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Approximate controllability for impulsive differential equations with local Lipschitz continuity

by

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Graduate School of Education

Pukyong National University

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Approximate controllability for impulsive differential equations with local Lipschitz continuity
(극소적인 립시츠연속을 가진 충동미분방정식에 대한 근사적인 제어성)

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극소적인 립시츠연속을 가진 충동미분방정식에 대한 근사적인 제어성

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요 약

이 논문은 주어진 Hilbert 공간에서 준선형 충동미분방정식에 대한 해의 정규성 문제를 다룬 후 가제어성을 증명하고자 한다. 먼저 다음과 같이 주어진 지연항을 포함한 방정식:

(1)
$$\begin{cases} x'(t)+Ax(t)=f(t,x(t))+(Bu)(t), \quad t\in(0,T], \quad t\neq t_{k,} \\ k=1,2,\cdots,m, \\ \Delta x(t_{k)}=x(t_k^{+)}-x(t_k^{-)}=I_{k(x(t_k^{-})),} \; k=1,2,\cdots,m, \\ x(0)=x_{0.} \end{cases}$$

먼저 (1)식을 추상적인 함수미분방정식으로 전환하여 주어진 비선형항의 Lipschitz 연속의 가정과 함수 성질을 이용한 부동점정리를 응용하여 해의 존재성과 정규성을 밝히고 가제어성의 충분조건을 유도하였다

(주결과) 1) H 와 V 를 Hilbert 공간으로 하고 V 가 조밀한 공간으로서 그의 공액공간을 V^*로 하자. 다음과 같이 각각의 조건:

1) I_k 함수의 국소적인 Lipschitz 조건

2) 비선형항의 Lipschitz 조건

3) $f \in L^2(0,T;V^*)$

4) 주작용소의 해석적 반군의 생성자으로 주어지면 두 방정식의 초기치 문제의 해는 유일하게 존재하며, 아울러

$$x\in L^2(0,T;V)\cap W^{1,1/2}(0,T;V^*)\subset C([0,T];H).$$

2) $x(T;u)$를 시간 T에서 제어 $u\in L^2(0,T;U)$ 에 대응하는 자취라고하면 $\{x(T;u) : u\in L^2(0,T;U)\}$의 집합이 전 공간 H상에서 조밀성을 보여 가제어성을 증명하였다.

Approximate controllability for impulsive differential equations with local Lipschitz continuity

Ah-Ran Park

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Abstract

In this paper, we first consider the existence and regularity of solutions of the semilinear impulsive differential equation under natural assumptions such as the local Lipschtiz continuity of nonlinear term. Thereafter, we will also establish the approximate controllability for the equation when the corresponding linear system is approximately controllable.

Keywords: approximate controllability, semilinear equation, ,impulsive differential equation, local lipschtiz continuity, controller operator, reachable set

AMS Classification Primary 35B37; Secondary 93C20

1 Introduction

In this paper, we are concerned with the global existence of solution and the approximate controllability for the semilinear impulsive control system:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in (0, T], \quad t \neq t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{1.1}$$

Let H and V be real Hilbert spaces such that V is a dense subspace in H. Let A be the operator associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying

Gårding's inequality:

$$(Au, v) = a(u, v), \quad u,\ v \in V$$

where V is a Hilbert space such that $V \subset H \subset V^*$. Then $-A$ generates an analytic semigroup in both H and V^*(see [1, Theorem 3.6.1]) and so the equation (1.1) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0, T] \times V$ to H is assumed to be locally Lipschitz continuous with respect to the second variable. Let U be a Banach space of control variables and the controller operator B be a bounded linear operator from the Banach space $L^2(0, T; U)$ to $L^2(0, T; H)$. The impulsive condition

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m,$$

is a combination of traditional evolution systems and short term perturbations whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc. Let $x(t; f, u)$ be a solution of the equation (1.1) associated with a nonlinear term f and a control u. We will show the approximate controllability for the equation (1.1), namely that the reachable set $R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\}$ is a dense subset of H. This kind of equations arise naturally in biology, in physics, control engineering problem, etc.

In the first part of this paper we establish the wellposedness and regularity property for the following equation:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T], \quad t \neq t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{1.2}$$

The existence of solutions for a class of semilinear functional differential equations has been studied by many authors. Recently, Kobayashi et el. [2] introduced the notion of semigroups of locally Lipschitz operators which provide us with mild solutions to the Cauchy problem for semilinear evolution equations. The regularity for the semilinear heat equations has been developed as seen in section 4.3.1 of Barbu [3] and [4, 5].

