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Hesitant Probabilistic Fuzzy Einstein Aggregation Operators

by

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Graduate School of Education

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Hesitant Probabilistic Fuzzy Einstein Aggregation Operators
(Hesitant 확률 퍼지 Einstein 집성연산자)

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by

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요　　약

본 논문은 hesitant 확률 퍼지 Einstein 집성연산자를 연구한 것으로 주요내용은 아래와 같다.
첫째, Einstein 합, Einstein 곱, Einstein 스칼라 곱셈과 같은 hesitant 확률 퍼지 Einstein 원에 대한 Einstein연산이 정의하고 그들의 특성을 논의 한다. 둘째, Hesitant 확률 퍼지 Einstein 가중 산술 연산자와 hesitant 확률 퍼지 Einstein 가중 기하 연산자 등을 포함한 몇 가지의 hesitant 확률 퍼지 Einstein 집성연산자를 제안하고, 이 연산자들의 바람직한 특성과 특별한 경우를 조사한다. 특히 기존의 일부 hesitant 확률 퍼지 집성연산자와 hesitant 확률 퍼지 Einstein 집성연산자가 제안된 연산자의 특별 예인 것을 보인다.

1 Introduction

Decision making problems generally consist of finding the most desirable alternative(s) from a given alternative set. Because of increasing of vagueness and complexity of socio-economic environment, it is difficult to acquire accurate and sufficient data in practical decision making. So it is necessary to deal with uncertainty data in real decision making process, and thus, several different methodologies and theories have been proposed, among which the fuzzy set theory[1] is outstanding and has been widely used in many fields in real life.[2–5] Since then many extensions of fuzzy set (FS) such as intuitionistic fuzzy set (IFS),[6] interval-valued fuzzy set (IVFS),[7] interval-valued intuitionistic fuzzy set (IVIFS),[8] hesitant fuzzy set (HFS),[9,10] dual hesitant fuzzy set (DFS),[11] and generalized hesitant fuzzy set (GHFS)[12] allowed people to deal with uncertainty and information in much broader perspective. In particular, as a new development of FS, the concept of HFS has been receiving increased attention and has recently become a popular research topic.[9,10,13–16]

HFS is a significant extension of the FS that models the uncertainty roused by hesitancy, which is a common phenomenon in decision making. Several possible values can be used to indicate the membership degree or an evaluation value under hesitant fuzzy environment. Thus, it is suitable and convenient for describing the hesitancy experienced by the decision makers during the decision making process. The original definition of the HFS was provided by Torra.[9] Xia and Xu[13] defined the hesitant fuzzy element (HFE), which is a set of values in the unit interval $[0, 1]$, and proposed and investigated the score function and comparison law of the HFEs as the basis for its calculation and application. Many authors[17–23] developed the aggregation operators and used to fuse the hesitant fuzzy information. However, we find that the occurring probabilities of the possible values in the HFE are equal, which is obviously impractical, and thus HFE is unsuitable for the hesitant judgments and evaluations of the decision makers in real decision making process under the hesitant fuzzy environment. To overcome this drawback, Xu and Zhou[24] proposed the hesitant probabilistic fuzzy set (HPFS) and hesitant probabilistic fuzzy element (HPFE), which are developed by introducing probabilities into the HFS and the HFE, respectively. For example, a decision maker provides a HFE $(0.3, 0.4, 0.5)$ to evaluate the "comport" of a house because he/she is hesitant to do this evaluation. However, he/she thinks 0.4 is more suitable than other values in this HFE, and 0.3 is less of possibility than others. Then the HFE $(0.3, 0.4, 0.5)$ cannot fully represent his/her evaluation, but the HPFE $(0.3|0.2, 0.4|0.5, 0.5|0.3)$ can describe this dilemma vividly and it is more convenient and reasonable than the HFE. Thus, the HPFE is more generalized HFE and can be used to depict hesitant fuzzy information with probabilities. They[24] also applied the HPFS and HPFE to group decision making and built the consensus of the decision makers based on the perspective of the aggregation operator. By combining the HPFE and the weighted operator, they developed basic weighted operators including the hesitant probabilistic fuzzy weighted averaging/geometric (HPFWA or HPFWG) operators and the hesitant probabilistic fuzzy ordered weighted averaging/geometric (HPFOWA or HPFOWG) operators.

On the other hand, the all aggregation operators introduced previously are based on the algebraic product and algebraic sum of HPFEs. In fact, Einstein operations including the Einstein product and Einstein sum are also good alternatives for structuring aggregation operators, and they have been used to aggregate the intuitionistic fuzzy values or the HFEs by many authors.[21−23,25−27] Thus, it is meaningful to use Einstein operations to aggregate hesitant probabilistic fuzzy information. However, it seems that in the literature there is little investigation on aggregation techniques using the Einstein operations to aggregate hesitant probabilistic fuzzy information. In this thesis, motivated by Xu and Zhou[24] and Yu,[21] we propose the hesitant probabilistic fuzzy Einstein weighted aggregation operators with the help of Einstein operations, and apply them to multiple attribute group decision making (MAGDM) under hesitant probabilistic fuzzy environment.

The remainder of this thesis is organized as follows. The following chapter recalls briefly some basic concepts and notions related to the HPFSs and HPFEs. In Chapter 3, based on the hesitant probabilistic fuzzy weighted aggregation operator and the Einstein operations, we propose the hesitant probabilistic fuzzy Einstein weighted aggregation operators including the hesitant probabilistic fuzzy Einstein weighted averaging/geometric (HPFEWA or HPFEWG) operators and the hesitant probabilistic fuzzy Einstein ordered weighted averaging/geometric (HPFEOWA or HPFEOWG) operators. Chapter 4 gives some concluding remarks.

2 Preliminaries

2.1 HPFS and HPFE

The HPFS and HPFE are defined to represent hesitant fuzzy information with probabilities as follows:

Definition 1.[24] Let R be a fixed set, then a HPFS on R is expressed by a mathematical symbol:

$$H_P = \left\{ \bar{h}(\gamma_i|p_i)|\gamma_i, p_i \right\}, \tag{1}$$

where $\bar{h}(\gamma_i|p_i)$ is a set of some elements $\gamma_i|p_i$ denoting the hesitant fuzzy information with probabilities to the set H_P, $\gamma_i \in R$, $0 \leq \gamma_i \leq 1$, $i = 1, 2, \ldots, \#\bar{h}$, where $\#\bar{h}$ is the number of possible elements in $\bar{h}(\gamma_i|p_i)$, $p_i \in [0, 1]$ is the hesitant probability of γ_i, and $\sum_{i=1}^{\#\bar{h}} p_i = 1$.

For convenience, Xu and Zhou[24] called $\bar{h}(\gamma_i|p_i)$ a HPFE, and H_P the set of HPFSs. In addition, they gave the following score function, deviation function and comparison law to compare different HPFEs.

Definition 2.[24] Let $\bar{h}(\gamma_i|p_i)$ $(i = 1, 2, , \ldots, \#\bar{h})$ be a HPFE, then

(1) $s(\bar{h}) = \sum_{i=1}^{\#\bar{h}} \gamma_i p_i$ is called the score function of $\bar{h}(\gamma_i|p_i)$, where $\#\bar{h}$ is the number of possible elements in $\bar{h}(\gamma_i|p_i)$;

(2) $d(\bar{h}) = \sum_{i=1}^{\#\bar{h}} (\gamma_i - s(\bar{h}))^2 p_i$ is called the deviation function of $\bar{h}(\gamma_i|p_i)$, where $s(\bar{h}) = \sum_{i=1}^{\#\bar{h}} \gamma_i p_i$ is the score function of $\bar{h}(\gamma_i|p_i)$ and $\#\bar{h}$ is the number of possible elements in $\bar{h}(\gamma_i|p_i)$.

If all probabilities are equal, i.e., $p_1 = p_2 = \cdots = p_{\#\bar{h}}$, then the HPFE is reduced to the HFE. So, in this case, the score function of the HPFE is consistent with that of the HFE.

Definition 3.[24] Let $\bar{h}_1(\gamma_i|p_i)$ and $\bar{h}_2(\gamma_j|p_j)$ be two HPFEs, $i = 1, 2, , \ldots, \#\bar{h}_1$, $j = 1, 2, , \ldots, \#\bar{h}_2$, $s(\bar{h}_1)$ and $s(\bar{h}_2)$ are the score functions of $\bar{h}_1$ and $\bar{h}_2$, respectively, and $d(\bar{h}_1)$ and $d(\bar{h}_2)$ are the deviation functions of $\bar{h}_1$ and $\bar{h}_2$, respectively, then

(1) If $s(\bar{h}_1) < s(\bar{h}_2)$, then $\bar{h}_1$ is smaller than $\bar{h}_2$, denoted by $\bar{h}_1 < \bar{h}_2$;

(2) If $s(\bar{h}_1) = s(\bar{h}_2)$, then

(a) If $d(\bar{h}_1) > d(\bar{h}_2)$, then $\bar{h}_1$ is smaller than $\bar{h}_2$, denoted by $\bar{h}_1 < \bar{h}_2$;

(b) If $d(\bar{h}_1) = d(\bar{h}_2)$, then $\bar{h}_1$ and $\bar{h}_2$ represent the same information, denoted by $\bar{h}_1 = \bar{h}_2$.

Some operations to aggregate HPFEs based on the operations of HFEs[9,13] are defined as follows:

Definition 4.[24] Let $\bar{h}(\gamma_i|p_i)$, $\bar{h}_1(\dot{\gamma}_j|\dot{p}_j)$ and $\bar{h}_2(\ddot{\gamma}_k|\ddot{p}_k)$ be three HPFEs, $i = 1, 2, \ldots, \#\bar{h}$, $j = 1, 2, \ldots, \#\bar{h}_1$, $k = 1, 2, \ldots, \#\bar{h}_2$, and $\lambda > 0$, then

(1) $(\bar{h})^c = \cup_{i=1,2,\ldots,\#\bar{h}} \{(1-\gamma_i)|p_i\}$;

(2) $\lambda\bar{h} = \cup_{i=1,2,\ldots,\#\bar{h}} \left\{1-(1-\gamma_i)^\lambda|p_i\right\}$;

(3) $\bar{h}^\lambda = \cup_{i=1,2,\ldots,\#\bar{h}} \left\{(\gamma_i)^\lambda|p_i\right\}$;

(4) $\bar{h}_1 \oplus \bar{h}_2 = \cup_{j=1,2,\ldots,\#\bar{h}_1, k=1,2,\ldots,\#\bar{h}_2} \{(\dot{\gamma}_j + \ddot{\gamma}_k - \dot{\gamma}_j\ddot{\gamma}_k)|\dot{p}_j\ddot{p}_k\}$;

(5) $\bar{h}_1 \otimes \bar{h}_2 = \cup_{j=1,2,\ldots,\#\bar{h}_1, k=1,2,\ldots,\#\bar{h}_2} \{\dot{\gamma}_j\ddot{\gamma}_k|\dot{p}_j\ddot{p}_k\}$.

Theorem 1. Let $\bar{h}(\gamma_i|p_i)$, $\bar{h}_1(\dot{\gamma}_j|\dot{p}_j)$ and $\bar{h}_2(\ddot{\gamma}_k|\ddot{p}_k)$ be three HPFEs, $i = 1, 2, \ldots, \#\bar{h}$, $j = 1, 2, \ldots, \#\bar{h}_1$, $k = 1, 2, \ldots, \#\bar{h}_2$, $\lambda > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$, then

(1) $\bar{h}_1 \oplus \bar{h}_2 = \bar{h}_2 \oplus \bar{h}_1$;

(2) $\bar{h} \oplus (\bar{h}_1 \oplus \bar{h}_2) = (\bar{h} \oplus \bar{h}_1) \oplus \bar{h}_2$;

(3) $\lambda(\bar{h}_1 \oplus \bar{h}_2) = (\lambda\bar{h}_1) \oplus (\lambda\bar{h}_2)$;

(4) $\lambda_1(\lambda_2\bar{h}) = (\lambda_1\lambda_2)\bar{h}$;

(5) $\bar{h}_1 \otimes \bar{h}_2 = \bar{h}_2 \otimes \bar{h}_1$;

(6) $\bar{h} \otimes (\bar{h}_1 \otimes \bar{h}_2) = (\bar{h} \otimes \bar{h}_1) \otimes \bar{h}_2$;

(7) $(\bar{h}_1 \otimes \bar{h}_2)^\lambda = \bar{h}_1^\lambda \otimes \bar{h}_2^\lambda$;

(8) $(\bar{h}^{\lambda_1})^{\lambda_2} = \bar{h}^{(\lambda_1\lambda_2)}$.

Proof. We only prove (3) and the other are trivial or similar to (3).

