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A Study on Approximate Solutions in Robust Minimax Programming

by

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August 24, 2018

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로버스트 최소 최대계획 문제에서의 근사해에 관한 연구

Advisor: Prof. Do Sang Kim

by

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요 약

본 논문에서는 로버스트 볼록 최소최대 최적화 문제에서의 최적해와 의사 α-해의 몇 가지 특성에 대한 최적조건과 쌍대정리에 대해 연구하였다. 이로부터 얻은 결과를 이용하여 로버스트 다목적 최적화 문제에서의 약 Pareto 해와 약 의사 Pareto 근사해에 대한 최적조건과 쌍대정리를 정립하였다. 그리고 최악의 경우 로버스트 최적화 접근법을 이용하여 데이터 불확실성에 직면한 로버스트 볼록 최소최대 분수계획 문제를 다루었다. 또한 로버스트 볼록 최소최대 분수 계획 문제를 Slater 조건하에서 최적조건과 근사 쌍대정리를 확립하였다.

Chapter 1
Introduction and Organization

1.1 Introduction

Minimax programming problems have been the subject of immense interest in the past few years. Some of the basic results of minimax programming problems can be found in books by Danskin [17] and Demyanov and Molozemov [18]. It is well known that optimality and duality lay down the foundation of algorithms for a solution of an optimization problem and hence constitute an important portion in the study of mathematical programming. The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [40]. After the work of Schmitendorf [40], many researchers have worked in this direction; see, for example, Antczak [1], Lai *et al.* [31], Yang and Hou [47] and the references therein.

Mathematically, a minimax programming problem is the problem:

$$\text{(P)}\ \min_{x\in\mathbb{R}^n}\max_{k\in\mathrm{K}} f_k(x) \text{ subject to } g_i(x) \leqq 0,\ i=1,\ldots,m,$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in \mathrm{K} := \{1,\ldots,l\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\ldots,m$ are given functions.

In addition, a minimax fractional programming problem is the one:

$$\text{(FP)}\ \min_{x\in\mathbb{R}^n}\max_{k\in\mathrm{K}} \frac{p_k(x)}{q_k(x)} \ \text{ subject to }\ g_i(x) \leqq 0,\ i=1,\ldots,m,$$

where $p_k,\ -q_k : \mathbb{R}^n \to \mathbb{R}$, $k \in \mathrm{K} := \{1, \ldots, l\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are given functions.

As we mentioned above, with regards to the problems, both (P) and (FP) have been studied by many researcher; see, for example, [1, 12, 31–33, 40, 47] and the references therein.

On the other hand, the data of many real-world optimization problems are often uncertain (that is, they are not known exactly at the time of the decision) due to lack of information, estimation errors or prediction errors. Recently, robust optimization approach, which associates an uncertain mathematical programming with its robust counterpart (see, for example, [2, 4, 8, 23, 45]), has emerged as a powerful deterministic approach for studying mathematical (both scalar and multiobjective) optimization with data uncertainty. Moreover, a robust fractional optimization problem is to optimize a fractional function over the constrained set defined by functions with data uncertainty.

The minimax programming problem (P) and the minimax fractional programming problem (FP) in the face of data uncertainty in the constraints can be captured by the problems

$$\text{(UP)} \ \min_{x \in \mathbb{R}^n} \max_{k \in \mathrm{K}} \ f_k(x) \ \text{ subject to } \ g_i(x, v_i) \leqq 0,\ i = 1, \ldots, m,$$

and

$$\text{(UFP)} \ \min_{x \in \mathbb{R}^n} \max_{k \in \mathrm{K}} \ \frac{p_k(x)}{q_k(x)} \ \text{ subject to } \ g_i(x, v_i) \leqq 0,\ i = 1, \ldots, m,$$

respectively, where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$.

The robust programming approach tells us to seek for a solution which simultaneously satisfies all possible realizations of the constraints. Throughout the thesis, we explore optimality and duality theorems for the uncertain minimax programming problem (UP) and the uncertain minimax fractional programming problem (UFP) by examining their robust (worst-case) counterparts:

$$\text{(RP)} \ \min_{x\in\mathbb{R}^n} \max_{k\in\mathrm{K}} \ \ f_k(x) \ \text{ subject to } \ g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m,$$

and

$$\text{(RFP)} \ \min_{x\in\mathbb{R}^n} \max_{k\in\mathrm{K}} \frac{p_k(x)}{q_k(x)} \ \text{ subject to } \ g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m,$$

respectively.

It is worth noting that, from the computational point of view, it may be more meaningful to find not exact solutions but approximate ones. Indeed, one can consider approximate solutions with a small error while solving optimization problem by a numerical method and; moreover, in some problems, if error value tends to zero, the limit of approximate solution is an exact solution, if it exists. It is meaningful not only to find solutions but establish necessary and sufficient conditions. It is well known that optimization problems may be viewed from either of two perspectives, i.e. the primal problem or the dual problem. Moreover, for convex optimization problems, the duality gap, i.e. the difference between optimal values of the primal and dual problems, is zero under a constraint qualification.

1.2 Organization of the Dissertation

This dissertation consists of three main chapters.

Chapter 2 presents some characterizations of an optimal solution and a quasi α-solution for the robust convex minimax optimization problem (RP), a dual model in the sense of Wolfe is established, and duality relations are also discussed; in addition, a nontrivial example is given.

Chapter 3 can be treated as applications of Chapter 2; namely, with the help of the results obtained by Chapter 2, we study optimality conditions and duality theorems both for a weak Pareto solution and a weak quasi ϵ-Pareto solution to the robust multiobjetive optimization problem.

In Chapter 4, we study a robust convex minimax fractional programming problem in the face of data uncertainty. Again, using the robust optimization approach (worst-case approach), optimality conditions and approximate duality theorems for the robust convex minimax fractional programming problem are explored under the Slater condition.

Finally, the Conclusions are given in the end of the dissertation.

Chapter 2
Optimality Conditions and Duality for Optimal and Approximate Solutions in Robust Minimax Programming

2.1 Introduction

The study of optimality conditions and duality relations for optimal solutions of minimax programming problems has been done by many researchers; see, for example, [12, 32, 33] and the references therein.

Along with optimality conditions, we propose a dual problem to the primal one and examine weak and strong duality relations.

In addition, we employ the (necessary/sufficient) optimality conditions obtained for the mimimax programming problem to derive the corresponding ones for a multiobjective optimization problem. This approach seems to be new in the literature, and we hope it will provide a useful opportunity to learn about a multiobjective optimization problem from the related mimimax programming problem, a *scalar* one.

The rest of the paper is organized as follows. Section 2 contains some basic definitions from variational analysis and several auxiliary results. In Section 3, we first establish necessary conditions for (local) optimal solutions of a mimimax programming problem. Then we provide sufficient conditions for the existence of such (global) solutions. Section 4 is devoted to studying duality relations in mimimax programming. Applications to multiobjective optimization problems are performed in Section 5.

2.2 Preliminaries

We use the following notation and terminology. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\| \cdot \|$. We say that a set Γ in $\mathbb{R}^n$ is *convex* whenever $\mu a_1 + (1-\mu)a_2 \in \Gamma$ for all $\mu \in [0,1]$, $a_1, a_2 \in \Gamma$. We denote the domain of f by dom f, that is, dom $f := \{x \in \mathbb{R}^n \colon f(x) < +\infty\}$. f is said to be *convex* if for all $\lambda \in [0,1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$. The function f is said to be *concave* whenever $-f$ is convex. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle \leq f(y) - f(x),\ \forall y \in \mathbb{R}^n\}, & \text{if } x \in \text{dom} f, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Proposition 2.2.1 (Cauchy–Schwartz inequality)**.** *For any two vectors* $x, y \in \mathbb{R}^n$, $|\langle x, y \rangle| \leqq \|x\|\|y\|$. *The above inequality holds as equality if and only if* $x = \alpha y$ *for some scalar* $\alpha \in \mathbb{R}$.

Lemma 2.2.1 (Moreau–Rockafellar sum rule)**.** *Consider two proper convex functions* $f_1, f_2 : \mathbb{R}^n \to \bar{\mathbb{R}}$ *such that* ri dom $f_1 \cap$ ri dom $f_2 \neq \emptyset$. *Then*

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$$

for every $x \in$ dom $(f_1 + f_2)$.

Proposition 2.2.2 (max-function rule)**.** *Consider convex functions* $f_k : \mathbb{R}^n \to \mathbb{R}, k = 1, \ldots, l,$ *and let* $\varphi(x) = \max\{f_1(x), \ldots, f_l(x)\}$*. then*

$$\partial\varphi(\bar{x}) = \text{co} \bigcup_{k \in \mathrm{K}(\bar{x})} \partial f_k(\bar{x}),$$

where $\mathrm{K}(\bar{x}) := \{k \in \mathrm{K} := \{1, \ldots, l\} : \varphi(\bar{x}) = f_k(\bar{x})\}$ *denotes the active index set.*

2.3 Mathematical Model and Representation of the Normal Cone

A standard form of minimax programming problem is the problem:

$$\text{(P)} \;\; \min_{x \in \mathbb{R}^n} \max_{k \in \mathrm{K}} f_k(x) \;\; \text{subject to} \;\; g_i(x) \leqq 0, \;\; i = 1, \ldots, m,$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in \mathrm{K} := \{1, \ldots, l\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are convex functions.

The minimax programming problem (P) in the face of data uncertainty in the constraints can be captured by the problem

$$\text{(UP)} \;\; \min_{x \in \mathbb{R}^n} \max_{k \in \mathrm{K}} \;\; f_k(x) \;\; \text{subject to} \;\; g_i(x, v_i) \leqq 0, \;\; i = 1, \ldots, m,$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$. The problem (UP) is to optimize an optimization problem with data uncertainty (incomplete data), which means that input parameters of these problems are not known exactly at the time when solution has to be determined [7]. Indeed, there are two main approaches to deal with constrained optimization under data uncertainty, namely *robust programming approach* and *stochastic programming approach*; in stochastic programming, one works with the probabilistic distribution of uncertainty and the constraints are required to be satisfied up to prescribed level of probability [22], while robust programming approach seeks for a solution which simultaneously satisfies all possible realizations of the constraints. It seems to be more convenient to use the robust approach to study optimization problems with data uncertainty, comparing with stochastic programming approach.

Throughout the thesis, we explore optimality and duality theorems for the uncertain minimax programming problem (UP) by examining its robust (worst-case) counterpart [7]:

$$\text{(RP)} \quad \min_{x\in\mathbb{R}^n} \max_{k\in \mathrm{K}} \; f_k(x) \;\; \text{subject to} \;\; g_i(x,v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i = 1,\ldots,m.$$

Denote by $F := \{x \in \mathbb{R}^n \colon g_i(x,v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i = 1,\ldots,m\}$ as the feasible set of (RP).

Definition 2.3.1. *We say the Slater condition holds for* (RP) *if there exists* $\bar{x} \in \mathbb{R}^n$ *such that*

$$g_i(\bar{x}, v_i) < 0, \forall v_i \in \mathcal{V}_i, i = 1, 2, \ldots, m.$$

Now, we establish optimality theorems for (RP) under the Slater condition. Then, by using the obtained results, we study the optimality condition for a quasi ϵ-solution to (RP) under the Slater condition. Moreover, we formulate a Wolfe type dual problem for the primal one and propose weak duality between the primal problem and its Wolfe type dual problem as well as strong duality which holds under the Slater condition. As a consequence, we study the behaviours of a quasi ϵ-solution to the dual problem. Before that, we first give the following notions of an optimal solution and a quasi ϵ-solution to the problem (RP).

Definition 2.3.2. *Let* $\varphi(x) := \max_{k \in \mathrm{K}} f_k(x), \ x \in \mathbb{R}^n$.

(i) *A point* $\bar{x} \in F$ *is said to be an optimal solution of problem* (RP) *if and only if*

$$\varphi(\bar{x}) \leqq \varphi(x), \quad \forall x \in F.$$

(ii) *Given* $\epsilon \geqq 0$. *A point* $\bar{x} \in F$ *is said to be a quasi* ϵ*-solution of problem* (RP) *if*

$$\varphi(\bar{x}) \leqq \varphi(x) + \sqrt{\epsilon}\|x - \bar{x}\|, \quad \forall x \in F.$$

It is worth noting that some characterizations of a quasi ϵ-solution to the problem (RP) has been minutely studied in [11, 28, 35].

In order to obtain Karush–Kuhn–Tucker (KKT) optimality condition in terms of the constraint functions $g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m$, the *normal cone* must be explicitly expressed in their terms. Below, we present such a result under the Slater condition. The proof is motivated by [20, Proposition 3.3] and [43, Proposition 2.3].

