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Some Hesitant Fuzzy Hamacher Power Weighted Aggregation Operators

by

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Some Hesitant Fuzzy Hamacher Power Weighted Aggregation Operators (Hesitant 퍼지 Hamacher 멱 가중 집성 연산자들)

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A Dissertation

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요 약

Hesitant 퍼지집합은 다양한 값의 모호성으로 인해 집합에 대한 원소의 여러 가지 가능한 소속도를 결정하는 것을 허락하는 상황을 모델링하는데 효과적으로 사용된다.

본 논문에서는, hesitant 퍼지 Hamacher 연산규칙에 기초하여, 먼저 hesitant 퍼지 정보를 집성하기 위해 몇 가지 hesitant 퍼지 Hamacher 멱 집성 연산자들을 제안하고, 이러한 제안된 연산자들의 기본적인 특성과 상호간의 관계를 조사한다. 또한 제안된 집성 연산자와 기존 집성 연산자들 간의 상호관계를 논의한다. 마지막으로, 제안된 연산자들에 기초하여, hesitant 퍼지 정보를 갖는 다속성 의사결정 문제를 해결하기 위한 새로운 기법을 제안한다.

1 Introduction

Decision making is generally made to find the most desirable alternative(s) in a given set of alternatives. As the complexity of the socio-economic environment increases, it is difficult to obtain accurate and sufficient data in real decision making. Thus, it is necessary to deal with uncertainties in the actual decision-making process. Thus, many different methodologies and theories are presented, and in particular, fuzzy set (FS) [1] is widely used in many fields in real life [2, 3, 4, 5]. Since then, many extensions of the fuzzy sets such as intuitionistic fuzzy set [6], interval-valued fuzzy set [7], interval-valued intuitionistic fuzzy set [8], hesitant fuzzy set (HFS) [9,10], dual hesitant fuzzy set [11], and generalized hesitant fuzzy set [12] have allowed people to deal with uncertainty and information much more extensively. In particular, as a new development in FS, the concept of HFS has been gaining attention and has recently become a popular subject for research [9, 10, 13, 14, 15, 16].

HFS is an important extension of the FS modelling that modelled the uncertainty caused by a common phenomenon of decision making. Several possible values can be used to indicate the membership degree or an evaluation value under hesitant fuzzy environment. Thus, it is appropriate and convenient to explain the hesitancy experienced by decision-makers during the decision-making process. The hesitant fuzzy aggregation operator is one of the core issues. Many authors [2, 17, 18, 19, 20, 21, 22, 23, 24, 25] developed the hesitant fuzzy aggregation operators and used to fuse the hesitant fuzzy information based on the algebraic product and algebraic sum, or the Einstein product and Einstein sum operational rules of HFEs. Recently, Tan et al. [26] extended the Hamacher t-norm and t-conorm to HFS and proposed a family of hesitant fuzzy Hamacher operators that allow decision makers have more choice in multiple attribute decision making problem. In this paper, motivated power average (PA) operator [27] and power geometric (PG) operator [28], we develop hesitant fuzzy power aggregation operators based on Hamacher t-norm and t-conorm for aggregating hesitant fuzzy information.

In order to do so, this paper is organized as follows. In Section 2, some basic concepts of some power aggregation operators and HFSs are reviewed. Some properties of the Hamacher operational rules on HFEs are investigated. In Section 3, based on the Hamacher operational rules on HFEs, we introduce some hesitant fuzzy Hamacher power weighted aggregation operators for hesitant fuzzy information. Some of their desirable properties are investigated and the relations between the various existing operators are discussed. In Section 4, based on the proposed operators, we develop a technique for hesitant fuzzy multiple attribute decision making.

2 Basic concepts and operations

2.1 Triangular norms and conorms

An important notion in fuzzy set theory is that of triangular norms and conorms which are used to define the generalized intersection and union of fuzzy sets, which is defined as follows:

Definition 1. [29] A triangular norm (t-norm) is a binary operation T on the unit interval [0,1], i.e., a function $T:[0,1]\times[0,1]\to[0,1]$, such that for all $x,y,z\in[0,1]$, the following four axioms are satisfied:

(1) (Boundary condition) $T(1,x)=x$;

(2) (Commutativity) $T(x,y)=T(y,x)$;

(3) (Associativity) $T(x,T(y,z))=T(T(x,y),z)$;

(4) (Monotonicity) $T(x_1,y_1)\le T(x_2,y_2)$ if $x_1\le x_2$ and $y_1\le y_2$.

The corresponding triangular conorm (t-conorm) of T (or the dual of T) is the function $S:[0,1]\times[0,1]\to[0,1]$ defined by $S(x,y)=1-T(1-x,1-y)$ for each $x,y\in[0,1]$.

For many t-norms and t-conorms, there are basic t-norms and t-conorms, namely, minimum T_M and maximum S_M, algebraic product T_A and algebraic sum S_A, Einstein product T_E and Einstein sum S_E, bounded difference T_B and bounded sum S_B, and drastic product T_D and drastic sum S_D, given respectively as follows:

$$T_M(x,y)=\min(x,y),\ S_M(x,y)=\max(x,y);$$

$$T_A(x,y)=xy,\ S_A(x,y)=x+y-xy;$$

$$T_E(x,y)=\frac{xy}{1+(1-x)(1-y)},\ S_E(x,y)=\frac{x+y}{1+xy};$$

$$T_B(x,y)=\max(0,x+y-1),\ S_B(x,y)=\min(1,x+y);$$

$$T_D(x,y)=\begin{cases}0, & \text{if}(x,y)\in[0,1)^2\\ \min(x,y), & \text{otherwise}\end{cases},\ S_D(x,y)\begin{cases}1, & \text{if}(x,y)\in(0,1]^2\\ \max(x,y), & \text{otherwise}\end{cases}.$$

These t-norms and t-conorms are ordered as follows:

$$T_D\le T_B\le T_E\le T_A\le T_M\ \text{, and}\ S_M\le S_A\le S_E\le S_B\le S_D. \tag{1}$$

From (1), since the drastic product T_D and minimum T_M are the smallest and the largest t-norms, respectively, it can be seen that $T_D\le T\le T_M$ for any t-

norm T. Whereas the algebraic T_A and the Einstein product T_E are two prototypical examples of the class of strict Archimedean t-norms, which are continuous, strictly monotone, and have the Archimedean property [29].

Hamacher [30] proposed a more generalized t-norm and t-conorm, called the Hamacher t-norm and t-conorm, which are defined as follows:

$$T_H^{\zeta}(x,y)=\frac{xy}{\zeta+(1-\zeta)(x+y-xy)},\ S_H^{\zeta}(x,y)=\frac{x+y-xy-(1-\zeta)xy}{\zeta+(1-\zeta)xy},\zeta>0. \quad (2)$$

Especially, when $\zeta=1$, then the Hamacher t-norm and t-conorm reduce to the algebraic t-norm T_A and t-conorm S_A, respectively; when $\zeta=2$, then the Hamacher t-norm and t-conorm reduce to the Einstein t-norm T_E and t-conorm S_E, respectively.

2.2 Power aggregation operators

The power average (PA) operator was originally introduced by Yager [27]. It used a nonlinear weighted average aggregation tool to aid and provide more versatility in the data aggregation process. In the following, we review some proposed power aggregation operators.

(1) The power average (PA) operator [27] is a mapping $\mathrm{PA}: R^{\mathrm{n}} \rightarrow R$, that is given by the following formula:

$$\mathrm{PA}(a_1,a_{2,}\cdots,a_n)=\frac{\sum_{i=1}^{n}(1+T(a_i))a_i}{\sum_{i=1}^{n}(1+T(a_i))}, \quad (3)$$

where

$$T(a_i)=\sum_{j=1,j\neq i}^{n}\mathrm{Sup}(a_i,a_j) \quad (4)$$

and $\mathrm{Sup}(a,b)$ is the support for a from b, which satisfies the following three properties:

(i) $\mathrm{Sup}(a,b)\in[0,1]$;

(ii) $\mathrm{Sup}(a,b)=\mathrm{Sup}(b,a)$;

(iii) $\mathrm{Sup}(a,b)\geq\mathrm{Sup}(x,y)$ if $|a-b|<|x-y|$.

(2) The power geometric (PG) operator [28] is a mapping $\mathrm{PG}: R^{\mathrm{n}} \rightarrow R$, which is given by the following formula:

$$\mathrm{PG}(a_1,a_{2,}\cdots,a_n)=\prod_{i=1}^{n}a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}}, \quad (5)$$

where $T(a_i)$ satisfies the condition (4).

(3) The power ordered weighted average (POWA) operator [27] is a mapping $\mathrm{POWA}: R^{\mathrm{n}} \rightarrow R$, which is given by the following formula:

$$\mathrm{POWA}(a_1, a_2, \cdots, a_n) = \sum_{i=1}^{n} u_i a_{\sigma(i)} \ , \qquad (6)$$

where

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right) \ , \ R_i = \sum_{j=1}^{i} V_{\sigma(j)} \ , TV = \sum_{i=1}^{n} V_{\sigma(i)} \ ,$$

$$V_{\sigma(i)} = 1 + T(a_{\sigma(i)}), T(a_{\sigma(i)}) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(a_{\sigma(i)}, a_{\sigma(j)}) \qquad (7)$$

where $a_{\sigma(i)}$ is the ith largest argument among of all the arguments a_i ($i = 1,2,\cdots,n$), $T(a_{\sigma(i)})$ denotes the support of ith largest argument by all of the other arguments, $\mathrm{Sup}(a_{\sigma(i)}, a_{\sigma(j)})$ indicates the support of the ith largest argument for the jth largest argument, and $g: [0,1] \rightarrow [0,1]$ is a basic unit-interval monotone (BUM) function having the following properties:
(1) $g(0) = 0$; (2) $g(1) = 1$; and (3) $g(x) \geq g(y)$ if $x > y$.

(4) The power ordered weighted geometric (POWG) operator [28] is a mapping $\mathrm{POWG}: R^{\mathrm{n}} \rightarrow R$, which is given by the following formula:

$$\mathrm{POWG}(a_1, a_2, \cdots, a_n) = \prod_{i=1}^{n} a_{\sigma(i)}^{u_i}, \qquad (8)$$

where u_i is a collection of weights satisfying the condition (7).

2.3 Hesitant fuzzy sets and hesitant fuzzy elements

In the following, some basic concepts of hesitant fuzzy set and hesitant fuzzy element are briefly reviewed [9, 10, 13].

Definition 2. [9, 10] Let X be a fixed set, a hesitant fuzzy set (HFS) on X is defined in terms of function h that returns a subset of [0,1] when applied to X. The HFS can be represented as the following mathematical symbol:

$$E = \{\langle x, h_E(x)\rangle | x \in X\}, \qquad (9)$$

where $h_E(x)$ is a set of values in [0,1] that denote the possible membership degrees of the element $x \in X$ to the set E. For convenience, we refer to $h = h_E(x)$ as a hesitant fuzzy element (HFE) and to H the set of all HFEs.

Given three HFEs h, h_1 and h_2, Torra and Narukawa [9,10] defined the following HFE operations:

(1) $h^c = \cup_{\gamma\in h} \{1-\gamma\}$;

(2) $h_1 \cup h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2} \{\gamma_1 \vee \gamma_2\}$;

(3) $h_1 \cap h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2} \{\gamma_1 \wedge \gamma_2\}$.

For the aggregation of hesitant fuzzy information, Xia and Xu [13] defined the following new operations:

(1) $h^\lambda = \cup_{\gamma\in h} \{\gamma^\lambda\}$, $\lambda > 0$;

(2) $\lambda h = \cup_{\gamma\in h} \{1-(1-\gamma)^\lambda\}$, $\lambda > 0$;

(3) $h_1 \oplus h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}$;

(4) $h_1 \otimes h_2 = \cup_{\gamma_1\in h_1,\gamma_2\in h_2} \{\gamma_1\gamma_2\}$.

Xia and Xu [13] also defined the following comparison rules for HFEs:

Definition 3. [13] For a HFE h, $s(h) = \frac{\sum_{\gamma\in h}\gamma}{l(h)}$ is called the score function of h, where $l(h)$ is the number of elements in h. For two HFEs h_1 and h_2,

● if $s(h_1) > s(h_2)$, then h_1 is superior to h_2, denoted by $h_1 > h_2$;

● if $s(h_1) = s(h_2)$, then h_1 is indifferent to h_2, denoted by $h_1 = h_2$.