(3) Since $\bar{h}_1 \oplus \bar{h}_2 = \cup_{j=1,2,\ldots,\#\bar{h}_1, k=1,2,\ldots,\#\bar{h}_2} \{\dot{\gamma}_j + \ddot{\gamma}_k - \dot{\gamma}_j\ddot{\gamma}_k|\dot{p}_j\ddot{p}_k\}$, by the operational law (2) in Definition 4, we have

$$\lambda(\bar{h}_1 \oplus \bar{h}_2) = \cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{1-(1-(\dot{\gamma}_j + \ddot{\gamma}_k - \dot{\gamma}_j\ddot{\gamma}_k))^\lambda|\dot{p}_j\ddot{p}_k\right\}$$

$$= \cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{1-((1-\dot{\gamma}_j)(1-\ddot{\gamma}_k))^{\lambda}\,|\dot{p}_j\ddot{p}_k\right\}.$$

Since $\lambda\bar{h}_1 = \cup_{j=1,2,\ldots,\#\bar{h}_1}\left\{1-(1-\dot{\gamma}_j)^{\lambda}|\dot{p}_j\right\}$ and $\lambda\bar{h}_2 = \cup_{k=1,2,\ldots,\#\bar{h}_2}\left\{1-(1-\dot{\gamma}_k)^{\lambda}|\ddot{p}_k\right\}$, we have

$$\begin{aligned}
&(\lambda\bar{h}_1)\oplus(\lambda\bar{h}_2)\\
&\quad= \cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{1-(1-\dot{\gamma}_j)^{\lambda}+1-(1-\dot{\gamma}_k)^{\lambda}-(1-(1-\dot{\gamma}_j)^{\lambda})(1-(1-\dot{\gamma}_k)^{\lambda})|\dot{p}_j\ddot{p}_k\right\}\\
&\quad= \cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{1-(1-\dot{\gamma}_j)^{\lambda}(1-\ddot{\gamma}_k)^{\lambda}|\dot{p}_j\ddot{p}_k\right\}.
\end{aligned}$$

Hence $\lambda(\bar{h}_1\oplus\bar{h}_2) = (\lambda\bar{h}_1)\oplus(\lambda\bar{h}_2)$. □

However, for a HPFE $\bar{h}(\gamma_i|p_i)$, $i = 1,2,\ldots,\#\bar{h}$, $\lambda_1 > 0$ and $\lambda_2 > 0$, the operational laws $(\lambda_1\bar{h})\oplus(\lambda_2\bar{h}) = (\lambda_1+\lambda_2)\bar{h}$ and $\bar{h}^{\lambda_1}\otimes\bar{h}^{\lambda_2} = \bar{h}^{(\lambda_1+\lambda_2)}$ do not hold in general. To illustrate this case, we give an example as follows:

Example 1. Let $\bar{h}(\gamma_i|p_i) = (0.7|0.5, 0.2|0.5)$ and $\lambda_1 = \lambda_2 = 1$, then

$$\begin{aligned}
(\lambda_1\bar{h})\oplus(\lambda_2\bar{h}) &= \bar{h}\oplus\bar{h} = \cup_{i,j=1,2}\left\{\gamma_i+\gamma_j-\gamma_i\gamma_j|0.25\right\}\\
&= (0.91|0.25, 0.76|0.25, 0.76|0.25, 0.36|0.25),\\
(\lambda_1+\lambda_2)\bar{h} &= 2\bar{h} = \cup_{i=1,2}\left\{1-(1-\gamma_i)^2|0.5\right\} = (0.91|0.5, 0.36|0.5)
\end{aligned}$$

and $s((\lambda_1\bar{h})\oplus(\lambda_2\bar{h})) = 0.6975 > 0.635 = s((\lambda_1+\lambda_2)\bar{h})$ and hence $(\lambda_1\bar{h})\oplus(\lambda_2\bar{h}) > (\lambda_1+\lambda_2)\bar{h}$. Similarly, we have $s(\bar{h}^{\lambda_1}\otimes\bar{h}^{\lambda_2}) = 0.2025 < 0.265 = s(\bar{h}^{(\lambda_1+\lambda_2)})$ and thus $\bar{h}^{\lambda_1}\otimes\bar{h}^{\lambda_2} < \bar{h}^{(\lambda_1+\lambda_2)}$.

Based on Definition 4, in order to aggregate the HPFEs, Xu and Zhou[24] developed some hesitant probabilistic fuzzy aggregation operators as follows:

Definition 5.[24] Let $\bar{h}_t$ $(t = 1,2,\ldots,T)$ be a collection of HPFEs, $w = (w_1, w_2, \ldots, w_T)^T$ be the weight vector of $\bar{h}_t$ with $w_t \in [0,1]$ and $\sum_{t=1}^{T} w_t = 1$, and p_t be the probability of γ_t in the HPFE $\bar{h}_t$, then

(1) the hesitant probabilistic fuzzy weighted averaging (HPFWA) operator:

$$\begin{aligned}
\text{HPFWA}(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T) &= (w_1\bar{h}_1)\oplus(w_2\bar{h}_2)\oplus\cdots\oplus(w_T\bar{h}_T)\\
&= \cup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2,\ldots,\gamma_T\in\bar{h}_T}\left\{1-\prod_{t=1}^{T}(1-\gamma_t)^{w_t}|p_1p_2\cdots p_T\right\}. \qquad (2)
\end{aligned}$$

(2) the hesitant probabilistic fuzzy weighted geometric (HPFWG) operator:

$$\begin{aligned}\mathrm{HPFWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\bar{h}_1)^{w_1} \otimes (\bar{h}_2)^{w_2} \otimes \cdots \otimes (\bar{h}_T)^{w_T} \\ &= \cup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \prod_{t=1}^{T} (\gamma_t)^{w_t} | p_1 p_2 \cdots p_T \right\}. \end{aligned} \quad (3)$$

Definition 6.[24] Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, $\bar{h}_{\sigma(t)}$ be the tth largest of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$, and $p_{\sigma(t)}$ be the probability of $\gamma_{\sigma(t)}$ in the HPFE $\bar{h}_{\sigma(t)}$, then the following two aggregation operators, which are based on the mapping $H_P^T \to H_P$ with an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_T)^T$ such that $\omega_t \in [0, 1]$ and $\sum_{t=1}^{T} \omega_t = 1$, are given by:

(1) the hesitant probabilistic fuzzy ordered weighted averaging (HPFOWA) operator:

$$\begin{aligned}\mathrm{HPFOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\omega_1 \bar{h}_{\sigma(1)}) \oplus (\omega_2 \bar{h}_{\sigma(2)}) \oplus \cdots \oplus (\omega_T \bar{h}_{\sigma(T)}) \\ &= \cup_{\gamma_{\sigma(1)} \in \bar{h}_{\sigma(1)}, \gamma_{\sigma(2)} \in \bar{h}_{\sigma(2)}, \ldots, \gamma_{\sigma(T)} \in \bar{h}_{\sigma(T)}} \left\{ 1 - \prod_{t=1}^{T} (1 - \gamma_{\sigma(t)})^{w_t} \middle| p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(T)} \right\}. \end{aligned} \quad (4)$$

(2) the hesitant probabilistic fuzzy ordered weighted geometric (HPFOWG) operator:

$$\begin{aligned}\mathrm{HPFOWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\bar{h}_{\sigma(1)})^{\omega_1} \otimes (\bar{h}_{\sigma(2)})^{\omega_2} \otimes \cdots \otimes (\bar{h}_{\sigma(T)})^{\omega_T} \\ &= \cup_{\gamma_{\sigma(1)} \in \bar{h}_{\sigma(1)}, \gamma_{\sigma(2)} \in \bar{h}_{\sigma(2)}, \ldots, \gamma_{\sigma(T)} \in \bar{h}_{\sigma(T)}} \left\{ \prod_{t=1}^{T} (\gamma_{\sigma(t)})^{w_t} \middle| p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(T)} \right\}. \end{aligned} \quad (5)$$

2.2 Einstein operations on HPFEs

It is well known that the t-norms and t-conorms are general concepts satisfying the requirements of the conjunction and disjunction operators. Einstein operations includes the Einstein sum $\oplus_\varepsilon$ and Einstein product $\otimes_\varepsilon$, which are examples of

t-conorms and t-norms, respectively. They are defined by Klement et al.[28] as follows:

$$x \otimes_\varepsilon y = \frac{xy}{1+(1-x)(1-y)},\ x \oplus_\varepsilon y = \frac{x+y}{1+xy},\quad x, y \in [0,1].$$

Based on the above Einstein operations, we give some new operations on HPFEs as follows:

Definition 7. Let $\bar{h}(\gamma_i|p_i)$, $\bar{h}_1(\dot{\gamma}_j|\dot{p}_j)$ and $\bar{h}_2(\ddot{\gamma}_k|\ddot{p}_k)$ be three HPFEs, $i = 1, 2, \ldots, \#\bar{h}$, $j = 1, 2, \ldots, \#\bar{h}_1$, $k = 1, 2, \ldots, \#\bar{h}_2$, and $\lambda > 0$, then

(1) $\bar{h}_1 \oplus_\varepsilon \bar{h}_2 = \cup_{j=1,2,\ldots,\#\bar{h}_1, k=1,2,\ldots,\#\bar{h}_2} \left\{ \frac{\dot{\gamma}_j + \ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k} | \dot{p}_j\ddot{p}_k \right\}$;

(2) $\bar{h}_1 \otimes_\varepsilon \bar{h}_2 = \cup_{j=1,2,\ldots,\#\bar{h}_1, k=1,2,\ldots,\#\bar{h}_2} \left\{ \frac{\dot{\gamma}_j\ddot{\gamma}_k}{1+(1-\dot{\gamma}_j)(1-\ddot{\gamma}_k)} | \dot{p}_j\ddot{p}_k \right\}$;

(3) $\lambda \cdot_\varepsilon \bar{h} = \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{(1+\gamma_i)^\lambda - (1-\gamma_i)^\lambda}{(1+\gamma_i)^\lambda + (1-\gamma_i)^\lambda} | p_i \right\}$;

(4) $\bar{h}^{\wedge_\varepsilon \lambda} = \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{2\gamma_i^\lambda}{(2-\gamma_i)^\lambda + \gamma_i^\lambda} | p_i \right\}$.

Thus, the above four operations on the HPFEs can be suitable for the HPFSs. Moreover, some relationships can be discussed for operations on HPFEs given in Definitions 4 and 7 as follows:

Theorem 2. Let $\bar{h}(\gamma_i|p_i)$, $\bar{h}_1(\dot{\gamma}_j|\dot{p}_j)$ and $\bar{h}_2(\ddot{\gamma}_k|\ddot{p}_k)$ be three HPFEs, $i = 1, 2, \ldots, \#\bar{h}$, $j = 1, 2, \ldots, \#\bar{h}_1$, $k = 1, 2, \ldots, \#\bar{h}_2$, and $\lambda > 0$, then

(1) $((\bar{h})^c)^{\wedge_\varepsilon \lambda} = (\lambda \cdot_\varepsilon \bar{h})^c$;

(2) $\lambda \cdot_\varepsilon (\bar{h})^c = (\bar{h}^{\wedge_\varepsilon \lambda})^c$;

(3) $(\bar{h}_1)^c \oplus_\varepsilon (\bar{h}_2)^c = (\bar{h}_1 \otimes_\varepsilon \bar{h}_2)^c$;

(4) $(\bar{h}_1)^c \otimes_\varepsilon (\bar{h}_2)^c = (\bar{h}_1 \oplus_\varepsilon \bar{h}_2)^c$.

Proof. (1)

$$\begin{aligned}
((\bar{h})^c)^{\wedge_\varepsilon \lambda} &= \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{2(1-\gamma_i)^\lambda}{(2-(1-\gamma_i))^\lambda + (1-\gamma_i)^\lambda} \Big| p_i \right\} \\
&= \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{2(1-\gamma_i)^\lambda}{(1+\gamma_i))^\lambda + (1-\gamma_i)^\lambda} \Big| p_i \right\} \\
&= \left(\cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{(1+\gamma_i)^\lambda - (1-\gamma_i)^\lambda}{(1+\gamma_i))^\lambda + (1-\gamma_i)^\lambda} \Big| p_i \right\} \right)^c \\
&= (\lambda \cdot_\varepsilon \bar{h})^c.
\end{aligned}$$

(2)

$$\begin{aligned}\lambda \cdot_\varepsilon (\bar{h})^c &= \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{(1+(1-\gamma_i))^\lambda - (1-(1-\gamma_i))^\lambda}{(1+(1-\gamma_i))^\lambda + (1-(1-\gamma_i))^\lambda} \Big| p_i \right\} \\ &= \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ 1 - \frac{2\gamma_i^\lambda}{(2-\gamma_i)^\lambda + \gamma_i^\lambda} \Big| p_i \right\} \\ &= \left(\cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{2\gamma_i^\lambda}{(2-\gamma_i)^\lambda + \gamma_i^\lambda} \Big| p_i \right\} \right)^c \\ &= (\bar{h}^{\wedge_\varepsilon \lambda})^c.\end{aligned}$$

(3)

$$\begin{aligned}(\bar{h}_1)^c \oplus_\varepsilon (\bar{h}_2)^c &= \cup_{i=1,2,\ldots,\#\bar{h}_1} \{(1-\dot{\gamma}_j)|\dot{p}_j\} \oplus_\varepsilon \cup_{i=1,2,\ldots,\#\bar{h}_2} \{(1-\ddot{\gamma}_k)|\ddot{p}_k\} \\ &= \cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{(1-\dot{\gamma}_j)+(1-\ddot{\gamma}_k)}{1+(1-\dot{\gamma}_j)(1-\ddot{\gamma}_k)} \Big| \dot{p}_j\ddot{p}_k \right\} \\ &= \left(\cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{\dot{\gamma}_j\ddot{\gamma}_k}{1+(1-\dot{\gamma}_j)(1-\ddot{\gamma}_k)} \Big| \dot{p}_j\ddot{p}_k \right\} \right)^c \\ &= (\bar{h}_1 \otimes_\varepsilon \bar{h}_2)^c.\end{aligned}$$

(4)

$$\begin{aligned}(\bar{h}_1)^c \otimes_\varepsilon (\bar{h}_2)^c &= \cup_{i=1,2,\ldots,\#\bar{h}_1} \{(1-\dot{\gamma}_j)|\dot{p}_j\} \otimes_\varepsilon \cup_{i=1,2,\ldots,\#\bar{h}_2} \{(1-\ddot{\gamma}_k)|\ddot{p}_k\} \\ &= \cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{(1-\dot{\gamma}_j)(1-\ddot{\gamma}_k)}{1-\dot{\gamma}_j\ddot{\gamma}_k} \Big| \dot{p}_j\ddot{p}_k \right\} \\ &= \left(\cup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{\dot{\gamma}_j+\ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k} \Big| \dot{p}_j\ddot{p}_k \right\} \right)^c \\ &= (\bar{h}_1 \oplus_\varepsilon \bar{h}_2)^c.\end{aligned}$$

□

Theorem 3. Let $\bar{h}(\gamma_i|p_i)$, $\bar{h}_1(\dot{\gamma}_j|\dot{p}_j)$ and $\bar{h}_2(\ddot{\gamma}_k|\ddot{p}_k)$ be three HPFEs, $i = 1,2,\ldots,\#\bar{h}$, $j = 1,2,\ldots,\#\bar{h}_1$, $k = 1,2,\ldots,\#\bar{h}_2$, $\lambda > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$, then

(1) $\bar{h}_1 \oplus_\varepsilon \bar{h}_2 = \bar{h}_2 \oplus_\varepsilon \bar{h}_1$;

(2) $\bar{h} \oplus_\varepsilon (\bar{h}_1 \oplus_\varepsilon \bar{h}_2) = (\bar{h} \oplus_\varepsilon \bar{h}_1) \oplus_\varepsilon \bar{h}_2$;
(3) $\lambda \cdot_\varepsilon (\bar{h}_1 \oplus_\varepsilon \bar{h}_2) = (\lambda \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (\lambda \cdot_\varepsilon \bar{h}_2)$;
(4) $\lambda_1 \cdot_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h}) = (\lambda_1\lambda_2) \cdot_\varepsilon \bar{h}$;
(5) $\bar{h}_1 \otimes_\varepsilon \bar{h}_2 = \bar{h}_2 \otimes_\varepsilon \bar{h}_1$;
(6) $\bar{h} \otimes_\varepsilon (\bar{h}_1 \otimes_\varepsilon \bar{h}_2) = (\bar{h} \otimes_\varepsilon \bar{h}_1) \otimes_\varepsilon \bar{h}_2$;
(7) $(\bar{h}_1 \otimes_\varepsilon \bar{h}_2)^{\wedge_\varepsilon \lambda} = \bar{h}_1^{\wedge_\varepsilon \lambda} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon \lambda}$;
(8) $(\bar{h}^{\wedge_\varepsilon \lambda_1})^{\wedge_\varepsilon \lambda_2} = \bar{h}^{\wedge_\varepsilon (\lambda_1\lambda_2)}$.