Lemma 2.3.1. *Let* $\bar{x} \in C := \{x \in \mathbb{R}^n \colon g(\cdot, v) \leqq 0, v \in \mathcal{V}\}$, *where* $\mathcal{V}$ *is a certain convex compact uncertain subset in* $\mathbb{R}^q$, *and* $g(\cdot, v)$ *is convex functions* *for all* $v \in \mathcal{V}$*. Suppose that the Slater condition holds for* (RP)*. Then* $\xi \in N_C(\bar{x})$ *if and only if there exist* $\bar{\lambda} \geqq 0$ *and* $\bar{v} \in \mathcal{V}$ *such that*

$$\xi \in \bar{\lambda}\partial g(\bar{x}, \bar{v}) \ \ \textit{and} \ \ \bar{\lambda} g(\bar{x}, \bar{v}) = 0.$$

Proof. Since $\mathcal{V}$ is convex and compact, we may let $\phi(x) = \sup_{v \in \mathcal{V}} g(x, v) = \max_{v \in \mathcal{V}} g(x, v)$, and the function $\phi(x)$ is convex as pointwise maxima of convex functions [2, 39]. By the definition of normal cone to the convex set C [the convexity of C is clear, since C is the 0-level set of $\phi(x)$], we have

$$\begin{aligned} N_C(\bar{x}) &= \{\xi \in \mathbb{R}^n \colon \langle \xi, x - \bar{x} \rangle \leqq 0, \ \forall x \in C\} \\ &= \{\xi \in \mathbb{R}^n \colon \langle -\xi, \bar{x} \rangle \leqq \langle -\xi, x \rangle, \ \forall x \in C\}. \end{aligned} \tag{2.1}$$

Observe that from (2.1), we see $\bar{x}$ is an optimal solution of the following convex problem with a linear objective function:

$$\text{(LP)} \ \min \ \langle -\xi, x \rangle \ \ \text{subject to} \ \ \phi(x) \leqq 0.$$

Since the Slater condition holds, by the standard KKT condition, we have there exist $\bar{\lambda} \geqq 0$ such that

$$\xi \in \bar{\lambda}\partial \phi(\bar{x}) \ \ \text{and} \ \ \bar{\lambda}\phi(\bar{x}) = 0;$$

furthermore, the compactness of $\mathcal{V}$ tells us there exist $\bar{\lambda} \geqq 0$ and $\bar{v} \in \mathcal{V}$ such that

$$\xi \in \bar{\lambda}\partial g(\bar{x}, \bar{v}) \ \ \text{and} \ \ \bar{\lambda} g(\bar{x}, \bar{v}) = 0.$$

Thus, the proof is complete. □

2.4 Optimality Conditions

In this section, we establish optimality conditions for both an optimal solution and a quasi ϵ-solution to the problem (RP).

2.4.1 For an optimal solution

The following theorem gives a KKT necessary condition for optimal solutions of the problem (RP).

Theorem 2.4.1. *Consider the problem* (RP), *suppose that the Slater condition holds for* (RP). *If* $\bar{x}$ *is an optimal solution of problem* (RP), *then there exist* $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ *and* $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, *such that*

$$\begin{aligned} &0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}), \\ &\tau_k \Big(f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x}) \Big) = 0, \quad k \in \mathrm{K}, \\ &\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \ldots, m. \end{aligned} \tag{2.2}$$

Proof. Let $\bar{x}$ be an optimal solution of the problem (RP). Then, $\bar{x}$ is a minimizer of the following problem:

$$\min_{x \in F} \varphi(x),$$

where $\varphi(x) := \max_{k \in \mathrm{K}} f_k(x)$. Observe that $\varphi(x)$ is a convex function, since $f_k(x), k \in \mathrm{K}$ is convex [39]. Thus, $\bar{x}$ is a minimizer of the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \{\varphi(x) + \delta_F(x)\}. \tag{2.3}$$

Applying the standard optimality condition to the unconstrained optimization problem (2.3), we have

$$0 \in \partial\Big(\varphi + \delta_F(\cdot)\Big)(\bar{x}).$$

Since the function φ is convex and the function $\delta_F(\cdot)$ is also convex, it follows from Lemma 2.2.1 that

$$0 \in \partial\varphi(\bar{x}) + \partial\delta_F(\bar{x}),$$

which by the fact that $\partial\delta_F(\bar{x}) = N_F(\bar{x})$ leads to

$$0 \in \partial\varphi(\bar{x}) + N_F(\bar{x}).$$

On the one hand, employing the formula for the convex subdifferential of maximum functions (see Proposition 2.2.2) and the Moreau–Rochafellar sum rule (see Lemma 2.2.1) we obtain

$$\begin{aligned}\partial\varphi(\bar{x}) = \partial(\max_{k\in \mathrm{K}} f_k)(\bar{x}) = \operatorname{co} \bigcup_{k\in \mathrm{K}(\bar{x})} \partial f_k(\bar{x})\\ = \left\{ \sum_{k\in \mathrm{K}(\bar{x})} \tau_k \partial f_k(\bar{x}) \,\middle|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k\in \mathrm{K}(\bar{x})} \tau_k = 1 \right\},\end{aligned}$$

where $\mathrm{K}(\bar{x}) := \{k \in \mathrm{K} \colon f_k(\bar{x}) = \varphi(\bar{x})\} \neq \emptyset$. On the other hand, since the Slater condition holds, and from Lemma 2.3.1, we have

$$\begin{aligned}0 \in \left\{ \sum_{k\in \mathrm{K}(\bar{x})} \tau_k \partial f_k(\bar{x}) \,\middle|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k\in \mathrm{K}(\bar{x})} \tau_k = 1 \right\}\\ + \left\{ \sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}, \bar{v}_i) \,\middle|\, \lambda_i \geqq 0, i \in I(\bar{x}) \right\} \qquad (2.4)\end{aligned}$$

Now, letting $\tau_k := 0$ for $k \in \mathrm{K} \setminus \mathrm{K}(\bar{x})$ and $\lambda_i := 0$ for $i \in \{1, \ldots, m\} \setminus I(\bar{x})$, we see that (2.4) clearly implies (2.2), which completes the proof of the theorem. □

Theorem 2.4.2 (sufficient KKT condition)**.** *Consider the problem* (RP), *assume that* $\bar{x} \in F$ *satisfy the conditions in Theorem 2.4.1, then* $\bar{x}$ *is an optimal solution to problem* (RP).

Proof. Put $\phi(x) := \max\limits_{k \in \mathrm{K}} f_k(x)$ for $x \in \mathbb{R}^n$. Since $\bar{x}$ satisfies the conditions in Theorem 2.4.1, then there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\xi_k \in \partial f_k(\bar{x}), k \in \mathrm{K}$, and $\eta_i \in \partial g_i(\bar{x}, \bar{v}_i), i = 1, \ldots, m$ such that

$$0 = \sum_{k \in \mathrm{K}} \tau_k \xi_k + \sum_{i=1}^{m} \lambda_i \eta_i, \tag{2.5}$$

$$\tau_k \Big(f_k(\bar{x}) - \phi_k(\bar{x}) \Big) = 0, \quad k \in \mathrm{K}, \tag{2.6}$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \ldots, m. \tag{2.7}$$

Assume to the contrary that $\bar{x}$ is not an optimal solution of problem (RP). Then there is $\hat{x} \in F$ such that

$$\phi(\bar{x}) > \phi(\hat{x}) \tag{2.8}$$

On the other hand, by definition of the subdifferential,

$$f_k(x) - f_k(\bar{x}) \geqq \langle \xi_k, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n, \ k \in \mathrm{K}, \tag{2.9}$$

$$g_i(x, \bar{v}_i) - g_i(\bar{x}, \bar{v}_i) \geqq \langle \eta_i, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n, \ i = 1, \ldots, m, \tag{2.10}$$

Combining the inequalities (2.9) and (2.10) along with (2.5) implies

$$\sum_{k\in \mathrm{K}} \tau_k f_k(x) - \sum_{k\in \mathrm{K}} \tau_k f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(x, \bar{v}_i) - \sum_{i=1}^{m} \lambda_i g_i(\bar{x}, \bar{v}_i) \geqq 0,\ \forall x \in \mathbb{R}^n.$$

For any x feasible to the problem (RP), $g_i(x, \bar{v}_i) \leqq 0, i = 1, \ldots, m$, which along with the complementary slackness condition (2.7) and the fact that $\lambda_i \geqq 0, i = 1, \ldots, m$ reduces the above inequality to

$$\sum_{k\in \mathrm{K}} \tau_k f_k(x) - \sum_{k\in \mathrm{K}} \tau_k f_k(\bar{x}) \geqq 0,\ \forall x \in F. \tag{2.11}$$

On the other side, by (2.6), it holds that

$$\sum_{k\in \mathrm{K}} \tau_k \phi(\bar{x}) = \sum_{k\in \mathrm{K}} \tau_k f_k(\bar{x}). \tag{2.12}$$

Now, taking (2.11) and (2.12) into account, we arrive at

$$\sum_{k\in \mathrm{K}} \tau_k \phi(\bar{x}) \leqq \sum_{k\in \mathrm{K}} \tau_k \phi(\hat{x}).$$

This implies that

$$\phi(\bar{x}) \leqq \phi(\hat{x}) \tag{2.13}$$

due to $\sum_{k\in \mathrm{K}} \tau_k > 0$. Obviously, (2.13) contradicts (2.8), which completes the proof of the theorem. □

2.4.2 For a quasi ϵ-solution

Below, we gives a KKT necessary condition for a quasi ϵ-solution of problem (RP), the proof is similar to Theorem 2.4.1.

Theorem 2.4.3 (necessary KKT condition)**.** *Consider the problem* (RP), *suppose that the Slater condition holds for* (RP). *If* $\bar{x}$ *is a quasi* ϵ*-solution of problem* (RP), *then there exist* $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ *and* $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, *such that*

$$0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) + \sqrt{\epsilon}\mathbb{B},$$

$$\tau_k \Big(f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x}) \Big) = 0, \quad k \in \mathrm{K},$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \ldots, m. \tag{2.14}$$

Proof. Let $\bar{x}$ be a quasi ϵ-solution of the problem (RP). Then, $\bar{x}$ is a minimizer of the following problem:

$$\min_{x \in F} \{\varphi(x) + \sqrt{\epsilon}\| \cdot - \bar{x}\|\},$$

where $\varphi(x) := \max_{k \in \mathrm{K}} f_k(x)$. Again, $\varphi(x)$ is a convex function, since $f_k(x), k \in \mathrm{K}$ is convex [39]. Thus, $\bar{x}$ is a minimizer of the following unconstrained optimization problem

$$\min_{x \in \mathbb{R}^n} \{\varphi(x) + \sqrt{\epsilon}\| \cdot - \bar{x}\| + \delta_F(x)\}. \tag{2.15}$$

Again, applying the standard optimality condition to the unconstrained optimization problem (2.15), we have

$$0 \in \partial \Big(\varphi + \sqrt{\epsilon}\| \cdot - \bar{x}\| + \delta_F(\cdot) \Big)(\bar{x}).$$

Since the function φ, $\delta_F(\cdot)$ and $\|\cdot - \bar{x}\|$ are convex, it follows from Lemma 2.2.1 that

$$0 \in \partial \varphi(\bar{x}) + \partial \delta_F(\bar{x}) + \sqrt{\epsilon}\partial \| \cdot - \bar{x}\|,$$

which by the facts that $\partial\delta_F(\bar{x}) = N_F(\bar{x})$ and $\partial\|\cdot - \bar{x}\| = \mathbb{B}$ leads to

$$0 \in \partial\varphi(\bar{x}) + N_F(\bar{x}) + \sqrt{\epsilon}\mathbb{B}.$$

On the one hand, employing the formula for the convex subdifferential of maximum functions (see Proposition 2.2.2) and the Moreau–Rochafellar sum rule (see Lemma 2.2.1) we obtain

$$\begin{aligned}\partial\varphi(\bar{x}) &= \partial(\max_{k\in \mathrm{K}} f_k)(\bar{x}) = \mathrm{co} \bigcup_{k\in \mathrm{K}(\bar{x})} \partial f_k(\bar{x}) \\ &= \left\{ \sum_{k\in \mathrm{K}(\bar{x})} \tau_k \partial f_k(\bar{x}) \,\middle|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k\in \mathrm{K}(\bar{x})} \tau_k = 1 \right\},\end{aligned}$$

where $\mathrm{K}(\bar{x}) := \{k \in \mathrm{K}\colon f_k(\bar{x}) = \varphi(\bar{x})\} \neq \emptyset$. On the other hand, since the Slater condition holds, and from Lemma 2.3.1, we have

$$\begin{aligned}0 \in &\left\{ \sum_{k\in \mathrm{K}(\bar{x})} \tau_k \partial f_k(\bar{x}) \,\middle|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k\in \mathrm{K}(\bar{x})} \tau_k = 1 \right\} \\ &+ \left\{ \sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}, \bar{v}_i) \,\middle|\, \lambda_i \geqq 0, i \in I(\bar{x}) \right\} + \sqrt{\epsilon}\mathbb{B} \qquad (2.16)\end{aligned}$$

Now, letting $\tau_k := 0$ for $k \in \mathrm{K} \setminus \mathrm{K}(\bar{x})$ and $\lambda_i := 0$ for $i \in \{1, \ldots, m\} \setminus I(\bar{x})$, we see that (2.16) clearly implies (2.14), which completes the proof of the theorem. □

Theorem 2.4.4 (sufficient KKT condition)**.** *Consider the problem* (RP), *assume that* $\bar{x} \in F$ *satisfy the conditions in Theorem 2.4.3 with* $\sum_{k\in \mathrm{K}} \tau_k = 1$, *then* $\bar{x}$ *is a quasi* ϵ*-solution of problem* (RP).