Let h_1 and h_2 be two HFEs. In the most case, $l(h_1) \neq l(h_2)$; for convenience, let $l = \max\{l(h_1), l(h_2)\}$. To compare h_1 and h_2, Xu and Xia [31] extended the shorter HFE until the length of both HFEs was the same. The simplest way to extend the shorter HFE is to add the same value repeatedly. In fact, we can extend the shorter ones by adding any values in them. The selection of these values mainly depends on the decision makers risk preferences. Optimists anticipate desirable outcomes and may add the maximum value, while pessimists expect unfavorable outcomes and may add the minimum value [31]. In this paper, we assume that the decision makers are all pessimistic (other situation can also be studied similarly).

Xu and Xia [31] proposed a variety of distance measures for HFEs, including the hesitant normalized Hamming distance, which is defined as follows:

$$d(h_1, h_2) = \frac{1}{l}\sum_{i=1}^{l}\left|{h_1}^{\sigma(i)} - {h_2}^{\sigma(i)}\right| , \quad (10)$$

where ${h_1}^{\sigma(i)}$ and ${h_2}^{\sigma(i)}$ are the largest values in h_1 and h_2, respectively.

Intrinsically, the addition and multiplication operators proposed by Xia and Xu [13] are algebraic sum and algebraic product operational rules on HFEs, respectively, a special pair of dual t-norm and t-conorm. Recently, Tan et al.

[26] extended these operations to obtain more general operations on HFEs by means of the Hamacher t-norm and Hamacher t-conorm as follows:

Definition 4. For any given three HFEs h, h_1, h_2, and $\zeta > 0$, the Hamacher operations on HFEs are defined as follows:

(1) $h_1 \oplus_H h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\zeta)\gamma_1\gamma_2}{1-(1-\zeta)\gamma_1\gamma_2} \right\}$;

(2) $h_1 \otimes_H h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1\gamma_2}{\zeta+(1-\zeta)(\gamma_1+\gamma_2-\gamma_1\gamma_2)} \right\}$;

(3) $\lambda \cdot_H h = \cup_{\gamma \in h} \left\{ \frac{(1+(\zeta-1)\gamma)^\lambda-(1-\gamma)^\lambda}{(1+(\zeta-1)\gamma)^\lambda+(\zeta-1)(1-\gamma_i)^\lambda} \right\}$, $\lambda > 0$;

(4) $h^{\wedge_H \lambda} = \cup_{\gamma \in h} \left\{ \frac{\zeta\gamma^\lambda}{(1+(\zeta-1)(1-\gamma))^\lambda+(\zeta-1)\gamma^\lambda} \right\}$, $\lambda > 0$.

Especially, if $\zeta = 1$, then these operations on HFEs reduce to those proposed by Xia and Xu [13]; if $\zeta = 2$, then these operations on HFEs reduce to the following:

(1) $h_1 \oplus_\varepsilon h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1+\gamma_2}{1-\gamma_1\gamma_2} \right\}$;

(2) $h_1 \otimes_\varepsilon h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1\gamma_2}{1+(1-\gamma_1)(1-\gamma_2)} \right\}$;

(3) $\lambda \cdot_\varepsilon h = \cup_{\gamma \in h} \left\{ \frac{(1+\gamma)^\lambda-(1-\gamma)^\lambda}{(1+\gamma)^\lambda+(1-\gamma_i)^\lambda} \right\}$, $\lambda > 0$;

(4) $h^{\wedge_H \lambda} = \cup_{\gamma \in h} \left\{ \frac{2\gamma^\lambda}{(2-\gamma)^\lambda+\gamma^\lambda} \right\}$, $\lambda > 0$,

which are defined as Einstein operations on HFEs by Yu [21].

Theorem 1. [26] For three HFEs h, h_1, h_2, and $\lambda > 0$, the following properties hold:

(1) ${h_1}^C \oplus_H {h_2}^C = (h_1 \otimes_H h_2)^C$;

(2) ${h_1}^C \otimes_H {h_2}^C = (h_1 \oplus_H h_2)^C$;

(3) $\lambda \cdot_H (h^C) = (h^{\wedge_H \lambda})^C$;

(4) $(h^C)^{\wedge_H \lambda} = (\lambda \cdot_H h)^C$.

Theorem 2. Let h, h_1 and h_2 be three HFEs, $\lambda > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$, then

(1) $h_1 \oplus_H h_2 = h_2 \oplus_H h_1$;

(2) $h \oplus_H (h_1 \oplus_H h_2) = (h \oplus_H h_1) \oplus_H h_2$;

(3) $\lambda \cdot_H (h_1 \oplus_H h_2) = (\lambda \cdot_H h_1) \oplus_H (\lambda \cdot_H h_2)$;

(4) $\lambda_1 \cdot_\varepsilon (\lambda_2 \cdot_\varepsilon \bar{h}) = (\lambda_1\lambda_2) \cdot_\varepsilon \bar{h}$;

(5) $h_1 \otimes_H h_2 = h_2 \otimes_H h_1$;

(6) $h \otimes_H (h_1 \otimes_H h_2) = (h \otimes_H h_1) \otimes_H h_2$;

(7) $(h_1 \otimes_H h_2)^{\wedge_H \lambda} = h_1^{\wedge_H \lambda} \otimes_H h_2^{\wedge_H \lambda}$;

(8) $(h^{\wedge_H \lambda_1})^{\wedge_H \lambda_2} = h^{\wedge_H (\lambda_1\lambda_2)}$.

Proof. Since (1), (2), (5) and (6) are trivial, we prove (3), (4), (7) and (8).

(3) Since $h_1 \oplus_H h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\zeta)\gamma_1\gamma_2}{1-(1-\zeta)\gamma_1\gamma_2} \right\}$, by the operational law (3) in Definition 4, we have

$\lambda \cdot_H (h_1 \oplus_H h_2)$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\left(1+(\zeta-1)\frac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\zeta)\gamma_1\gamma_2}{1-(1-\zeta)\gamma_1\gamma_2}\right)^\lambda - \left(1-\frac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\zeta)\gamma_1\gamma_2}{1-(1-\zeta)\gamma_1\gamma_2}\right)^\lambda}{\left(1+(\zeta-1)\frac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\zeta)\gamma_1\gamma_2}{1-(1-\zeta)\gamma_1\gamma_2}\right)^\lambda + (\zeta-1)\left(1-\frac{\gamma_1+\gamma_2-\gamma_1\gamma_2-(1-\zeta)\gamma_1\gamma_2}{1-(1-\zeta)\gamma_1\gamma_2}\right)^\lambda} \right\}$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{((1+(\zeta-1)\gamma_1)(1+(\zeta-1)\gamma_2))^\lambda - ((1-\gamma_1)(1-\gamma_2))^\lambda}{((1+(\zeta-1)\gamma_1)(1+(\zeta-1)\gamma_2))^\lambda + (\zeta-1)((1-\gamma_1)(1-\gamma_2))^\lambda} \right\}.$$

Since $\lambda \cdot_H h_1 = \cup_{\gamma_1 \in h_1} \left\{ \frac{(1+(\zeta-1)\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+(\zeta-1)\gamma_1)^\lambda+(\zeta-1)(1-\gamma_1)^\lambda} \right\}$ and $\lambda \cdot_H h_2 = \cup_{\gamma_2 \in h_2} \left\{ \frac{(1+(\zeta-1)\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+(\zeta-1)\gamma_2)^\lambda+(\zeta-1)(1-\gamma_2)^\lambda} \right\}$, we have

$(\lambda \cdot_H h_1) \oplus_H (\lambda \cdot_H h_2)$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\left[\begin{array}{c} \frac{(1+(\zeta-1)\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+(\zeta-1)\gamma_1)^\lambda+(\zeta-1)(1-\gamma_1)^\lambda} + \frac{(1+(\zeta-1)\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+(\zeta-1)\gamma_2)^\lambda+(\zeta-1)(1-\gamma_2)^\lambda} \\ - \frac{(1+(\zeta-1)\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+(\zeta-1)\gamma_1)^\lambda+(\zeta-1)(1-\gamma_1)^\lambda} \times \frac{(1+(\zeta-1)\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+(\zeta-1)\gamma_2)^\lambda+(\zeta-1)(1-\gamma_2)^\lambda} \\ -(1-\zeta)\frac{(1+(\zeta-1)\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+(\zeta-1)\gamma_1)^\lambda+(\zeta-1)(1-\gamma_1)^\lambda} \cdot \frac{(1+(\zeta-1)\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+(\zeta-1)\gamma_2)^\lambda+(\zeta-1)(1-\gamma_2)^\lambda} \end{array}\right]}{1-(1-\zeta)\frac{(1+(\zeta-1)\gamma_1)^\lambda-(1-\gamma_1)^\lambda}{(1+(\zeta-1)\gamma_1)^\lambda+(\zeta-1)(1-\gamma_1)^\lambda} \cdot \frac{(1+(\zeta-1)\gamma_2)^\lambda-(1-\gamma_2)^\lambda}{(1+(\zeta-1)\gamma_2)^\lambda+(\zeta-1)(1-\gamma_2)^\lambda}} \right\}$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{((1+(\zeta-1)\gamma_1)(1+(\zeta-1)\gamma_2))^{\lambda} - \left((1-\gamma_1)(1-\gamma_2)\right)^{\lambda}}{((1+(\zeta-1)\gamma_1)(1+(\zeta-1)\gamma_2))^{\lambda} + (\zeta-1)\left((1-\gamma_1)(1-\gamma_2)\right)^{\lambda}} \right\}.$$

Hence $\lambda \cdot_H (h_1 \oplus_H h_2) = (\lambda \cdot_H h_1) \oplus_H (\lambda \cdot_H h_2)$.

(4) Since $\lambda_2 \cdot_H h = \cup_{r \in h} \left\{ \frac{(1+(\zeta-1)\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+(\zeta-1)\gamma)^{\lambda_2} + (\zeta-1)(1-\gamma_i)^{\lambda_2}} \right\}$, then we have

$\lambda_1 \cdot_H (\lambda_2 \cdot_H h)$

$$= \cup_{r \in h} \left\{ \frac{\left(1+(\zeta-1)\frac{(1+(\zeta-1)\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+(\zeta-1)\gamma)^{\lambda_2} + (\zeta-1)(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1} - \left(1-\frac{(1+(\zeta-1)\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+(\zeta-1)\gamma)^{\lambda_2} + (\zeta-1)(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1}}{\left(1+(\zeta-1)\frac{(1+(\zeta-1)\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+(\zeta-1)\gamma)^{\lambda_2} + (\zeta-1)(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1} + (\zeta-1)\left(1-\frac{(1+(\zeta-1)\gamma)^{\lambda_2} - (1-\gamma)^{\lambda_2}}{(1+(\zeta-1)\gamma)^{\lambda_2} + (\zeta-1)(1-\gamma_i)^{\lambda_2}}\right)^{\lambda_1}} \right\}$$

$$= \cup_{r \in h} \left\{ \frac{(1+(\zeta-1)\gamma)^{(\lambda_1\lambda_2)} - (1-\gamma)^{(\lambda_1\lambda_2)}}{(1+(\zeta-1)\gamma)^{(\lambda_1\lambda_2)} + (\zeta-1)(1-\gamma_i)^{(\lambda_1\lambda_2)}} \right\}$$

$= (\lambda_1\lambda_2) \cdot_H h$.