Proof. Since (1), (2), (5) and (6) are trivial, and (7) and (8) are similar to (3) and (4), respectively, we only prove (3) and (4).

(3) Since $\bar{h}_1 \oplus_\varepsilon \bar{h}_2 = \cup_{j=1,2,\ldots,\#\bar{h}_1, k=1,2,\ldots,\#\bar{h}_2} \left\{ \frac{\dot{\gamma}_j+\ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k} | \dot{p}_j\ddot{p}_k \right\}$, by the operational law (3) in Definition 7, we have

$$\lambda \cdot_\varepsilon (\bar{h}_1 \oplus_\varepsilon \bar{h}_2) = \bigcup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{\left(1+\frac{\dot{\gamma}_j+\ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k}\right)^\lambda - \left(1-\frac{\dot{\gamma}_j+\ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k}\right)^\lambda}{\left(1+\frac{\dot{\gamma}_j+\ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k}\right)^\lambda + \left(1-\frac{\dot{\gamma}_j+\ddot{\gamma}_k}{1+\dot{\gamma}_j\ddot{\gamma}_k}\right)^\lambda} \middle| \dot{p}_j\ddot{p}_k \right\}$$
$$= \bigcup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{(1+\dot{\gamma}_j)^\lambda(1+\ddot{\gamma}_k)^\lambda - (1-\dot{\gamma}_j)^\lambda(1-\ddot{\gamma}_k)^\lambda}{(1+\dot{\gamma}_j)^\lambda(1+\ddot{\gamma}_k)^\lambda + (1-\dot{\gamma}_j)^\lambda(1-\ddot{\gamma}_k)^\lambda} \middle| \dot{p}_j\ddot{p}_k \right\}.$$

Since $\lambda \cdot_\varepsilon \bar{h}_1 = \cup_{j=1,2,\ldots,\#\bar{h}_1} \left\{ \frac{(1+\dot{\gamma}_j)^\lambda-(1-\dot{\gamma}_j)^\lambda}{(1+\dot{\gamma}_j)^\lambda+(1-\dot{\gamma}_j)^\lambda} | \dot{p}_j \right\}$ and $\lambda \cdot_\varepsilon \bar{h}_2 = \cup_{k=1,2,\ldots,\#\bar{h}_2} \left\{ \frac{(1+\ddot{\gamma}_k)^\lambda-(1-\ddot{\gamma}_k)^\lambda}{(1+\ddot{\gamma}_k)^\lambda+(1-\ddot{\gamma}_k)^\lambda} | \ddot{p}_k \right\}$, we have

$$(\lambda \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (\lambda \cdot_\varepsilon \bar{h}_2) = \bigcup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{\frac{(1+\dot{\gamma}_j)^\lambda-(1-\dot{\gamma}_j)^\lambda}{(1+\dot{\gamma}_j)^\lambda+(1-\dot{\gamma}_j)^\lambda} + \frac{(1+\ddot{\gamma}_k)^\lambda-(1-\ddot{\gamma}_k)^\lambda}{(1+\ddot{\gamma}_k)^\lambda+(1-\ddot{\gamma}_k)^\lambda}}{1+\frac{(1+\dot{\gamma}_j)^\lambda-(1-\dot{\gamma}_j)^\lambda}{(1+\dot{\gamma}_j)^\lambda+(1-\dot{\gamma}_j)^\lambda} \cdot \frac{(1+\ddot{\gamma}_k)^\lambda-(1-\ddot{\gamma}_k)^\lambda}{(1+\ddot{\gamma}_k)^\lambda+(1-\ddot{\gamma}_k)^\lambda}} \middle| \dot{p}_j\ddot{p}_k \right\}$$
$$= \bigcup_{\substack{j=1,2,\ldots,\#\bar{h}_1,\\ k=1,2,\ldots,\#\bar{h}_2}} \left\{ \frac{(1+\dot{\gamma}_j)^\lambda(1+\ddot{\gamma}_k)^\lambda - (1-\dot{\gamma}_j)^\lambda(1-\ddot{\gamma}_k)^\lambda}{(1+\dot{\gamma}_j)^\lambda(1+\ddot{\gamma}_k)^\lambda + (1-\dot{\gamma}_j)^\lambda(1-\ddot{\gamma}_k)^\lambda} \middle| \dot{p}_j\ddot{p}_k \right\}.$$

Hence $\lambda \cdot_\varepsilon (\bar{h}_1 \oplus_\varepsilon \bar{h}_2) = (\lambda \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (\lambda \cdot_\varepsilon \bar{h}_2)$.

(4) Since $\lambda_2 \cdot_\varepsilon \bar{h} = \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{(1+\gamma_i)^{\lambda_2}-(1-\gamma_i)^{\lambda_2}}{(1+\gamma_i)^{\lambda_2}+(1-\gamma_i)^{\lambda_2}} | p_i \right\}$, then we have

$$\lambda_1 \cdot_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h})$$
$$= \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{\left(1+\frac{(1+\gamma_i)^{\lambda_2}-(1-\gamma_i)^{\lambda_2}}{(1+\gamma_i)^{\lambda_2}+(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1} - \left(1-\frac{(1+\gamma_i)^{\lambda_2}-(1-\gamma_i)^{\lambda_2}}{(1+\gamma_i)^{\lambda_2}+(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1}}{\left(1+\frac{(1+\gamma_i)^{\lambda_2}-(1-\gamma_i)^{\lambda_2}}{(1+\gamma_i)^{\lambda_2}+(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1} + \left(1-\frac{(1+\gamma_i)^{\lambda_2}-(1-\gamma_i)^{\lambda_2}}{(1+\gamma_i)^{\lambda_2}+(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1}} \middle| p_i \right\}$$

$$= \cup_{i=1,2,\ldots,\#\bar{h}} \left\{ \frac{(1+\gamma_i)^{(\lambda_1\lambda_2)} - (1-\gamma_i)^{(\lambda_1\lambda_2)}}{(1+\gamma_i)^{(\lambda_1\lambda_2)} + (1-\gamma_i)^{(\lambda_1\lambda_2)}} \middle| p_i \right\}$$
$$= (\lambda_1\lambda_2) \cdot_\varepsilon \bar{h}.$$

□

For a HPFE $\bar{h}(\gamma_i|p_i)$, $i = 1, 2, \ldots, \#\bar{h}$, $\lambda_1 > 0$ and $\lambda_2 > 0$, the operational laws $(\lambda_1 \cdot_\varepsilon \bar{h}) \oplus_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h}) = (\lambda_1 + \lambda_2) \cdot_\varepsilon \bar{h}$ and $\bar{h}^{\wedge_\varepsilon \lambda_1} \otimes_\varepsilon \bar{h}^{\wedge_\varepsilon \lambda_2} = \bar{h}^{\wedge_\varepsilon (\lambda_1+\lambda_2)}$ do not hold in general. To illustrate this case, we give an example as follows:

Example 2. Let $\bar{h}(\gamma_i|p_i) = (0.3|0.5, 0.5|0.5)$ and $\lambda_1 = \lambda_2 = 1$, then

$$(\lambda_1 \cdot_\varepsilon \bar{h}) \oplus_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h}) = \bar{h} \oplus_\varepsilon \bar{h} = \cup_{i,j=1,2} \left\{ \frac{\gamma_i + \gamma_j}{1 + \gamma_i\gamma_j} \middle| 0.25 \right\}$$
$$= (0.5505|0.25, 0.6957|0.25, 0.6957|0.25, 0.8|0.25),$$
$$(\lambda_1 + \lambda_2) \cdot_\varepsilon \bar{h} = 2 \cdot_\varepsilon \bar{h} = \cup_{i=1,2} \left\{ \frac{(1+\gamma_i)^2 - (1-\gamma_i)^2}{(1+\gamma_i)^2 + (1-\gamma_i)^2} \middle| 0.5 \right\}$$
$$= (0.5505|0.5, 0.8|0.5).$$

Clearly, $s((\lambda_1 \cdot_\varepsilon \bar{h}) \oplus_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h})) = 0.6856 > 0.6752 = s((\lambda_1 + \lambda_2) \cdot_\varepsilon \bar{h})$. Hence $(\lambda_1 \cdot_\varepsilon \bar{h}) \oplus_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h}) < (\lambda_1 + \lambda_2) \cdot_\varepsilon \bar{h}$.

Similarly, we have $s(\bar{h}^{\wedge_\varepsilon \lambda_1} \otimes_\varepsilon \bar{h}^{\wedge_\varepsilon \lambda_2}) = 0.2566 > 0.13 = s(\bar{h}^{\wedge_\varepsilon (\lambda_1+\lambda_2)})$ and thus $\bar{h}^{\wedge_\varepsilon \lambda_1} \otimes_\varepsilon \bar{h}^{\wedge_\varepsilon \lambda_2} < \bar{h}^{\wedge_\varepsilon (\lambda_1+\lambda_2)}$.

3 Some HPFE weighted aggregation operators based on Einstein operation

One important issue is the question of how to extend Einstein operations to aggregate the HPFE information provided by the decision makers. The optimal approach is weighted aggregation operators, in which the widely used technologies are the weighted averaging (WA) operator, the ordered weighted averaging (OWA) operator, and their extended forms.[29,30] Yu[21] proposed the hesitant fuzzy Einstein weighted averaging (HFEWA) operator, the hesitant fuzzy Einstein ordered weighted averaging (HFEOWA) operator, the hesitant fuzzy Einstein weighted geometric (HFEWG) operator, and the hesitant fuzzy Einstein ordered weighted geometric (HFEOWG) operator based on those operators. Similar to these hesitant fuzzy information aggregation operators, we propose the corresponding hesitant probabilistic fuzzy Einstein weighted and ordered operators, to aggregate the HPFEs.

Definition 8. Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, then a hesitant probabilistic fuzzy Einstein weighted averaging (HPFEWA) operator is a mapping $H_P^T \to H_P$ such that

$$\text{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = (w_1 \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (w_2 \cdot_\varepsilon \bar{h}_2) \oplus_\varepsilon \cdots \oplus_\varepsilon (w_T \cdot_\varepsilon \bar{h}_T), \quad (6)$$

where $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ with $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, and p_t is the probability of γ_t in HPFE $\bar{h}_t$. In particular, if $w = \left(\frac{1}{T}, \frac{1}{T}, \ldots, \frac{1}{T}\right)^T$, then the HPFEWA operator is reduced to the hesitant probabilistic fuzzy Einstein averaging (HPFEA) operator:

$$\text{HPFEA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = (\frac{1}{T} \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (\frac{1}{T} \cdot_\varepsilon \bar{h}_2) \oplus_\varepsilon \cdots \oplus_\varepsilon (\frac{1}{T} \cdot_\varepsilon \bar{h}_T). \quad (7)$$

By Definitions 7 and 8, we can get the following result by using mathematical induction.

Theorem 4. Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, then their aggregated value by using HPFEWA operator is also a HPFE and

$$\begin{aligned} &\text{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \\ &= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{\prod_{t=1}^T (1 + \gamma_t)^{w_t} - \prod_{t=1}^T (1 - \gamma_t)^{w_t}}{\prod_{t=1}^T (1 + \gamma_t)^{w_t} + \prod_{t=1}^T (1 - \gamma_t)^{w_t}} \Big| p_1 p_2 \cdots p_T \right\}, \quad (8) \end{aligned}$$

where $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ with $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, and p_t is the probability of γ_t in HPFE $\bar{h}_t$.