In this paper, we propose a different approach of the earlier works (briefly introduced in [1, 15, 7]) about the mild, strong, and classical solutions of Cauchy problems. Our approach is that results of the linear cases of Di Blasio [8] on the L^2-regularity remain valid under the above formulation of the semilinear problem (1.2).

Next, based on the regularity for (1.2), we intend to establish the approximate controllability for (1.1). Approximate controllability for semilinear control systems can be founded in [4.9-15]. Similar considerations of linear and semilinear systems have been dealt with in many references, linear problems in the book [16] and Nakagiri [17] , the system (1.1) with the uniform bounded nonlinear term in [18], the system (1.1) with the uniform Lipschtz continuous nonlinear term in [4, 19, 20, 21]. However there are few papers treating the systems with local Lipschipz continuity, we can just find a recent article Wang [22]. Among these literatures, in [19, 22], they assumed that the semigroup $S(t)$ generated by A is compact in order to guarantee the compactness of the solution mapping, and the approximate controllability for the equation (1.1) was investigated.

In this paper, in order to show that the main result of [19] is extended to the nonlinear differential equation, we assume that the embedding $D(A) \subset V$ is compact. Then by virtue of the result in Aubin [23], we can take advantage of the fact that the solution mapping $u \in L^2(0, T; U) \mapsto x(T; f, u)$ is compact.

Under natural assumptions such as the local Lipschtiz continuity of nonlinear term, we obtain the approximate controllability for the equation (1.1) when the corresponding linear system is approximately controllable.

The paper is organized as follows. In section 2, the results of general linear evolution equations besides notations and assumptions are stated. In section 3, we will obtain that the regularity for parabolic linear equations can also be applicable to (1.2) with nonlinear terms satisfying local Lipschitz continuity. The approach used here is similar to that developed in [1, 4] on the general semilnear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations. Thereafter, we investigate the approximate controllability for the problem (1.1) in Section 4. In the proofs of the main theorems, we need some compactness hypothesis. So we make the natural assumption that the embedding $D(A) \subset V$ is compact instead of the compact property of semigroup used in [9, 19]. Finally we give a simple example to which our main result can be applied.

2 Regularity for linear equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u, \ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. From the following inequalities

$$\omega_1 ||u||^2 \le \text{Re}\, a(u,u) + \omega_2 |u|^2 \le C|Au|\,|u| + \omega_2 |u|^2 \le \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one which continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^**(Section 1.3.3 of [24]).*

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0,T;X)$ is the collection of all strongly measurable square integrable functions from $(0,T)$ into X and $W^{1,2}(0,T;X)$ is the set of all absolutely continuous functions on $[0,T]$ such that their derivative belongs to $L^2(0,T;X)$. $C([0,T];X)$ will denote the set of all continuously functions from $[0,T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X,Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X,X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0,T;D(A)) \cap W^{1,2}(0,T;H),$$
$$\mathcal{W}_1(T) = L^2(0,T;V) \cap W^{1,2}(0,T;V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0,T];V), \quad \mathcal{W}_1(T) \subset C([0,T];H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \leq M_0||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \leq M_0||x||_{\mathcal{W}_1(T)}. \tag{2.5}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \leq M, \quad ||s(t)||_* \leq M.$$

The following Lemma is from Lemma 3.6.2 of [1].

Lemma 2.2. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:*

$$|S(t)x| \leq Mt^{-1/2}||x||_*, \quad ||S(t)x|| \leq Mt^{-1/2}|x|.$$

Lemma 2.3. *(a) A^α is a closed operator with its domain dense.*

(b) If $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$.

(c) *For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$.*

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \le \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(H,V)} \le \frac{C_\alpha}{t^{3\alpha/2}}.$$

Proof. *From [1,Lemma 3.6.2] it follows that there exists a positive constant C such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^* :*

$$|AS(t)x| \le \frac{C}{t}|x|, \quad ||AS(t)x|| \le \frac{C}{t^{3/2}}|x|.$$

First of all, consider the following linear system

$$\begin{cases} x'(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.6}$$

By virtue of Theorem 3.3 of [8](or Theorem 3.1 of [4], [1]), we have the following result on the corresponding linear equation of (2.6).

Lemma 2.4. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.6) belonging to $\mathcal{W}(T) \subset C([0,T];V)$ and satisfying

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.7}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (2.6) belonging to $\mathcal{W}_1(T) \subset C([0,T];H)$ and satisfying

$$||x||_{\mathcal{W}_1(T)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.8}$$

where C_1 is a constant depending on T.