(7) Since $h_1 \otimes_H h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\gamma_1\gamma_2}{\zeta+(1-\zeta)(\gamma_1+\gamma_2-\gamma_1\gamma_2)} \right\}$, by the operational law (4) in Definition 4, we have

$$(h_1 \otimes_H h_2)^{\wedge_H \lambda} = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\zeta\left(\frac{\gamma_1\gamma_2}{\zeta+(1-\zeta)(\gamma_1+\gamma_2-\gamma_1\gamma_2)}\right)^{\lambda}}{\left(1+(\zeta-1)\left(1-\frac{\gamma_1\gamma_2}{\zeta+(1-\zeta)(\gamma_1+\gamma_2-\gamma_1\gamma_2)}\right)\right)^{\lambda} + (\zeta-1)\left(\frac{\gamma_1\gamma_2}{\zeta+(1-\zeta)(\gamma_1+\gamma_2-\gamma_1\gamma_2)}\right)^{\lambda}} \right\}$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \left\{ \frac{\zeta\gamma_1^{\lambda}\gamma_2^{\lambda}}{\left((1+(\zeta-1)(1-\gamma_1))(1+(\zeta-1)(1-\gamma_2))\right)^{\lambda} + (\zeta-1)\gamma_1^{\lambda}\gamma_2^{\lambda}} \right\}.$$

Since $h_1^{\wedge_H \lambda} = \cup_{\gamma_1 \in h_1} \left\{ \frac{\zeta\gamma_1^{\lambda}}{(1+(\zeta-1)(1-\gamma_1))^{\lambda} + (\zeta-1)\gamma_1^{\lambda}} \right\}$ and

$h_2^{\wedge_H \lambda} = \cup_{\gamma_2 \in h_2} \left\{ \frac{\zeta\gamma_2^{\lambda}}{(1+(\zeta-1)(1-\gamma_2))^{\lambda} + (\zeta-1)\gamma_2^{\lambda}} \right\}$, we have

$h_1{}^{\wedge_H{}^{\lambda}} \otimes_H h_2{}^{\wedge_H{}^{\lambda}}$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\frac{\frac{\zeta\gamma_1{}^{\lambda}}{(1+(\zeta-1)(1-\gamma_1))^{\lambda}+(\zeta-1)\gamma_1{}^{\lambda}}\cdot\frac{\zeta\gamma_2{}^{\lambda}}{(1+(\zeta-1)(1-\gamma_2))^{\lambda}+(\zeta-1)\gamma_2{}^{\lambda}}}{\left[\zeta+(1-\zeta)\left(\begin{array}{c}\frac{\zeta\gamma_1{}^{\lambda}}{(1+(\zeta-1)(1-\gamma_1))^{\lambda}+(\zeta-1)\gamma_1{}^{\lambda}}+\frac{\zeta\gamma_2{}^{\lambda}}{(1+(\zeta-1)(1-\gamma_2))^{\lambda}+(\zeta-1)\gamma_2{}^{\lambda}}\\-\frac{\zeta\gamma_1{}^{\lambda}}{(1+(\zeta-1)(1-\gamma_1))^{\lambda}+(\zeta-1)\gamma_1{}^{\lambda}}\cdot\frac{\zeta\gamma_2{}^{\lambda}}{(1+(\zeta-1)(1-\gamma_2))^{\lambda}+(\zeta-1)\gamma_2{}^{\lambda}}\end{array}\right)\right]}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2}\left\{\frac{\zeta\gamma_1{}^{\lambda}\gamma_2{}^{\lambda}}{\left((1+(\zeta-1)(1-\gamma_1))(1+(\zeta-1)(1-\gamma_2))\right)^{\lambda}+(\zeta-1)\gamma_1{}^{\lambda}\gamma_2{}^{\lambda}}\right\}.$$

Hence $(h_1 \otimes_H h_2)^{\wedge_H{}^{\lambda}} = h_1{}^{\wedge_H{}^{\lambda}} \otimes_H h_2{}^{\wedge_H{}^{\lambda}}$.

(8) Since $h^{\wedge_H{}^{\lambda_1}} = \cup_{r\in h}\left\{\frac{\zeta\gamma^{\lambda_1}}{(1+(\zeta-1)(1-\gamma))^{\lambda_1}+(\zeta-1)\gamma^{\lambda_1}}\right\}$, we have

$(h^{\wedge_H{}^{\lambda_1}})^{\wedge_H{}^{\lambda_2}}$

$$= \cup_{r\in h}\left\{\frac{\zeta\left(\frac{\zeta\gamma^{\lambda_1}}{(1+(\zeta-1)(1-\gamma))^{\lambda_1}+(\zeta-1)\gamma^{\lambda_1}}\right)^{\lambda_2}}{\begin{array}{c}\left(1+(\zeta-1)\left(1-\frac{\zeta\gamma^{\lambda_1}}{(1+(\zeta-1)(1-\gamma))^{\lambda_1}+(\zeta-1)\gamma^{\lambda_1}}\right)\right)^{\lambda_2}\\+(\zeta-1)\left(\frac{\zeta\gamma^{\lambda_1}}{(1+(\zeta-1)(1-\gamma))^{\lambda_1}+(\zeta-1)\gamma^{\lambda_1}}\right)^{\lambda_2}\end{array}}\right\}$$

$$= \cup_{r\in h}\left\{\frac{\zeta\gamma^{(\lambda_1\lambda_2)}}{(1+(\zeta-1)(1-\gamma))^{(\lambda_1\lambda_2)}+(\zeta-1)\gamma^{(\lambda_1\lambda_2)}}\right\}$$

$= h^{\wedge_H{}^{(\lambda_1\lambda_2)}}$.

3. Hesitant fuzzy Hamacher power weighted aggregation operator

The above-mentioned power aggregation operators have usually been used in situations where the input arguments are the exact values. In this section, we shall extend the power aggregation operators to accommodate the situations where the input arguments are hesitant fuzzy information based on the Hamacher operations.

3.1 Hesitant fuzzy Hamacher power weighted average operator and hesitant fuzzy Hamacher power weighted geometric operators

On the basis of the hesitant fuzzy Hamacher weighted average (HFHWA) operator [12] and PA operator, we firstly give the definition of the hesitant fuzzy Hamacher power weighted average (HFHPWA) operator as follows.

Definition 5. Let $h_i(i = 1,2,\cdots,n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. A hesitant fuzzy Hamacher power weighted average (HFHPWA) operator is a function $H^n \to H$ such that

$$\mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_n) = \oplus_{H_{i=1}^{n}} \left(\frac{w_i(1 + T(h_i)) \cdot_H h_i}{\sum_{i=1}^{n} w_i(1 + T(h_i))} \right), \tag{11}$$

where parameter $\zeta > 0$, $T(h_i) = \sum_{j=1, j\neq i}^{n} w_j \mathrm{Sup}(h_i, h_j)$ and $\mathrm{Sup}(h_i, h_j)$ is the support for h_i from h_j, satisfying the following conditions:

(1) $\mathrm{Sup}(h_i, h_j) \in [0,1]$;
(2) $\mathrm{Sup}(h_i, h_j) = \mathrm{Sup}(h_j, h_i)$;
(3) $\mathrm{Sup}(h_i, h_j) \geq \mathrm{Sup}(h_s, h_t)$ if $d(h_i, h_j) \leq d(h_s, h_t)$, where d is the hesitant normalized Hamming distance measure between two HFEs given in Eq. (10).

Theorem 3. Let $h_i(i = 1,2,\cdots,n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots w_n)^T$ be the weight vector of $h_i(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by HFHPWA operator is also a HFE, and

$\mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_n)$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}} - \prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}\right\}. \quad (12)$$

Proof. Eq. (12) can be proved by mathematical induction on n as follows.
For $n = 1$, the result of Eq. (12) is clear.
Suppose that Eq. (12) holds for $n = k$, that is

$\mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_k)$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k}\left\{\frac{\prod_{i=1}^{k}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k}w_i(1+T(h_i))}} - \prod_{i=1}^{k}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k}w_i(1+T(h_i))}}}{\prod_{i=1}^{k}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k}w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{k}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k}w_i(1+T(h_i))}}}\right\}.$$

Then, when $n = k + 1$, by Definitions 4 and 5, we have

$\mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_{k+1})$

$$= \oplus_{H_{i=1}^{k}}\left(\frac{w_i(1+T(h_i))\cdot_H h_i}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}\right) \oplus_H \left(\frac{w_{k+1}(1+T(h_{k+1}))\cdot_H h_{k+1}}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}\right)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k}\left\{\frac{\prod_{i=1}^{k}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}} - \prod_{i=1}^{k}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}}{\prod_{i=1}^{k}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{k}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}}\right\}$$

$$\oplus_H \cup_{\gamma_{k+1}\in h_{k+1}}\left\{\frac{(1+(\zeta-1)\gamma_{k+1})^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}} - (1-\gamma_{k+1})^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}}{(1+(\zeta-1)\gamma_{k+1})^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}} + (\zeta-1)(1-\gamma_{k+1})^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}}\right\}.$$

Let $a_1 = \prod_{i=1}^{k}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}$, $b_1 = \prod_{i=1}^{k}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}$,

$a_2 = (1+(\zeta-1)\gamma_{k+1})^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}$ and $b_2 = (1-\gamma_{k+1})^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1}w_i(1+T(h_i))}}$, then

$\mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_{k+1})$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_k \in h_k} \left\{ \frac{a_1 - b_1}{a_1 + (\zeta - 1) b_1} \right\} \oplus_H \cup_{\gamma_{k+1} \in h_{k+1}} \left\{ \frac{a_2 - b_2}{a_2 + (\zeta - 1) b_2} \right\}$$

$$= \cup_{\gamma_1 \in h_1, \cdots, \gamma_k \in h_k, \gamma_{k+1} \in h_{k+1}} \left\{ \frac{\left[\begin{array}{c} \frac{a_1 - b_1}{a_1 + (\zeta-1) b_1} + \frac{a_2 - b_2}{a_2 + (\zeta-1) b_2} - \frac{a_1 - b_1}{a_1 + (\zeta-1) b_1} \cdot \frac{a_2 - b_2}{a_2 + (\zeta-1) b_2} \\ -(1-\zeta) \frac{a_1 - b_1}{a_1 + (\zeta-1) b_1} \cdot \frac{a_2 - b_2}{a_2 + (\zeta-1) b_2} \end{array} \right]}{1 - (1-\zeta) \frac{a_1 - b_1}{a_1 + (\zeta-1) b_1} \cdot \frac{a_2 - b_2}{a_2 + (\zeta-1) b_2}} \right\}$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_{k+1} \in h_{k+1}} \left\{ \frac{a_1 a_2 - b_1 b_2}{a_1 a_2 + (\zeta - 1) b_1 b_2} \right\}$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_{k+1} \in h_{k+1}} \left\{ \frac{\begin{array}{c} \prod_{i=1}^{k+1} (1 + (\zeta - 1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} \\ - \prod_{i=1}^{k+1} (1 - \gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} \end{array}}{\begin{array}{c} \prod_{i=1}^{k+1} (1 + (\zeta - 1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} \\ +(\zeta - 1) \prod_{i=1}^{k+1} (1 - \gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} \end{array}} \right\},$$

i.e., Eq. (12) holds for $n = k + 1$. Thus Eq. (12) holds for all n. □

Remark 1. (1) If $\mathrm{Sup}(h_i, h_j) = k$, for all $i \neq j$, then

$$\mathrm{HFHPWA}_\zeta(h_1, h_2, \cdots, h_n) = \oplus_{H_{i=1}^n} (w_i \cdot_H h_i)$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\prod_{i=1}^n (1 + (\zeta - 1)\gamma_i)^{w_i} - \prod_{i=1}^n (1 - \gamma_i)^{w_i}}{\prod_{i=1}^n (1 + (\zeta - 1)\gamma_i)^{w_i} + (\zeta - 1) \prod_{i=1}^n (1 - \gamma_i)^{w_i}} \right\}, \quad (13)$$

which indicates that when all supports are the same, the HFHPWA operator reduces to the hesitant fuzzy Hamacher weighted average (HFHWA) operator [12]. Especially, if $\mathrm{Sup}(h_i, h_j) = 0$ for all $i \neq j$, i.e., all the supports are zero, then there is no support in the aggregation process, and in this case, we have $T(h_i) = 0$, $i = 1,2, \cdots, n$, then

$$\frac{w_i(1 + T(h_i))}{\sum_{i=1}^n w_i(1 + T(h_i))} = w_i, \; i = 1,2, \cdots, n$$

and thus, it is clear that the HFHPWA operator reduces to the HFHWA operator [12].

(2) For the HFHPWA operator, if $\zeta = 1$, then the HFHPWA operator reduces to the following:

$$\mathrm{HFPWA}_1(h_1, h_2, \cdots, h_n) = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ 1 - \prod_{i=1}^{n} (1 - \gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right\} \quad (14)$$

which is called the hesitant fuzzy power weighted average (HFPWA) operator and if ζ= 2, then the HFHPWA operator reduces to the hesitant fuzzy Einstein power weighted average (HFEPWA) operator [40]:

$$\mathrm{HFEPWA}_2(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + \prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}} \right\}. \quad (15)$$

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the HFHPWA operator reduces to the hesitant fuzzy Hamacher power average (HFHPA) operator:

$$\mathrm{HFHPA}_\zeta(h_1, h_2, \cdots, h_n) = \oplus_{H_{i=1}^{n}} \left(\frac{(1+T'(h_i)) \cdot_H h_i}{\sum_{i=1}^{n}(1+T'(h_i))} \right)$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}} - \prod_{i=1}^{n}(1-\gamma_i)^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}} + (\zeta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}} \right\}, (16)$$

where $T'(h_i) = \frac{1}{n}\sum_{j=1, j\neq i}^{n} \mathrm{Sup}(h_i, h_j)$.

In order to analyze the relationship between the HFHPWA operator and the HFPWA operator, we introduce the following lemma.