Proof. We prove Eq. (8) by mathematical induction. For $T = 2$: Since $w_1 \cdot_\varepsilon \bar{h}_1 = \cup_{\gamma_1 \in \bar{h}_1} \left\{ \frac{(1+\gamma_1)^{w_1} - (1-\gamma_1)^{w_1}}{(1+\gamma_1)^{w_1} + (1-\gamma_1)^{w_1}} | p_1 \right\}$ and $w_2 \cdot_\varepsilon \bar{h}_2 = \cup_{\gamma_2 \in \bar{h}_2} \left\{ \frac{(1+\gamma_2)^{w_2} - (1-\gamma_2)^{w_2}}{(1+\gamma_2)^{w_2} + (1-\gamma_2)^{w_2}} | p_2 \right\}$, then

$$(w_1 \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (w_2 \cdot_\varepsilon \bar{h}_2) = \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2} \left\{ \frac{\frac{(1+\gamma_1)^{w_1}-(1-\gamma_1)^{w_1}}{(1+\gamma_1)^{w_1}+(1-\gamma_1)^{w_1}} + \frac{(1+\gamma_2)^{w_2}-(1-\gamma_2)^{w_2}}{(1+\gamma_2)^{w_2}+(1-\gamma_2)^{w_2}}}{1 + \frac{(1+\gamma_1)^{w_1}-(1-\gamma_1)^{w_1}}{(1+\gamma_1)^{w_1}+(1-\gamma_1)^{w_1}} \cdot \frac{(1+\gamma_2)^{w_2}-(1-\gamma_2)^{w_2}}{(1+\gamma_2)^{w_2}+(1-\gamma_2)^{w_2}}} \middle| p_1 p_2 \right\}$$

$$= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2} \left\{ \frac{\prod_{t=1}^{2}(1+\gamma_t)^{w_t} - \prod_{t=1}^{2}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{2}(1+\gamma_t)^{w_t} + \prod_{t=1}^{2}(1-\gamma_t)^{w_t}} \middle| p_1 p_2 \right\}.$$

If Eq. (8) holds for $T = k$, that is

$$(w_1 \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (w_2 \cdot_\varepsilon \bar{h}_2) \oplus_\varepsilon \cdots \oplus_\varepsilon (w_k \cdot_\varepsilon \bar{h}_k) = \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_k \in \bar{h}_k} \left\{ \frac{\prod_{t=1}^{k}(1+\gamma_t)^{w_t} - \prod_{t=1}^{k}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{k}(1+\gamma_t)^{w_t} + \prod_{t=1}^{k}(1-\gamma_t)^{w_t}} \middle| p_1 p_2 \cdots p_k \right\},$$

then when $T = k+1$, by the Einstein operations of HPFEs, we have

$$\begin{aligned}
&(w_1 \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (w_2 \cdot_\varepsilon \bar{h}_2) \oplus_\varepsilon \cdots \oplus_\varepsilon (w_{k+1} \cdot_\varepsilon \bar{h}_{k+1}) \\
&= \Big((w_1 \cdot_\varepsilon \bar{h}_1) \oplus_\varepsilon (w_2 \cdot_\varepsilon \bar{h}_2) \oplus_\varepsilon \cdots \oplus_\varepsilon (w_k \cdot_\varepsilon \bar{h}_k)\Big) \oplus_\varepsilon (w_{k+1} \cdot_\varepsilon \bar{h}_{k+1}) \\
&= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_k \in \bar{h}_k} \left\{ \frac{\prod_{t=1}^{k}(1+\gamma_t)^{w_t} - \prod_{t=1}^{k}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{k}(1+\gamma_t)^{w_t} + \prod_{t=1}^{k}(1-\gamma_t)^{w_t}} \middle| p_1 p_2 \cdots p_k \right\} \\
&\quad \oplus_\varepsilon \bigcup_{\gamma_{k+1} \in \bar{h}_{k+1}} \left\{ \frac{(1+\gamma_{k+1})^{w_{k+1}} - (1-\gamma_{k+1})^{w_{k+1}}}{(1+\gamma_{k+1})^{w_{k+1}} + (1-\gamma_{k+1})^{w_{k+1}}} \middle| p_{k+1} \right\} \\
&= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_k \in \bar{h}_k, \gamma_{k+1} \in \bar{h}_{k+1}} \left\{ \frac{\prod_{t=1}^{k+1}(1+\gamma_t)^{w_t} - \prod_{t=1}^{k+1}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{k+1}(1+\gamma_t)^{w_t} + \prod_{t=1}^{k+1}(1-\gamma_t)^{w_t}} \middle| p_1 p_2 \cdots p_k p_{k+1} \right\},
\end{aligned}$$

i.e., Eq. (8) holds for $T = k+1$. Hence Eq. (8) holds for all T. Thus

$$\mathrm{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{\prod_{t=1}^{T}(1+\gamma_t)^{w_t} - \prod_{t=1}^{T}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{T}(1+\gamma_t)^{w_t} + \prod_{t=1}^{T}(1-\gamma_t)^{w_t}} \middle| p_1 p_2 \cdots p_T \right\},$$

which completes the proof of theorem. □

Based on Theorem 4, we have basic properties of the HPFEWA operator as follows:

Theorem 5. Let $\bar{h}_t(\gamma_i^{(t)}|p_t)$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, $w = (w_1, w_2, \ldots, w_T)^T$ be the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ such that $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, and p_t be the corresponding probability of $\gamma_i^{(t)}$ in HPFE $\bar{h}_t$, then we have the followings:

(1) (Boundary):

$$\bar{h}^- \leq \text{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \leq \bar{h}^+, \tag{9}$$

where $\bar{h}^- = (\min_{1\leq t\leq T} \min_{t\in\bar{h}_t} \gamma_t | p_1 \cdots p_T)$ and $\bar{h}^+ = (\max_{1\leq t\leq T} \max_{t\in\bar{h}_t} \gamma_t | p_1 \cdots p_T)$.

(2) (Monotonicity): Let $\bar{h}_t^*(\dot{\gamma}_i^{(t)}|p_t)$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs with $\#_t = \#\bar{h}_t = \#\bar{h}_t^*$ for $t = 1, 2, \ldots, T$, $w = (w_1, w_2, \ldots, w_T)^T$ be the weight vector of $\bar{h}_t^*$ $(t = 1, 2, \ldots, T)$ such that $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, and p_t be the probability of $\dot{\gamma}_i^{(t)}$ in HPFE $\bar{h}_t^*$. If $\gamma_i^{(t)} \leq \dot{\gamma}_i^{(t)}$ for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$, then

$$\text{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \leq \text{HPFEWA}(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*). \tag{10}$$

Proof. (1) Let $f(x) = \frac{1-x}{1+x}$, $x \in [0, 1]$, then $f'(x) = \frac{-2}{(1+x)^2} < 0$, i.e., $f(x)$ is a decreasing function. Let $\max \gamma_t = \max_{1\leq t\leq T} \max_{t\in\bar{h}_t} \gamma_t$ and $\min \gamma_t = \min_{1\leq t\leq T} \min_{t\in\bar{h}_t} \gamma_t$. For any $\gamma_t \in \bar{h}_t$ $(t = 1, 2, \ldots, T)$, since $\min_{\gamma_t\in\bar{h}_t} \gamma_t \leq \gamma_t \leq \max_{\gamma_t\in\bar{h}_t} \gamma_t$, then $f(\max_{\gamma_t\in\bar{h}_t} \gamma_t) \leq f(\gamma_t) \leq f(\min_{\gamma_t\in\bar{h}_t} \gamma_t)$, and so

$$\frac{1-\max\gamma_t}{1+\max\gamma_t} \leq \frac{1-\max_{\gamma_t\in\bar{h}_t}\gamma_t}{1+\max_{\gamma_t\in\bar{h}_t}\gamma_t} \leq \frac{1-\gamma_t}{1+\gamma_t} \leq \frac{1-\min_{\gamma_t\in\bar{h}_t}\gamma_t}{1+\min_{\gamma_t\in\bar{h}_t}\gamma_t} \leq \frac{1-\min\gamma_t}{1+\min\gamma_t}.$$

Since $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ with $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, we have

$$\prod_{t=1}^T \left(\frac{1-\max\gamma_t}{1+\max\gamma_t}\right)^{w_t} \leq \prod_{t=1}^T \left(\frac{1-\gamma_t}{1+\gamma_t}\right)^{w_t} \leq \prod_{t=1}^T \left(\frac{1-\min\gamma_t}{1+\min\gamma_t}\right)^{w_t}.$$

Since $\prod_{t=1}^T \left(\frac{1-\max\gamma_t}{1+\max\gamma_t}\right)^{w_t} = \left(\frac{1-\max\gamma_t}{1+\max\gamma_t}\right)^{\sum_{t=1}^T w_t} = \frac{1-\max\gamma_t}{1+\max\gamma_t}$ and $\prod_{t=1}^T \left(\frac{1-\min\gamma_t}{1+\min\gamma_t}\right)^{w_t} = \left(\frac{1-\min\gamma_t}{1+\min\gamma_t}\right)^{\sum_{t=1}^T w_t} = \frac{1-\min\gamma_t}{1+\min\gamma_t}$, we get

$$\frac{1-\max\gamma_t}{1+\max\gamma_t} \leq \prod_{t=1}^T \left(\frac{1-\gamma_t}{1+\gamma_t}\right)^{w_t} \leq \frac{1-\min\gamma_t}{1+\min\gamma_t}$$

$$\Leftrightarrow \frac{2}{1+\max\gamma_t} \le 1+\prod_{t=1}^{T}\left(\frac{1-\gamma_t}{1+\gamma_t}\right)^{w_t} \le \frac{2}{1+\min\gamma_t}$$
$$\Leftrightarrow \frac{1+\min\gamma_t}{2} \le \frac{1}{1+\prod_{t=1}^{T}\left(\frac{1-\gamma_t}{1+\gamma_t}\right)^{w_t}} \le \frac{1+\max\gamma_t}{2}$$
$$\Leftrightarrow \min\gamma_t \le \frac{2}{1+\prod_{t=1}^{T}\left(\frac{1-\gamma_t}{1+\gamma_t}\right)^{w_t}} - 1 \le \max\gamma_t,$$

i.e.,

$$\min\gamma_t \le \frac{\prod_{t=1}^{T}(1+\gamma_t)^{w_t} - \prod_{t=1}^{T}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{T}(1+\gamma_t)^{w_t} + \prod_{t=1}^{T}(1-\gamma_t)^{w_t}} \le \max\gamma_t. \tag{11}$$

Let HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}(\gamma_i|p_1p_2\cdots p_T)$, $i = 1, 2, \ldots, \#\bar{h}$, where $\#\bar{h} = \#\bar{h}_1 \times \#\bar{h}_2 \times \cdots \times \#\bar{h}_T$, $\bar{h}^- = (\min\gamma_t|p_1p_2\cdots p_T)$ and $\bar{h}^+ = (\max\gamma_t|p_1p_2\cdots p_T)$, then Eq. (11) is transformed into the following forms: $\min\gamma_t \le \gamma_i \le \max\gamma_t$ for all $i = 1, 2, \ldots, \#\bar{h}$. Thus $s(\bar{h}^-) = \min\gamma_t p_1p_2\cdots p_T \le \sum_{i=1}^{\#\bar{h}}\gamma_i p_1p_2\cdots p_T = s(\bar{h})$ and $s(\bar{h}) = \sum_{i=1}^{\#\bar{h}}\gamma_i p_1p_2\cdots p_T \le \max\gamma_t p_1p_2\cdots p_T = s(\bar{h}^+)$.

If $s(\bar{h}^-) < s(\bar{h})$ and $s(\bar{h}) < s(\bar{h}^+)$, then by Definition 3, we have $\bar{h}^- <$ HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) < \bar{h}^+$. If $s(\bar{h}) = s(\bar{h}^+)$, i.e., $\max\gamma_t = \sum_{i=1}^{\#\bar{h}}\gamma_i$, then $d(\bar{h}) = \sum_{i=1}^{\#\bar{h}}(\gamma_i - s(\bar{h}))^2 p_1p_2\cdots p_T = (\max\gamma_t - s(\bar{h}))^2 p_1p_2\cdots p_T = d(\bar{h}^+)$, in this case, from Definition 3, it follows that HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}^+$. If $s(\bar{h}) = s(\bar{h}^-)$, then by the similar way, we have HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}^-$.

(2) Let $f(x) = \frac{1-x}{1+x}$, $x \in [0,1]$, then $f(x)$ is a decreasing function. If $\gamma_i^{(t)} \le \dot{\gamma}_i^{(t)}$ for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$, then $f(\gamma_i^{(t)}) \ge f(\dot{\gamma}_i^{(t)})$, for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$, i.e., $\frac{1-\gamma_i^{(t)}}{1+\gamma_i^{(t)}} \ge \frac{1-\dot{\gamma}_i^{(t)}}{1+\dot{\gamma}_i^{(t)}}$, for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$. For any $\gamma_i^{(t)} \in \bar{h}_t$ $(t = 1, 2, \ldots, T)$, since $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ such that $w_t \in [0,1]$, $t = 1, 2, \ldots, T$ and $\sum_{t=1}^{T} w_t = 1$, we have

$$\left(\frac{1-\gamma_i^{(t)}}{1+\gamma_i^{(t)}}\right)^{w_t} \ge \left(\frac{1-\dot{\gamma}_i^{(t)}}{1+\dot{\gamma}_i^{(t)}}\right)^{w_t}, \ t = 1, 2, \ldots, T.$$

Then

$$\prod_{t=1}^{T}\left(\frac{1-\gamma_i^{(t)}}{1+\gamma_i^{(t)}}\right)^{w_t} \ge \prod_{t=1}^{T}\left(\frac{1-\dot{\gamma}_i^{(t)}}{1+\dot{\gamma}_i^{(t)}}\right)^{w_t}$$

$$\Leftrightarrow 1+\prod_{t=1}^{T}\left(\frac{1-\gamma_i^{(t)}}{1+\gamma_i^{(t)}}\right)^{w_t} \geq 1+\prod_{t=1}^{T}\left(\frac{1-\dot{\gamma}_i^{(t)}}{1+\dot{\gamma}_i^{(t)}}\right)^{w_t}$$
$$\Leftrightarrow \frac{1}{1+\prod_{t=1}^{T}\left(\frac{1-\gamma_i^{(t)}}{1+\gamma_i^{(t)}}\right)^{w_t}} \leq \frac{1}{1+\prod_{t=1}^{T}\left(\frac{1-\dot{\gamma}_i^{(t)}}{1+\dot{\gamma}_i^{(t)}}\right)^{w_t}}$$
$$\Leftrightarrow \frac{2}{1+\prod_{t=1}^{T}\left(\frac{1-\gamma_i^{(t)}}{1+\gamma_i^{(t)}}\right)^{w_t}}-1 \leq \frac{2}{1+\prod_{t=1}^{T}\left(\frac{1-\dot{\gamma}_i^{(t)}}{1+\dot{\gamma}_i^{(t)}}\right)^{w_t}}-1,$$

i.e.,

$$\frac{\prod_{t=1}^{T}(1+\gamma_i^{(t)})^{w_t}-\prod_{t=1}^{T}(1-\gamma_i^{(t)})^{w_t}}{\prod_{t=1}^{T}(1+\gamma_i^{(t)})^{w_t}+\prod_{t=1}^{T}(1-\gamma_i^{(t)})^{w_t}} \leq \frac{\prod_{t=1}^{T}(1+\dot{\gamma}_i^{(t)})^{w_t}-\prod_{t=1}^{T}(1-\dot{\gamma}_i^{(t)})^{w_t}}{\prod_{t=1}^{T}(1+\dot{\gamma}_i^{(t)})^{w_t}+\prod_{t=1}^{T}(1-\dot{\gamma}_i^{(t)})^{w_t}}. \quad (12)$$

Let HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}(\gamma_i|p_1p_2\cdots p_T)$ and HPFEWA$(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*) = \bar{h}^*(\dot{\gamma}_i|p_1p_2\cdots p_T)$, where $i = 1, 2, \ldots, \#$, and $\# = \#_1 \times \#_2 \times \cdots \times \#_T$ is the number of possible elements in $\bar{h}(\gamma_i|p_1p_2\cdots p_T)$ and $\bar{h}^*(\dot{\gamma}_i|p_1p_2\cdots p_T)$, respectively, then the Eq. (12) is transformed into the form: $\gamma_i \leq \dot{\gamma}_i$ $(i = 1, 2, \ldots, \#)$. Thus $s(\bar{h}) = \sum_{i=1}^{\#}\gamma_i p_1p_2\cdots p_T \leq \sum_{i=1}^{\#}\dot{\gamma}_i p_1p_2\cdots p_T = s(\bar{h}^*)$.