Lemma 2.5. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}, \tag{2.9}$$
$$||x||_{L^2(0,T;H)} \le C_2 T||k||_{L^2(0,T;H)}, \tag{2.10}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.11}$$

Proof. The assertion (2.9) is immediately obtained by (2.7). Since

$$\begin{aligned}||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \le M\int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M\int_0^T t\int_0^t |k(s)|^2 dsdt \le M\frac{T^2}{2}\int_0^T |k(s)|^2 ds\end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.3), (2.9), and (2.10) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

3 Semilinear differential equations

Let f be a nonlinear mapping from V into H.

Assumption (F). There exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(r_1) \le L(r_2)$ for $r_1 \le r_2$ and

$$|f(t,x)| \le L(r), \quad |f(t,x) - f(t,y)| \le L(r)||x-y||$$

hold for any $t \in [0,T]$, $||x|| \le r$ and $||y|| \le r$.

Assumption (I). The functions $I_k : V \to H$ are continuous and there exist positive constants $L(I_k)$ and $\beta \in (1/3, 1]$ such that

$$|A^\beta I_k(x)| \le L(I_k)||x||, \quad |A^\beta I_k(x) - I_k(y)| \le L(I_k)||x-y||, \quad k = 1,2,\cdots,m$$

for each $x, y \in V$, and

$$||x(t_k^-)|| \le K, \quad k = 1,2,\cdots,m.$$

From now on, we establish the following results on the local solvability of the following equation;

$$\begin{cases} x'(t) + Ax(t) = f(t,x(t)) + k(t), \quad t \in (0,T], \quad t \ne t_k, \\ \quad k = 1,2,\cdots,m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)),\ k = 1,2,\cdots,m, \\ x(0) = x_0. \end{cases} \tag{3.1}$$

Let us rewrite $(Fx)(t) = f(t, x(t))$ for each $x \in L^2(0,T;V)$. Then there is a constant, denoted again by $L(r)$, such that

$$||Fx||_{L^2(0,T;H)} \le L(r)\sqrt{T}, \quad ||Fx_1 - Fx_2||_{L^2(0,T;H)} \le L(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0,T;V) : ||x||_{L^2(0,T;V)} \le r\}$.

Here, we note that by using interpolation theory, we have that for any $t > 0$,

$$L^2(0,t;V) \cap W^{1,2}(0,t;V^*) \subset C([0,t];H).$$

Thus, for any $t > 0$, there exists a constant $c > 0$ such that

$$||x||_{C([0,t];H)} \le c||x||_{L^2(0,t;V)\cap W^{1,2}(0,t;V^*)}. \tag{3.2}$$

Let

$$0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m = T.$$

Then by Assumption (I) and (3.1), it is immediately seen that

$$x \in W^{1,2}(t_i, t_{i+1}; V^*), \quad i = 0, \cdots, m-1.$$

Thus by virtue of Assumption (I) and (3.2), we may consider that there exists a constant $C_3 > 0$ such that

$$\max_{0\le t\le T}\{|x(t)| : x \text{ is a solution of } (3.1)\} \le C_3||x||_{L^2(0,T;V)}. \tag{3.3}$$

From now on, we establish the following results on the solvability of the equation(3.1).

Theorem 3.1. *Let Assumption (F) be satisfied. Assume that $x_0 \in H$, $k \in L^2(0,T;V^*)$. Then, there exists a time $T_0 \in (0,T)$ such that the equation (3.1) admits a solution*

$$x \in W_1(T_0) \subset C([0,T_0];H). \tag{3.4}$$

Proof. For a solution of (3.1) in the wider sense, we are going to find a local solution of the following integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\{(Fx)(s) + k(s)\}ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)). \tag{3.5}$$

To prove a local solution, we will use the successive iteration method. First, put

$$x_0(t) = S(t)x_0 + \int_0^t S(t-s)k(s)ds$$

and define $x_{j+1}(t)$ as

$$x_{j+1}(t) = x_0(t) + \int_0^t S(t-s)(Fx_j)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x_j(t_k^-)). \tag{3.6}$$

By virtue of Lemma 2.4, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0(\cdot)||_{\mathcal{W}_1(t)} \leq C_1(|x_0| + ||k||_{L^2(0,t;V^*)}), \tag{3.7}$$

where C_1 is a constant in Lemma 2.4. Choose $r > C_1(|x_0| + ||k||_{L^2(0,t;V^*)})$. Putting $p(t) = \int_0^t S(t-s)(Fx_0)(s)ds$, by (2.11) of Lemma 2.5, we have

$$||p||_{L^2(0,t;V)} \leq C_2\sqrt{t}||Fx_0||_{L^2(0,t;H)} \leq C_2 L(r)t. \tag{3.8}$$

Putting $g(t) := S(t-t_k)I_k(x(t_k^-))$, by Assumption (I) and Lemma 2.3, we have

$$||g(t)||_{L^2(0,t;V)} \leq 2(3\beta)^{-1/2}(3\beta-1)^{-1}C_{1-\beta}KL(I_k)t^{3\beta/2}. \tag{3.9}$$