Lemma 1. [32,33] Let $x_i > 0$, $w_i > 0$, $i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} w_i = 1$, then $\prod_{i=1}^{n} {x_i}^{w_i} \leq \sum_{i=1}^{n} w_i x_i$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Theorem 4. Let $h_i (i = 1, 2, \cdots, n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of h_i such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then

$$\mathrm{HFHPWA}_\zeta(h_1, h_2, \cdots, h_n) \leq \mathrm{HFPWA}(h_1, h_2, \cdots, h_n).$$

Proof. For any $\gamma_i \in h_i (i = 1, 2, \cdots, n)$, by Lemma 1, we have

$$\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}$$

$$\leq \sum_{i=1}^{n} \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}(1+(\zeta-1)\gamma_i) + (\zeta-1)\sum_{i=1}^{n} \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}(1-\gamma_i) = \zeta.$$

Then,

$$\frac{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}$$

$$=1-\frac{\zeta\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}$$

$$\leq 1-\frac{\zeta\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}{\zeta}=1-\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}},$$

which implies that $\oplus_{H_{i=1}^{n}}\left(\frac{w_i(1+T(h_i))\cdot_H h_i}{\sum_{i=1}^{n}w_i(1+T(h_i))}\right)\leq\oplus_{i=1}^{n}\left(\frac{w_i(1+T(h_i))h_i}{\sum_{i=1}^{n}w_i(1+T(h_i))}\right)$. Thus we obtain $\text{HFHPWA}_\zeta(h_1,h_2,\cdots,h_n)\leq\text{HFPWA}(h_1,h_2,\cdots,h_n)$. □

Theorem 4 shows that the values aggregated by the HFHPWA operator are not larger than those obtained by the HFPWA operator. That is to say, the HFHPWA operator reflects the decision maker's pessimistic attitude than the HFPWA operator in aggregation process. Furthermore, based on Theorem 3, we have the properties of the HFHPWA operator as follows.

Theorem5. Let $h_i(i=1,2,\cdots,n)$ be a collection of HFEs and $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of h_i such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then we have the followings:

(1) Boundedness: If $h^-=\min\{\gamma_i|\gamma_i\in h_i\}$ and $h^+=\max\{\gamma_i|\gamma_i\in h_i\}$, then

$$h^-\leq\text{HFHPWA}_\zeta(h_1,h_2,\cdots,h_n)\leq h^+.$$

(2) Monotonicity: Let $h_i'(i=1,2,\cdots,n)$ be a collection of HFEs, if $w=(w_1,w_2,\cdots,w_n)^T$ is also the weight vector of h_i', and $\gamma_i\leq\gamma_i'$ for any h_i and h_i' $(i=1,2,\cdots,n)$, then

$$\text{HFHPWA}_\zeta(h_1,h_2,\cdots,h_n)\leq\text{HFHPWA}_\zeta(h_1',h_2',\cdots,h_n').$$

Proof. (1) Let $f(x)=\frac{1+(\zeta-1)x}{1-x}, x\in[0,1)$, then $f'(x)=\frac{\zeta}{(1-x)^2}>0$, thus $f(x)$ is an increasing function. Since $h^-\leq\gamma_i\leq h^+$ for all i, then $f(h^-)\leq f(\gamma_i)\leq f(h^+)$, i.e., $\frac{1+(\zeta-1)h^-}{1-h^-}\leq\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\leq\frac{1+(\zeta-1)h^+}{1-h^+}$. Since $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of h_i satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then for all i, we have

$$\left(\frac{1+(\zeta-1)h^-}{1-h^-}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \leq \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \leq \left(\frac{1+(\zeta-1)h^+}{1-h^+}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}$$

$$\Leftrightarrow \left(1+\frac{\zeta h^-}{1-h^-}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \leq \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \leq \left(1+\frac{\zeta h^+}{1-h^+}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}$$

$$\Leftrightarrow \prod_{i=1}^n \left(1+\frac{\zeta h^-}{1-h^-}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}$$

$$\leq \prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \leq \prod_{i=1}^n \left(1+\frac{\zeta h^+}{1-h^+}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}$$

$$\Leftrightarrow 1+\frac{\zeta h^-}{1-h^-} \leq \prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \leq 1+\frac{\zeta h^+}{1-h^+}$$

$$\Leftrightarrow \zeta+\frac{\zeta h^-}{1-h^-} \leq \prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta-1) \leq \zeta+\frac{\zeta h^+}{1-h^+}$$

$$\Leftrightarrow \frac{1}{\zeta+\frac{\zeta h^+}{1-h^+}} \leq \frac{1}{\prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta-1)} \leq \frac{1}{\zeta+\frac{\zeta h^-}{1-h^-}}$$

$$\Leftrightarrow \frac{1-h^+}{\zeta} \leq \frac{\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n (1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}} \leq \frac{1-h^-}{\zeta}$$

$$\Leftrightarrow 1-h^+ \leq \frac{\zeta\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n (1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}} \leq 1-h^-$$

$$\Leftrightarrow h^- \leq 1-\frac{\zeta\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n (1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}} \leq h^+$$

$$\Leftrightarrow h^- \leq \frac{\prod_{i=1}^n (1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} - \prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n (1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^n (1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}} \leq h^+.$$

Thus, we have $h^- \leq \mathrm{HFHPWA}_\zeta(h_1, h_2, \cdots, h_n) \leq h^+$.

(2) Let $f(x) = \frac{1+(\zeta-1)x}{1-x}$, $x \in [0,1)$, then by (1), $f(x)$ is an increasing funcntion. If

for all h_i and h_i', $\gamma_i \le \gamma_i'$, then $\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i} \le \frac{1+(\zeta-1)\gamma_i'}{1-\gamma_i'}$. For convenience, let $t_i = \frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}$, then we have

$$\left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{t_i} \le \left(\frac{1+(\zeta-1)\gamma_i'}{1-\gamma_i'}\right)^{t_i}$$

$$\Leftrightarrow \prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{t_i} + (\zeta-1) \le \prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i'}{1-\gamma_i'}\right)^{t_i} + (\zeta-1)$$

$$\Leftrightarrow \frac{1}{\prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i}{1-\gamma_i}\right)^{t_i} + (\zeta-1)} \ge \frac{1}{\prod_{i=1}^n \left(\frac{1+(\zeta-1)\gamma_i'}{1-\gamma_i'}\right)^{t_i} + (\zeta-1)}$$

$$\Leftrightarrow \frac{\zeta\prod_{i=1}^n(1-\gamma_i)^{t_i}}{\prod_{i=1}^n(1+(\zeta-1)\gamma_i)^{t_i} + (\zeta-1)\prod_{i=1}^n(1-\gamma_i)^{t_i}} \ge \frac{\zeta\prod_{i=1}^n(1-\gamma_i')^{t_i}}{\prod_{i=1}^n(1+(\zeta-1)\gamma_i')^{t_i} + (\zeta-1)\prod_{i=1}^n(1-\gamma_i')^{t_i}}$$

$$\Leftrightarrow 1 - \frac{\zeta\prod_{i=1}^n(1-\gamma_i)^{t_i}}{\prod_{i=1}^n(1+(\zeta-1)\gamma_i)^{t_i} + (\zeta-1)\prod_{i=1}^n(1-\gamma_i)^{t_i}} \le 1 - \frac{\zeta\prod_{i=1}^n(1-\gamma_i')^{t_i}}{\prod_{i=1}^n(1+(\zeta-1)\gamma_i')^{t_i} + (\zeta-1)\prod_{i=1}^n(1-\gamma_i')^{t_i}}$$

$$\Leftrightarrow \frac{\prod_{i=1}^n(1+(\zeta-1)\gamma_i)^{t_i} - \prod_{i=1}^n(1-\gamma_i)^{t_i}}{\prod_{i=1}^n(1+(\zeta-1)\gamma_i)^{t_i} + (\zeta-1)\prod_{i=1}^n(1-\gamma_i)^{t_i}} \le \frac{\prod_{i=1}^n(1+(\zeta-1)\gamma_i')^{t_i} - \prod_{i=1}^n(1-\gamma_i')^{t_i}}{\prod_{i=1}^n(1+(\zeta-1)\gamma_i')^{t_i} + (\zeta-1)\prod_{i=1}^n(1-\gamma_i')^{t_i}}.$$

Thus, by Theorem 3 $\text{HFHPWA}_\zeta(h_1, h_2, \cdots, h_n) \le \text{HFHPWA}_\zeta(h_1', h_2', \cdots, h_n')$. □

Based on the HFHWG operator [12] and PG operator, we develop the hesitant fuzzy Hamacher power weighted geometric operator as follows.

Definition 6. Let $h_i (i = 1,2,\cdots,n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i (i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$. A hesitant fuzzy Hamacher power weighted geometric (HFHPWG) operator is a function $H^n \to H$ such that

$$\text{HFHPWG}_\zeta(h_1, h_2, \cdots, h_n) = \otimes_{H_{i=1}^n}\left(h_i^{\wedge_H \frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right), \tag{17}$$

where parameter $\zeta > 0$, $T(h_i) = \sum_{j=1, j\ne i}^n w_j \text{Sup}(h_i, h_j)$ and $\text{Sup}(h_i, h_j)$ is the support for h_i from h_j.

Theorem 6. Let $h_i(i = 1,2,\cdots,n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by HFHPWG operator is also a HFE, and

$$\mathrm{HFHPWG}_\zeta(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n} \left\{ \frac{\zeta \prod_{i=1}^{n} (\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n} (1 + (\zeta-1)(1-\gamma_i))^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1) \prod_{i=1}^{n} (\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}} \right\}. \quad (18)$$

Proof. Similar to the proof of Theorem 3, Eq. (18) can be proved by mathematical induction on n. □

Remark 2. (1) If $\mathrm{Sup}(h_i, h_j) = k$, for all $i \neq j$, then

$$\mathrm{HFHPG}_\zeta(h_1, h_2, \cdots, h_n) = \otimes_{H_{i=1}^{n}} \left(h_i^{\wedge_H w_i} \right) \quad (19)$$

which indicates that when all supports are the same, the HFHPWG operator reduces to the hesitant fuzzy Hamacher weighted geometric (HFHWG) operator [12]. Especially, if $\mathrm{Sup}(h_i, h_j) = 0$ for all $i \neq j$, then there is no support in the aggregation process, and in this case, the HFHPWG operator reduces to the HFHWG operator [12].

(2) If $\zeta = 1$, then the HFHPWG operator reduces to the hesitant fuzzy power weighted geometric (HFPWG) operator:

$$\mathrm{HFPWG}_1(h_1, h_2, \cdots, h_n) = \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n} \left\{ \prod_{i=1}^{n} (\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} \right\}. \quad (20)$$

and if $\zeta = 2$, then the HFHPWG operator reduces to the hesitant fuzzy Einstein power weighted geometric (HFEPWG) operator [40]:

$$\mathrm{HFEPWG}_2(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n} \left\{ \frac{2 \prod_{i=1}^{n} (\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n} (2-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + \prod_{i=1}^{n} (\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}} \right\}. \quad (21)$$

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the HFHPWG operator reduces to the hesitant fuzzy Hamacher power geometric (HFHPG) operator:

$$\mathrm{HFHPG}_{\zeta}(h_1,h_2,\cdots,h_n)=\otimes_{H_{i=1}^{n}}\left(h_i^{\wedge_H\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}\right)$$

$$=\cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\prod_{i=1}^{n}(\gamma_i)^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}+(\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}}\right\}, \qquad (22)$$

where $T'(h_i)=\frac{1}{n}\sum_{j=1,j\neq i}^{n}\mathrm{Sup}(h_i,h_j)$.

Theorem 7. Let $h_i(i=1,2,\cdots,n)$ be a collection of HFEs and $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of h_i such that $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, then

$$\mathrm{HFHPWG}_{\zeta}(h_1,h_2,\cdots,h_n)\geq \mathrm{HFPWG}(h_1,h_2,\cdots,h_n).$$

Proof. For any $\gamma_i\in h_i(i=1,2,\cdots,n)$, by Lemma 1, we have

$$\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}$$
$$\leq\sum_{i=1}^{n}\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}(1+(\zeta-1)(1-\gamma_i))$$
$$+(\zeta-1)\sum_{i=1}^{n}\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}\gamma_i=\zeta.$$

Then,

$$\frac{\zeta\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}$$
$$\geq\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}},$$

which implies that $\otimes_{H_{i=1}^{n}}\left(h_i^{\wedge_H\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}\right)\geq\otimes_{i=1}^{n}\left(h_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}\right)$. That is,

$$\mathrm{HFHPWG}_{\zeta}(h_1,h_2,\cdots,h_n)\geq \mathrm{HFPWG}(h_1,h_2,\cdots,h_n). \qquad \square$$

Theorem 7 shows that the values aggregated by the HFHPWG operator are not smaller than those obtained by the HFPWG operator. That is to say, the HFHPWG operator reflects the decision maker's more optimisitc attitude than the HFPWG operator in aggregation process. Furthermore, similar to Theorem 3, we have the properties of the HFHPWG operator as follows.