Proof. Put $\phi(x) := \max_{k \in \mathrm{K}} f_k(x)$ for $x \in \mathbb{R}^n$. Since $\bar{x}$ satisfies the conditions in Theorem 2.4.3, then there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\xi_k \in \partial f_k(\bar{x}), k \in \mathrm{K}$, $\eta_i \in \partial g_i(\bar{x}, \bar{v}_i), i = 1, \ldots, m$ and $b \in \mathbb{B}$ such that

$$0 = \sum_{k \in \mathrm{K}} \tau_k \xi_k + \sum_{i=1}^{m} \lambda_i \eta_i + \sqrt{\epsilon} b, \tag{2.17}$$

$$\tau_k \Big(f_k(\bar{x}) - \phi_k(\bar{x}) \Big) = 0, \quad k \in \mathrm{K}, \tag{2.18}$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \ldots, m. \tag{2.19}$$

On the other hand, by definition of the subdifferential,

$$f_k(x) - f_k(\bar{x}) \geqq \langle \xi_k, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n, \ k \in \mathrm{K}, \tag{2.20}$$

$$g_i(x, \bar{v}_i) - g_i(\bar{x}, \bar{v}_i) \geqq \langle \eta_i, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n, \ i = 1, \ldots, m, \tag{2.21}$$

and by the Cauchy–Schwartz inequality (see Proposition 2.2.1),

$$\|b\| \|x - \bar{x}\| \geqq \langle b, x - \bar{x} \rangle, \ \forall x \in \mathbb{R}^n. \tag{2.22}$$

Combining the inequalities (2.20), (2.21), and (2.22) along with (2.17) implies

$$\sum_{k \in \mathrm{K}} \tau_k f_k(x) - \sum_{k \in \mathrm{K}} \tau_k f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i g_i(x, \bar{v}_i) - \sum_{i=1}^{m} \lambda_i g_i(\bar{x}, \bar{v}_i) + \sqrt{\epsilon} \|b\| \|x - \bar{x}\|$$
$$\geqq 0, \ \forall x \in \mathbb{R}^n.$$

For any x feasible to the problem (RP), $g_i(x, \bar{v}_i) \leqq 0, i = 1, \ldots, m$, which along with the complementary slackness condition (2.19) and the fact that $\lambda_i \geqq 0, i = 1, \ldots, m$ reduces the above inequality to

$$\sum_{k \in \mathrm{K}} \tau_k f_k(x) - \sum_{k \in \mathrm{K}} \tau_k f_k(\bar{x}) + \sqrt{\epsilon} \|b\| \|x - \bar{x}\| \geqq 0, \ \forall x \in F.$$

As $b \in \mathbb{B}$, $\|b\| \leqq 1$, thereby leading to

$$\sum_{k \in \mathrm{K}} \tau_k f_k(x) - \sum_{k \in \mathrm{K}} \tau_k f_k(\bar{x}) + \sqrt{\epsilon}\|x - \bar{x}\| \geqq 0, \ \forall x \in F. \tag{2.23}$$

On the other side, by (2.18), it holds that

$$\sum_{k \in \mathrm{K}} \tau_k \phi(\bar{x}) = \sum_{k \in \mathrm{K}} \tau_k f_k(\bar{x}).$$

Since $\sum_{k \in \mathrm{K}} \tau_k = 1$, we have $\phi(\bar{x}) = \sum_{k \in \mathrm{K}} \tau_k f_k(\bar{x})$. Finally, from (2.23), the requisite results are yielded. □

2.5 Duality Relations

In this section we formulate a dual problem to the primal one in the sense of Wolfe [46], and explore weak and strong duality relations between them, for both an optimal solution and a quasi ϵ-solution.

2.5.1 For an optimal solution

In connection with the robust minimax programming problem (RP), denote $\varphi(y) := \max_{k\in \mathrm{K}} f_k(y)$, we consider a dual problem in the following form:

$$
\begin{aligned}
(\mathrm{RD})_{\mathrm{W}} \qquad & \text{Maximize}_{(y,\tau,v,\lambda)} \quad && \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
& \text{subject to} && 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) \\
& && \tau_k(f_k(y) - \varphi(y)) = 0,\ k \in \mathrm{K} \\
& && \tau_k \geqq 0,\ \sum_{k\in \mathrm{K}} \tau_k = 1 \\
& && \lambda_i \geqq 0,\ v_i \in \mathcal{V}_i,\ i = 1, \ldots, m.
\end{aligned}
$$

Let F_D be the feasible set of $(\mathrm{RD})_{\mathrm{W}}$, where $F_D = \{(y, \tau, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y), \tau_k(f_k(y) - \varphi(y)) = 0, k \in \mathrm{K}, \tau_k \geqq 0, \sum_{k\in \mathrm{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i = 1, \ldots, m\}$. We should note that a point $(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_D$ is called an *optimal solution* of problems $(\mathrm{RD})_{\mathrm{W}}$ if for all $(y, \tau, v, \lambda) \in F_D$,

$$\varphi(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

The following theorem describes a weak duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_{\mathrm{W}}$.

Theorem 2.5.1 (weak duality). *For any feasible solution x of* (RP) *and any feasible solution* (y, τ, v, λ) *of* $(\mathrm{RD})_{\mathrm{W}}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

Proof. Since $(y, \tau, v, \lambda) \in F_D$, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \tau_k = 1$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(y), k \in \mathrm{K}$ and $\bar{\zeta}_i \in \partial g_i(\cdot, v_i)(y), i = 1, \ldots, m$ such that

$$\sum_{k \in \mathrm{K}} \tau_k \bar{\xi}_k + \sum_{i=1}^{m} \lambda_i \bar{\zeta}_i = 0, \tag{2.24}$$

$$\tau_k(f_k(y) - \varphi(y)) = 0, \ k \in \mathrm{K}, \tag{2.25}$$

thus from (2.24), we have

$$\sum_{k \in \mathrm{K}} \tau_k \langle \bar{\xi}_k, x - y \rangle + \sum_{i=1}^{m} \lambda_i \langle \bar{\zeta}_i, x - y \rangle = 0,$$

by the convexity of $f_k(\cdot), k \in \mathrm{K}$ and $g_i(\cdot, v_i), i = 1, \ldots, m$,

$$\sum_{k \in \mathrm{K}} \tau_k (f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i (g_i(x, v_i) - g_i(y, v_i)) \geqq 0. \tag{2.26}$$

Finally, from (2.25) and (2.26), and the fact $\lambda_i g_i(x, v_i) = 0$, due to $\sum_{k \in \mathrm{K}} \tau_k = 1$, we obtain

$$\varphi(x) = \sum_{k \in \mathrm{K}} \tau_k \max_{k \in \mathrm{K}} f_k(x) \geqq \sum_{k \in \mathrm{K}} \tau_k f_k(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

Thus the proof of the theorem is completed. □

In what follows, a strong duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_{\mathrm{W}}$ is given.

Theorem 2.5.2 (strong duality)**.** *Let $\bar{x} \in F$ be an optimal solution of the robust problem* (RP) *such that the Slater condition holds at this point. Then there exists $(\bar{\tau}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_D$ is an optimal solution of problem* $(\mathrm{RD})_{\mathrm{W}}$.

Proof. Let $\bar{x} \in F$ be an optimal solution of (RP) such that the Slater condition holds at this point. By Theorem 2.4.1, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i$ and $\lambda_i \geqq 0, i = 1, \ldots, m$ such that

$$
\begin{aligned}
& 0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}), \\
& \tau_k (f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x})) = 0, \ k \in \mathrm{K}, \\
& \lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ i = 1, \ldots, m.
\end{aligned} \tag{2.27}
$$

Putting

$$
\bar{\tau}_k := \frac{\tau_k}{\sum_{k \in \mathrm{K}} \tau_k}, \ k \in \mathrm{K}, \quad \bar{\lambda}_i := \frac{\lambda_i}{\sum_{k \in \mathrm{K}} \tau_k}, \ i = 1, \ldots, m,
$$

we then have $\bar{\tau}_i + (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \bar{\tau}_k = 1$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that the assertion in (2.27) is still valid when τ_k's and λ_i's are replaced by $\bar{\tau}_k$'s and $\bar{\lambda}_k$'s, respectively. Consequently, $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a feasible solution of $(\mathrm{RD})_{\mathrm{W}}$.

Now, by Theorem 2.5.1 (weak duality), for any feasible (y, τ, v, λ) of $(\text{RD})_{\text{W}}$,

$$\varphi(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = \varphi(\bar{x}) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i),$$

which means that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is an optimal solution of problem $(\text{RD})_{\text{W}}$. $\square$

Here comes an example to illustrate our duality results. Note that this example is modified by [35, Example 2].

Example 2.5.1. Consider the following minimax optimization problem with data uncertainty:

$$(\text{RP})^1 \qquad \min_{(x_1,x_2)\in\mathbb{R}^2} \max_{k\in\{1,2\}} \{f_1(x_1, x_2), f_2(x_1, x_2)\}$$

$$\text{subject to } \ x_1^2 - 2v_1x_1 - 3 \leqq 0, \ v_1 \in [-1, 1].$$

Let

$$f_1(x_1, x_2) = x_1 + x_2^2,$$

$$f_2(x_1, x_2) = -x_1 + x_2^2,$$

$$g_1\Big((x_1, x_2), v_1\Big) = x_1^2 - 2v_1x_1 - 3.$$

Then, the feasible set of $(\text{RP})^1$ is

$$\begin{aligned} F^1 &= \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1^2 - 2v_1x_1 - 3 \leqq 0, \ v_1 \in [-1, 1]\} \\ &= \{(x_1, x_2) \in \mathbb{R}^2 \mid -1 \leqq x_1 \leqq 1, \ x_2 \in \mathbb{R}\}, \end{aligned}$$

and $\{(0, 0)\}$ is the set of optimal solutions of $(\text{RP})^1$; moreover, it is clear that the Slater condition holds for $(\text{RP})^1$.

Now, we formulate a robust dual problem $(\mathrm{RD})^1_{\mathrm{W}}$ for $(\mathrm{RP})^1$ as follows:

$$(\mathrm{RD})^1_W \qquad \max_{(y,\tau,v,\lambda)} \quad \varphi(y_1,y_2)+\lambda_1 g_1\Big((y_1,y_2),v_1\Big)$$

$$\text{subject to} \quad 0\in \tau_1\partial f_1(y_1,y_2)+\tau_2\partial f_2(y_1,y_2)+\lambda_1\partial g_1(\cdot,v_1)(y_1,y_2)$$

$$\tau_1\Big(f_1(y_1,y_2)-\varphi(y_1,y_2)\Big)=0$$

$$\tau_2\Big(f_2(y_1,y_2)-\varphi(y_1,y_2)\Big)=0$$

$$\tau_1\geqq 0,\ \tau_2\geqq 0,\ \tau_1+\tau_2=1,\ \lambda_1\geqq 0,\ v_1\in[-1,1],$$

where $\varphi(y_1,y_2)=\max\limits_{\{1,2\}}\{f_1(y_1,y_2),f_2(y_1,y_2)\}$.

By calculation, we have the set of all feasible solutions of $(\mathrm{RD})^1_{\mathrm{W}}$ is $F^1_D :=\Big\{\Big((0,0),(\frac{1+2\lambda_1 v_1}{2},\frac{1-2\lambda_1 v_1}{2}),v_1,\lambda_1\Big) : \lambda_1\in[0,\frac{1}{2}],v_1\in[-1,1]\Big\}$. It is not difficult to see the validness of Theorem 2.5.1 (weak duality) and Theorem 2.5.2 (strong duality).

2.5.2 For a quasi ϵ-solution

Denote again $\varphi(y) := \max_{k\in \mathrm{K}} f_k(y)$, here we consider a dual problem that enjoys the following form:

$$
\begin{aligned}
(\mathrm{RD})_{\mathrm{Q}} \qquad \text{Maximize}_{(y,\tau,v,\lambda)} \quad & \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
\text{subject to} \quad & 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) + \sqrt{\epsilon}\mathbb{B} \\
& \tau_k(f_k(y) - \varphi(y)) = 0, \ k \in \mathrm{K} \\
& \tau_k \geqq 0, \ \sum_{k\in \mathrm{K}} \tau_k = 1 \\
& \lambda_i \geqq 0, \ v_i \in \mathcal{V}_i, \ i = 1, \ldots, m, \epsilon \geqq 0.
\end{aligned}
$$

Let F_Q be the feasible set of $(\mathrm{RD})_{\mathrm{Q}}$, where $F_Q = \{(y, \tau, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) + \sqrt{\epsilon}\mathbb{B}, \tau_k(f_k(y) - \varphi(y)) = 0, k \in \mathrm{K}, \tau_k \geqq 0, \sum_{k\in \mathrm{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i = 1, \ldots, m, \epsilon \geqq 0\}$.