Put

$$M_1 := \max\{C_2L(r)t, 2(3\beta)^{-1/2}(3\beta-1)^{-1}C_{1-\beta}KL(I_k)t^{3\beta/2}\} \tag{3.10}$$

then for any t satisfying, $M_1 < r$, from (3.4) and (3.5).
so that, from(3.7) and (3.8) and (3.9),

$$||x_1||_{L^2(0,t;V)} \leq r + C_2L(r)t + 2(3\beta)^{-1/2}(3\beta-1)^{-1}C_{1-\beta}K \sum_{0<t_k<t} L(I_k)t^{3\beta/2} \leq 3r.$$

By induction, it can be shown that for all $j = 1, 2, ...$

$$||x_j||_{L^2(0,t;V)} \leq 3r, 0 \leq t \leq M_1. \tag{3.11}$$

Hence, from the equation

$$x_{j+1}(t) - x_j(t) = \int_0^t S(t-s)\{f(t, x_j(s)) - f(t, x_{j-1}(s))\}ds + \sum_{0<t_k<t} S(t-t_k)\{I_k(x_j(t_k^-)) - I_k(x_{j-1}(t_k^-)\}.$$

Set

$$h(t) := S(t-t_k)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}.$$

Then from (3.2) and (3.3) it follows that

$$
\begin{aligned}
||h||_{L^2(0,T;V)} &= [\int_0^T || \int_{t_k}^t S'(s-t_k)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}ds||^2 dt]^{1/2} \\
&\le [\int_0^T || \int_{t_k}^t AS(s-t_k)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}ds||^2 dt]^{1/2} \\
&\le [\int_0^T \{\int_{t_k}^t \frac{C_{1-\beta}}{(s-t_k)^{3(1-\beta)/2}} L(I_k)|(x_1(t_k^-) - x_2(t_k^-))|ds\}^2 dt]^{1/2} \\
&\le (3\beta)^{-1/2} 2(3\beta-1)^{-1} C_{1-\beta} C_3 L(I_k) T^{3\beta/2} ||x_1 - x_2||_{L^2(0,T;V)}.
\end{aligned}
$$

Hence, from the equation

$$
\begin{aligned}
x_{j+1}(t) - x_j(t) = &\int_0^t S(t-s)\{(Fx_j)(s) - (Fx_{j-1})(s)\}ds \\
&+ \sum_{0<t_k<t} S(t-t_k)\{I_k(x_j(t_k^-)) - I_k(x_{j-1}(t_k^-))\}.
\end{aligned}
$$

Put

$$
M_2 := C_2 L(3r)\sqrt{t} + (3\beta)^{-1/2} 2(3\beta-1)^{-1} C_{1-\beta} C_3 \sum_{0<t_k<t} L(I_k) t^{3\beta/2}. \tag{3.12}
$$

Then from (2.11), (3.11) and Assumption (F), we can observe that the inequality

$$
\begin{aligned}
||x_{j+1} - x_j||_{L^2(0,t;V)} &\le C_2 L(3r)\sqrt{t}||x_j - x_{j-1}||_{L^2(0,t;V)} \\
&\quad + (3\beta)^{-1/2} 2(3\beta-1)^{-1} C_{1-\beta} C_3 \sum_{0<t_k<t} L(I_k) t^{3\beta/2} ||x_j - x_{j-1}||_{L^2(0,t;V)} \\
&\le M_2 ||x_j - x_{j-1}||_{L^2(0,t;V)} \\
&\le (M_2)^j ||x_1 - x_0||_{L^2(0,t;V)}.
\end{aligned}
$$

Choose $T_0 > 0$ satisfying $max\{M_1, M_2\} < 1$. Then $\{x_j\}$ is strongly convergent to a function x in $L^2(0, T_0; V)$ uniformly on $0 \le t \le T_0$. By letting $j \to \infty$ in (3.6) has a unique solution x in $\mathcal{W}_1(T)$. □

From now on, we give a norm estimation of the solution of (3.1) and establish the global existence of solutions with the aid of norm estimations.

Theorem 3.2. *Under the assumption (F) for the nonlinear mapping f, there exists a unique solution x of (3.1) such that*

$$
x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H), \quad T > 0.
$$

for any $x_0 \in H$, $k \in L^2(0,T;V^*)$. *Moreover, there exists a constant* C_3 *such that*

$$||x||_{\mathcal{W}_1(T)} \le C_4(1 + |x_0| + ||k||_{L^2(0,T;V^*)}), \tag{3.13}$$

where C_4 *is a constant depending on* T.