Theorem 8. Let $h_i(i=1,2,\cdots,n)$ be a collection of HFEs and $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of h_i such that $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, then we have followings:

(1) Boundedness: If $h^- = \min\{\gamma_i | \gamma_i \in h_i\}$ and $h^+ = \max\{\gamma_i | \gamma_i \in h_i\}$, then

$$h^- \le \text{HFHPWG}_\zeta(h_1,h_2,\cdots,h_n) \le h^+.$$

(2) Monotonicity: Let $h_i'(i=1,2,\cdots,n)$ be a collection of HFEs, if $w=(w_1,w_2,\cdots,w_n)^T$ is also the weight vector of h_i', and $\gamma_i \le \gamma_i'$ for any h_i and $h_i'(i=1,2,\cdots,n)$, then

$$\text{HFHPWG}_\zeta(h_1,h_2,\cdots,h_n) \le \text{HFHPWG}_\zeta(h_1',h_2',\cdots,h_n').$$

Proof. (1) Let $g(x)=\frac{1+(\zeta-1)(1-x)}{x}$, $x \in (0,1]$, then $g'(x)=\frac{-\zeta}{x^2}<0$, thus $g(x)$ is a decreasing function. Since $h^- \le \gamma_i \le h^+$ for all i, then $g(h^-) \ge g(\gamma_i) \ge g(h^+)$, i.e., $\frac{1+(\zeta-1)(1-h^+)}{h^+} \le \frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i} \le \frac{1+(\zeta-1)(1-h^-)}{h^-}$. Since $w=(w_1,w_2,\cdots,w_n)^T$ is the weight vector of h_i satisfying $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, then for all i, let $t_i = \frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}$, we have

$$\left(\frac{1+(\zeta-1)(1-h^+)}{h^+}\right)^{t_i} \le \left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} \le \left(\frac{1+(\zeta-1)(1-h^-)}{h^-}\right)^{t_i}$$

$$\Leftrightarrow \prod_{i=1}^n \left(\frac{1+(\zeta-1)(1-h^+)}{h^+}\right)^{t_i} \le \prod_{i=1}^n \left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} \le \prod_{i=1}^n \left(\frac{1+(\zeta-1)(1-h^-)}{h^-}\right)^{t_i}$$

$$\Leftrightarrow \frac{\zeta}{h^+} - (\zeta-1) \le \prod_{i=1}^n \left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} \le \frac{\zeta}{h^-} - (\zeta-1)$$

$$\Leftrightarrow \frac{\zeta}{h^+} \le \prod_{i=1}^n \left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} + (\zeta-1) \le \frac{\zeta}{h^-}$$

$$\Leftrightarrow \frac{h^-}{\zeta} \le \frac{1}{\prod_{i=1}^n \left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} + (\zeta-1)} \le \frac{h^+}{\zeta}$$

$$\Leftrightarrow \frac{h^-}{\zeta} \leq \frac{\prod_{i=1}^{n}(\gamma_i)^{t_i}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{t_i} + (\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{t_i}} \leq \frac{h^+}{\zeta}$$

$$\Leftrightarrow h^- \leq \frac{\zeta\prod_{i=1}^{n}(\gamma_i)^{t_i}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{t_i} + (\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{t_i}} \leq h^+ \quad .$$

Thus, we have $h^- \leq \mathrm{HFHPWG}_\zeta(h_1, h_2, \cdots, h_n) \leq h^+$.

(2) Let $g(x) = \frac{1+(\zeta-1)(1-x)}{x}$, $x \in (0,1]$, then by (1), $g(x)$ is a decreasing function. Then for all i, $\gamma_i \leq \gamma_i'$, we have $\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i} \geq \frac{1+(\zeta-1)(1-\gamma_i')}{\gamma_i'}$. For convenience, let $t_i = \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}$, then we have

$$\left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} \geq \left(\frac{1+(\zeta-1)(1-\gamma_i')}{\gamma_i'}\right)^{t_i}$$

$$\Leftrightarrow \prod_{i=1}^{n}\left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} \geq \prod_{i=1}^{n}\left(\frac{1+(\zeta-1)(1-\gamma_i')}{\gamma_i'}\right)^{t_i}$$

$$\Leftrightarrow \prod_{i=1}^{n}\left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} + (\zeta-1) \geq \prod_{i=1}^{n}\left(\frac{1+(\zeta-1)(1-\gamma_i')}{\gamma_i'}\right)^{t_i} + (\zeta-1)$$

$$\Leftrightarrow \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\zeta-1)(1-\gamma_i)}{\gamma_i}\right)^{t_i} + (\zeta-1)} \leq \frac{1}{\prod_{i=1}^{n}\left(\frac{1+(\zeta-1)(1-\gamma_i')}{\gamma_i'}\right)^{t_i} + (\zeta-1)}$$

$$\Leftrightarrow \frac{\zeta\prod_{i=1}^{n}(\gamma_i)^{t_i}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{t_i} + (\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{t_i}} \leq \frac{\zeta\prod_{i=1}^{n}(\gamma_i')^{t_i}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i'))^{t_i} + (\zeta-1)\prod_{i=1}^{n}(\gamma_i')^{t_i}}.$$

Thus, by Theorem 6 $\mathrm{HFHPWG}_\zeta(h_1, h_2, \cdots, h_n) \leq \mathrm{HFHPWG}_\zeta(h_1', h_2', \cdots, h_n')$. □

Theorem 9. Let $h_i (i = 1,2,\cdots,n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of h_i such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then we have

(1) $\mathrm{HFHPWA}_\zeta(h_1^c, h_2^c, \cdots, h_n^c) = (\mathrm{HFHPWG}_\zeta(h_1, h_2, \cdots, h_n))^c$;

(2) $\mathrm{HFHPWG}_\zeta(h_1^c, h_2^c, \cdots, h_n^c) = (\mathrm{HFHPWA}_\zeta(h_1, h_2, \cdots, h_n))^c$.

Proof. Since (2) is similar (1), we only prove (1).

$\mathrm{HFHPWA}_\zeta(h_1^{\complement}, h_2^{\complement}, \cdots, h_n^{\complement})$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{1 - \frac{\zeta\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)(1-\gamma_i))^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n}(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}\right\}$$

$$= (\mathrm{HFHPWG}_\zeta(h_1, h_2, \cdots, h_n))^{\complement}. \qquad \square$$

In what follows, we define the generalized hesitant fuzzy Hamacher power weighted average (GHFHPWA) operator and the generalized hesitant fuzzy Hamacher power weighted geometric (GHFHPWG) operator.

Definition 7. Let $h_i(i = 1,2,\cdots,n)$ be a collection of HFEs, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. For a parameter $\lambda > 0$, a generalized hesitant fuzzy Hamacher power weighted average (GHFHPWA) operator is a function $H^n \to H$ such that

$$\mathrm{GHFHPWA}_\zeta(h_1, h_2, \cdots, h_n) = \left(\oplus_{H_{i=1}^{n}}\left(\frac{w_i(1+T(h_i))\cdot_H h_i^{\wedge_H \lambda}}{\sum_{i=1}^{n} w_i(1+T(h_i))}\right)\right)^{\wedge_H \frac{1}{\lambda}}, \qquad (23)$$

where parameter $\zeta > 0$, $T(h_i) = \sum_{j=1, j\neq i}^{n} w_j \mathrm{Sup}(h_i, h_j)$ and $\mathrm{Sup}(h_i, h_j)$ is the support for h_i from h_j.

Theorem 10. Let $h_i(i = 1,2,\cdots,n)$ be a collection of HFEs, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i(i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then the aggregated value by GHFHPWA operator is also a HFE, and

$\mathrm{GHFHPWA}_{\zeta}(h_1, h_2, \cdots, h_n)$

$$= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}}{\left[\left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta^2-1)\prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}} + (\zeta-1)\left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}\right]}\right\}, \quad (24)$$

where $a_i = (1+(\zeta-1)(1-\gamma_i))^{\lambda} + (\zeta^2-1){\gamma_i}^{\lambda}$ and $b_i = (1+(\zeta-1)(1-\gamma_i))^{\lambda} - {\gamma_i}^{\lambda}$.

Proof. We first use the mathematical induction on n to prove

$$\oplus_{H_{i=1}^{n}}\left(\frac{w_i(1+T(h_i))\cdot_H {h_i}^{\wedge_H \lambda}}{\sum_{i=1}^{n} w_i(1+T(h_i))}\right) = \left\{\frac{\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}\right\}. \quad (25)$$

(1) When $n=1$, since $\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))} = 1$, we have

$$\oplus_{H_{i=1}^{n}}\left(\frac{w_i(1+T(h_i))\cdot_H {h_i}^{\wedge_H \lambda}}{\sum_{i=1}^{n} w_i(1+T(h_i))}\right) = {h_1}^{\wedge_H \lambda}$$

$$= \bigcup_{\gamma_1\in h_1}\left\{\frac{\zeta{\gamma_1}^{\lambda}}{(1+(\zeta-1)(1-\gamma_1))^{\lambda} + (\zeta-1){\gamma_1}^{\lambda}}\right\}$$

$$= \bigcup_{\gamma_1\in h_1}\left\{\frac{a_1-b_1}{a_1+(\zeta-1)b_1}\right\}.$$

Thus, Eq. (25) holds for $n=1$.

(2) Suppose that Eq. (25) holds for $n=k$, that is

$$\oplus_{H_{i=1}^{k}}\left(\frac{w_i(1+T(h_i))\cdot_H {h_i}^{\wedge_H \lambda}}{\sum_{i=1}^{k} w_i(1+T(h_i))}\right)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k}\left\{\frac{\prod_{i=1}^{k} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k} w_i(1+T(h_i))}} - \prod_{i=1}^{k} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k} w_i(1+T(h_i))}}}{\prod_{i=1}^{k} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{k} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k} w_i(1+T(h_i))}}}\right\},$$

then, when $n = k+1$, by the operational laws in Definition 4, we have

$$\oplus_{H_{i=1}^{k+1}}\left(\frac{w_i(1+T(h_i))\cdot_H h_i^{\wedge_H\lambda}}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}\right)$$

$$= \oplus_{H_{i=1}^{k}}\left(\frac{w_i(1+T(h_i))\cdot_H h_i^{\wedge_H\lambda}}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}\right) \oplus_H \left(\frac{w_{k+1}(1+T(h_{k+1}))\cdot_H h_{k+1}^{\wedge_H\lambda}}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}\right)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k}\left\{\frac{\prod_{i=1}^{k} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} - \prod_{i=1}^{k} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}}}{\prod_{i=1}^{k} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{k} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}}}\right\}$$

$$\oplus_H \cup_{\gamma_{k+1}\in h_{k+1}}\left\{\frac{a_{k+1}^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} - b_{k+1}^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}}}{a_{k+1}^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} + (\zeta-1) b_{k+1}^{\frac{w_{k+1}(1+T(h_{k+1}))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}}}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_k\in h_k,\gamma_{k+1}\in h_{k+1}}\left\{\frac{\prod_{i=1}^{k+1} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} - \prod_{i=1}^{k+1} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}}}{\prod_{i=1}^{k+1} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{k+1} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{k+1} w_i(1+T(h_i))}}}\right\},$$

i.e., Eq. (25) holds for $n = k+1$. Thus, Eq. (25) holds for all n.

