma$.

**Proof** Let $\mathcal{P} = \{\langle H, A\rangle \in \mathcal{SS}(U)_A : \langle H, A\rangle\delta\langle G, A\rangle\}$. From Lemma 4.3.8, there exists an ultra soft filter $\mathcal{F}$ such that $\langle F, A\rangle \in \mathcal{F} \subseteq \mathcal{P}$. The soft cluster $\sigma$ determined by $\mathcal{F}$ contains both $\langle F, A\rangle$ and $\langle G, A\rangle$. □

We now prove a result similar to: if $\mathcal{F}$ is an ultra soft filter on $U_A$ and $\langle F, A\rangle\widetilde{\subseteq}U_A$, then the trace of $\mathcal{F}$ on $U_A$ is an ultra soft filter on $\langle F, A\rangle$ iff $\langle F, A\rangle \in \mathcal{F}$.

**Theorem 4.3.11.** Let $\sigma$ be a soft cluster in a soft proximity space $(U_A, \delta)$ and let $\langle F, A\rangle \in \sigma$. Then there exists a unique soft cluster $\sigma'$ in $(\langle F, A\rangle, \delta_{\langle F,A\rangle})$ contained in $\sigma$, namely $\sigma' = \{\langle G, A\rangle\widetilde{\subseteq}\langle F, A\rangle : \langle G, A\rangle \in \sigma\}$.

**Proof** By Theorem 4.2.4, $\sigma$ is determined by an ultra soft filter $\mathcal{F}$ containing $\langle F, A\rangle$. Then $\mathcal{F}_{\langle F,A\rangle} = \{\langle G, A\rangle \Cap \langle F, A\rangle : \langle G, A\rangle \in \mathcal{F}\}$, the trace of $\mathcal{F}$ on $\langle F, A\rangle$, is an ultra soft filter on $\langle F, A\rangle$ and so generates a soft cluster $\sigma'$ on $\langle F, A\rangle$. If $\langle H, A\rangle \in \sigma'$, then $\langle H, A\rangle\delta[\langle G, A\rangle \Cap \langle F, A\rangle]$ for each $\langle G, A\rangle \in \mathcal{F}$. This implies $\langle H, A\rangle\delta\langle G, A\rangle$ for each $\langle G, A\rangle \in \mathcal{F}$, i.e., $(H, A) \in \sigma$. Thus $\sigma' \subseteq \sigma$, and clearly

$$\begin{aligned}\sigma' &= \{\langle G, A\rangle\widetilde{\subseteq}\langle F, A\rangle : \langle G, A\rangle\delta\langle H, A\rangle \text{ for every } \langle H, A\rangle \in \mathcal{F}\}\\ &= \{\langle G, A\rangle\widetilde{\subseteq}\langle F, A\rangle : \langle G, A\rangle \in \sigma\}.\end{aligned}$$

That $\sigma'$ is the only soft cluster on $\langle F, A\rangle$ contained in $\sigma$ is shown by a method similar to that used in Lemma 4.3.4. □

Let $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ be a mapping. The following results are well known:

If $\mathcal{F}$ is an ultra soft filterbase on $U_A$, then $f_{pu}(\mathcal{F}) = \{f_{pu}(\langle G, A\rangle) : \langle G, A\rangle \in \mathcal{F}\}$ is an ultra soft filterbase on $V_B$.

We find the following analogue in soft proximity spaces:

**Theorem 4.3.12.** If $f_{pu} : \mathcal{SS}(U)_A \to \mathcal{SS}(V)_B$ is a mapping, $f_{pu}$ is a soft proximally continuous mapping from $(U_A, \delta_1)$ to $(V_B, \delta_2)$, then each soft cluster $\sigma_1$ on $U_A$, there corresponds a soft cluster $\sigma_2$ on $V_B$ such that

$$\sigma_2 = \{\langle H, B\rangle \in \mathcal{SS}(V)_B : \langle H, B\rangle \delta_2 f_{pu}(\langle G, A\rangle) \text{ for each } \langle G, A\rangle \in \sigma_1\}.$$

**Proof** $\sigma_1$ is determined by an ultra soft filter $\mathcal{F}$ on $U_A$. Now $f_{pu}(\mathcal{F})$ is an ultra soft filterbase on $V_B$ and generates a soft cluster $\sigma_2$ on $V_B$. If $\langle H, B\rangle \delta_2 f_{pu}(\langle G, A\rangle)$ for every $\langle G, A\rangle \in \sigma_1$, then $\langle H, B\rangle \delta_2 f_{pu}(\langle F, A\rangle)$ for every $\langle F, A\rangle \in \mathcal{F}$, so that $\langle H, B\rangle \in \sigma_2$. To prove the reverse inclusion, we first note that

$$f_{pu}(\sigma_1) \subseteq \sigma_2.$$

This follows from the fact that if $\langle G, A\rangle \in \sigma_1$, then $\langle G, A\rangle \delta_1 \langle F, A\rangle$ for every $\langle F, A\rangle \in \mathcal{F}$, and $f_{pu}$ being a soft proximally continuous mapping implies $f_{pu}(\langle G, A\rangle)\delta_2 f_{pu}(\langle F, A\rangle)$ for each $\langle F, A\rangle \in \mathcal{F}$, i.e., $f_{pu}(\langle G, A\rangle) \in \sigma_2$. Thus if $\langle H, B\rangle \in \sigma_2$, then $\langle H, A\rangle \delta_2 f_{pu}(\langle G, A\rangle)$ for every $\langle G, A\rangle \in \sigma_1$.

## 4.4 Conclusions

In soft topology, a soft proximity space is an axiomatization of notions of "nearness" that hold soft set-to-soft set, as opposed to the better known soft point-to-soft set notions that characterize soft topological spaces. The concepts of $\delta$-neighborhood and soft proximally continuity be introduced in a soft proximity space and furnish approaches to the study of sot proximity spaces. We consider soft clusters in soft proximity spaces and show that ultra soft filters and soft clusters are closely related, and used this relationship to drive several important results in the theory of soft proximity spaces.

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앞으로 더욱더 열심히 정진하여 진정한 이학박사가 되도록 노력할 것을 다짐합니다.