Definition 2.5.1. *Let $\epsilon \geqq 0$ be given, a point $(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_Q$ is called a quasi ϵ-solution of the problem* $(\mathrm{RD})_{\mathrm{Q}}$ *if for all $(y, \tau, v, \lambda) \in F_Q$,*

$$\varphi(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon}\|y - \bar{y}\|.$$

Remark 2.5.1. [28, Remark 4.1] The notion of a quasi ϵ-solution of $(\mathrm{RD})_{\mathrm{Q}}$ is motivated by Ekeland Variational Principle [21] as we have mentioned, and for the notion of a quasi ϵ-solution of $(\mathrm{RD})_{\mathrm{Q}}$, which is motivated by [19] where the author introduced the notion of the quasi ϵ-saddle point. It is worth noting here that we consider the notion of a quasi ϵ-solution over the feasible set, and it is not necessary to mention how explicitly the feasible set is defined by.

The following theorem shows a weak duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_{\mathrm{Q}}$.

Theorem 2.5.3 (weak duality)**.** *For any feasible solution x of* (RP) *and any feasible solution (y, τ, v, λ) of* $(\mathrm{RD})_{\mathrm{Q}}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon}\|x - y\|.$$

Proof. Since $(y, \tau, v, \lambda) \in F_Q$, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \tau_k = 1$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(y), k \in \mathrm{K}$ and $\bar{\zeta}_i \in \partial g_i(\cdot, v_i)(y), i = 1, \ldots, m$, $b \in \mathbb{B}$ such that

$$\sum_{k \in \mathrm{K}} \tau_k \bar{\xi}_k + \sum_{i=1}^{m} \lambda_i \bar{\zeta}_i + \sqrt{\epsilon} b = 0, \tag{2.28}$$

$$\tau_k (f_k(y) - \varphi(y)) = 0, \ k \in \mathrm{K}, \tag{2.29}$$

thus from (2.28), we have

$$\sum_{k \in \mathrm{K}} \tau_k \langle \bar{\xi}_k, x - y \rangle + \sum_{i=1}^{m} \lambda_i \langle \bar{\zeta}_i, x - y \rangle + \sqrt{\epsilon} \langle b, x - y \rangle = 0,$$

by the convexity of $f_k(\cdot), k \in \mathrm{K}$ and $g_i(\cdot, v_i), i = 1, \ldots, m$,

$$\sum_{k\in \mathrm{K}} \tau_k(f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i(g_i(x, v_i) - g_i(y, v_i)) + \sqrt{\epsilon}\|x - y\|$$

$$\geqq \sum_{k\in \mathrm{K}} \tau_k(f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i(g_i(x, v_i) - g_i(y, v_i)) + \sqrt{\epsilon}\|b\|\|x - y\|$$

$$\geqq 0. \tag{2.30}$$

Finally, from (2.29) and (2.30), and the fact $\lambda_i g_i(x, v_i) = 0$, due to $\sum_{k\in \mathrm{K}} \tau_k = 1$, we obtain

$$\varphi(x) = \sum_{k\in \mathrm{K}} \tau_k \max_{k\in \mathrm{K}} f_k(x) \geqq \sum_{k\in \mathrm{K}} \tau_k f_k(x)$$

$$\geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon}\|x - y\|.$$

Thus, we complete the proof. □

Below, a strong duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_\mathrm{Q}$ is proposed.

Theorem 2.5.4 (strong duality)**.** *Let* $\bar{x} \in F$ *be a quasi* ϵ*-solution of the robust problem* (RP) *such that the Slater condition holds at this point. Then there exists* $(\bar{\tau}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ *such that* $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_Q$ *is a quasi* ϵ*-solution of problem* $(\mathrm{RD})_\mathrm{Q}$.

Proof. Let $\bar{x} \in F$ be a quasi ϵ-solution of (RP) such that the Slater condition holds at this point. By Theorem 2.4.3, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$,

$\bar{v}_i \in \mathcal{V}_i$ and $\lambda_i \geqq 0, i = 1, \ldots, m$, $b \in \mathbb{B}$ such that

$$
\begin{aligned}
&0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) + \sqrt{\epsilon} b, \\
&\tau_k(f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x})) = 0, \;\; k \in \mathrm{K}, \\
&\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \;\; i = 1, \ldots, m. \qquad (2.31)
\end{aligned}
$$

Putting

$$
\bar{\tau}_k := \frac{\tau_k}{\sum_{k \in \mathrm{K}} \tau_k}, \;\; k \in \mathrm{K}, \;\; \bar{\lambda}_i := \frac{\lambda_i}{\sum_{k \in \mathrm{K}} \tau_k}, \;\; i = 1, \ldots, m,
$$

we then have $\bar{\tau}_i + (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \bar{\tau}_k = 1$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that the assertion in (2.31) is still valid when τ_k's and λ_i's are replaced by $\bar{\tau}_k$'s and $\bar{\lambda}_k$'s, respectively. Consequently, $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a feasible solution of $(\mathrm{RD})_{\mathrm{Q}}$.

Now, by Theorem 2.5.3 (weak duality), for any feasible (y, τ, v, λ) of $(\mathrm{RD})_{\mathrm{Q}}$,

$$
\begin{aligned}
\varphi(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) &= \varphi(\bar{x}) \\
&\geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon} \|\bar{x} - y\|,
\end{aligned}
$$

which means that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a quasi ϵ-solution of problem $(\mathrm{RD})_{\mathrm{W}}$. □

Chapter 3
Robust Multiobjective Optimization Problems via Minimax Programming

3.1 Introduction and Mathematical Modelling

It is well-known that mathematical optimization problems in the face of data uncertainty have been treated by the worst case approach or the stochastic approach. The worst case approach for optimization problems, which has emerged as a powerful deterministic approach for studying optimization problems with data uncertainty, associates an uncertain optimization problem with its robust counterpart. Recently, the study of convex programs that are affected by data uncertainty is becoming increasingly important in optimization [2, 3, 5–7, 24–27].

Many researchers [2, 25, 37, 44] have investigated optimality and duality theories for linear or convex optimization problems under data uncertainty with the worst-case approach (the robust approach). It was shown that the value of the robust counterpart of primal problem is equal to the value of the optimistic counterpart of the dual primal ("primal worst equals dual best") [2, 25, 27].

Recently, many researchers [9, 10, 29, 30] have studied optimality and duality theories for robust multiobjective optimization programming problems under different suitable constrained qualifications. In this chapter, we investigate optimality conditions and duality theorems for robust multiobjective programming problems under data uncertainty via minimax programing; namely, applying some results of the robust mimimax programming problem obtained by Chapter 2 to a robust multiobjective optimization problem.

Let us consider the following multiobjective convex optimization problem in the absence of data uncertainty:

$$\text{(MP)} \ \min \ \Big(f_1(x), \ldots, f_l(x)\Big) \ \text{ subject to } \ g_i(x) \leqq 0, \ i = 1, \ldots, m,$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in \mathrm{K}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$ are convex functions.

The multiobjective onvex optimization problem (MP) in the face of data uncertainty in the constraints can be captured by the problem:

$$\text{(UMP)} \ \min \ \Big(f_1(x), \ldots, f_l(x)\Big) \ \text{ subject to } \ g_i(x, v_i) \leqq 0, \ i = 1, \ldots, m,$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \ldots, m$.

The robust counterpart of (UMP) is

$$\text{(RMP)} \ \min \ \Big(f_1(x), \ldots, f_l(x)\Big) \ \text{ subject to } \ g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m.$$

The robust feasible set of (RMP) is defined by

$$F := \{x \in \mathbb{R}^n \colon g_i(x, v_i) \leqq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\}. \tag{3.1}$$

With the notation given at the beginning of Chapter 2 and for convenience, we label the above constrained *multiobjective robust optimization problem* as follows:

$$\mathrm{Min}_{\mathbb{R}^l_+} \Big\{ f(x) : \ x \in F \Big\}, \tag{RMP}$$

where the robust feasible set F is given by (3.1) and $\mathbb{R}^l_+$ denotes the nonnegative orthant of $\mathbb{R}^l$.

Note that "$\mathrm{Min}_{\mathbb{R}^l_+}$" in the above problem is understood with respect to the ordering cone $\mathbb{R}^l_+$. Now, we recall the notions of (robust) weak Pareto solutions, which can be seen in [45, Section 4].

Definition 3.1.1. A point $\bar{x} \in F$ is a *weak Pareto solution* of problem (RMP) [or a robust weak Pareto solution of problem (UMP)] if and only if

$$f(x) - f(\bar{x}) \notin -\mathrm{int}\, \mathbb{R}^l_+ \quad \forall x \in F,$$

where $\mathrm{int}\, \mathbb{R}^l_+$ stands for the topological interior of $\mathbb{R}^l_+$.

3.2 Optimality Conditions

The following result is a Karush–Kuhn–Tucker (KKT) necessary condition for weak Pareto solutions of problem (RMP).

Theorem 3.2.1 (necessary KKT condition for a weak Pareto solution)**.** *Let the Slater condition be satisfied at $\bar{x} \in F$. If $\bar{x}$ is a weak Pareto solution of problem* (RMP), *then there exist $\tau := (\tau_1, ..., \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ and $\lambda := (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ such that*

$$0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) \text{ and } \lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ i = 1, \ldots, m. \tag{3.2}$$

Proof. Let $\bar{x}$ be a weak Pareto solution of problem (RMP) and let

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}),\ k \in \mathrm{K},\ x \in \mathbb{R}^n.$$

We will show that $\bar{x}$ is an optimal solution of the robust minimax programming problem:

$$\min_{x\in F}\max_{k\in \mathrm{K}} \widehat{f}_k(x). \tag{3.3}$$

To do this, let us put $\widehat{\varphi}(x) := \max_{k\in\mathrm{K}} \widehat{f}_k(x)$ and prove that

$$\widehat{\varphi}(\bar{x}) \leq \widehat{\varphi}(x), \quad \forall x \in F. \tag{3.4}$$

Indeed, if (3.4) is not valid, then there exists $x_0 \in F$ such that $\widehat{\varphi}(x_0) < \widehat{\varphi}(\bar{x})$. Since $\widehat{\varphi}(\bar{x}) = 0$, it holds that $\max_{k\in\mathrm{K}} \{f_k(x_0) - f_k(\bar{x})\} < 0$. Thus,

$$f(x_0) - f(\bar{x}) \in -\mathrm{int}\ \mathbb{R}^l_+,$$

which contradicts the fact that $\bar{x}$ is a weak Pareto solution of the problem (RMP). So, we can employ the necessary KKT condition in Theorem 2.4.1, but applied to problem (3.3). Then we find $\tau := (\tau_1, ..., \tau_l) \in \mathbb{R}^l_+\backslash\{0\}$, $\bar{v}_i \in \mathcal{V}_i,\ i = 1, \ldots, m$ and $\lambda := (\lambda_1, ..., \lambda_n) \in \mathbb{R}^m_+$ such that

$$\begin{aligned}
&0 \in \sum_{k\in\mathrm{K}} \tau_k \partial \widehat{f}_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}),\\
&\tau_k\Big(\widehat{f}_k(\bar{x}) - \max_{k\in\mathrm{K}} \widehat{f}_k(\bar{x})\Big) = 0, \quad k \in \mathrm{K},\\
&\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \ldots, m.
\end{aligned} \tag{3.5}$$

It is now clear that (3.5) implies (3.2) and thus, the proof is complete. □

The forthcoming theorem describes the KKT optimality condition for a weak quasi ϵ-Pareto solution of problem (RMP). Before that, let us recall the notion of a weak quasi ϵ-Pareto solution, with regard to this notion, one may refer to [38, 41, 42].

Definition 3.2.1. *Let $\epsilon = (\epsilon_1, \ldots, \epsilon_l) \in \mathbb{R}^l_{++}$ with $\mathbb{R}^l_{++}$ denoting the positive orthant of $\mathbb{R}^l$. A point $z \in F$ is said to be a weak quasi ϵ-Pareto solution of* (RMP), *if there exists no $x \in F$ such that*

$$f_k(x) + \sqrt{\epsilon_k}\|x - z\| < f_k(z), k \in \mathrm{K}.$$

Theorem 3.2.2 (necessary KKT condition for a weak quasi ϵ-Pareto solution)**.** *Let $\epsilon = (\epsilon_1, \ldots, \epsilon_l) \in \mathbb{R}^l_{++}$ be given, and the Slater condition be satisfied at $\bar{x} \in F$. If $\bar{x}$ is a weak quasi ϵ-Pareto solution of problem* (RMP), *then there exist $\tau := (\tau_1, ..., \tau_l) \in \mathbb{R}^l_+\backslash\{0\}$, $\bar{v}_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ and $\lambda := (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ such that*

$$\begin{aligned}
&0 \in \sum_{k\in\mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + \sum_{k\in\mathrm{K}} \tau_k \sqrt{\epsilon_k}\mathbb{B}, \\
&\lambda_i g_i(\bar{x}, \bar{v}_i) = 0,\ i = 1, \ldots, m.
\end{aligned} \tag{3.6}$$

Proof. Let $\bar{x}$ be a weak quasi ϵ-Pareto solution of problem (RMP), and let

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}) + \sqrt{\epsilon_k}\|x - \bar{x}\|,\ k \in \mathrm{K},\ x \in \mathbb{R}^n.$$

We will show that $\bar{x}$ is an optimal solution of the robust minimax programming problem:

$$\min_{x\in F} \max_{k\in\mathrm{K}} \widehat{f}_k(x). \tag{3.7}$$

To this end, let us take $\widehat{\varphi}(x) := \max_{k\in \mathrm{K}} \widehat{f}_k(x)$ and prove that

$$\widehat{\varphi}(\bar{x}) \le \widehat{\varphi}(x), \quad \forall x \in F. \tag{3.8}$$

Actually, if (3.8) is not true, then there exists $x_0 \in F$ such that

$$\widehat{\varphi}(x_0) < \widehat{\varphi}(\bar{x}).$$

Since $\widehat{\varphi}(\bar{x}) = 0$, it holds that

$$\max_{k\in \mathrm{K}} \{f_k(x_0) - f_k(\bar{x})\} < 0.$$

Thus,

$$f(x_0) - f(\bar{x}) + \sqrt{\epsilon}\|x - \bar{x}\| \in -\mathrm{int}\, \mathbb{R}^l_+, \ \sqrt{\epsilon} = (\sqrt{\epsilon_1}, \dots, \sqrt{\epsilon_l}),$$

which contradicts the fact that $\bar{x}$ is a weak quasi ϵ-Pareto solution of the problem (RMP).