For convenience, let $\beta = \frac{\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}$ and $t_i = \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}$. By the operational laws in Definition 4, we have

$$\mathrm{GHFHPWA}_{\zeta}(h_1,h_2,\cdots,h_n) = \left(\oplus_{H_{i=1}^{n}}\left(t_i\cdot_H h_i^{\wedge_H\lambda}\right)\right)^{\wedge_H\frac{1}{\lambda}}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\beta^{\frac{1}{\lambda}}}{(1+(\zeta-1)(1-\beta))^{\frac{1}{\lambda}} + (\zeta-1)\beta^{\frac{1}{\lambda}}}\right\}$$

$$= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\left(\frac{\prod_{i=1}^{n}a_i{}^{t_i}-\prod_{i=1}^{n}b_i{}^{t_i}}{\prod_{i=1}^{n}a_i{}^{t_i}+(\zeta-1)\prod_{i=1}^{n}b_i{}^{t_i}}\right)^{\frac{1}{\lambda}}}{\left(1+(\zeta-1)\left(1-\frac{\prod_{i=1}^{n}a_i{}^{t_i}-\prod_{i=1}^{n}b_i{}^{t_i}}{\prod_{i=1}^{n}a_i{}^{t_i}+(\zeta-1)\prod_{i=1}^{n}b_i{}^{t_i}}\right)\right)^{\frac{1}{\lambda}}+(\zeta-1)\left(\frac{\prod_{i=1}^{n}a_i{}^{t_i}-\prod_{i=1}^{n}b_i{}^{t_i}}{\prod_{i=1}^{n}a_i{}^{t_i}+(\zeta-1)\prod_{i=1}^{n}b_i{}^{t_i}}\right)^{\frac{1}{\lambda}}}\right\}$$

$$= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\left(\prod_{i=1}^{n}a_i{}^{t_i}-\prod_{i=1}^{n}b_i{}^{t_i}\right)^{\frac{1}{\lambda}}}{\left(\prod_{i=1}^{n}a_i{}^{t_i}+(\zeta^2-1)\prod_{i=1}^{n}b_i{}^{t_i}\right)^{\frac{1}{\lambda}}+(\zeta-1)\left(\prod_{i=1}^{n}a_i{}^{t_i}-\prod_{i=1}^{n}b_i{}^{t_i}\right)^{\frac{1}{\lambda}}}\right\},$$

which completes the proof of the theorem. □

Remark 3. (1) If $\lambda=1$, then $a_i=\zeta(1+(\zeta-1)\gamma_i)$ and $b_i=\zeta(1-\gamma_i)$, and the GHFHPWA operator reduces to the HFHPWA operator. In fact, by Eq. (24), we have

$$\mathrm{GHFHPWA}_{\zeta}(h_1,h_2,\cdots,h_n)$$

$$= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\left(\prod_{i=1}^{n}a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}-\prod_{i=1}^{n}b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}\right)}{\left[\prod_{i=1}^{n}a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta^2-1)\prod_{i=1}^{n}b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta-1)\left(\prod_{i=1}^{n}a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}-\prod_{i=1}^{n}b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}\right)\right]}\right\}$$

$$= \bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\prod_{i=1}^{n}a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}-\prod_{i=1}^{n}b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}{\prod_{i=1}^{n}a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}+(\zeta-1)\prod_{i=1}^{n}b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n}w_i(1+T(h_i))}}}\right\}$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n}(1+(\zeta-1)\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n}(1-\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}} \right\}$$

$$= \mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_n).$$

(2) If $\mathrm{Sup}(h_i, h_j) = k$, for all $i \neq j$, then

$$\mathrm{GHFHPWA}_{\zeta}(h_1, h_2, \cdots, h_n) = \left(\oplus_{H_{i=1}^{n}} \left(w_i \cdot_H h_i{}^{\wedge_H \lambda}\right)\right)^{\wedge_H \frac{1}{\lambda}} \tag{26}$$

and thus the HFHPWA operator reduces to the generalized hesitant fuzzy Hamacher weighted average (GHFHWA) operator [12]. Especially, if $\mathrm{Sup}(h_i, h_j) = 0$ for all $i \neq j$, then there is no support in the aggregation process, and thus it is clear that the GHFHPWA operator reduces to the GHFHWA operator [12].

(3) If $\zeta = 1$, then the GHFHPWA operator reduces to the generalized hesitant fuzzy power weighted average (GHFPWA) operator:

$\mathrm{GHFPWA}_1(h_1, h_2, \cdots, h_n)$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \left(1 - \prod_{i=1}^{n} (1-\gamma_i{}^{\lambda})^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}} \right\}. \tag{27}$$

and if $\zeta = 2$, then the GHFHPWA operator reduces to the generalized hesitant fuzzy Einstein power weighted average (GHFEPWA) operator [40]:

$\mathrm{GHFEPWA}_2(h_1, h_2, \cdots, h_n)$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{2\left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}}{\left[\left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + 3\prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}} + \left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}\right]} \right\}, \quad (28)$$

where $a_i = (2-\gamma_i)^\lambda + 3{\gamma_i}^\lambda$, $b_i = (2-\gamma_i)^\lambda - {\gamma_i}^\lambda$.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the GHFHPWA operator reduces to the generalized hesitant fuzzy Hamacher power average (GHFHPA) operator:

$$\text{GHFHPA}_\zeta(h_1, h_2, \cdots, h_n) = \left(\oplus_{H_{i=1}^{n}} \left(\frac{\left(1+T'(h_i)\right) \cdot_H {h_i}^{\wedge_H \lambda}}{\sum_{i=1}^{n}(1+T'(h_i))}\right)\right)^{\wedge_H \frac{1}{\lambda}}$$

$$= \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\zeta\left(\prod_{i=1}^{n} a_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}\right)^{\frac{1}{\lambda}}}{\left[\left(\prod_{i=1}^{n} a_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}} + (\zeta^2-1)\prod_{i=1}^{n} b_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^{n}(1+T'(h_i))}}\right)^{\frac{1}{\lambda}} + (\zeta-1)\left(\prod_{i=1}^{n} a_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} - \prod_{i=1}^{n} b_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}\right]} \right\} \quad (29)$$

where $a_i = (1+(\zeta-1)(1-\gamma_i))^\lambda + (\zeta^2-1){\gamma_i}^\lambda$, $b_i = (1+(\zeta-1)(1-\gamma_i))^\lambda - {\gamma_i}^\lambda$ and $T'(h_i) = \frac{1}{n}\sum_{j=1, j\neq i}^{n} \text{Sup}(h_i, h_j)$.

Definition 8. Let $h_i (i = 1,2,\cdots,n)$ be a collection of HFEs $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i (i = 1,2,\cdots,n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. For $\lambda > 0$, a generalized hesitant fuzzy Hamacher power weighted geometric

(GHFHPWG) operator is a function $H^n \to H$ such that

$$\mathrm{GHFHPWG}_\zeta(h_1, h_2, \cdots, h_n) = \frac{1}{\lambda} \cdot_H \left(\otimes_{H_{i=1}^n} \left((\lambda \cdot_H h_i\ )^{\wedge_H \frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right) \right), \quad (30)$$

where $\zeta > 0$, $T(h_i) = \sum_{j=1, j\neq i}^n w_j \mathrm{Sup}(h_i, h_j)$ and $\mathrm{Sup}(h_i, h_j)$ is the support for h_i from h_j.

Theorem 11. Let $h_i (i = 1,2, \cdots, n)$ be a collection of HFEs $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of $h_i (i = 1,2, \cdots, n)$ such that $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, then the aggregated value by GHFHPWG operator is also a HFE, and

$$\mathrm{GHFHPWG}_\zeta(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\left[\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta^2 - 1) \prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right)^{\frac{1}{\lambda}} - \left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} - \prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right)^{\frac{1}{\lambda}} \right]}{\left[\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta^2 - 1) \prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right)^{\frac{1}{\lambda}} + (\zeta - 1) \left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} - \prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right)^{\frac{1}{\lambda}} \right]} \right\}, \quad (31)$$

where $c_i = (1 + (\zeta - 1)\gamma_i)^\lambda + (\zeta^2 - 1)(1 - \gamma_i)^\lambda$ and $d_i = (1 + (\zeta - 1)\gamma_i)^\lambda - (1 - \gamma_i)^\lambda$.

Proof. We first use the mathematical induction on n to prove

$$\otimes_{H_{i=1}^n} \left((\lambda \cdot_H h_i)^{\wedge_H \frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} \right)$$

$$= \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \cdots, \gamma_n \in h_n} \left\{ \frac{\zeta \prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} + (\zeta - 1) \prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}} \right\}. \quad (32)$$

(1) When $n = 1$, then we have

$$\otimes_{H_{i=1}^{k}}\left(\left(\lambda\cdot_{H}h_{i}\right)^{\wedge_{H}\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{n}w_{i}(1+T(h_{i}))}}\right)=\lambda\cdot_{H}h_{1}$$

$$=\ \cup_{\gamma_{1}\in h_{1}}\left\{\frac{(1+(\zeta-1)\gamma_{1})^{\lambda}-(1-\gamma_{1})^{\lambda}}{(1+(\zeta-1)\gamma_{1})^{\lambda}+(\zeta-1)(1-\gamma_{1})^{\lambda}}\right\}$$

$$=\ \cup_{\gamma_{1}\in h_{1}}\left\{\frac{\zeta d_{1}}{c_{1}+(\zeta-1)d_{1}}\right\}\ .$$

Thus, Eq. (32) holds for $n=1.$

(2) Suppose that Eq. (32) holds for $n=k,$ that is

$$\otimes_{H_{i=1}^{k}}\left(\left(\lambda\cdot_{H}h_{i}\right)^{\wedge_{H}\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k}w_{i}(1+T(h_{i}))}}\right)$$

$$=\ \cup_{\gamma_{1}\in h_{1},\gamma_{2}\in h_{2},\cdots,\gamma_{k}\in h_{k}}\left\{\frac{\zeta\prod_{i=1}^{k}d_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k}w_{i}(1+T(h_{i}))}}}{\prod_{i=1}^{k}c_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k}w_{i}(1+T(h_{i}))}}+(\zeta-1)\prod_{i=1}^{k}d_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k}w_{i}(1+T(h_{i}))}}}\right\}\ ,$$

then, when $n=k+1,$ by the operational laws in Definition 4, we have

$$\otimes_{H_{i=1}^{k+1}}\left(\left(\lambda\cdot_{H}h_{i}\right)^{\wedge_{H}\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}\right)$$

$$=\otimes_{H_{i=1}^{k}}\left(\left(\lambda\cdot_{H}h_{i}\right)^{\wedge_{H}\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}\right)\otimes_{H}\left(\left(\lambda\cdot_{H}h_{k+1}\right)^{\wedge_{H}\frac{w_{k+1}\left(1+T(h_{k+1})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}\right)$$

$$=\ \cup_{\gamma_{1}\in h_{1},\gamma_{2}\in h_{2},\cdots,\gamma_{k}\in h_{k}}\left\{\frac{\zeta\prod_{i=1}^{k}d_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}}{\prod_{i=1}^{k}c_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}+(\zeta-1)\prod_{i=1}^{k}d_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}}\right\}$$

$$\otimes_{H}\ \cup_{\gamma_{k+1}\in h_{k+1}}\left\{\frac{\zeta d_{k+1}^{\frac{w_{k+1}\left(1+T(h_{k+1})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}}{c_{k+1}^{\frac{w_{k+1}\left(1+T(h_{k+1})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}+(\zeta-1)d_{k+1}^{\frac{w_{k+1}\left(1+T(h_{k+1})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}}\right\}$$

$$=\ \cup_{\gamma_{1}\in h_{1},\gamma_{2}\in h_{2},\cdots,\gamma_{k}\in h_{k},\gamma_{k+1}\in h_{k+1}}\left\{\frac{\zeta\prod_{i=1}^{k+1}d_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}}{\prod_{i=1}^{k+1}c_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}+(\zeta-1)\prod_{i=1}^{k+1}d_{i}^{\frac{w_{i}\left(1+T(h_{i})\right)}{\sum_{i=1}^{k+1}w_{i}(1+T(h_{i}))}}}\right\}\ ,$$

i.e., Eq. (32) holds for $n = k + 1$. Thus, Eq. (32) holds for all n.