Proof. Let x be a solution of (3.1) on $[0, T_0]$, $T_0 > 0$ satisfies $\max\{M_1, M_2\} < 1$. Here M_1 and M_2 be constants in (3.10) and (3.12), respectively. Then by virtue of Theorem3.1, the solution x is represented as

$$x(t) = x_0(t) + \int_0^t S(t-s)(Fx)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)).$$

where

$$x_0(t) = S(t)x_0 + \int_0^t S(t-s)k(s)ds.$$

By (3.7), we have $x_0(\cdot) \in \mathcal{W}_1(T_0)$, so that

$$||x_0||_{\mathcal{W}_1(T_0)} \le C_1(|x_0| + ||k||_{L^2(0,T_0;V^*)}),$$

where C_1 is constant in Lemma2.4. Moreover, from (3.7)-(3.9), it follow that

$$||x||_{\mathcal{W}_1(T_0)} \le C_1(|x_0| + ||k||_{L^2(0,T_0;V^*)}) + \max\{M_1, M_2\}||x||_{\mathcal{W}_1(T_0)}. \tag{3.14}$$

Thus, Moreover, there exists a constant C_4 such that

$$||x||_{\mathcal{W}_1(T_0)} \le C_4(1 + |x_0| + ||k||_{L^2(0,T_0;V^*)}).$$

Now from

$$|S(t)x_0 + \int_0^t S(t-s)\{(Fx)(s) + k(s)\}ds| \le M|x_0| + MtL(r) + M\sqrt{t}||k||_{L^2(0,t;H)},$$

$$|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))| \le MK|A^{-\beta}|\sum_{0<t_k<t} L(I_k).$$

it follow

$$|x| \le M|x_0| + MT_0L(r) + M\sqrt{T_0}||k||_{L^2(0,T_0;H)} + MK|A^{-\beta}|\sum_{0<t_k<T_0} L(I_k) < \infty.$$

Hence, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$ and obtain an analogous estimate to (3.14). Since the condition (3.10),(3.12) is independent of initial value, the solution can be extended to the interval $[0, nT_0]$ for any natural number n, i.e., for the initial $u(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (3.14) holds for the solution in $[0, (n+1)T_0]$. □

From the following result, we obtain that the solution mapping is continuous, which is useful for physical application of the given equation.

Theorem 3.3. *Let the assumption (F) and (I) be satisfied and* $(x_0, k) \in H \times L^2(0,T;V)$*. Then the solution* x *of the equation (3.1) belongs to* $x \in \mathcal{W}_1 \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ *and the mapping*

$$H \times L^2(0,T;V^*) \ni (x_0, k) \mapsto x \in \mathcal{W}_1(T) \tag{3.15}$$

is continuous.

Proof. From Theorem 3.2, it follows that if $(x_0, k) \in H \times L^2(0,T;V^*)$ then x belongs to $\mathcal{W}_1(T)$. Let $(x_{0i}, k_i) \in H \times L^2(0,T;V^*)$ and $x_i \in \mathcal{W}_1(T)$ be the solution of (3.1) with (x_{0i}, k_i) in place of (x_0, k) for $i = 1, 2$. Hence, we assume that x_i belongs to a ball $B_r(T) = \{y \in \mathcal{W}_1(T) : ||y||_{\mathcal{W}_1(T)} \le r\}$.

Let

$$(px_j)(t) = \int_0^t S(t-s)Fx_j(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x_j(t_k^-)).$$

Then, by virtue 2) of Lemma2.4, we get

$$||x_1 - x_2||_{W_1(T)} = C_1\{|x_1 - x_2| + ||k_1 - k_2||_{L^2(0,T;V^*)} + ||px_1 - px_2||_{L^2(0,T;V^*)}\}. \tag{3.16}$$

Set $||\cdot||_{L^2(0,T_0;V)} = ||\cdot||_{L^2}$ for brevity, where $T_0 > 0$ satisfies $max\{M_1, M_2\} < 1$. Then, we have

$$\begin{aligned} &||px_1 - px_2||_{L^2(0,T_0;V^*)} \le ||px_1 - px_2||_{L^2} \\ &= ||\int_0^t S(t-s)\{Fx_1 - Fx_2\}ds||_{L^2} \\ &+ ||\sum_{0<t_k<t} S(t-t_k)\{I_k(x_i(t_k^-)) - I_k(x_2(t_k^-))||_{L^2} \\ &\le M_2||x_1 - x_2||_{L^2}. \end{aligned} \tag{3.17}$$

Hence, by (3.16), (3.17), we see that

$$x_n \mapsto x \in \mathcal{W}_1(T_0) \equiv L^2(0,T_0;V) \cap W^{1,2}(0,T_0;V^*).$$

This implies that $(x_n(T_0), (x_n)_{T_0}) \mapsto (x(T_0), x_{T_0})$ in $H \times L^2(0,T;V^*)$. Hence the same argument show that $x_n \mapsto x$ in

$$L^2(0, min\{2T_0, T\}; V) \cap W^{1,2}(0, min\{2T_0, T\}; V^*).$$

Repeating this process we conclude that $x_n \mapsto x$ in $\mathcal{W}_1(T)$.