Thereby, we now employ the necessary KKT condition in Theorem 2.4.1, but applied to problem (3.7). Then we find $\tau := (\tau_1, ..., \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i, \ i = 1, \dots, m$, and $\lambda := (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ such that

$$\begin{aligned}
&0 \in \sum_{k\in \mathrm{K}} \tau_k \partial \widehat{f}_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}) + \sum_{k\in \mathrm{K}} \tau_k \sqrt{\epsilon_k}\mathbb{B},\\
&\tau_k\Big(\widehat{f}_k(\bar{x}) - \max_{k\in \mathrm{K}} \widehat{f}_k(\bar{x})\Big) = 0, \quad k \in \mathrm{K},\\
&\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \dots, m.
\end{aligned} \tag{3.9}$$

It is now clear that (3.9) implies (3.6) and thus, the proof is complete. □

3.3 Duality Theorems

In this section, we formulate a dual problem to the primal one in the sense of Wolfe [46], and explore weak and strong duality relations between them, for both a weak Pareto solution and a weak quasi ϵ-Pareto solution.

3.3.1 For a weak Pareto solution

In connection with the robust multiobjective programming problem (RMP), we consider a dual problem in the following form:

$$
\begin{aligned}
(\mathrm{RMD})_{\mathrm{W}} \qquad & \text{Maximize}_{(y,\tau,v,\lambda)} && f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)e \\
& \text{subject to} && 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) \\
& && \tau_k \geqq 0, \ \sum_{k\in \mathrm{K}} \tau_k = 1, e = (1, \ldots, 1) \\
& && \lambda_i \geqq 0, \ v_i \in \mathcal{V}_i, \ i = 1, \ldots, m.
\end{aligned}
$$

Let F_{MD} be the feasible set of $(\mathrm{RMD})_{\mathrm{W}}$, where $F_{MD} = \{(y, \tau, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y), \tau_k \geqq 0, \sum_{k\in \mathrm{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i = 1, \ldots, m\}$.

In addition, let $L(y, \tau, v, \lambda) := f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)e$.

Definition 3.3.1. A point $(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_{MD}$ is said to be a *weak Pareto solution* of problems $(\mathrm{RMD})_{\mathrm{W}}$ if

$$L(y, \tau, v, \lambda) - L(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \notin \operatorname{int} \mathbb{R}^l_+, \quad \forall (y, \tau, v, \lambda) \in F_{MD}$$

Now, we give a weak duality relation between the primal problem (RMP) and the dual problem $(\text{RMD})_{\text{W}}$ in the following.

Theorem 3.3.1 (weak duality)**.** *For any feasible solution x of* (RP) *and any feasible solution (y, τ, v, λ) of* $(\text{RD})_{\text{W}}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

Proof. Since $(y, \tau, v, \lambda) \in F_D$, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \text{K}} \tau_k = 1$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(y), k \in \text{K}$ and $\bar{\zeta}_i \in \partial g_i(\cdot, v_i)(y), i = 1, \ldots, m$ such that

$$\sum_{k \in \text{K}} \tau_k \bar{\xi}_k + \sum_{i=1}^{m} \lambda_i \bar{\zeta}_i = 0, \tag{3.10}$$

$$\tau_k (f_k(y) - \varphi(y)) = 0, \;\; k \in \text{K}, \tag{3.11}$$

thus from (3.10), we have

$$\sum_{k \in \text{K}} \tau_k \langle \bar{\xi}_k, x - y \rangle + \sum_{i=1}^{m} \lambda_i \langle \bar{\zeta}_i, x - y \rangle = 0,$$

by the convexity of $f_k(\cdot), k \in \text{K}$ and $g_i(\cdot, v_i), i = 1, \ldots, m$,

$$\sum_{k \in \text{K}} \tau_k (f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i (g_i(x, v_i) - g_i(y, v_i)) \geqq 0. \tag{3.12}$$

Finally, from (3.11) and (3.12), and the fact $\lambda_i g_i(x, v_i) = 0$, due to $\sum_{k \in \text{K}} \tau_k = 1$, we obtain

$$\varphi(x) = \sum_{k \in \text{K}} \tau_k \max_{k \in \text{K}} f_k(x) \geqq \sum_{k \in \text{K}} \tau_k f_k(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

Thus the proof of the theorem is completed. □

In what follows, a strong duality relation between the primal problem (RMP) and the dual problem $(\mathrm{RMD})_{\mathrm{W}}$ is given.

Theorem 3.3.2 (strong duality)**.** *Let $\bar{x} \in F$ be an optimal solution of the robust problem* (RMP) *such that the Slater condition holds at this point. Then there exists $(\bar{\tau}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_{MD}$ is an optimal solution of problem* $(\mathrm{RMD})_{\mathrm{W}}$.

Proof. Let $\bar{x} \in F$ be an optimal solution of (RMP) such that the Slater condition holds at this point. By Theorem 3.2.1, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i$ and $\lambda_i \geqq 0, i = 1, \ldots, m$ such that

$$
\begin{aligned}
&0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}), \\
&\tau_k(f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x})) = 0, \ k \in \mathrm{K}, \\
&\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ i = 1, \ldots, m.
\end{aligned} \tag{3.13}
$$

Putting

$$
\bar{\tau}_k := \frac{\tau_k}{\sum_{k \in \mathrm{K}} \tau_k}, \ k \in \mathrm{K}, \quad \bar{\lambda}_i := \frac{\lambda_i}{\sum_{k \in \mathrm{K}} \tau_k}, \ i = 1, \ldots, m,
$$

we then have $\bar{\tau}_i + (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \bar{\tau}_k = 1$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that the assertion in (3.13) is still valid when τ_k's and λ_i's are replaced by $\bar{\tau}_k$'s and $\bar{\lambda}_k$'s, respectively. Consequently, $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a feasible solution of $(\mathrm{RD})_{\mathrm{W}}$.

Now, by Theorem 3.3.1 (weak duality), for any feasible (y, τ, v, λ) of $(\mathrm{RD})_{\mathrm{W}}$,

$$\varphi(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = \varphi(\bar{x}) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i),$$

which means that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is an optimal solution of problem $(\mathrm{RD})_{\mathrm{W}}$. $\square$

3.3.2 For a weak quasi ϵ-Pareto solution

In connection with the robust multiobjective programming problem (RMP), we consider a dual problem in the following form:

$$
\begin{aligned}
(\mathrm{RMD})_{\mathrm{Q}} \quad \mathrm{Max}_{(y,\tau,v,\lambda)} \quad & f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) e \\
\text{subject to} \quad & 0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) + \sum_{k \in \mathrm{K}} \tau_k \sqrt{\epsilon} \mathbb{B} \\
& \tau_k \geqq 0, \ \sum_{k \in \mathrm{K}} \tau_k = 1, e = (1, \ldots, 1) \\
& \lambda_i \geqq 0, \ v_i \in \mathcal{V}_i, \ i = 1, \ldots, m.
\end{aligned}
$$

Let F_{QD} be the feasible set of $(\mathrm{RMD})_{\mathrm{Q}}$, where $F_{QD} = \{(y, \tau, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) + \sum_{k \in \mathrm{K}} \tau_k \sqrt{\epsilon} \mathbb{B}, \tau_k \geqq 0, \sum_{k \in \mathrm{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i = 1, \ldots, m\}$.

In addition, let $L(y, \tau, v, \lambda) := f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) e$.

Definition 3.3.2. A point $(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_{QD}$ is said to be a *weak Pareto solution* of problems $(\text{RMD})_\text{Q}$ if

$$L(y, \tau, v, \lambda) - L(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \notin \operatorname{int} \mathbb{R}^l_+, \quad \forall (y, \tau, v, \lambda) \in F_{QD}$$

The following theorem shows a weak duality relation between the primal problem (RMP) and the dual problem $(\text{RMD})_\text{Q}$.

Theorem 3.3.3 (weak duality)**.** *For any feasible solution x of* (RMP) *and any feasible solution (y, τ, v, λ) of* $(\text{RMD})_\text{Q}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon}\|x - y\|.$$

Proof. Since $(y, \tau, v, \lambda) \in F_{QD}$, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \text{K}} \tau_k = 1$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(y), k \in \text{K}$ and $\bar{\zeta}_i \in \partial g_i(\cdot, v_i)(y), i = 1, \ldots, m$, $b \in \mathbb{B}$ such that

$$\sum_{k \in \text{K}} \tau_k \bar{\xi}_k + \sum_{i=1}^{m} \lambda_i \bar{\zeta}_i + \sqrt{\epsilon} b = 0, \tag{3.14}$$

$$\tau_k (f_k(y) - \varphi(y)) = 0, \ k \in \text{K}, \tag{3.15}$$

thus from (3.14), we have

$$\sum_{k \in \text{K}} \tau_k \langle \bar{\xi}_k, x - y \rangle + \sum_{i=1}^{m} \lambda_i \langle \bar{\zeta}_i, x - y \rangle + \sqrt{\epsilon} \langle b, x - y \rangle = 0,$$

by the convexity of $f_k(\cdot), k \in \mathrm{K}$ and $g_i(\cdot, v_i), i = 1, \ldots, m$,

$$\begin{aligned}
&\sum_{k\in \mathrm{K}} \tau_k(f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i(g_i(x, v_i) - g_i(y, v_i)) + \sqrt{\epsilon}\|x - y\| \\
\geqq &\sum_{k\in \mathrm{K}} \tau_k(f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i(g_i(x, v_i) - g_i(y, v_i)) + \sqrt{\epsilon}\|b\|\|x - y\| \\
\geqq &0. \qquad (3.16)
\end{aligned}$$

Finally, from (3.15) and (3.16), and the fact $\lambda_i g_i(x, v_i) = 0$, due to $\sum_{k\in \mathrm{K}} \tau_k = 1$, we obtain

$$\begin{aligned}
\varphi(x) = \sum_{k\in \mathrm{K}} \tau_k \max_{k\in \mathrm{K}} f_k(x) &\geqq \sum_{k\in \mathrm{K}} \tau_k f_k(x) \\
&\geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon}\|x - y\|.
\end{aligned}$$

Thus, we complete the proof. □

Below, a strong duality relation between the primal problem (RMP) and the dual problem $(\mathrm{RMD})_{\mathrm{Q}}$ is proposed.

Theorem 3.3.4 (strong duality)**.** *Let $\bar{x} \in F$ be a quasi ϵ-solution of the robust problem* (RMP) *such that the Slater condition holds at this point. Then there exists $(\bar{\tau}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_{QD}$ is a quasi ϵ-solution of problem* $(\mathrm{RMD})_{\mathrm{Q}}$.

Proof. Let $\bar{x} \in F$ be a quasi ϵ-solution of (RMP) such that the Slater condition holds at this point. By Theorem 3.2.2, there exist $\tau := (\tau_1, \ldots, \tau_l) \in$

$\mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i$ and $\lambda_i \geqq 0, i = 1, \ldots, m$, $b \in \mathbb{B}$ such that

$$0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) + \sqrt{\epsilon} b,$$

$$\tau_k(f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x})) = 0, \ k \in \mathrm{K},$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ i = 1, \ldots, m. \tag{3.17}$$

Putting

$$\bar{\tau}_k := \frac{\tau_k}{\sum_{k \in \mathrm{K}} \tau_k}, \ k \in \mathrm{K}, \ \ \bar{\lambda}_i := \frac{\lambda_i}{\sum_{k \in \mathrm{K}} \tau_k}, \ i = 1, \ldots, m,$$

we then have $\bar{\tau}_i + (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \bar{\tau}_k = 1$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that the assertion in (3.17) is still valid when τ_k's and λ_i's are replaced by $\bar{\tau}_k$'s and $\bar{\lambda}_k$'s, respectively. Consequently, $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a feasible solution of $(\mathrm{RMD})_{\mathrm{Q}}$.