If $s(\bar{h}) < s(\bar{h}^*)$, then by Definition 3, we have HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) <$ HPFEWA$(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*)$. If $s(\bar{h}) = s(\bar{h}^*)$, i.e., $\sum_{i=1}^{\#}\gamma_i = \sum_{i=1}^{\#}\dot{\gamma}_i$, then $d(\bar{h}) = \sum_{i=1}^{\#}(\gamma_i - s(\bar{h}))^2 p_1p_2\cdots p_T = \sum_{i=1}^{\#}(\dot{\gamma}_i - s(\bar{h}^*))^2 p_1p_2\cdots p_T = d(\bar{h}^*)$, in this case, by Definition 3, it follows HPFEWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) =$ HPFEWA$(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*)$. □

However, the HPFEWA operator does not satisfy the idempotency. To illustrate this, we give an example as follows:

Example 3. Let $\bar{h}_1 = \bar{h}_2 = (0.3|0.5, 0.7|0.5)$ and $w = (0.2, 0.8)^T$ be the weight vector $\bar{h}_t$ $(t = 1, 2)$, then

$$\text{HPFEWA}(\bar{h}_1, \bar{h}_2) = \cup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2}\left\{\frac{\prod_{t=1}^{2}(1+\gamma_t)^{w_t}-\prod_{t=1}^{2}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{2}(1+\gamma_t)^{w_t}+\prod_{t=1}^{2}(1-\gamma_t)^{w_t}}|p_1p_2\right\}$$
$$= (0.3|0.25, 0.398|0.25, 0.639|0.25, 0.7|0.25)$$

and thus HPFEWA$(\bar{h}_1, \bar{h}_2) \neq (0.3|0.5, 0.7|0.5)$.

Based on the HPFWG operator and Einstein operation, we develop the hesitant probabilistic fuzzy Einstein weighted geometric operator as follows:

Definition 9. Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, then a hesitant probabilistic fuzzy Einstein weighted geometric (HPFEWG) operator is a mapping $H_P^T \to H_P$ such that

$$\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon w_2} \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{h}_T^{\wedge_\varepsilon w_T}, \tag{13}$$

where $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ with $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, and p_t is the probability of γ_t in HPFE $\bar{h}_t$. In particular, if $w = \left(\frac{1}{T}, \frac{1}{T}, \ldots, \frac{1}{T}\right)^T$, then the HPFEWG operator is reduced to the hesitant probabilistic fuzzy Einstein geometric (HPFEG) operator:

$$\text{HPFEG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_1^{\wedge_\varepsilon \frac{1}{T}} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon \frac{1}{T}} \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{h}_T^{\wedge_\varepsilon \frac{1}{T}}. \tag{14}$$

Theorem 6. Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, then their aggregated value by using HPFEWG operator is also a HPFE and

$$\begin{aligned} &\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \\ &= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{2 \prod_{t=1}^T \gamma_t^{w_t}}{\prod_{t=1}^T (2 - \gamma_t)^{w_t} + \prod_{t=1}^T \gamma_t^{w_t}} \middle| p_1 p_2 \cdots p_T \right\}, \end{aligned} \tag{15}$$

where $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ with $w_t \in [0, 1]$ and $\sum_{t=1}^T w_t = 1$, and p_t is the probability of γ_t in HPFE $\bar{h}_t$.

Proof. We prove Eq. (15) by mathematical induction on T. When $T = 2$, since $\bar{h}_1^{\wedge_\varepsilon w_1} = \cup_{\gamma_1 \in \bar{h}_1} \left\{ \frac{2\gamma_1^{w_1}}{(2-\gamma_1)^{w_1} + \gamma_1^{w_1}} \middle| p_1 \right\}$ and $\bar{h}_2^{\wedge_\varepsilon w_2} = \cup_{\gamma_2 \in \bar{h}_2} \left\{ \frac{2\gamma_2^{w_2}}{(2-\gamma_2)^{w_2} + \gamma_2^{w_2}} \middle| p_2 \right\}$, we have

$$\begin{aligned} \bar{h}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon w_2} &= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2} \left\{ \frac{\frac{2\gamma_1^{w_1}}{(2-\gamma_1)^{w_1} + \gamma_1^{w_1}} \cdot \frac{2\gamma_2^{w_2}}{(2-\gamma_2)^{w_2} + \gamma_2^{w_2}}}{1 + \left(1 - \frac{2\gamma_1^{w_1}}{(2-\gamma_1)^{w_1} + \gamma_1^{w_1}}\right)\left(1 - \frac{2\gamma_2^{w_2}}{(2-\gamma_2)^{w_2} + \gamma_2^{w_2}}\right)} \middle| p_1 p_2 \right\} \\ &= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2} \left\{ \frac{2 \prod_{t=1}^2 \gamma_t^{w_t}}{\prod_{t=1}^2 (2 - \gamma_t)^{w_t} + \prod_{t=1}^2 \gamma_t^{w_t}} \middle| p_1 p_2 \right\}. \end{aligned}$$

Assume that Eq. (15) holds for $T = k$, i.e.,

$$\begin{aligned}&\bar{h}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon w_2} \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{h}_k^{\wedge_\varepsilon w_k} \\ &\quad = \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \cdots, \gamma_k \in \bar{h}_k} \left\{ \frac{2\prod_{t=1}^{k} \gamma_t^{w_t}}{\prod_{t=1}^{k}(2-\gamma_t)^{w_t} + \prod_{t=1}^{k} \gamma_t^{w_t}} \Big| p_1 p_2 \cdots p_k \right\}.\end{aligned}$$

By the Einstein operational laws of HPFEs for $T = k+1$, we have

$$\begin{aligned}&\bar{h}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon w_2} \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{h}_{k+1}^{\wedge_\varepsilon w_{k+1}} \\ &\quad = \left(\bar{h}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon w_2} \otimes_\varepsilon \cdots \otimes_\varepsilon \bar{h}_k^{\wedge_\varepsilon w_k}\right) \otimes_\varepsilon \bar{h}_{k+1}^{\wedge_\varepsilon w_{k+1}} \\ &\quad = \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_k \in \bar{h}_k} \left\{ \frac{2\prod_{t=1}^{k} \gamma_t^{w_t}}{\prod_{t=1}^{k}(2-\gamma_t)^{w_t} + \prod_{t=1}^{k} \gamma_t^{w_t}} \Big| p_1 p_2 \cdots p_k \right\} \\ &\qquad \otimes_\varepsilon \bigcup_{\gamma_{k+1} \in \bar{h}_{k+1}} \left\{ \frac{2\gamma_{k+1}^{w_{k+1}}}{(2-\gamma_{k+1})^{w_{k+1}} + \gamma_{k+1}^{w_{k+1}}} \big| p_{k+1} \right\} \\ &\quad = \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_k \in \bar{h}_k, \gamma_{k+1} \in \bar{h}_{k+1}} \left\{ \frac{2\prod_{t=1}^{k+1} \gamma_t^{w_t}}{\prod_{t=1}^{k+1}(2-\gamma_t)^{w_t} + \prod_{t=1}^{k+1} \gamma_t^{w_t}} \Big| p_1 p_2 \cdots p_k p_{k+1} \right\},\end{aligned}$$

i.e., Eq. (15) holds for $T = k+1$. Then, Eq. (15) holds for all T. Hence we complete the proof of the theorem. □

Based on Theorem 6, we have basic properties of the HPFEWG operator as follows:

Theorem 7. Let $\bar{h}_t(\gamma_i^{(t)}|p_t)$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, $w = (w_1, w_2, \ldots, w_T)^T$ be the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ such that $w_t \in [0, 1]$ and $\sum_{t=1}^{T} w_t = 1$, and p_t be the corresponding probability of $\gamma_i^{(t)}$ in HPFE $\bar{h}_t$, then we have the followings:

(1) (Boundary):

$$\bar{h}^- \leq \text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \leq \bar{h}^+, \tag{16}$$

where $\bar{h}^- = (\min_{1 \leq t \leq T} \min_{t \in \bar{h}_t} \gamma_t | p_1 \cdots p_T)$ and $\bar{h}^+ = (\max_{1 \leq t \leq T} \max_{t \in \bar{h}_t} \gamma_t | p_1 \cdots p_T)$.

(2) (Monotonicity): Let $\bar{h}_t^*(\dot{\gamma}_i^{(t)}|p_t)$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs with $\#_t = \#\bar{h}_t = \#\bar{h}_t^*$ for $t = 1, 2, \ldots, T$, $w = (w_1, w_2, \ldots, w_T)^T$ be the weight

vector of $\bar{h}_t^*$ $(t = 1, 2, \ldots, T)$ such that $w_t \in [0, 1]$ and $\sum_{t=1}^{T} w_t = 1$, and p_t be the probability of $\dot{\gamma}_i^{(t)}$ in HPFE $\bar{h}_t^*$. If $\gamma_i^{(t)} \leq \dot{\gamma}_i^{(t)}$ for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$, then

$$\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \leq \text{HPFEWG}(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*). \tag{17}$$

Proof. (1) Let $g(x) = \frac{2-x}{x}$, $x \in (0, 1]$, then $g'(x) = \frac{-2}{x^2} < 0$, i.e., $g(x)$ is a decreasing function. Let $\max \gamma_t = \max_{1 \leq t \leq T} \max_{t \in \bar{h}_t} \gamma_t$ and $\min \gamma_t = \min_{1 \leq t \leq T} \min_{t \in \bar{h}_t} \gamma_t$. For any $\gamma_t \in \bar{h}_t$ $(t = 1, 2, \ldots, T)$, since $\min_{\gamma_t \in \bar{h}_t} \gamma_t \leq \gamma_t \leq \max_{\gamma_t \in \bar{h}_t} \gamma_t$, then $g(\max_{\gamma_t \in \bar{h}_t} \gamma_t) \leq g(\gamma_t) \leq g(\min_{\gamma_t \in \bar{h}_t} \gamma_t)$, and so

$$\frac{2 - \max \gamma_t}{\max \gamma_t} \leq \frac{2 - \max_{\gamma_t \in \bar{h}_t} \gamma_t}{\max_{\gamma_t \in \bar{h}_t} \gamma_t} \leq \frac{2 - \gamma_t}{\gamma_t} \leq \frac{2 - \min_{\gamma_t \in \bar{h}_t} \gamma_t}{\min_{\gamma_t \in \bar{h}_t} \gamma_t} \leq \frac{2 - \min \gamma_t}{\min \gamma_t}.$$

Since $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ with $w_t \in [0, 1]$ and $\sum_{t=1}^{T} w_t = 1$, we have

$$\prod_{t=1}^{T} \left(\frac{2 - \max \gamma_t}{\max \gamma_t} \right)^{w_t} \leq \prod_{t=1}^{T} \left(\frac{2 - \gamma_t}{\gamma_t} \right)^{w_t} \leq \prod_{t=1}^{T} \left(\frac{2 - \min \gamma_t}{\min \gamma_t} \right)^{w_t}.$$

Since $\prod_{t=1}^{T} \left(\frac{2-\max \gamma_t}{\max \gamma_t} \right)^{w_t} = \left(\frac{2-\max \gamma_t}{\max \gamma_t} \right)^{\sum_{t=1}^{T} w_t} = \frac{2-\max \gamma_t}{\max \gamma_t}$ and $\prod_{t=1}^{T} \left(\frac{2-\min \gamma_t}{\min \gamma_t} \right)^{w_t} = \left(\frac{2-\min \gamma_t}{\min \gamma_t} \right)^{\sum_{t=1}^{T} w_t} = \frac{2-\min \gamma_t}{\min \gamma_t}$, we obtain

$$\begin{aligned} \frac{2 - \max \gamma_t}{\max \gamma_t} &\leq \prod_{t=1}^{T} \left(\frac{2 - \gamma_t}{\gamma_t} \right)^{w_t} \leq \frac{2 - \min \gamma_t}{\min \gamma_t} \\ \Leftrightarrow \frac{2}{\max \gamma_t} &\leq 1 + \prod_{t=1}^{T} \left(\frac{2 - \gamma_t}{\gamma_t} \right)^{w_t} \leq \frac{2}{\min \gamma_t} \\ \Leftrightarrow \frac{\min \gamma_t}{2} &\leq \frac{1}{1 + \prod_{t=1}^{T} \left(\frac{2-\gamma_t}{\gamma_t} \right)^{w_t}} \leq \frac{\max \gamma_t}{2} \\ \Leftrightarrow \min \gamma_t &\leq \frac{2}{1 + \prod_{t=1}^{T} \left(\frac{2-\gamma_t}{\gamma_t} \right)^{w_t}} \leq \max \gamma_t, \end{aligned}$$

i.e.,

$$\min \gamma_t \leq \frac{2 \prod_{t=1}^{T} \gamma_t^{w_t}}{\prod_{t=1}^{T} (2 - \gamma_t)^{w_t} + \prod_{t=1}^{T} \gamma_t^{w_t}} \leq \max \gamma_t. \tag{18}$$