Now, for convenience, let $\delta = \frac{\zeta \prod_{i=1}^{n} d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}{\prod_{i=1}^{n} c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}} + (\zeta-1)\prod_{i=1}^{n} d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}}$ and $t_i = \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}$. By the operational laws in Definition 4, we have

$$\mathrm{GHFHPWG}_{\zeta}(h_1, h_2, \cdots, h_n) = \frac{1}{\lambda}\left(\otimes_{H_{i=1}^{n}}\left((\lambda \cdot_H h_i)^{\wedge_H \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}}\right)\right)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{(1+(\zeta-1)\delta)^{\frac{1}{\lambda}} - (1-\delta)^{\frac{1}{\lambda}}}{(1+(\zeta-1)\delta)^{\frac{1}{\lambda}} + (\zeta-1)(1-\delta)^{\frac{1}{\lambda}}}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\left(1+(\zeta-1)\frac{\zeta\prod_{i=1}^{n} d_i^{t_i}}{\prod_{i=1}^{n} c_i^{t_i}+(\zeta-1)\prod_{i=1}^{n} d_i^{t_i}}\right)^{\frac{1}{\lambda}} - \left(1-\frac{\zeta\prod_{i=1}^{n} d_i^{t_i}}{\prod_{i=1}^{n} c_i^{t_i}+(\zeta-1)\prod_{i=1}^{n} d_i^{t_i}}\right)^{\frac{1}{\lambda}}}{\left(1+(\zeta-1)\frac{\zeta\prod_{i=1}^{n} d_i^{t_i}}{\prod_{i=1}^{n} c_i^{t_i}+(\zeta-1)\prod_{i=1}^{n} d_i^{t_i}}\right)^{\frac{1}{\lambda}} + (\zeta-1)\left(1-\frac{\zeta\prod_{i=1}^{n} d_i^{t_i}}{\prod_{i=1}^{n} c_i^{t_i}+(\zeta-1)\prod_{i=1}^{n} d_i^{t_i}}\right)^{\frac{1}{\lambda}}}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\left(\prod_{i=1}^{n} c_i^{t_i} + (\zeta^2-1)\prod_{i=1}^{n} d_i^{t_i}\right)^{\frac{1}{\lambda}} - \left(\prod_{i=1}^{n} c_i^{t_i} - \prod_{i=1}^{n} d_i^{t_i}\right)^{\frac{1}{\lambda}}}{\left(\prod_{i=1}^{n} c_i^{t_i} + (\zeta^2-1)\prod_{i=1}^{n} d_i^{t_i}\right)^{\frac{1}{\lambda}} + (\zeta-1)\left(\prod_{i=1}^{n} c_i^{t_i} - \prod_{i=1}^{n} d_i^{t_i}\right)^{\frac{1}{\lambda}}}\right\},$$

which completes the proof of the theorem. □

Remark 4. (1) If $\lambda = 1$, then $c_i = \zeta(1+(\zeta-1)(1-\gamma_i))$ and $d_i = \zeta\gamma_i$, and the GHFHPWG operator reduces to the HFHPWG operator. In fact, by Eq. (31), we have

$\mathrm{GHFHPWG}_{\zeta}(h_1, h_2, \cdots, h_n)$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\left[\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}+(\zeta^2-1)\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} -\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}-\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)\right]}{\left[\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}+(\zeta^2-1)\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} +(\zeta-1)\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}-\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)\right]}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}+(\zeta-1)\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}\right\}$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\zeta\prod_{i=1}^n (\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}{\prod_{i=1}^n\left(1+(\zeta-1)(1-\gamma_i)\right)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}} +(\zeta-1)\prod_{i=1}^n(\gamma_i)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}}\right\}$$

$$= \mathrm{HFHPWG}_\zeta(h_1,h_2,\cdots,h_n).$$

(2) If $\mathrm{Sup}(h_i,h_j)=k$, for all $i\neq j$, then

$$\mathrm{GHFHPWG}_\zeta(h_1,h_2,\cdots,h_n)=\frac{1}{\lambda}\cdot_H\left(\otimes_{H_{i=1}^n}(\lambda\cdot_H h_i)^{\wedge_H w_i}\right) \tag{33}$$

and thus the HFHPWG operator reduces to the generalized hesitant fuzzy Hamacher weighted geometric (GHFHWG) operator [12]. Especially, if $\mathrm{Sup}(h_i,h_j)=0$ for all $i\neq j$, then the GHFHPWG operator also reduces to the GHFHWG operator [12].

(3) If $\zeta=1$, then the GHFHPWG operator reduces to the generalized hesitant fuzzy power weighted geometric (GHFPWG) operator:

$$\mathrm{GHFPWG}_1(h_1,h_2,\cdots,h_n)= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(\prod_{i=1}^n(\gamma_i{}^\lambda)^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}\right\}. \tag{34}$$

and if $\zeta=2$, then the GHFHPWG operator reduces to the generalized hesitant fuzzy Einstein power weighted geometric (GHFEPWG) operator [40]:

$\mathrm{GHFEPWG}_2(h_1, h_2, \cdots, h_n)$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\left[\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}+3\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}} - \left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}-\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}\right]}{\left[\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}+3\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}} +(\zeta-1)\left(\prod_{i=1}^n c_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}-\prod_{i=1}^n d_i^{\frac{w_i(1+T(h_i))}{\sum_{i=1}^n w_i(1+T(h_i))}}\right)^{\frac{1}{\lambda}}\right]}\right\}, \quad (35)$$

where $c_i = (1+\gamma_i)^\lambda + 3(1-\gamma_i)^\lambda$, $d_i = (1+\gamma_i)^\lambda - (1-\gamma_i)^\lambda$.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^T$, then the GHFHPWA operator reduces to the generalized hesitant fuzzy Hamacher power average (GHFHPA) operator:

$$\mathrm{GHFHPA}_\zeta(h_1, h_2, \cdots, h_n) = \frac{1}{\lambda} \cdot_H \left(\otimes_{H_{i=1}^n}\left((\lambda \cdot_H h_i)^{\wedge_H \frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}}\right)\right)$$

$$= \cup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\frac{\left[\left(\prod_{i=1}^n c_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}} +(\zeta^2-1)\prod_{i=1}^n d_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}}\right)^{\frac{1}{\lambda}} - \left(\prod_{i=1}^n c_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}} -\prod_{i=1}^n d_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}}\right)^{\frac{1}{\lambda}}\right]}{\left[\left(\prod_{i=1}^n c_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}} +(\zeta^2-1)\prod_{i=1}^n d_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}}\right)^{\frac{1}{\lambda}} +(\zeta-1)\left(\prod_{i=1}^n c_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}} -\prod_{i=1}^n d_i^{\frac{(1+T'(h_i))}{\sum_{i=1}^n (1+T'(h_i))}}\right)^{\frac{1}{\lambda}}\right]}\right\}, \quad (36)$$

where $c_i = (1 + (\zeta - 1)\gamma_i)^\lambda + (\zeta^2 - 1)(1 - \gamma_i)^\lambda$, $d_i = (1 + (\zeta - 1)\gamma_i)^\lambda - (1 - \gamma_i)^\lambda$ and $T'(h_i) = \frac{1}{n}\sum_{j=1, j\neq i}^{n} \mathrm{Sup}(h_i, h_j)$.

Theorem 12. Let $h_i (i = 1,2,\cdots,n)$ be a collection of HFEs and $w = (w_1, w_2, \cdots, w_n)^T$ be the weight vector of h_i such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then we have

(1) $\mathrm{GHFHPWA}_\zeta(h_1^C, h_2^C, \cdots, h_n^C) = (\mathrm{GHFHPWG}_\zeta(h_1, h_2, \cdots, h_n))^C$;

(2) $\mathrm{GHFHPWG}_\zeta(h_1^C, h_2^C, \cdots, h_n^C) = (\mathrm{GHFHPWA}_\zeta(h_1, h_2, \cdots, h_n))^C$.

Proof. Since (2) is similar (1), we only prove (1). For convenience, let $t_i = \frac{w_i(1+T(h_i))}{\sum_{i=1}^{n} w_i(1+T(h_i))}$, then $\mathrm{GHFHPWA}_\zeta(h_1^C, h_2^C, \cdots, h_n^C)$.

3.2 Hesitant fuzzy Hamacher power ordered weighted average operator and hesitant fuzzy Hamacher power ordered weighted geometric operator

Motivated by the idea of the POWA operator [27], POWG operator [28] and Hamacher operations, we define the hesitant fuzzy Hamacher power ordered weighted average operator and hesitant fuzzy Hamacher power ordered weighted geometric operator as follows.

Definition 8. Let $h_i (i = 1,2,\cdots,n)$ be a collection of HFEs. A hesitant fuzzy Hamacher power ordered weighted average (HFHPOWA) operator is a function $H^n \to H$ such that

$$\mathrm{HFHPOWA}_\zeta(h_1, h_2, \cdots, h_n) = \oplus_{H_{i=1}^{n}} \left(u_i \cdot_H h_{\sigma(i)}\right) , \tag{37}$$

where parameter $\zeta > 0$, $h_{\sigma(i)}$ is the ith largest HFE of $h_j (j = 1,2,\cdots,n)$, and $u_i (i = 1,2,\cdots,n)$ is a collection of weights such that

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right), \; R_i = \sum_{j=1}^{i} V_{\sigma(j)}, \; TV = \sum_{i=1}^{n} V_{\sigma(i)} ,$$

$$V_{\sigma(i)} = 1 + T\left(h_{\sigma(i)}\right), T\left(h_{\sigma(i)}\right) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(h_{\sigma(i)}, h_{\sigma(j)}) , \tag{38}$$

where $T\left(h_{\sigma(i)}\right)$ denotes the support of ith largest HFE by all of the other HFEs, $\mathrm{Sup}(h_{\sigma(i)}, h_{\sigma(j)})$ indicates the support of the ith largest HFE for the jth largest HFE, and $g : [0,1] \to [0,1]$ is a basic unit-interval monotone (BUM) function having the following properties: (1) $g(0) = 0$; (2) $g(1) = 1$; and (3) $g(x) \geq g(y)$ if $x > y$.

Theorem 13. Let $h_i(i=1,2,\cdots,n)$ be a collection of HFEs, then the aggregated value by HFHPOWA operator is also a HFE, and

$$\mathrm{HFHPOWA}_{\zeta}(h_1,h_2,\cdots,h_n)$$

$$= \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}} \left\{ \frac{\prod_{i=1}^{n}\left(1+(\zeta-1)\gamma_{\sigma(i)}\right)^{u_i} - \prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}\right)^{u_i}}{\prod_{i=1}^{n}\left(1+(\zeta-1)\gamma_{\sigma(i)}\right)^{u_i} + (\zeta-1)\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}\right)^{u_i}} \right\}, \tag{39}$$

where $u_i(i=1,2,\cdots,n)$ is a collection of weights satisfying the condition (38).

Proof. Similar to the proof of Theorem 3, Eq. (39) can be proved by mathematical induction on n. □

Remark 5. (1) If $g(x)=x$, then the HFHPOWA reduces to the HFHPA operator. In fact, by Eq. (39), we have

$$\begin{aligned}
\mathrm{HFHPOWA}_{\zeta}(h_1,h_2,\cdots,h_n) &= \oplus_{H_{i=1}^{n}}\left(u_i \cdot_H h_{\sigma(i)}\right) \\
&= \oplus_{H_{i=1}^{n}}\left(\left(g\left(\frac{R_i}{TV}\right)-g\left(\frac{R_{i-1}}{TV}\right)\right)\cdot_H h_{\sigma(i)}\right) \\
&= \oplus_{H_{i=1}^{n}}\left(\frac{V_{\sigma(i)}}{TV}\cdot_H h_{\sigma(i)}\right) \\
&= \oplus_{H_{i=1}^{n}}\left(\frac{1+T\left(h_{\sigma(i)}\right)}{\sum_{i=1}^{n}\left(1+T\left(h_{\sigma(i)}\right)\right)}\cdot_H h_{\sigma(i)}\right) \\
&= \oplus_{H_{i=1}^{n}}\left(\frac{1+T\left(h_{\sigma(i)}\right)}{\sum_{i=1}^{n}\left(1+T(h_i)\right)}\cdot_H h_i\right) \\
&= \mathrm{HFHPA}_{\zeta}(h_1,h_2,\cdots,h_n).
\end{aligned}$$

(2) If $\mathrm{Sup}(h_i,h_j)=k$, for all $i\neq j$, and $g(x)=x$, then

$$\mathrm{HFHPOWA}_{\zeta}(h_1,h_2,\cdots,h_n) = \oplus_{H_{i=1}^{n}}\left(\frac{1}{n}\cdot_H h_i\right)$$

which indicates that the HFHPOWA operator reduces to the hesitant fuzzy Hamacher average (HFHA) operator [26]. Especially, if $\mathrm{Sup}(h_i,h_j)=0$ for all $i\neq j$, and $g(x)=x$, then $u_i=\frac{1}{n}$, $i=1,2,\cdots,n$, and thus the HFHPOWA operator also reduces to the HFHA operator [26].

(3) If $\zeta=1$, then the HFHPOWA operator reduces to the hesitant fuzzy power ordered weighted average (HFPOWA) operator [20]:

$$\mathrm{HFPOWA}_1(h_1,h_2,\cdots,h_n)=\oplus_{i=1}^{n}(u_i h_{\sigma(i)})$$

$$=\ \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{1-\prod_{i=1}^{n}(1-\gamma_{\sigma(n)})^{u_i}\right\}, \quad (40)$$

where $u_i(i=1,2,\cdots,n)$ is a collection of weights satisfying the condition (38). If $\zeta=2,$ then the HFHPOWA operator reduces to the hesitant fuzzy Einstein power ordered weighted average (HFEPOWA) operator [24]:

$$\mathrm{HFEPOWA}_2(h_1,h_2,\cdots,h_n)$$

$$=\ \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\frac{\prod_{i=1}^{n}(1+\gamma_{\sigma(n)})^{u_i}\ -\prod_{i=1}^{n}(1-\gamma_{\sigma(n)})^{u_i}}{\prod_{i=1}^{n}(1+\gamma_{\sigma(n)})^{u_i}\ +\prod_{i=1}^{n}(1-\gamma_{\sigma(n)})^{u_i}}\right\}, \quad (41)$$

where $u_i(i=1,2,\cdots,n)$ is a collection of weights satisfying the condition (38).