Now, by Theorem 3.3.3 (weak duality), for any feasible (y, τ, v, λ) of $(\mathrm{RMD})_{\mathrm{Q}}$,

$$\begin{aligned} \varphi(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) &= \varphi(\bar{x}) \\ &\geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon} \|\bar{x} - y\|, \end{aligned}$$

which means that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a quasi ϵ-solution of problem $(\mathrm{RMD})_{\mathrm{W}}$. □

Chapter 4
Optimality Conditions and Duality for Optimal and Approximate Solutions in Robust Minimax Fractional Programming

4.1 Introduction

In this chapter, we study the optimality conditions and duality for an optimal solution and an approximate solution in robust minimax fractional programming. First, let us recall that a standard form of minimax fractional programming problem is the one:

$$\text{(FP)}\ \min_{x\in\mathbb{R}^n}\max_{k\in\mathrm{K}}\frac{p_k(x)}{q_k(x)}\ \text{ subject to }\ g_i(x)\leqq 0,\ i=1,\ldots,m,$$

where $p_k,\ -q_k:\mathbb{R}^n\to\mathbb{R}$, $k\in\mathrm{K}:=\{1,\ldots,l\}$ and $g_i:\mathbb{R}^n\to\mathbb{R}$, $i=1,\ldots,m$ are convex functions.

The minimax fractional programming problem (FP) in the face of data uncertainty in the constraints can be captured by the one

$$\text{(UFP)}\ \min_{x\in\mathbb{R}^n}\max_{k\in\mathrm{K}}\frac{p_k(x)}{q_k(x)}\ \text{ subject to }\ g_i(x,v_i)\leqq 0,\ i=1,\ldots,m,$$

where $g_i:\mathbb{R}^n\times\mathbb{R}^q\to\mathbb{R}$, $g_i(\cdot,v_i)$ is convex and $v_i\in\mathbb{R}^q$ is an uncertain parameter which belongs to the set $\mathcal{V}_i\subset\mathbb{R}^q$, $i=1,\ldots,m$.

The robust counterpart of the problem (UFP) is as follows:

$$\text{(RFP)} \;\; \min_{x\in\mathbb{R}^n} \max_{k\in \mathrm{K}} \frac{p_k(x)}{q_k(x)} \;\; \text{subject to} \;\; g_i(x,v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i=1,\ldots,m.$$

Denote again by $F := \{x \in \mathbb{R}^n \colon g_i(x,v_i) \leqq 0, \; \forall v_i \in \mathcal{V}_i, \; i=1,\ldots,m\}$ as the feasible set of (RFP).

Moreover, we let $f_k(x) := \frac{p_k(x)}{q_k(x)}$, and $\varphi(x) =: \max_{k\in \mathrm{K}} f_k(x)\Big(=: \max_{k\in \mathrm{K}} \frac{p_k(x)}{q_k(x)}\Big)$ for convenience. Note that a very remarkable phenomenon of a (robust) fractional programming problem is that its objective function is, in general, not convex functions, even under more restrictive convexity/concavity assumptions. Hence, $f_k(x)$ is generally nonconvex.

4.2 Preliminaries

In this section, we recall some notations and give preliminary results for next sections. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm $\| \cdot \|$. We say that a set Γ in $\mathbb{R}^n$ is *convex* whenever $\mu a_1 + (1-\mu)a_2 \in \Gamma$ for all $\mu \in [0,1]$, $a_1, a_2 \in \Gamma$. We denote the domain of f by dom f, that is, dom $f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. f is said to be *convex* if for all $\lambda \in [0,1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$. The function f is said to be *concave* whenever $-f$ is convex. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y-x\rangle \leq f(y) - f(x), \; \forall y \in \mathbb{R}^n\}, \; \text{if } x \in \mathrm{dom} f, \\ \emptyset, \; \text{otherwise.} \end{cases}$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U and a constant $L > 0$ such that for all y and z in U,

$$|g(y) - g(z)| \leq L\|y - z\|.$$

Definition 4.2.1. *For each $d \in \mathbb{R}^n$, the* Clarke directional derivative *of g at $x \in \mathbb{R}^n$ in the direction d, denoted by $g^\circ(x; d)$, is given by*

$$g^\circ(x; d) \ = \limsup_{h \to 0,\ t \to 0+} \frac{g(x + h + td) - g(x + h)}{t}.$$

We also denote the usual one-sided directional derivative *of g at x by $g'(x; d)$. Thus*

$$g'(x; d) = \lim_{t \to 0+} \frac{g(x + td) - g(x)}{t},$$

whenever this limit exists.

Definition 4.2.2. *The* Clarke subdifferential *of g at x, denoted by $\partial^\circ g(x)$, is the (nonempty) set of all ξ in $\mathbb{R}^n$ satisfying the following condition:*

$$g^\circ(x; d) \geq \langle \xi, d \rangle, \quad \textit{for all} \ \ d \in \mathbb{R}^n.$$

We summarize some fundamental results in the calculus of the Clarke subdifferential (for more details, see [13–16, 34]):

- $\partial^\circ g(x)$ is a nonempty, convex, compact subset of $\mathbb{R}^n$;

- The function $d \mapsto g^\circ(x; d)$ is convex;

- For every d in $\mathbb{R}^n$, one has

$$g^\circ(x; d) = \max\{\langle \xi, d \rangle : \xi \in \partial^\circ g(x)\}.$$

Let $\mathcal{V} \subset \mathbb{R}^q$ be a compact set and let $g \colon \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ be a given function. Here after all, we assume that the following assumptions hold:

- (A1) $g(x, v)$ is upper semicontinuous in (x, v).
- (A2) g is locally Lipschitz in x, uniformly for v in $\mathcal{V}$, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in \mathcal{V}$,

$$|g(y, v) - g(z, v)| \leq L\|y - z\|.$$

- (A3) $g_x^\circ(x, v; \cdot) = g_x'(x, v; \cdot)$, the derivatives being with respect to x.

We define a function $\psi \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(x) := \max\{g(x, v) : v \in \mathcal{V}\},$$

and observe that our assumptions (A1)-(A2) imply that ψ is defined and finite (with the maximum defining ψ attained) on $\mathbb{R}^n$. Let

$$\mathcal{V}(x) := \{v \in \mathcal{V} : g(x, v) = \psi(x)\},$$

then for each $x \in \mathbb{R}^n$, $\mathcal{V}(x)$ is a nonempty closed set.

The following lemma, which is a nonsmooth version of Danskin's theorem [17] for max-functions, makes connection between the functions $\psi'(x; d)$ and $g_x^\circ(x, v; d)$.

Lemma 4.2.1. *Under the assumptions* (A1)–(A3), *the usual one-sided directional derivative* $\psi'(x;d)$ *exists, and satisfies*

$$\begin{aligned}\psi'(x;d) = \psi^\circ(x;d) &= \max\{g_x^\circ(x,v;d) : v \in \mathcal{V}(x)\} \\ &= \max\{\langle \xi, d\rangle : \xi \in \partial_x^\circ g(x,v),\ v \in \mathcal{V}(x)\}.\end{aligned}$$

Proof. See [15, Theorem 2] (see also [13, Theorem 2.1], [17]). □

The following result will be useful in the sequel.

Lemma 4.2.2. [36] *In addition to the basic assumptions* (A1)–(A3), *suppose that* $\mathcal{V}$ *is convex, and that* $g(x,\cdot)$ *is concave on* $\mathcal{V}$, *for each* $x \in U$. *Then the following statements hold*:

(i) *The set* $\mathcal{V}(x)$ *is convex and compact.*

(ii) *The set*

$$\partial_x^\circ g(x,\mathcal{V}(x)) := \{\xi : \exists v \in \mathcal{V}(x) \ \ \text{such that} \ \ \xi \in \partial_x^\circ g(x,v)\}$$

is convex and compact.

(iii) $\partial^\circ \psi(x) = \{\xi : \exists v \in \mathcal{V}(x) \ \ \text{such that} \ \ \xi \in \partial_x^\circ g(x,v)\}$.

Proposition 4.2.1. [16, Proposition 2.3.3] *If* $f_i, i = 1,\dots,l$ *is a finite family of functions each of which is Lipschitz near* x, *it follows easily that their sum* $f = \sum_{i=1}^{l} f_i$ *is also Lipschitz near* x. *Moreover, one has*

$$\partial^\circ\Big(\sum_{i=1}^{l} f_i\Big)(x) \subset \sum_{i=1}^{l} \partial^\circ f_i(x).$$

Proposition 4.2.2. [16, Proposition 2.3.12] *Let $f_i, i \in I := \{1, \dots, l\}$ be Lipschitz near x, then one has*

$$\partial^\circ f(x) \subset \text{co}\,\{\partial^\circ f_i(x) \mid i \in I(x)\},$$

where $I(x) := \{i \in I \mid f_i(x) = 0\}$. and if f_i is regular at x for each i in $I(x)$, then equality holds and f is regular at x.

Proposition 4.2.3. [16, Proposition 2.3.14] *Let ψ_1, ψ_2 be Lipschitz near x, and suppose $\psi_2 \neq 0$. Then $\frac{\psi_1}{\psi_2}$ is Lipschitz near x, and one has*

$$\partial^\circ\Big(\frac{\psi_1}{\psi_2}\Big)(x) \subset \frac{\psi_2(x)\partial^\circ\psi_1(x) - \psi_1(x)\partial^\circ\psi_2(x)}{\psi_2^2(x)}.$$

If in addition $\psi_1(x) \geq 0, \psi_2(x) > 0$ and if ψ_1 and $-\psi_2$ are regular at x, then equality holds and ψ_1/ψ_2 is regular at x.

4.3 Optimality Conditions

Theorem 4.3.1 (KKT condition for an optimal solution)**.** Consider the problem (RFP), assume that the Slater condition holds. If $\bar{x}$ is an optimal solution of the problem (RFP), then there exist $\tau := (\tau_1, \dots, \tau_l) \in \mathbb{R}^l_+ \backslash \{0\}$, $\bar{v}_i \in \mathcal{V}_i,\ i = 1, \dots, m$ and $\lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m_+$, such that

$$0 \in \sum_{k \in \mathrm{K}} \tau_k \Big(\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\Big) + \sum_{i \in M} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}),$$

$$\tau_k\Big(\frac{p_k(\bar{x})}{q_k(\bar{x})} - \max_{k \in \mathrm{K}} \frac{p_k(\bar{x})}{q_k(\bar{x})}\Big) = 0, \quad k \in \mathrm{K},$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \dots, m.$$

Proof. Let $\bar{x}$ be an optimal solution of the problem (RFP), and let $f_k(x) := \frac{p_k(x)}{q_k(x)}$, furthermore, $\varphi(x) =: \max_{k \in \mathrm{K}} f_k(x)$. Then, $\bar{x}$ is a minimizer of the following problem:

$$\min_{x \in F} \varphi(x), \tag{4.1}$$

observe that $\varphi(x)$ is a locally Lipschitz function. Applying the standard optimality condition [16, Propostion 2.4.2] to the problem (4.1), we have

$$0 \in \partial^{\circ}\varphi(\bar{x}) + N_F(\bar{x}).$$

On the one hand, employing Proposition 4.2.2 and Proposition 4.2.3, we obtain

$$\begin{aligned}
&\partial^{\circ}\varphi(\bar{x}) = \partial^{\circ}(\max_{k \in \mathrm{K}} f_k)(\bar{x}) = \mathrm{co} \bigcup_{k \in \mathrm{K}(\bar{x})} \partial^{\circ} f_k(\bar{x}) \\
&= \mathrm{co} \bigcup_{k \in \mathrm{K}(\bar{x})} \{\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\} \\
&= \left\{ \sum_{k \in \mathrm{K}(\bar{x})} \tau_k \Big(\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\Big) \,\Big|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k \in \mathrm{K}(\bar{x})} \tau_k = 1 \right\},
\end{aligned}$$

where $\mathrm{K}(\bar{x}) := \{k \in \mathrm{K} \colon f_k(\bar{x}) = \varphi(\bar{x})\} \neq \emptyset$. On the other hand, since the Slater condition holds, and from Lemma 2.3.1, we have

$$\begin{aligned}
0 \in \\
&\left\{ \sum_{k \in \mathrm{K}(\bar{x})} \tau_k \partial\Big(\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\Big) \,\Big|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k \in \mathrm{K}(\bar{x})} \tau_k = 1 \right\} \\
&\quad + \left\{ \sum_{i \in I(\bar{x})} \lambda_i \partial g_i(\bar{x}, \bar{v}_i) \,\Big|\, \lambda_i \geqq 0, i \in I(\bar{x}) \right\}.
\end{aligned} \tag{4.2}$$

Now, letting $\tau_k := 0$ for $k \in \mathrm{K} \setminus \mathrm{K}(\bar{x})$ and $\lambda_i := 0$ for $i \in \{1, \ldots, m\} \setminus I(\bar{x})$, we see that (4.2) clearly implies the conditions in the theorem, which completes the proof of the theorem. $\square$