Let HPFEWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}(\gamma_i|p_1p_2\cdots p_T)$, $i = 1, 2, \ldots, \#\bar{h}$, where $\#\bar{h} = \#\bar{h}_1 \times \#\bar{h}_2 \times \cdots \times \#\bar{h}_T$, $\bar{h}^- = (\min \gamma_t|p_1p_2\cdots p_T)$ and $\bar{h}^+ = (\max \gamma_t|p_1p_2\cdots p_T)$, then Eq. (18) is transformed into the following forms: $\min \gamma_t \leq \gamma_i \leq \max \gamma_t$ for all $i = 1, 2, \ldots, \#\bar{h}$. Thus $s(\bar{h}^-) = \min \gamma_t p_1p_2\cdots p_T \leq \sum_{i=1}^{\#\bar{h}} \gamma_i p_1p_2\cdots p_T = s(\bar{h})$ and $s(\bar{h}) = \sum_{i=1}^{\#\bar{h}} \gamma_i p_1p_2\cdots p_T \leq \max \gamma_t p_1p_2\cdots p_T = s(\bar{h}^+)$. If $s(\bar{h}^-) < s(\bar{h})$ and $s(\bar{h}) < s(\bar{h}^+)$, then by Definition 3, we have $\bar{h}^- <$ HPFEWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) < \bar{h}^+$. If $s(\bar{h}) = s(\bar{h}^+)$, i.e., $\max \gamma_t = \sum_{i=1}^{\#\bar{h}} \gamma_i$, then $d(\bar{h}) = \sum_{i=1}^{\#\bar{h}} (\gamma_i - s(\bar{h}))^2 p_1p_2\cdots p_T = (\max \gamma_t - s(\bar{h}))^2 p_1p_2\cdots p_T = d(\bar{h}^+)$, in this case, from Definition 3, it follows that HPFEWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}^+$. If $s(\bar{h}) = s(\bar{h}^-)$, then by the similar way, we have HPFEWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}^-$.

(2) Let $g(x) = \frac{2-x}{x}$, $x \in (0, 1]$, then $g(x)$ is a decreasing function. If $\gamma_i^{(t)} \leq \dot{\gamma}_i^{(t)}$ for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$, then $g(\gamma_i^{(t)}) \geq g(\dot{\gamma}_i^{(t)})$, for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$, i.e., $\frac{2-\gamma_i^{(t)}}{\gamma_i^{(t)}} \geq \frac{2-\dot{\gamma}_i^{(t)}}{\dot{\gamma}_i^{(t)}}$, for each $i = 1, 2, \ldots, \#_t$, $t = 1, 2, \ldots, T$. For any $\gamma_i^{(t)} \in \bar{h}_t$ $(t = 1, 2, \ldots, T)$, since $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ such that $w_t \in [0, 1]$, $t = 1, 2, \ldots, T$ and $\sum_{t=1}^T w_t = 1$, we have

$$\left(\frac{2-\gamma_i^{(t)}}{\gamma_i^{(t)}}\right)^{w_t} \geq \left(\frac{2-\dot{\gamma}_i^{(t)}}{\dot{\gamma}_i^{(t)}}\right)^{w_t}, \; i = 1, 2, \ldots, \#_t, \; t = 1, 2, \ldots, T.$$

Then

$$\begin{aligned}
&\prod_{t=1}^{T}\left(\frac{2-\gamma_i^{(t)}}{\gamma_i^{(t)}}\right)^{w_t} \geq \prod_{t=1}^{T}\left(\frac{2-\dot{\gamma}_i^{(t)}}{\dot{\gamma}_i^{(t)}}\right)^{w_t} \\
&\quad\Leftrightarrow 1 + \prod_{t=1}^{T}\left(\frac{2-\gamma_i^{(t)}}{\gamma_i^{(t)}}\right)^{w_t} \geq 1 + \prod_{t=1}^{T}\left(\frac{2-\dot{\gamma}_i^{(t)}}{\dot{\gamma}_i^{(t)}}\right)^{w_t} \\
&\quad\Leftrightarrow \frac{1}{1 + \prod_{t=1}^{T}\left(\frac{2-\gamma_i^{(t)}}{\gamma_i^{(t)}}\right)^{w_t}} \leq \frac{1}{1 + \prod_{t=1}^{T}\left(\frac{2-\dot{\gamma}_i^{(t)}}{\dot{\gamma}_i^{(t)}}\right)^{w_t}} \\
&\quad\Leftrightarrow \frac{2}{1 + \prod_{t=1}^{T}\left(\frac{2-\gamma_i^{(t)}}{\gamma_i^{(t)}}\right)^{w_t}} - 1 \leq \frac{2}{1 + \prod_{t=1}^{T}\left(\frac{2-\dot{\gamma}_i^{(t)}}{\dot{\gamma}_i^{(t)}}\right)^{w_t}} - 1,
\end{aligned}$$

i.e.,

$$\frac{2\prod_{t=1}^{T}(\gamma_i^{(t)})^{w_t}}{\prod_{t=1}^{T}(2-\gamma_i^{(t)})^{w_t}+\prod_{t=1}^{T}(\gamma_i^{(t)})^{w_t}} \le \frac{2\prod_{t=1}^{T}(\gamma_i^{(t)})^{w_t}}{\prod_{t=1}^{T}(2-\gamma_i^{(t)})^{w_t}+\prod_{t=1}^{T}(\gamma_i^{(t)})^{w_t}}. \tag{19}$$

Let $\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}(\gamma_i|p_1p_2\cdots p_T)$ and $\text{HPFEWG}(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*) = \bar{h}^*(\dot{\gamma}_i|p_1p_2\cdots p_T)$, where $i = 1, 2, \ldots, \#$, and $\# = \#_1 \times \#_2 \times \cdots \times \#_T$ is the number of possible elements in $\bar{h}(\gamma_i|p_1p_2\cdots p_T)$ and $\bar{h}^*(\dot{\gamma}_i|p_1p_2\cdots p_T)$, respectively, then the Eq. (19) is transformed into the form: $\gamma_i \le \dot{\gamma}_i$ $(i = 1, 2, \ldots, \#)$. Thus $s(\bar{h}) = \sum_{i=1}^{\#} \gamma_i p_1p_2\cdots p_T \le \sum_{i=1}^{\#} \dot{\gamma}_i p_1p_2\cdots p_T = s(\bar{h}^*)$. If $s(\bar{h}) < s(\bar{h}^*)$, then by Definition 3, $\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) < \text{HPFEWG}(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*)$. If $s(\bar{h}) = s(\bar{h}^*)$, i.e., $\sum_{i=1}^{\#} \gamma_i = \sum_{i=1}^{\#} \dot{\gamma}_i$, then $d(\bar{h}) = \sum_{i=1}^{\#}(\gamma_i - s(\bar{h}))^2 p_1p_2\cdots p_T = \sum_{i=1}^{\#}(\dot{\gamma}_i - s(\bar{h}^*))^2 p_1p_2\cdots p_T = d(\bar{h}^*)$, in this case, from Definition 3, it follows that $\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \text{HPFEWG}(\bar{h}_1^*, \bar{h}_2^*, \ldots, \bar{h}_T^*)$. □

If all probabilities of values in each HPFE are equal, i.e., $p_1 = p_2 = \cdots = p_{\#\bar{h}_t}$ $(t = 1, 2, \ldots, T)$, then the HPFE is reduced to the HFE. In this case, the score function of the HPFEWA (resp. HPFEWG) operator is consistent with that of the HFEWA (resp. HFEWG) operator.[21] So we can conclude that the HPFEWA (resp. HPFEWG) operator is reduced to the HFEWA (resp. HFEWG) operator.[21] In order to analyze the relationship between the HPFEWA (resp. HPFEWG) operator and the HPFWA (resp. HPFWG) operator,[24] we introduce the following lemma.

Lemma 1.[31,32] Let $x_i > 0$, $w_i > 0$, $i = 1, 2, \ldots, N$, and $\sum_{i=1}^{N} w_i = 1$, then $\prod_{i=1}^{N} x_i^{w_i} \le \sum_{i=1}^{N} w_i x_i$, with equality if and only if $x_1 = x_2 = \cdots = x_N$.

Theorem 8. If $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ are a collection of HPFEs and $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$, with $w_t \in [0, 1]$ and $\sum_{t=1}^{T} w_t = 1$, and p_t is the probability of γ_t in HPFE $\bar{h}_t$, then

(1) $\text{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \le \text{HPFWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T)$;

(2) $\text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \ge \text{HPFWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T)$.

Proof. (1) For any $\gamma_t \in \bar{h}_t$ $(t = 1, 2, \ldots, T)$, by Lemma 1, we obtain the inequality $\prod_{t=1}^{T}(1+\gamma_t)^{w_t} + \prod_{t=1}^{T}(1-\gamma_t)^{w_t} \le \sum_{t=1}^{T} w_t(1+\gamma_t) + \sum_{t=1}^{T} w_t(1-\gamma_t) = 2$

and then

$$\frac{\prod_{t=1}^{T}(1+\gamma_t)^{w_t}-\prod_{t=1}^{T}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{T}(1+\gamma_t)^{w_t}+\prod_{t=1}^{T}(1-\gamma_t)^{w_t}} = 1-\frac{2\prod_{t=1}^{T}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{T}(1+\gamma_t)^{w_t}+\prod_{t=1}^{T}(1-\gamma_t)^{w_t}} \leq 1-\prod_{t=1}^{T}(1-\gamma_t)^{w_t}.$$

Hence we can obtain the inequality:

$$\bigcup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2,\ldots,\gamma_T\in\bar{h}_T}\left\{\frac{\prod_{t=1}^{T}(1+\gamma_t)^{w_t}-\prod_{t=1}^{T}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{T}(1+\gamma_t)^{w_t}+\prod_{t=1}^{T}(1-\gamma_t)^{w_t}}\right\} \leq \bigcup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2,\ldots,\gamma_T\in\bar{h}_T}\left\{1-\prod_{t=1}^{T}(1-\gamma_t)^{w_t}\right\}. \tag{20}$$

Let HPFEWA$(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T)=\bar{h}(\gamma_i|p_i)$ and HPFWA$(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T)=\bar{h}^*(\gamma_i^*|p_i)$, $i=1,2,\ldots\#$, where $\#=\#\bar{h}=\#\bar{h}^*$ is the number of possible elements in $\bar{h}(\gamma_i|p_i)$ and $\bar{h}^*(\dot{\gamma}_i|p_i)$, respectively, then the Eq. (20) is transformed into the form: $\gamma_i\leq\gamma_i^*$ $(i=1,2,\ldots,\#)$. According to $s(\bar{h})=\sum_{i=1}^{\#\bar{h}}\gamma_i p_i$, we have HPFEWA$(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T)\leq$ HPFWA$(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T)$.

(2) For any $\gamma_t\in\bar{h}_t$ $(t=1,2,\ldots,T)$, by Lemma 1, we have $\prod_{t=1}^{T}(2-\gamma_t)^{w_t}+\prod_{t=1}^{T}\gamma_t^{w_t}\leq\sum_{t=1}^{T}w_t(2-\gamma_t)+\sum_{t=1}^{T}w_t\gamma_t=2$ and then

$$\frac{2\prod_{t=1}^{T}\gamma_t^{w_t}}{\prod_{t=1}^{T}(2-\gamma_t)^{w_t}+\prod_{t=1}^{T}\gamma_t^{w_t}}\geq\prod_{t=1}^{T}\gamma_t^{w_t}.$$

Hence by similar way to (1), HPFEWG$(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T)\geq$ HPFWG$(\bar{h}_1,\bar{h}_2,\ldots,\bar{h}_T)$. □

Example 4. Let $\bar{h}_1=(0.5|0.5,0.6|0.5)$ and $\bar{h}_2=(0.1|0.2,0.3|0.3,0.4|0.5)$ be two HPFEs and $w=(0.6,0.4)^T$ be the weight vector of them, then by Eq. (8), the aggregated values by the HPFEWA operator is

$$\begin{aligned}\text{HPFEWA}(\bar{h}_1,\bar{h}_2)&=(w_1\cdot_\varepsilon\bar{h}_1)\oplus_\varepsilon(w_2\cdot_\varepsilon\bar{h}_2)\\ &=\bigcup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2}\left\{\frac{\prod_{t=1}^{2}(1+\gamma_t)^{w_t}-\prod_{t=1}^{2}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{2}(1+\gamma_t)^{w_t}+\prod_{t=1}^{2}(1-\gamma_t)^{w_t}}\Big|p_1p_2\right\}\\ &=\{0.3537|0.1,0.4247|0.15,0.4614|0.25,0.4268|0.1,0.4928|0.15,0.5265|0.25\}.\end{aligned}$$

If we use the HPFWA operator (Eq. (2)) to aggregate two HPFEs, then we have

$$
\begin{aligned}
\mathrm{HPFWA}(\bar{h}_1, \bar{h}_2) &= (w_1\bar{h}_1) \oplus (w_2\bar{h}_2) \\
&= \bigcup_{\gamma_1\in\bar{h}_1, \gamma_2\in\bar{h}_2} \left\{1 - \prod_{t=1}^{2}(1-\gamma_t)^{w_t}\Big|p_1p_2\right\} \\
&= \{0.3675|0.1, 0.4280|0.15, 0.4622|0.25, 0.4467|0.1, 0.4996|0.15, 0.5296|0.25\}.
\end{aligned}
$$

Then $s(\mathrm{HPFEWA}(\bar{h}_1, \bar{h}_2)) = 0.4627$ and $s(\mathrm{HPFWA}(\bar{h}_1, \bar{h}_2)) = 0.4685$, and thus $\mathrm{HPFEWA}(\bar{h}_1, \bar{h}_2) < \mathrm{HPFWA}(\bar{h}_1, \bar{h}_2)$.