□

4 Approximate Controllability

Consider the following nonlinear equation. Let U be a Banach space of control variables. Here B is a linear bounded operator from $L^2(0,T;U)$ to $L^2(0,T;H)$, which is called a controller.

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)) + (Bu)(t), \quad t \in [0,T],\ t \ni t_k, \\ x(0) = x_0. \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)),\ k = 1, 2, \cdots, m. \end{cases} \tag{4.1}$$

Let $x(T; f, u)$ be a state value of the system (4.1) at time T corresponding to the nonlinear term f and the control u. Let $S(\cdot)$ be the analytic semigroup generated by $-A$. Then the solution $x(t; f, u)$ can be written as

$$x(t; f, u) = S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s, f, u)) + (Bu)(s)\}ds + \sum_{0<t_k<t} S(t-s)I_k(x(t_k^-)),$$

and in view of Theorem 3.2

$$||x(\cdot; f, u)||_{\mathcal{W}_1(T)} \le C_4(1 + |x_0| + ||B||||u||_{L^2(0,T;U)}). \tag{4.2}$$

We define the reachable sets for the system (4.1) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0,T;U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0,T;U)\}.$$

Definition 4.1. *The system (4.1) is said to be approximately controllable at time T if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0,T;U)$ such that the solution $x(T; f, u)$ of (4.1) satisfies $|x(T; f, u) - x_1| < \epsilon$, that is, $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H.*

We define a linear bounded operator $\hat{S}$ from $L^2(0,T;H)$ to H by

$$\hat{S}p = \int_0^T S(T-t)p(t)dt,$$

for $p(\cdot) \in L^2(0,T;H)$.

Assumption (B) For any $\varepsilon > 0$, $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}Bu| \le \varepsilon \\ ||Bu||_{L^2(0,t;H)} \le q_1||p||_{L^2(0,t;H)}, \quad 0 \le t \le T \end{cases}$$

where q is a constant independent of p.

Assumption (F1) The nonlinear operator f is a nonlinear mapping of $[0,T]\times H$ into H satisfying the following. There exists a constant $L_1 = L_1(r) > 0$ such that

$$|f(t,x) - f(t,y)| \le L_1||x-y||, \quad t \in [0,T],$$

hold for $||x|| \le r$ and $||y|| \le r$.

Assumption (H) We assume the following inequality condition:

$$max\{q,1\}\{1-M_2\}^{-1}C_2L_1\sqrt{T} < 1.$$

where C_2 is the constant in (2.11),

$$M_2 = C_2\sqrt{T}L_1 + (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3T^{3\beta/2}\sum_{0\le t_k \le T} L(I_k)$$

as seem in (3.12).

Lemma 4.1. *Let u_1 and u_2 be in $L^2(0,T;U)$. Then under Assumption(B) and Assumption$(F1)$, one has that, for $0 \le t \le T$,*

$$||x(t:f,u_1) - x(t:f,u_2)||_{L_2(0,T;V)} \le \{1-M_2\}^{-1}C_2\sqrt{t}||Bu_1 - Bu_2||_{L^2(0,T;H)}. \quad (4.3)$$

Proof. Let $x_1(t) = x(t:f,u_1)$ and $x_2(t) = x(t:f,u_2)$. Then for $0 \le t \le T$,we have

$$\begin{aligned} x_1(t) - x_2(t) = &\int_0^t S(t-s)\{f(s,x_1(s)) - f(s,x_2(s))\}ds \\ &+ \int_0^t S(t-s)\{Bu_1 - Bu_2\}ds \\ &+ \sum_{0\le t_k\le T} S(t-s)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}. \end{aligned} \quad (4.4)$$

By $Assumption(F1)$ and Lemma 2.5 of (2.11), we obtain

$$||\int_0^t S(t-s)\{f(s,x_1(s))-f(s,x_2(s))\}ds||_{L^2(0,t;V)} \le C_2\sqrt{t}L_1||x_1-x_2||_{L^2(0,t;V)}.$$