Theorem 4.3.2 (KKT condition for a quasi ϵ-solution)**.** Consider the problem (RFP), assume that the Slater condition holds. If $\bar{x}$ is a quasi ϵ-solution of the problem (RFP), then there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i,\ i = 1, \ldots, m$ and $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, such that

$$0 \in \sum_{k \in \mathrm{K}} \tau_k \Big(\frac{q_k(\bar{x}) \partial p_k(\bar{x}) - p_k(\bar{x}) \partial q_k(\bar{x})}{q_k^2(\bar{x})} \Big) + \sum_{i \in M} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) + \sqrt{\epsilon}\mathbb{B},$$

$$\tau_k \Big(\frac{p_k(\bar{x})}{q_k(\bar{x})} - \max_{k \in \mathrm{K}} \frac{p_k(\bar{x})}{q_k(\bar{x})} \Big) = 0, \quad k \in \mathrm{K},$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \quad i = 1, \ldots, m.$$

Proof. Let $\bar{x}$ be a quasi ϵ-solution of the problem (RFP). Then, $\bar{x}$ is a minimizer of the following problem:

$$\min_{x \in F} \{ \varphi(x) + \sqrt{\epsilon} \| \cdot - \bar{x} \| \}, \tag{4.3}$$

where $\varphi(x) =: \max_{k \in \mathrm{K}} f_k(x)$, and $f_k(x) := \frac{p_k(x)}{q_k(x)}$. Applying the standard optimality condition [16, Propostion 2.4.2] to the problem (4.3), we have

$$0 \in \partial^\circ (\varphi + \sqrt{\epsilon} \| \cdot - \bar{x} \|)(\bar{x}) + N_F(\bar{x}).$$

Since the function φ is Lipschitz, and $\| \cdot - \bar{x} \|$ is convex and hence Lipschitz, it follows from Proposition 4.2.1 that

$$0 \in \partial^\circ \varphi(\bar{x}) + \sqrt{\epsilon} \partial \| \cdot - \bar{x} \| + N_F(\bar{x}).$$

Moreover, the fact and $\partial\| \cdot - \bar{x}\| = \mathbb{B}$ leads to

$$0 \in \partial^\circ \varphi(\bar{x}) + N_F(\bar{x}) + \sqrt{\epsilon}\mathbb{B}.$$

On the one hand, employing Proposition 4.2.2 and Proposition 4.2.3, we obtain

$$\partial^\circ \varphi(\bar{x}) = \partial^\circ(\max_{k\in \mathrm{K}} f_k)(\bar{x}) = \mathrm{co} \bigcup_{k\in \mathrm{K}(\bar{x})} \partial^\circ f_k(\bar{x})$$

$$= \mathrm{co} \bigcup_{k\in \mathrm{K}(\bar{x})} \{\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\}$$

$$= \left\{ \sum_{k\in \mathrm{K}(\bar{x})} \tau_k \Big(\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\Big) \,\Big|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k\in \mathrm{K}(\bar{x})} \tau_k = 1 \right\},$$

where $\mathrm{K}(\bar{x}) := \{k \in \mathrm{K} \colon f_k(\bar{x}) = \varphi(\bar{x})\} \neq \emptyset$. On the other hand, since the Slater condition holds, and from Lemma 2.3.1, we have

$$0 \in \left\{ \sum_{k\in \mathrm{K}(\bar{x})} \tau_k \partial\Big(\frac{q_k(\bar{x})\partial p_k(\bar{x}) - p_k(\bar{x})\partial q_k(\bar{x})}{q_k^2(\bar{x})}\Big) \,\Big|\, \tau_k \geqq 0, k \in \mathrm{K}(\bar{x}), \sum_{k\in \mathrm{K}(\bar{x})} \tau_k = 1 \right\}$$

$$+ \left\{ \sum_{i\in I(\bar{x})} \lambda_i \partial g_i(\bar{x}, \bar{v}_i) \,\Big|\, \lambda_i \geqq 0, i \in I(\bar{x}) \right\} + \sqrt{\epsilon}\mathbb{B} \tag{4.4}$$

Now, letting $\tau_k := 0$ for $k \in \mathrm{K}\setminus \mathrm{K}(\bar{x})$ and $\lambda_i := 0$ for $i \in \{1, \ldots, m\}\setminus I(\bar{x})$, we see that (4.4) clearly implies the conditions in the theorem, which completes the proof of the theorem. □

4.4 Duality Theorems

In this section, we formulate a dual problem to the primal one in the sense of Wolfe [46], and explore weak and strong duality relations between them, for both an optimal solution and a quasi ϵ-solution.

4.4.1 For an optimal solution

In connection with the robust minimax fractional programming problem (RFP), denote $\varphi(y) := \max_{k\in \mathrm{K}} f_k(y)$, and $f_k(x) := \frac{p_k(x)}{q_k(x)}$, we consider a dual problem in the following form:

$$
\begin{aligned}
(\mathrm{RFD})_{\mathrm{W}} \qquad & \mathrm{Maximize}_{(y,\tau,v,\lambda)} \quad && \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
& \text{subject to} && 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) \\
& && \tau_k(f_k(y) - \varphi(y)) = 0, \ k \in \mathrm{K} \\
& && \tau_k \geqq 0, \ \sum_{k\in \mathrm{K}} \tau_k = 1 \\
& && \lambda_i \geqq 0, \ v_i \in \mathcal{V}_i, \ i = 1, \ldots, m.
\end{aligned}
$$

Let F_D be the feasible set of $(\mathrm{RD})_{\mathrm{W}}$, where $F_D = \{(y, \tau, v, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y), \tau_k(f_k(y) - \varphi(y)) = 0, k \in \mathrm{K}, \tau_k \geqq 0, \sum_{k\in \mathrm{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i = 1, \ldots, m\}$. We should note that a point

$(\bar{y}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_D$ is called an *optimal solution* of problems $(\mathrm{RD})_{\mathrm{W}}$ if for all $(y, \tau, v, \lambda) \in F_D$,

$$\varphi(\bar{y}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{y}, \bar{v}_i) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

The following theorem describes a weak duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_{\mathrm{W}}$.

Theorem 4.4.1 (weak duality)**.** *For any feasible solution x of* (RP) *and any feasible solution (y, τ, v, λ) of* $(\mathrm{RD})_{\mathrm{W}}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

Proof. Since $(y, \tau, v, \lambda) \in F_D$, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \tau_k = 1$, $\lambda := (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(y), k \in \mathrm{K}$ and $\bar{\zeta}_i \in \partial g_i(\cdot, v_i)(y), i = 1, \ldots, m$ such that

$$\sum_{k \in \mathrm{K}} \tau_k \bar{\xi}_k + \sum_{i=1}^{m} \lambda_i \bar{\zeta}_i = 0, \tag{4.5}$$

$$\tau_k (f_k(y) - \varphi(y)) = 0, \ k \in \mathrm{K}, \tag{4.6}$$

thus from (4.5), we have

$$\sum_{k \in \mathrm{K}} \tau_k \langle \bar{\xi}_k, x - y \rangle + \sum_{i=1}^{m} \lambda_i \langle \bar{\zeta}_i, x - y \rangle = 0,$$

by the convexity of $f_k(\cdot), k \in \mathrm{K}$ and $g_i(\cdot, v_i), i = 1, \ldots, m$,

$$\sum_{k \in \mathrm{K}} \tau_k (f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i (g_i(x, v_i) - g_i(y, v_i)) \geqq 0. \tag{4.7}$$

Finally, from (4.6) and (4.7), and the fact $\lambda_i g_i(x, v_i) = 0$, due to $\sum_{k\in \mathrm{K}} \tau_k = 1$, we obtain

$$\varphi(x) = \sum_{k\in \mathrm{K}} \tau_k \max_{k\in \mathrm{K}} f_k(x) \geqq \sum_{k\in \mathrm{K}} \tau_k f_k(x) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i).$$

Thus the proof of the theorem is completed. □

In what follows, a strong duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_{\mathrm{W}}$ is given.

Theorem 4.4.2 (strong duality). *Let $\bar{x} \in F$ be an optimal solution of the robust problem* (RP) *such that the Slater condition holds at this point. Then there exists $(\bar{\tau}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_D$ is an optimal solution of problem* $(\mathrm{RD})_{\mathrm{W}}$.

Proof. Let $\bar{x} \in F$ be an optimal solution of (RP) such that the Slater condition holds at this point. By Theorem 2.4.1, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i$ and $\lambda_i \geqq 0, i = 1, \ldots, m$ such that

$$\begin{aligned}
&0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}), \\
&\tau_k(f_k(\bar{x}) - \max_{k\in \mathrm{K}} f_k(\bar{x})) = 0, \ k \in \mathrm{K}, \\
&\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ i = 1, \ldots, m. \qquad (4.8)
\end{aligned}$$

Putting

$$\bar{\tau}_k := \frac{\tau_k}{\sum_{k\in \mathrm{K}} \tau_k}, \ k \in \mathrm{K}, \ \ \bar{\lambda}_i := \frac{\lambda_i}{\sum_{k\in \mathrm{K}} \tau_k}, \ i = 1, \ldots, m,$$

we then have $\bar{\tau}_i + (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+$ with $\sum_{k\in \mathrm{K}} \bar{\tau}_k = 1$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that the assertion in (4.8) is still valid when τ_k's and λ_i's are replaced by $\bar{\tau}_k$'s and $\bar{\lambda}_k$'s, respectively. Consequently, $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a feasible solution of $(\mathrm{RD})_{\mathrm{W}}$.

Now, by Theorem 4.4.1 (weak duality), for any feasible (y, τ, v, λ) of $(\mathrm{RD})_{\mathrm{W}}$,

$$\varphi(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = \varphi(\bar{x}) \geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i),$$

which means that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is an optimal solution of problem $(\mathrm{RD})_{\mathrm{W}}$. □

4.4.2 For a quasi ϵ-solution

Denote again $\varphi(y) := \max\limits_{k\in \mathrm{K}} f_k(y)$, here we consider a dual problem that enjoys the following form:

$$
\begin{aligned}
(\mathrm{RD})_{\mathrm{Q}} \qquad \mathrm{Maximize}_{(y,\tau,v,\lambda)} \quad & \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) \\
\text{subject to} \quad & 0 \in \sum_{k\in \mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, v_i)(y) + \sqrt{\epsilon}\mathbb{B} \\
& \tau_k(f_k(y) - \varphi(y)) = 0, \ k \in \mathrm{K} \\
& \tau_k \geqq 0, \ \sum_{k\in \mathrm{K}} \tau_k = 1 \\
& \lambda_i \geqq 0, \ v_i \in \mathcal{V}_i, \ i = 1, \ldots, m, \epsilon \geqq 0.
\end{aligned}
$$

Let F_Q be the feasible set of $(\mathrm{RD})_{\mathrm{Q}}$, where $F_Q = \{(y,\tau,v,\lambda) \in \mathbb{R}^n \times \mathbb{R}^l_+ \times \mathcal{V} \times \mathbb{R}^m_+ \colon 0 \in \sum_{k\in\mathrm{K}} \tau_k \partial f_k(y) + \sum_{i=1}^m \lambda_i \partial g_i(\cdot, v_i)(y) + \sqrt{\epsilon}\mathbb{B}, \tau_k(f_k(y) - \varphi(y)) = 0, k \in \mathrm{K}, \tau_k \geqq 0, \sum_{k\in\mathrm{K}} \tau_k = 1, \lambda_i \geqq 0, v_i \in \mathcal{V}_i, i = 1,\ldots,m, \epsilon \geqq 0\}$.

Definition 4.4.1. *Let $\epsilon \geqq 0$ be given, a point $(\bar{y},\bar{\tau},\bar{v},\bar{\lambda}) \in F_Q$ is called a quasi ϵ-solution of the problem* $(\mathrm{RD})_{\mathrm{Q}}$ *if for all $(y,\tau,v,\lambda) \in F_Q$,*

$$\varphi(\bar{y}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{y},\bar{v}_i) \geqq \varphi(y) + \sum_{i=1}^m \lambda_i g_i(y,v_i) - \sqrt{\epsilon}\|y-\bar{y}\|.$$

The following theorem shows a weak duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_{\mathrm{Q}}$.