On the other hand, by Eq. (15), the aggregated value by HPFEWG operator is

$$
\begin{aligned}
\mathrm{HPFEWG}(\bar{h}_1, \bar{h}_2) &= \bar{h}_1^{\wedge_\varepsilon w_1} \otimes_\varepsilon \bar{h}_2^{\wedge_\varepsilon w_2} \\
&= \bigcup_{\gamma_1\in\bar{h}_1, \gamma_2\in\bar{h}_2} \left\{\frac{2\prod_{t=1}^{2}\gamma_t^{w_t}}{\prod_{t=1}^{2}(2-\gamma_t)^{w_t} + \prod_{t=1}^{2}\gamma_t^{w_t}}\Big|p_1p_2\right\} \\
&= \{0.2748|0.1, 0.4108|0.15, 0.4581|0.25, 0.3126|0.1, 0.4622|0.15, 0.5135|0.25\}.
\end{aligned}
$$

If we use the HPFWG operator (Eq. (3)) to aggregate two HPFEs, then we get

$$
\begin{aligned}
\mathrm{HPFWG}(\bar{h}_1, \bar{h}_2) &= (\bar{h}_1)^{w_1} \otimes (\bar{h}_2)^{w_2} \\
&= \bigcup_{\gamma_1\in\bar{h}_1, \gamma_2\in\bar{h}_2} \left\{\prod_{t=1}^{2}(\gamma_t)^{w_t}|p_1p_2\right\} \\
&= \{0.2627|0.1, 0.4076|0.15, 0.4573|0.25, 0.2930|0.1, 0.4547|0.15, 0.5102|0.25\}.
\end{aligned}
$$

It is clear that $\mathrm{HPFEWG}(\bar{h}_1, \bar{h}_2) > \mathrm{HPFWG}(\bar{h}_1, \bar{h}_2)$.

Theorem 9. If $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ are a collection of HPFEs, $w = (w_1, w_2, \ldots, w_T)^T$ is the weight vector of $\bar{h}_t$ with $w_t \in [0, 1]$ and $\sum_{t=1}^{T} w_t = 1$, and p_t is the probability of γ_t in HPFE $\bar{h}_t$, then

(1) $\mathrm{HPFEWA}((\bar{h}_1)^c, (\bar{h}_2)^c, \ldots, (\bar{h}_T)^c) = (\mathrm{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T))^c$;

(2) $\mathrm{HPFEWG}((\bar{h}_1)^c, (\bar{h}_2)^c, \ldots, (\bar{h}_T)^c) = (\mathrm{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T))^c$.

Proof. Since (2) is similar (1), we only prove (1).

$$
\begin{aligned}
&\mathrm{HPFEWA}((\bar{h}_1)^c, (\bar{h}_2)^c, \ldots, (\bar{h}_T)^c) \\
&= \bigcup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2,\ldots,\gamma_T\in\bar{h}_T} \left\{ \frac{\prod_{t=1}^T (1+(1-\gamma_t))^{w_t} - \prod_{t=1}^T (1-(1-\gamma_t))^{w_t}}{\prod_{t=1}^T (1+(1-\gamma_t))^{w_t} + \prod_{t=1}^T (1-(1-\gamma_t))^{w_t}} \middle| p_1 p_2 \cdots p_T \right\} \\
&= \bigcup_{\gamma_1\in\bar{h}_1,\gamma_2\in\bar{h}_2,\ldots,\gamma_T\in\bar{h}_T} \left\{ 1 - \frac{2\prod_{t=1}^T \gamma_t^{w_t}}{\prod_{t=1}^T (2-\gamma_t)^{w_t} + \prod_{t=1}^T \gamma_t^{w_t}} \middle| p_1 p_2 \cdots p_T \right\} \\
&= (\mathrm{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T))^c.
\end{aligned}
$$

□

Theorem 8 shows that (1) the values aggregated by the HPFEWA operator are not larger than those obtained by the HPFWA operator. That is to say, the HPFEWA operator reflects the decision maker's pessimistic attitude than the HPFWA operator in aggregation process; (2) the values aggregated by the HPFWG operator are not larger than those obtained by the HPFEWG operator. Thus the HPFEWG operator reflects the decision maker's optimistic attitude than the HPFWG operator in aggregation process. Moreover, we develop the following ordered weighted operators based on the HPFOWA operator[24] and the HPFOWG operator,[24] to aggregate the HPFEs.

Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, $\bar{h}_{\sigma(t)}$ be the tth largest of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$, and $p_{\sigma(t)}$ be the probability of $\gamma_{\sigma(t)}$ in the HPFE $\bar{h}_{\sigma(t)}$, then we develop the following two aggregation operators, which are based on the mapping $H_P^T \to H_P$ with an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_T)^T$ such that $\omega_t \in [0, 1]$ and $\sum_{t=1}^T \omega_t = 1$:

(1) The hesitant probabilistic fuzzy Einstein ordered weighted averaging (HPFE-OWA) operator:

$$
\begin{aligned}
\mathrm{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\omega_1 \cdot_\varepsilon \bar{h}_{\sigma(1)}) \oplus_\varepsilon (\omega_2 \cdot_\varepsilon \bar{h}_{\sigma(2)}) \oplus_\varepsilon \cdots \oplus_\varepsilon (\omega_T \cdot_\varepsilon \bar{h}_{\sigma(T)}) \\
&= \bigcup_{\substack{\gamma_{\sigma(i)}\in\bar{h}_{\sigma(i)},\\ i=1,2,\ldots,T}} \left\{ \frac{\prod_{t=1}^T (1+\gamma_{\sigma(t)})^{\omega_t} - \prod_{t=1}^T (1-\gamma_{\sigma(t)})^{\omega_t}}{\prod_{t=1}^T (1+\gamma_{\sigma(t)})^{\omega_t} + \prod_{t=1}^T (1-\gamma_{\sigma(t)})^{\omega_t}} \middle| p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(T)} \right\}. \quad (21)
\end{aligned}
$$

(2) The hesitant probabilistic fuzzy Einstein ordered weighted geometric (HPFE-

OWG) operator:

$$\text{HPFEOWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = (\bar{h}_{\sigma(1)}^{\wedge_\varepsilon \omega_1}) \otimes_\varepsilon (\bar{h}_{\sigma(2)}^{\wedge_\varepsilon \omega_2}) \otimes_\varepsilon \cdots \otimes_\varepsilon (\bar{h}_{\sigma(T)}^{\wedge_\varepsilon \omega_T})$$

$$= \bigcup_{\substack{\gamma_{\sigma(i)} \in \bar{h}_{\sigma(i)}, \\ i=1,2,\ldots,T}} \left\{ \frac{2 \prod_{t=1}^T \gamma_{\sigma(t)}^{\omega_t}}{\prod_{t=1}^T (2-\gamma_{\sigma(t)})^{\omega_t} + \prod_{t=1}^T \gamma_{\sigma(t)}^{\omega_t}} \middle| p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(T)} \right\}. \quad (22)$$

Example 5. Let $\bar{h}_1 = (0.5|0.5, 0.6|0.5)$ and $\bar{h}_2 = (0.1|0.2, 0.3|0.3, 0.4|0.5)$ be two HPFEs and suppose that the associated aggregated vector is $\omega = (0.55, 0.45)^T$. Based on Definition 3, the score values of $\bar{h}_1$ and $\bar{h}_2$ are $s(\bar{h}_1) = 0.55$ and $s(\bar{h}_2) = 0.31$. Since $s(\bar{h}_1) > s(\bar{h}_2)$, then

$$\bar{h}_{\sigma(1)} = \bar{h}_1 = (0.5|0.5, 0.6|0.5), \ \bar{h}_{\sigma(2)} = \bar{h}_2 = (0.1|0.2, 0.3|0.3, 0.4|0.5).$$

From Eq. (21), the aggregated values by the HPFEOWA operator is

$$\text{HPFEOWA}(\bar{h}_1, \bar{h}_2) = (\omega_1 \cdot_\varepsilon \bar{h}_{\sigma(1)}) \oplus_\varepsilon (\omega_2 \cdot_\varepsilon \bar{h}_{\sigma(2)})$$
$$= \{0.3340|0.1, 0.4023|0.1, 0.4148|0.15, 0.4564|0.25, 0.4781|0.15, 0.5167|0.25\}.$$

On the other hand, from Eq. (22), the aggregated values by the HPFEOWG operator is

$$\text{HPFEOWG}(\bar{h}_1, \bar{h}_2) = (\bar{h}_{\sigma(1)}^{\wedge_\varepsilon \omega_1}) \otimes_\varepsilon (\bar{h}_{\sigma(2)}^{\wedge_\varepsilon \omega_2})$$
$$= \{0.2937|0.1, 0.2859|0.1, 0.4005|0.15, 0.4466|0.15, 0.4530|0.25, 0.5033|0.25\}.$$

In following, let us look at the HPFEOWA and HPFEOWG operators for some special cases of the associated vector ω:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\text{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_{\sigma(1)} = \max\{\bar{h}_i\},$$
$$\text{HPFEOWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n) = \bar{h}_{\sigma(1)} = \max\{\bar{h}_t\}.$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\text{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_{\sigma(T)} = \min\{\bar{h}_t\},$$
$$\text{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_{\sigma(T)} = \min\{\bar{h}_t\}.$$

(3) If $\omega_s = 1,\ w_t = 0,\ s \neq t$, then

$$\bar{h}_{\sigma(T)} \leq \mathrm{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_{\sigma(s)} \leq \bar{h}_{\sigma(1)},$$
$$\bar{h}_{\sigma(T)} \leq \mathrm{HPFEOWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) = \bar{h}_{\sigma(s)} \leq \bar{h}_{\sigma(1)},$$

where $\bar{h}_{\sigma(s)}$ is the sth largest of $\bar{h}_t$ $(t = 1, 2, \ldots, T)$.

(4) If $\omega = (\frac{1}{T}, \frac{1}{T}, \ldots, \frac{1}{T})^T$, then

$$\begin{aligned}
&\mathrm{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \\
&= \bigcup_{\substack{\gamma_{\sigma(i)} \in \bar{h}_{\sigma(i)}, \\ i=1,2,\ldots,T}} \left\{ \frac{\prod_{t=1}^{T}(1+\gamma_{\sigma(t)})^{\frac{1}{T}} - \prod_{t=1}^{T}(1-\gamma_{\sigma(t)})^{\frac{1}{T}}}{\prod_{t=1}^{T}(1+\gamma_{\sigma(t)})^{\frac{1}{T}} + \prod_{t=1}^{T}(1-\gamma_{\sigma(t)})^{\frac{1}{T}}} \Big| p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(T)} \right\} \\
&= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{\prod_{t=1}^{T}(1+\gamma_t)^{\frac{1}{T}} - \prod_{t=1}^{T}(1-\gamma_t)^{\frac{1}{T}}}{\prod_{t=1}^{T}(1+\gamma_t)^{\frac{1}{T}} + \prod_{t=1}^{T}(1-\gamma_t)^{\frac{1}{T}}} \Big| p_1 p_2 \cdots p_T \right\} \\
&= \mathrm{HPFEA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{HPFEOWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \\
&= \bigcup_{\substack{\gamma_{\sigma(i)} \in \bar{h}_{\sigma(i)}, \\ i=1,2,\ldots,T}} \left\{ \frac{2\prod_{t=1}^{T}\gamma_{\sigma(t)}^{\frac{1}{T}}}{\prod_{t=1}^{T}(2-\gamma_{\sigma(t)})^{\frac{1}{T}} + \prod_{t=1}^{T}\gamma_{\sigma(t)}^{\frac{1}{T}}} \Big| p_{\sigma(1)} p_{\sigma(2)} \cdots p_{\sigma(T)} \right\} \\
&= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{2\prod_{t=1}^{T}\gamma_t^{\frac{1}{T}}}{\prod_{t=1}^{T}(2-\gamma_t)^{\frac{1}{T}} + \prod_{t=1}^{T}\gamma_t^{\frac{1}{T}}} \Big| p_1 p_2 \cdots p_T \right\} \\
&= \mathrm{HPFEG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T),
\end{aligned}$$

i.e., the HPFEOWA (resp. HPFEOWG) operator is reduced to HPFEA (resp. HPFEG) operator.

Similar to Theorems 8 and 9, the above ordered weighted operators have the relationship below:

Theorem 10. If $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ are a collection of HPFEs, $\omega = (\omega_1, \omega_2, \ldots, \omega_T)^T$ is associated vector of the aggregation operator such that $\omega_t \in [0, 1]$ and $\sum_{t=1}^{T} \omega_t = 1$, then

(1) HPFEOWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \leq$ HPFOWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T)$;
(2) HPFEOWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) \geq$ HPFOWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T)$.

Theorem 11. If $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ are a collection of HPFEs, $\omega = (\omega_1, \omega_2, \ldots, \omega_T)^T$ is associated vector of the aggregation operator such that $\omega_t \in [0, 1]$ and $\sum_{t=1}^{T} \omega_t = 1$, then
(1) HPFEOWA$((\bar{h}_1)^c, (\bar{h}_2)^c, \ldots, (\bar{h}_T)^c) = ($HPFEOWG$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T))^c$;
(2) HPFEOWG$((\bar{h}_1)^c, (\bar{h}_2)^c, \ldots, (\bar{h}_T)^c) = ($HPFEOWA$(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T))^c$.

Clearly, the fundamental characteristic of the HPFEWA and HPFEWG operators are that they consider the importance of each given HPFE, whereas the fundamental characteristic of the HPFEOWA and HPFEOWG operators are the reordering step, and they weight all the ordered positions of the HPFEs instead of weighing the given HPFEs themselves. By combining the advantages of the HPFEWA (resp. HPFEWG) and HPFEOWA (resp. HPFEOWG) operators, in the following, we develop some hesitant probabilistic fuzzy hybrid aggregation operators that weight both the given HPFEs and their ordered positions.