Moreover, by Lemma 2.5 of (2.11) and Theorem 3.1, we have

$$||\int_0^t S(t-s)\{Bu_1-Bu_2\}ds||_{L^2(0,t;V)} \le C_2\sqrt{T}||Bu_1-Bu_2||_{L^2(0,t;H)}$$

and

$$\begin{aligned}&||\sum_{0\le t_k\le t} S(t-s)\{I_k(x_1(t_k^-))-I_k(x_2(t_k^-))\}||_{L^2(0,t;V)}\\ &\le (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3t^{3\beta/2}\sum_{0\le t_k\le t} L(I_k)||x_1(t_k^-)-x_2(t_k^-)||_{L^2(0,t;V)}.\end{aligned}$$

Thus, from (4.4) it follows that

$$\begin{aligned}&||x(t;f,u_1)-x(t;f,u_2)||_{L^2(0,T;V)}\\ &\le C_2\sqrt{T}||Bu_1-Bu_2||_{L^2(0,T;H)}+C_2\sqrt{T}L_1||x_1-x_2||_{L^2(0,T;V)}\\ &+(3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3t^{3\beta/2}\sum_{0\le t_k\le t} L(I_k)||x_1(t_k^-)-x_2(t_k^-)||_{L^2(0,T;V)}.\end{aligned}$$

Theorem 4.1. *Under Assumptions (B),(F1), and (H) the system(4.1) is approximately controllable on* $[0,T]$.

Proof. The reachable set for the system(4.1) is given by

$$R_T=\{x(T;f,u): u\in L^2(0,T;U)\}.$$

We will show that $D(A)\subset \overline{R_T(f)}$, i.e., for given $\varepsilon>0$ and $\xi_T\in D(A)$, there exists $u\in L^2(0,T;U)$ such that

$$|\xi_T - x(T;f,u)|<\varepsilon, \tag{4.5}$$

where

$$\begin{aligned}x(T;,f,u)=S(T)x_0+\int_0^T S(T-s)\{f(s,x(s,f,u))+(Bu)(s)\}ds\\ +\sum_{0<t_k<T} S(T-s)I_k(x(t_k^-)).\end{aligned} \tag{4.6}$$

As $\xi_T \in D(A)$ there exists a $p \in L^2(0,T;H)$such that

$$\hat{S}p = \xi_T - S(T)x_0,$$

for instance, take $p(s) = (\xi_T - sA\xi_T) - S(s)x_0/T$. Let $u_1 \in L^2(0,T;U)$be arbitrary fixed. Since by Assumption (B) there exists $u_2 \in L^2(0,T;U)$ such that

$$|\hat{S}(p - f(\cdot, x(\cdot; f, u_1))) - \hat{S}Bu_2| < \frac{\varepsilon}{4}, \tag{4.7}$$

it follows that

$$|\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_1)) - \hat{S}Bu_2| < \frac{\varepsilon}{4}. \tag{4.8}$$

We can also choose $w_2 \in L^2(0,T;U)$ by Assumption (B) such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_2)) - f(\cdot, x(\cdot; f, u_1))) - \hat{S}Bw_2| < \frac{\varepsilon}{8} \tag{4.9}$$

$$||Bw_2||_{L^2(0,T;H)} \leq q||f(\cdot, x(\cdot; f, u_2)) - f(\cdot, x(\cdot; f, u_1))||_{L^2(0,T;H)}.$$

Choose a constant r_1 satisfying

$$||x(\cdot; f, u_1)||_{C([0,T];H)} \leq r_1, ||x(\cdot; f, u_2)||_{C([0,T];H)} \leq r_1.$$

Therefor, in view of Lemma4.1 and Assumption (B)

$$\begin{aligned} ||Bw_2||_{L^2(0,T;H)} &\leq q||f(s, x(s; f, u_2)) - f(s, x(s; f, u_1))||_{L^2(0,T;H)} \\ &\leq qL_1||x(t; f, u_1) - x(t; f, u_2)||_{L^2(0,T;V)} \\ &\leq q\{1 - M_2\}^{-1}C_2L_1\sqrt{T}||Bu_1 - Bu_2||_{L^2(0,T;H)}. \end{aligned} \tag{4.10}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_3)) - f(\cdot, x(\cdot; f, u_2))) - \hat{S}Bw_3| < \frac{\varepsilon}{8}$$

$$||Bw_3||_{L^2(0,T;H)} \leq q||f(\cdot, x(\cdot; f, u_3)) - f(\cdot, x(\cdot; f, u_2))||_{L^2(0,T;H)}.$$