Theorem 4.4.3 (weak duality)**.** *For any feasible solution x of* (RP) *and any feasible solution (y,τ,v,λ) of* $(\mathrm{RD})_{\mathrm{Q}}$,

$$\varphi(x) \geqq \varphi(y) + \sum_{i=1}^m \lambda_i g_i(y,v_i) - \sqrt{\epsilon}\|x-y\|.$$

Proof. Since $(y,\tau,v,\lambda) \in F_Q$, there exist $\tau := (\tau_1,\ldots,\tau_l) \in \mathbb{R}^l_+$ with $\sum_{k\in\mathrm{K}} \tau_k = 1$, $\lambda := (\lambda_1,\ldots,\lambda_m) \in \mathbb{R}^m_+$, $\bar{\xi}_k \in \partial f_k(y), k \in \mathrm{K}$ and $\bar{\zeta}_i \in \partial g_i(\cdot,v_i)(y), i = 1,\ldots,m$, $b \in \mathbb{B}$ such that

$$\sum_{k\in\mathrm{K}} \tau_k \bar{\xi}_k + \sum_{i=1}^m \lambda_i \bar{\zeta}_i + \sqrt{\epsilon} b = 0, \tag{4.9}$$

$$\tau_k(f_k(y) - \varphi(y)) = 0,\ k \in \mathrm{K}, \tag{4.10}$$

thus from (2.28), we have

$$\sum_{k\in \mathrm{K}} \tau_k\langle\bar{\xi}_k, x-y\rangle + \sum_{i=1}^{m} \lambda_i\langle\bar{\zeta}_i, x-y\rangle + \sqrt{\epsilon}\langle b, x-y\rangle = 0,$$

by the convexity of $f_k(\cdot), k \in \mathrm{K}$ and $g_i(\cdot, v_i), i = 1, \ldots, m$,

$$\begin{aligned}
&\sum_{k\in \mathrm{K}} \tau_k(f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i(g_i(x, v_i) - g_i(y, v_i)) + \sqrt{\epsilon}\|x-y\| \\
\geqq &\sum_{k\in \mathrm{K}} \tau_k(f_k(x) - f_k(y)) + \sum_{i=1}^{m} \lambda_i(g_i(x, v_i) - g_i(y, v_i)) + \sqrt{\epsilon}\|b\|\|x-y\| \\
\geqq &0. \qquad (4.11)
\end{aligned}$$

Finally, from (4.10) and (4.11), and the fact $\lambda_i g_i(x, v_i) = 0$, due to $\sum_{k\in \mathrm{K}} \tau_k = 1$, we obtain

$$\begin{aligned}
\varphi(x) = \sum_{k\in \mathrm{K}} \tau_k \max_{k\in \mathrm{K}} f_k(x) &\geqq \sum_{k\in \mathrm{K}} \tau_k f_k(x) \\
&\geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon}\|x-y\|.
\end{aligned}$$

Thus, we complete the proof. □

Below, a strong duality relation between the primal problem (RP) and the dual problem $(\mathrm{RD})_\mathrm{Q}$ is proposed.

Theorem 4.4.4 (strong duality)**.** *Let $\bar{x} \in F$ be a quasi ϵ-solution of the robust problem* (RP) *such that the Slater condition holds at this point. Then there exists $(\bar{\tau}, \bar{v}, \bar{\lambda}) \in \mathbb{R}^l_+ \times \mathbb{R}^q \times \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda}) \in F_Q$ is a quasi ϵ-solution of problem* $(\mathrm{RD})_\mathrm{Q}$.

Proof. Let $\bar{x} \in F$ be a quasi ϵ-solution of (RP) such that the Slater condition holds at this point. By Theorem 2.4.3, there exist $\tau := (\tau_1, \ldots, \tau_l) \in \mathbb{R}^l_+ \setminus \{0\}$, $\bar{v}_i \in \mathcal{V}_i$ and $\lambda_i \geqq 0, i = 1, \ldots, m$, $b \in \mathbb{B}$ such that

$$0 \in \sum_{k \in \mathrm{K}} \tau_k \partial f_k(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\cdot, \bar{v}_i)(\bar{x}) + \sqrt{\epsilon} b,$$

$$\tau_k(f_k(\bar{x}) - \max_{k \in \mathrm{K}} f_k(\bar{x})) = 0, \ k \in \mathrm{K},$$

$$\lambda_i g_i(\bar{x}, \bar{v}_i) = 0, \ i = 1, \ldots, m. \tag{4.12}$$

Putting

$$\bar{\tau}_k := \frac{\tau_k}{\sum_{k \in \mathrm{K}} \tau_k}, \ k \in \mathrm{K}, \quad \bar{\lambda}_i := \frac{\lambda_i}{\sum_{k \in \mathrm{K}} \tau_k}, \ i = 1, \ldots, m,$$

we then have $\bar{\tau}_i + (\bar{\tau}_1, \ldots, \bar{\tau}_l) \in \mathbb{R}^l_+$ with $\sum_{k \in \mathrm{K}} \bar{\tau}_k = 1$ and $\bar{\lambda} := (\bar{\lambda}_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$. Observe that the assertion in (4.12) is still valid when τ_k's and λ_i's are replaced by $\bar{\tau}_k$'s and $\bar{\lambda}_k$'s, respectively. Consequently, $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a feasible solution of $(\mathrm{RD})_{\mathrm{Q}}$.

Now, by Theorem 4.4.3 (weak duality), for any feasible (y, τ, v, λ) of $(\mathrm{RD})_{\mathrm{Q}}$,

$$\varphi(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i) = \varphi(\bar{x})$$

$$\geqq \varphi(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i) - \sqrt{\epsilon} \|\bar{x} - y\|,$$

which means that $(\bar{x}, \bar{\tau}, \bar{v}, \bar{\lambda})$ is a quasi ϵ-solution of problem $(\mathrm{RD})_{\mathrm{W}}$. $\square$

Conclusions

In this dissertation, we studied some characterizations of an optimal solution and a quasi α-solution for the robust convex minimax optimization problem (RP). By using the obtained results, we then studied optimality conditions and duality theorems both for a weak Pareto solution and a weak quasi ϵ-Pareto solution to the robust multiobjetive optimization problem. Moreover, a robust convex minimax fractional programming problem in the face of data uncertainty is also discussed by using the robust optimization approach (worst-case approach). Optimality conditions and approximate duality theorems for such a robust convex minimax fractional programming problem were explored under the Slater condition.

References

[1] Antczak, T.: Parametric saddle point criteria in semi-infinite minimax fractional programming problems under $(p;r)$-invexity. Numer. Func. Anal. Optim., **36**, 1–28 (2015)

[2] Beck, A., Ben-Tal, A.: Duality in Robust Optimization: Primal Worst Equals Dual Best, Oper. Res. Lett., **37** (1), 1–9 (2009)

[3] Ben-Tal, A., Ghaoui, L. E., Nemirovski, A.: Robust optimzation. Princeton series in applied mathematics. Priceton, NJ: Priceton University Press. (2009)

[4] Ben-Tal, A., Nemirovski, A.: Robust convex optimization. Math. Oper. Res. **23**, 769–805 (1998)

[5] Ben-Tal, A., Nemirovski, A.: Robust solutions to uncertain linear programs. Oper. Res. Lett., **25**, 1–13 (1999)

[6] Ben-Tal, A., Nemirovski, A.: Robust optimization-methodology and applications. Math. Program., Ser B, **92**, 453–480 (2002)

[7] Ben-Tal, A., Nemirovski, A.: A selected topics in robust convex optimization, Math. Program., Ser B, **112**, 125–158 (2008)

[8] Ben-Tal, A., Ghaoui, L.E., Nemirovski, A.: Robust Optimization. Princeton and Oxford: Princeton University Press (2009)

[9] Chuong, T.D.: Optimality and duality for robust multiobjective optimization Problems. Nonlinear Anal., **134**, 127–143 (2016)

[10] Chuong, T.D., Kim, D.S. Nonsmooth semi-infinite multiobjective optimization problems. J. Optim. Theory Appl., 160, 748–762 (2014)

[11] Chuong, T.D., Kim, D.S.: Approximate solutions of multiobjective optimization problems, Positivity **20** (1), 187–207 (2016)

[12] Chuong, T.D., Kim, D.S.: Nondifferentiable minimax programming problems with applications, Ann. Oper. Res., **251**, 73–87 (2017)

[13] Clarke, F.H.: Generelized gradients and applications. Trans. Amer. Math. Soc., **205**, 247–262 (1975)

[14] Clarke, F.H.: A new approach to Lagrange multipliers. Math. Oper. Res., **1** 165–174 (1976)

[15] Clarke, F.H.: Generalized gradients of Lipschitz functions. Adv. in Math., **40** 52–67 (1981)

[16] Clarke, F.H.: Optimization and Nonsmooth Analysis. Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1990)

[17] Danskin, J.M.: The Theory of Max-Min and its Application to Weapons Allocation Problems. Springer-Verlag, New York (1967)

[18] Demyanov,V.F., Molozemov, V. N.: Introduction to Minmax, John Wiley and Sons, New York (1974)

[19] Dutta, J.: Necessary optimality conditions and saddle points for approximate optimization in banach space. Top 13, 127–143 (2005)

[20] Dhara, A., Dutta, J.: Optimality Conditions in Convex Optimization: a Finite-Dimensional View. CRC Press Taylor & Francis Group (2012)

[21] Ekeland, I.: On the variational principle. J. Math. Anal. Appl., 47, 324–353 (1974)

[22] Houda, M.: Comparison of approximations in stochastic and robust optimization programs, In: Hušková, M., Janžura, M. (eds.) Prague Stochastics 2006, pp. 418–425. Prague, Matfyzpress (2006)

[23] Ide, J., Schöbel, A.: Robustness for uncertain multi-objective optimization: a survey and analysis of different concepts. OR Spectrum. **38**(1), 235–271 (2016)

[24] Jeyakumar, V., Li, G.: Characterizing robust set containments and solutions of uncertain linear programs without qualifications. Oper. Res. Lett., **38**, 188–194 (2010)

[25] Jeyakumar V., Li, G.Y.: Strong duality in robust convex programming: complete characterizations, SIAM J. Optim., **20**, 3384-3407 (2010)

[26] Jeyakumar, V., Lee, G.M., Li, G.: Characterizing robust solutions sets convex programs under data uncertainty. J. Optim. Theory Appl., **64**, 407–435 (2015)

[27] Jeyakumar, V., Li, G., Lee, G.M.: Robust duality for generalized convex programming problems under data uncertainty. Nonlinear Anal., **75**, 1362–1373 (2012)

[28] Jiao, L.G., Lee, J.H.: Approximate Optimality and Approximate Duality for Quasi Approximate Solutions in Robust Convex Semidefinite Programs. J. Optim. Theory Appl., **176**, 74–93 (2018)

[29] Kuroiwa, D., Lee, G.M.: On robust multiobjective optimization. Vietnam J. Math., **40**, 305–317 (2012)

[30] Kuroiwa, D., Lee, G.M.: On robust convex multiobjective optimization. J. Nonlinear Convex Anal., **15**, 1125–1136 (2014)

[31] Lai, H.C., Liu, J.C., Tanaka, K.: Necessary and sufficient conditions for minimax fractional programming. J. Math. Anal. Appl., **230** 311–328 (1999)

[32] Lai, H.C., Huang, T.Y.: Optimality conditions for a nondifferentiable minimax programming in complex spaces, Nonlinear Anal., **71**, 1205–1212 (2009).

[33] Lai, H.C., Huang, T.Y.: Nondifferentiable minimax fractional programming in complex spaces with parametric duality, J. Global Optim., **53**, 243–254 (2012).

[34] Lebourg, G.: Generic differentiability of Lipschitzian functions. Trans. Amer. Math. Soc., **256**, 125–144 (1979)

[35] Lee, J.H., Jiao, L.G.: On quasi ϵ-solution for robust convex optimization problems. Optim. Lett., **11**, 1609–1622 (2017)

[36] Lee, G.M., Son, P.T.: On nonsmooth optimality theorems for robust optimization problems. Bull. Korean Math. Soc., **51**, 287–301 (2014)

[37] Li, G.Y., Jeyakumar, V., Lee, G.M.: Robust conjugate duality for convex optimization under uncertainty with application to data classification. Nonlinear Anal., **74**, 2327–2341 (2011).

[38] Liu, J.C.: ϵ-Duality theorem of nondifferentiable nonconvex multiobjective programming. J. Optim. Theory Appl., **69**, 153–167 (1991)

[39] Rockafellar, R.T.: Convex Analysis, Princeton Univ. Press, Princeton, N. J. (1970)

[40] Schmitendorf, W.E.: Necessary conditions and sufficient conditions for static minimax problems. J. Math. Anal. Appl., **57**, 683–693 (1977)

[41] Son, T.Q., Kim, D.S.: ϵ-mixed type duality for nonconvex multiobjective programs with an infinite number of constraints. J. Global Optim., **57**, 447–465 (2013)

[42] Son, T.Q., Kim, D.S., Jiao, L.G.: An Approach to ϵ-Duality Theorems for Nonconvex Semi-Infinite Multiobjective Optimization Problems. Taiwanese J. Math., to appear (2018)

[43] Strodiot, J.J., Nguyen, V.H., Heukemes, N.: ϵ-Optimal solutions in non-differentiable convex programming and some related questions. Math. Program., **25**, 307–328 (1983)

[44] Suzuki, S., Kuroiwa, D., Lee, G.M.: Surrogate duality for robust optimization. European J. Oper. Res., **231**, 257–262 (2013)

[45] Wiecek, M.M., Dranichak, G.M.: Robust Multiobjective Optimization for Decision Making Under Uncertainty and Conflict. In: Gupta, A., Capponi, A. (eds.) TutORials in Operations Research, Optimization Challenges in Complex, Networked, and Risky Systems, pp. 84–114. INFORMS (2016)

[46] Wolfe, P.: A duality theorem for nonlinear programming, Quart. Appl. Math., **19**, 239–244 (1961)

[47] Yang, X.M., Hou, S.H.: On minimax fractional optimality conditions and duality with generalized convexity. J. Glob. Optim., **31** 235–252 (2005)