Let $\bar{h}_t$ $(t = 1, 2, \ldots, T)$ be a collection of HPFEs, $w = (w_1, w_2, \ldots, w_T)^T$ be the weight vector of $\bar{h}_t$, with $w_t \in [0, 1]$ and $\sum_{t=1}^{T} w_t = 1$ and p_t be the probability of γ_t in the HPFE $\bar{h}_t$, then we develop the following two aggregation operators, which are based on the mapping $H_P^T \to H_P$ with an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_T)^T$ such that $\omega_t \in [0, 1]$ and $\sum_{t=1}^{T} \omega_t = 1$:

(1) The hesitant probabilistic fuzzy Einstein hybrid averaging (HPFEHA) operator:

$$\begin{aligned} \text{HPFEHA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\omega_1 \cdot_\varepsilon \dot{h}_{\sigma(1)}) \oplus_\varepsilon (\omega_2 \cdot_\varepsilon \dot{h}_{\sigma(2)}) \oplus_\varepsilon \cdots \oplus_\varepsilon (\omega_T \cdot_\varepsilon \dot{h}_{\sigma(T)}) \\ &= \bigcup_{\substack{\dot{\gamma}_{\sigma(i)} \in \dot{h}_{\sigma(i)}, \\ i=1,2,\ldots,T}} \left\{ \frac{\prod_{t=1}^{T}(1+\dot{\gamma}_{\sigma(t)})^{\omega_t} - \prod_{t=1}^{T}(1-\dot{\gamma}_{\sigma(t)})^{\omega_t}}{\prod_{t=1}^{T}(1+\dot{\gamma}_{\sigma(t)})^{\omega_t} + \prod_{t=1}^{T}(1-\dot{\gamma}_{\sigma(t)})^{\omega_t}} \middle| \dot{p}_{\sigma(1)}\dot{p}_{\sigma(2)} \cdots \dot{p}_{\sigma(T)} \right\}, \end{aligned} \quad (23)$$

where $\dot{h}_{\sigma(t)}$ is the tth largest of the weighted HPFEs $\dot{h}_t = Tw_t \cdot_\varepsilon \bar{h}_t$ $(t = 1, 2, \ldots, T)$, T is the balancing coefficient, and $\dot{p}_{\sigma(t)}$ be the probability of $\dot{\gamma}_{\sigma(t)}$ in the HPFE $\dot{h}_{\sigma(t)}$.

(2) The hesitant probabilistic fuzzy Einstein hybrid geometric (HPFEHG) operator:

$$\begin{aligned}\text{HPFEHG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\ddot{h}_{\sigma(1)}^{\wedge_\varepsilon \omega_1}) \otimes_\varepsilon (\ddot{h}_{\sigma(2)}^{\wedge_\varepsilon \omega_2}) \otimes_\varepsilon \cdots \otimes_\varepsilon (\ddot{h}_{\sigma(T)}^{\wedge_\varepsilon \omega_T}) \\ &= \bigcup_{\substack{\ddot{\gamma}_{\sigma(i)} \in \ddot{h}_{\sigma(i)}, \\ i=1,2,\ldots,T}} \left\{ \frac{2\prod_{t=1}^{T} \ddot{\gamma}_{\sigma(t)}^{\omega_t}}{\prod_{t=1}^{T}(2-\ddot{\gamma}_{\sigma(t)})^{\omega_t} + \prod_{t=1}^{T} \ddot{\gamma}_{\sigma(t)}^{\omega_t}} \Big| \ddot{p}_{\sigma(1)} \ddot{p}_{\sigma(2)} \cdots \ddot{p}_{\sigma(T)} \right\}, \quad (24)\end{aligned}$$

where $\ddot{h}_{\sigma(t)}$ is the tth largest of the weighted HPFEs $\ddot{h}_t = \bar{h}_t^{\wedge_\varepsilon T w_t}$ $(t = 1, 2, \ldots, T)$, T is the balancing coefficient, and $\ddot{p}_{\sigma(t)}$ be the probability of $\ddot{\gamma}_{\sigma(t)}$ in the HPFE $\ddot{h}_{\sigma(t)}$.

Especially, if $w = (\frac{1}{T}, \frac{1}{T}, \ldots, \frac{1}{T})^T$, then $\dot{h}_t = \ddot{h}_t = \bar{h}_t$ $(t = 1, 2, \ldots, T)$, in this case, the HPFEHA (resp. HPFEHG) operator is reduced to the HPFE-OWA (resp. HPFEOWG) operator. If $\omega = (\frac{1}{T}, \frac{1}{T}, \ldots, \frac{1}{T})^T$, then since $\frac{1}{T} \cdot_\varepsilon \dot{h}_t = \frac{1}{T} \cdot_\varepsilon (T w_t \cdot_\varepsilon \bar{h}_t) = \cup_{\gamma_t \in \bar{h}_t} \left\{ \frac{(1+\gamma_t)^{w_t} - (1-\gamma_t)^{w_t}}{(1+\gamma_t)^{w_t} + (1-\gamma_t)^{w_t}} \Big| p_t \right\}$ and $\ddot{h}_t^{\wedge_\varepsilon \frac{1}{T}} = (\bar{h}_t^{\wedge_\varepsilon T w_t})^{\wedge_\varepsilon \frac{1}{T}} = \cup_{\gamma_t \in \bar{h}_t} \left\{ \frac{2\gamma_t^{w_t}}{(2-\gamma_t)^{w_t} + \gamma_t^{w_t}} \Big| p_t \right\}$, we have

$$\begin{aligned}\text{HPFEHA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\frac{1}{T} \cdot_\varepsilon \dot{h}_{\sigma(1)}) \oplus_\varepsilon (\frac{1}{T} \cdot_\varepsilon \dot{h}_{\sigma(2)}) \oplus_\varepsilon \cdots \oplus_\varepsilon (\frac{1}{T} \cdot_\varepsilon \dot{h}_{\sigma(T)}) \\ &= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{\prod_{t=1}^{T}(1+\gamma_t)^{w_t} - \prod_{t=1}^{T}(1-\gamma_t)^{w_t}}{\prod_{t=1}^{T}(1+\gamma_t)^{w_t} + \prod_{t=1}^{T}(1-\gamma_t)^{w_t}} \Big| p_1 p_2 \cdots p_T \right\} \\ &= \text{HPFEWA}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T),\end{aligned}$$

$$\begin{aligned}\text{HPFEHG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T) &= (\ddot{h}_{\sigma(1)}^{\wedge_\varepsilon \frac{1}{T}}) \otimes_\varepsilon (\ddot{h}_{\sigma(2)}^{\wedge_\varepsilon \frac{1}{T}}) \otimes_\varepsilon \cdots \otimes_\varepsilon (\ddot{h}_{\sigma(T)}^{\wedge_\varepsilon \frac{1}{T}}) \\ &= \bigcup_{\gamma_1 \in \bar{h}_1, \gamma_2 \in \bar{h}_2, \ldots, \gamma_T \in \bar{h}_T} \left\{ \frac{2\prod_{t=1}^{T} \gamma_t^{w_t}}{\prod_{t=1}^{T}(2-\gamma_t)^{w_t} + \prod_{t=1}^{T} \gamma_t^{w_t}} \Big| p_1 p_2 \cdots p_T \right\} \\ &= \text{HPFEWG}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_T),\end{aligned}$$

i.e., the HPFEHA (resp. HPFEHG) operator is reduced to the HPFEWA (resp. HPFEWG) operator.

Example 6. Let $\bar{h}_1 = (0.5|0.5, 0.6|0.5)$ and $\bar{h}_2 = (0.1|0.2, 0.3|0.3, 0.5|0.5)$ be two HPFEs. Suppose that the weight vector of them is $w = (0.63, 0.37)^T$, and the aggregation associated vector is $\omega = (0.3, 0.7)^T$. Then

$$\begin{aligned}\dot{h}_1 &= \left(\frac{(1+0.5)^{2\times 0.63} - (1-0.5)^{2\times 0.63}}{(1+0.5)^{2\times 0.63} + (1-0.5)^{2\times 0.63}}|0.5, \frac{(1+0.6)^{2\times 0.63} - (1-0.6)^{2\times 0.63}}{(1+0.6)^{2\times 0.63} + (1-0.6)^{2\times 0.63}}|0.5\right)\\ &= (0.5993|0.5, 0.7031|0.5),\\ \dot{h}_2 &= \left(\frac{(1+0.1)^{2\times 0.37} - (1-0.1)^{2\times 0.37}}{(1+0.1)^{2\times 0.37} + (1-0.1)^{2\times 0.37}}|0.2, \frac{(1+0.3)^{2\times 0.37} - (1-0.3)^{2\times 0.37}}{(1+0.3)^{2\times 0.37} + (1-0.3)^{2\times 0.37}}|0.3,\right.\\ &\quad \left.\frac{(1+0.5)^{2\times 0.37} - (1-0.5)^{2\times 0.37}}{(1+0.5)^{2\times 0.37} + (1-0.5)^{2\times 0.37}}|0.2\right)\\ &= (0.7411|0.2, 0.2251|0.3, 0.3851|0.5)\end{aligned}$$

and $s(\dot{h}_1) = 0.6512$ and $s(\dot{h}_2) = 0.4083$. Since $s(\dot{h}_1) > s(\dot{h}_2)$, we have

$$\begin{aligned}\dot{h}_{\sigma(1)} &= \dot{h}_1 = (0.5993|0.5, 0.7031|0.5),\\ \dot{h}_{\sigma(2)} &= \dot{h}_2 = (0.7411|0.2, 0.2251|0.3, 0.3851|0.5).\end{aligned}$$

From Eq. (23), we have

$$\begin{aligned}&\text{HPFEHA}(\bar{h}_1, \bar{h}_2) = (\omega_1 \cdot_\varepsilon \dot{h}_{\sigma(1)}) \oplus_\varepsilon (\omega_2 \cdot_\varepsilon \dot{h}_{\sigma(2)})\\ &= \bigcup_{\dot{\gamma}_{\sigma(1)}\in\dot{h}_{\sigma(1)}, \dot{\gamma}_{\sigma(2)}\in\dot{h}_{\sigma(2)}} \left\{\frac{\prod_{t=1}^{2}(1+\dot{\gamma}_{\sigma(t)})^{\omega_t} - \prod_{t=1}^{2}(1-\dot{\gamma}_{\sigma(t)})^{\omega_t}}{\prod_{t=1}^{2}(1+\dot{\gamma}_{\sigma(t)})^{\omega_t} + \prod_{t=1}^{2}(1-\dot{\gamma}_{\sigma(t)})^{\omega_t}}\Big|\dot{p}_{\sigma(1)}\dot{p}_{\sigma(2)}\right\}\\ &= \{0.3715|0.15, 0.4175|0.15, 0.4557|0.25, 0.4977|0.25, 0.7037|0.1, 0.7302|0.1\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\ddot{h}_1 &= \left(\frac{2\times 0.5^{2\times 0.63}}{(2-0.5)^{2\times 0.63} + 0.5^{2\times 0.63}}|0.5, \frac{2\times 0.6^{2\times 0.63}}{(2-0.6)^{2\times 0.63} + 0.6^{2\times 0.63}}|0.5\right)\\ &= (0.4007|0.5, 0.5117|0.5),\\ \ddot{h}_2 &= \left(\frac{2\times 0.1^{2\times 0.37}}{(2-0.1)^{2\times 0.37} + 0.1^{2\times 0.37}}|0.2, \frac{2\times 0.3^{2\times 0.37}}{(2-0.3)^{2\times 0.37} + 0.3^{2\times 0.37}}|0.3,\right.\\ &\quad \left.\frac{2\times 0.5^{2\times 0.37}}{(2-0.5)^{2\times 0.37} + 0.5^{2\times 0.37}}|0.5\right)\\ &= (0.2033|0.2, 0.4339|0.3, 0.6145|0.5)\end{aligned}$$

and since $s(\ddot{h}_1) = 0.4562 > 0.4465 = s(\ddot{h}_2)$, we have $\ddot{h}_{\sigma(1)} = \ddot{h}_1$ and $\ddot{h}_{\sigma(2)} = \ddot{h}_2$. From Eq. (24), we have

$$\text{HPFEHG}(\bar{h}_1, \bar{h}_2) = (\ddot{h}_{\sigma(1)}^{\wedge_\varepsilon \omega_1}) \otimes_\varepsilon (\ddot{h}_{\sigma(2)}^{\wedge_\varepsilon \omega_2})$$

$$= \bigcup_{\ddot{\gamma}_{\sigma(1)} \in \ddot{h}_{\sigma(1)}, \ddot{\gamma}_{\sigma(2)} \in \ddot{h}_{\sigma(2)}} \left\{ \frac{2\prod_{t=1}^{2} \gamma_{\sigma(t)}^{\omega_t}}{\prod_{t=1}^{2}(2-\gamma_{\sigma(t)})^{\omega_t} + \prod_{t=1}^{2} \gamma_{\sigma(t)}^{\omega_t}} \middle| \ddot{p}_{\sigma(1)} \ddot{p}_{\sigma(2)} \right\}$$
$$= \{0.2512|0.1, 0.2728|0.1, 0.4237|0.15, 0.4563|0.25, 0.5441|0.15, 0.5825|0.25\}.$$

4 Conclusions

In this thesis, we have been defined some new operation laws of HPFEs such as Einstein sum, Einstein product and Einstein scalar multiplication, and have been developed some new hesitant probabilistic fuzzy Einstein aggregation operators, including the HPFEWA, HPFEWG, HPFEOWA, HPFEOWG, HPFEHA and HPFEHG operators, to accommodate the situations where the given arguments are HPFEs. We have investigated desirable properties of these operators and have given some numerical examples to illustrate the developed operators. Furthermore, we have compared the developed operators with the existing hesitant probabilistic fuzzy aggregation operators proposed by Xu and Zhou.[24]

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