Let r_2 be a constant satisfying $r_2 \geq r_1$ and

$$||x(\cdot; f, u+3)||_{C([0,T];H)} \leq r_2.$$

Then, in a similar way to (4.10) we have

$$\begin{aligned} ||Bw_3||_{L^2(0,T;H)} &\leq q||f(s, x(s; f, u_3)) - f(s, x(s; f, u_2))||_{L^2(0,T;H)} \\ &\leq qL_1||x(t; f, u_3) - x(t; f, u_2)||_{L^2(0.T;V)} \\ &\leq q\{1 - M_2\}^{-1}C_2L_1\sqrt{T}||Bu_2 - Bu_3||_{L^2(0,T;H)} \\ &\leq (q\{1 - M_2\}^{-1}C_2L_1\sqrt{T})^2||Bu_1 - Bu_2||_{L^2(0,T;H)}. \end{aligned}$$

By proceeding with this process and from

$$\begin{aligned}&||B(u_n - u_{n+1})||_{L^2(0,T;H)}\\&\quad = ||Bw_n||_{L^2(0,T;H)} \le (q\{1-M_2\}^{-1}C_2L_1\sqrt{T})^{n-1}||B(u_2-u_1)||_{L^2(0,T;H)}.\end{aligned}$$

Here, nothing that Assumption (H) is equivalent to

$$q\{1-M_2\}^{-1}C_2L_1\sqrt{T} < 1,$$

it follows that there exists $u^* \in L^2(0,T;H)$ such that

$$\lim_{n\to\infty} Bu_n = u^* \quad \text{in} \quad L^2(0,T;H).$$

From(4.8),(4.9) it follow that

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_2)) - \hat{S}Bu_3|\\&\quad = |\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_1)) - \hat{S}Bu_2 + \hat{S}Bw_2\\&\quad - [\hat{S}f(\cdot, x(\cdot; f, u_2)) - \hat{S}f(\cdot, x(\cdot; f, u_1))]|\\&\quad < (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.\end{aligned}$$

By choosing $w_n \in L^2(0,T;U)$ by Assumption (B), such that

$$|\hat{S}(f(\cdot, x(\cdot; f, u_n)) - f(\cdot, x(\cdot; f, u_{n-1}))) - \hat{S}Bw_n| < \frac{\varepsilon}{2^{n+1}}$$

putting $u_{n+1} = u_n - w_n$ we have

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_n)) - \hat{S}Bu_{n+1}|\\&\quad < (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1, 2, \ldots.\end{aligned}$$

Therefor, for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Bu_{N+1} - \hat{S}Bu_N| < \frac{\varepsilon}{2},$$

$$\begin{aligned}&|\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_N)) - \hat{S}Bu_N|\\&\quad \le |\xi_T - S(T)x_0 - \hat{S}f(\cdot, x(\cdot; f, u_N)) - \hat{S}Bu_{N+1}| + |\hat{S}Bu_{N+1} - \hat{S}Bu_N|\\&\quad \le (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.\end{aligned}$$

Thus the system (4.1) is approximately controllable on $[0, T]$ as N tends to infinity. $\square$

Example. We consider the semilinear heat equation dealt with by Naito [19] and Zhou [21]. Let

$$
\begin{aligned}
&H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),\\
&a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx
\end{aligned}
$$

and

$$A = -d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$
\begin{cases}
\frac{\partial}{\partial t}x(t,y) + Ax(t,y) = f(t, x(t,y)) + Bu(t), \quad t \in (0,T], \quad t \neq t_k,\\
k = 1, 2, \cdots, m,\\
\Delta x(t_k) = x(t_k^+, y) - x(t_k^-, y) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m,\\
x(t,0) = x(t,\pi) = 0, \quad t > 0\\
x(0,y) = x_0(y).
\end{cases}
\tag{4.8}
$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Let

$$
\begin{aligned}
&U = \{\sum_{n=2}^\infty u_n\phi_n : \sum_{n=2}^\infty u_n^2 < \infty\},\\
&Bu = 2u_2\phi_1 + \sum_{n=2}^\infty u_n\phi_n, \quad \text{for} \quad u = \sum_{n=2}^\infty u_n \in U,\\
&T > 0.
\end{aligned}
$$

In [19] Naito showed that the operator B is one to one, $\mathrm{R}(B)$ is closed and $L^2(0,T) = R(B) + N$, where $R(B)$ is the range of the operator B. It follows that the operator B satisfies Assumption (A).

We assume that the nonlinear operator $f : [0,T] \times V \to H$ is continuous and there is a constant $0 < \gamma < 1$ and a function $k \in L^2[0,T]$ such that

$$|f(t,x)| \leq k(s)||x||^\gamma, \quad \forall (t,x) \in [0,T] \times V.$$

Hence, Assumption (F) and (4.4) are satisfied. Therefore, by Theorem 4.1 with condition on Assumption (I), the semilinear system (4.8) is approximately controllable at time T.

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