Similar to Theorems 4 and 5, we have the properties of HFHPOWA operator as follows.

Theorem 14. If $h_i(i=1,2,\cdots,n)$ is a collection of HFEs and $u_i(i=1,2,\cdots,n)$ is the collection of the weights which satisfies the condition (38), then

$$\mathrm{HFHPOWA}_\zeta(h_1,h_2,\cdots,h_n)\le \mathrm{HFPOWA}(h_1,h_2,\cdots,h_n)\ .$$

Theorem 15. If $h_i(i=1,2,\cdots,n)$ is a collection of HFEs and $u_i(i=1,2,\cdots,n)$ is the collection of the weights which satisfies the condition (38), then we have the followings:

(1) Boundedness: If $h^-=\min\{\gamma_i|\gamma_i\in h_i\}$ and $h^+=\max\{\gamma_i|\gamma_i\in h_i\},$ then

$$h^-\le \mathrm{HFHPOWA}_\zeta(h_1,h_2,\cdots,h_n)\le h^+.$$

(2) Monotonicity: Let $h_i'(i=1,2,\cdots,n)$ be a collection of HFEs, if for any $h_{\sigma(i)}$ and $h'_{\sigma(i)}(i=1,2,\cdots,n),$ $\gamma_{\sigma(i)}\le\gamma'_{\sigma(i)},$ then

$$\mathrm{HFHPOWA}_\zeta(h_1,h_2,\cdots,h_n)\le \mathrm{HFHPWA}_\zeta(h_1',h_2',\cdots,h_n').$$

Definition 9. Let $h_i(i=1,2,\cdots,n)$ be a collection of HFEs. A hesitant fuzzy Hamacher power ordered weighted geometric (HFHPOWG) operator is a function $H^n\to H$ such that

$$\mathrm{HFHPOWG}_\zeta(h_1,h_2,\cdots,h_n)=\otimes_{H_{i=1}^{n}}\left(h_{\sigma(n)}^{\wedge_H u_i}\right)\ , \quad (42)$$

where parameter $\zeta>0,$ $h_{\sigma(i)}$ is the ith largest HFE of $h_j(j=1,2,\cdots,n),$ and $u_i(i=1,2,\cdots,n)$ is a collection of weights satisfying the condition (38).

Theorem 16. If $h_i(i=1,2,\cdots,n)$ is a collection of HFEs, then the aggregated value by HFHPOWG operator is also a HFE, and

$$\mathrm{HFHPOWG}_{\zeta}(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{ \frac{\zeta \prod_{i=1}^{n} (\gamma_{\sigma(i)})^{u_i}}{\prod_{i=1}^{n} \left(1 + (\zeta - 1)(1 - \gamma_{\sigma(i)})\right)^{u_i} + (\zeta - 1) \prod_{i=1}^{n} (\gamma_{\sigma(i)})^{u_i}} \right\}, \tag{43}$$

where $h_{\sigma(i)}$ is the ith largest HFE of $h_j (j = 1,2,\cdots,n)$ and $u_i (i = 1,2,\cdots,n)$ is the collection of the weights satisfying the condition (38).

Proof. Similar to the proof of Theorem 3, Eq. (43) can be proved by Mathematical induction on n. □

Remark 6. (1) If $g(x) = x$, then the HFHPOWG reduce to the HFHPG operator. In fact, by Eq. (42), we have

$$\begin{aligned} \mathrm{HFHPOWG}_{\zeta}(h_1, h_2, \cdots, h_n) &= \otimes_{H_{i=1}^{n}} \left(h_{\sigma(i)}^{\wedge_H u_i} \right) \\ &= \otimes_{H_{i=1}^{n}} \left(h_{\sigma(i)}^{\wedge_H \left(g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right) \right)} \right) \\ &= \oplus_{H_{i=1}^{n}} \left(h_{\sigma(i)}^{\wedge_H \frac{V_{\sigma(i)}}{TV}} \right) \\ &= \oplus_{H_{i=1}^{n}} \left(h_{\sigma(i)}^{\wedge_H \left(\frac{1 + T(h_{\sigma(i)})}{\sum_{i=1}^{n} (1 + T(h_{\sigma(i)}))} \right)} \right) \\ &= \oplus_{H_{i=1}^{n}} \left(h_{\sigma(i)}^{\wedge_H \left(\frac{1 + T(h_i)}{\sum_{i=1}^{n} (1 + T(h_i))} \right)} \right) \\ &= \mathrm{HFHPG}_{\zeta}(h_1, h_2, \cdots, h_n). \end{aligned}$$

(2) If $\mathrm{Sup}(h_i, h_j) = k$, for all $i \neq j$, and $g(x) = x$, then

$$\mathrm{HFHPOWG}_{\zeta}(h_1, h_2, \cdots, h_n) = \otimes_{H_{i=1}^{n}} \left(h_i^{\wedge_H \frac{1}{n}} \right)$$

which indicates that the HFHPOWG operator reduces to the hesitant fuzzy Hamacher geometric (HFHG) operator [26]. If $\mathrm{Sup}(h_i, h_j) = 0$ for all $i \neq j$, and $g(x) = x$, then the HFHPOWG operator also reduces to the HFHG operator [26].

(3) If $\zeta = 1$, then the HFHPOWG operator (43) reduces to the hesitant fuzzy power ordered weighted average (HFPOWG) operator [20]:

$$\mathrm{HFPOWG}_{1}(h_1, h_2, \cdots, h_n) = \otimes_{i=1}^{n} (h_{\sigma(i)})^{u_i}$$

$$= \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}} \left\{\prod_{i=1}^{n}(\gamma_{\sigma(i)})^{u_i}\right\}, \tag{44}$$

where $u_i(i = 1,2,\cdots,n)$ is a collection of weights satisfying the condition (38). If $\zeta = 2,$ then the HFHPOWG operator reduces to the hesitant fuzzy Einstein power ordered weighted geometric (HFEPOWG) operator [24]:

$$\text{HFEPOWG}_2(h_1, h_2, \cdots, h_n)$$
$$= \cup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}} \left\{\frac{2\prod_{i=1}^{n}(\gamma_{\sigma(i)})^{u_i}}{\prod_{i=1}^{n}(2-\gamma_{\sigma(i)})^{u_i} + \prod_{i=1}^{n}(\gamma_{\sigma(i)})^{u_i}}\right\}, \tag{45}$$

where $u_i(i = 1,2,\cdots,n)$ is a collection of weights satisfying the condition (38).

Similar to Theorems 7, 8 and 9, we have the properties of HFHPOWG operator as follows.

Theorem 17. If $h_i(i = 1,2,\cdots,n)$ is a collection of HFEs and $u_i(i = 1,2,\cdots,n)$ is the collection of the weights which satisfies the condition (38), then

$$\text{HFHPOWG}_\zeta(h_1, h_2, \cdots, h_n) \geq \text{HFPOWG}(h_1, h_2, \cdots, h_n).$$

Theorem 18. If $h_i(i = 1,2,\cdots,n)$ is a collection of HFEs and $u_i(i = 1,2,\cdots,n)$ is the collection of the weights which satisfies the condition (38), then we have the followings:

(1) Boundedness: If $h^- = \min\{\gamma_i|\gamma_i \in h_i\}$ and $h^+ = \max\{\gamma_i|\gamma_i \in h_i\},$ then

$$h^- \leq \text{HFHPOWG}_\zeta(h_1, h_2, \cdots, h_n) \leq h^+.$$

(2) Monotonicity: Let $h'_i(i = 1,2,\cdots,n)$ be a collection of HFEs, if for any $h_{\sigma(i)}$ and $h'_{\sigma(i)}(i = 1,2,\cdots,n),$ $\gamma_{\sigma(i)} \leq \gamma'_{\sigma(i)},$ then

$$\text{HFHPOWG}_\zeta(h_1, h_2, \cdots, h_n) \leq \text{HFHPOWG}_\zeta(h'_1, h'_2, \cdots, h'_n).$$

Theorem 19. If $h_i(i = 1,2,\cdots,n)$ is a collection of HFEs and $u_i(i = 1,2,\cdots,n)$ is the collection of the weights satisfying the condition (38), then we have

(1) $\text{HFHPOWA}_\zeta(h_1^C, h_2^C, \cdots, h_n^C) = (\text{HFHPOWG}_\zeta(h_1, h_2, \cdots, h_n))^C;$

(2) $\text{HFHPOWG}_\zeta(h_1^C, h_2^C, \cdots, h_n^C) = (\text{HFHPOWA}_\zeta(h_1, h_2, \cdots, h_n))^C.$

4 An approach to multiple attribute decision making based on hesitant fuzzy Hamacher power aggregation operators

In this section, we use hesitant fuzzy Hamacher power aggregation operators to develop an approach to multiple attribute decision making (MADM) with hesitant fuzzy information.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of n alternatives and $G = \{g_1, g_2, \cdots, g_m\}$ be a set of m attributes. Suppose the decision maker provides the evaluating values that the alternatives $x_j(i = 1, 2, \cdots, n)$ satisfy the attributes $g_i(j = 1, 2, \cdots, m)$ represented by the HFEs $h_{ij}(i = 1, 2, \cdots, m;\ j = 1, 2, \cdots, n)$. All these HFEs are contained in the hesitant fuzzy decision matrix $D = (h_{ij})_{m \times n}$ (see Table 1).

Table 1. Hesitant fuzzy decision matrix D

	x_1	x_2	$\cdots$	x_n
g_1	h_{11}	h_{12}	$\cdots$	h_{1n}
g_2	h_{21}	h_{22}	$\cdots$	h_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
g_m	h_{m1}	h_{m2}	$\cdots$	h_{mn}

The following steps can be used to solve the MADM problem under the hesitant fuzzy environment, and obtain an optimal alternative:

step 1 : Obtain the normalized hesitant fuzzy decision matrix. In general, the attribute set G can be divided two subsets: G_1 and G_2, where G_1 and G_2 are the set of benefit attributes and cost attributes, respectively. If all the attributes are of the same type, then the evaluation values do not need normalization, whereas if there are benefit attributes and cost attributes in MADM, in such cases, we may transform the evaluation values of cost type into the evaluation values of the benefit type by the following normalization formula:

$$r_{ij} = \begin{cases} h_{ij}, & j \in G_1 \\ h_{ij}^{C}, & j \in G_2 \end{cases}, \tag{46}$$

where $h_{ij}^{C} = \cup_{\gamma_{ij} \in \bar{h}_{ij}} \{1 - \gamma_{ij}\}$ is the complement of h_{ij}. Then we obtain the normalized hesitant fuzzy decision matrix $H = (r_{ij})_{m \times n}$ (see Table 2).

Table 2. Normalized hesitant fuzzy decision matrix H

	x_1	x_2	$\cdots$	x_n
g_1	r_{11}	r_{12}	$\cdots$	r_{1n}
g_2	r_{21}	r_{22}	$\cdots$	r_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
g_m	r_{m1}	r_{m2}	$\cdots$	r_{mn}

step 2: Calculate the supports

$$\mathrm{Sup}(r_{ij}, r_{kj}) = 1 - d(r_{ij}, r_{kj}),\ j = 1,2,\cdots,n,\ i,k = 1,2,\cdots,m \tag{47}$$

which satisfy conditions (1)-(3) in Definition 5, Here we assume that $d(r_{ij}, r_{kj})$ is the hesitant normalized Hamming distance between r_{ij} and r_{kj} given in Eq. (10).

step 3: Calculate the weights of evaluating values. Calculate the support $T(r_{ij})$ of the HFE r_{ij} by the other HFEs $r_{kj}(k = 1,2,\cdots,m,\text{and } k \neq i)$:

$$T(r_{ij}) = \sum_{k=1,k\neq i}^{m} \mathrm{Sup}(r_{ij}, r_{kj}) \tag{48}$$

and then calculate the weights $\rho_{ij}(i = 1,2,\cdots,m)$ that are associated with HFEs $r_{ij}(i = 1,2,\cdots,m)$:

$$\rho_{ij} = \frac{(1 + T(r_{ij}))}{\sum_{i=1}^{m}(1 + T(r_{ij}))},\ i = 1,2,\cdots,m, \tag{49}$$

where $\rho_{ij} \geq 0$, $i = 1,2,\cdots,m$, and $\sum_{i=1}^{m} \rho_{ij} = 1$.

step 4: Compute overall assessments of alternative. Utilize the HFHPWA operator (Eq. (12)):

$$r_j = \mathrm{HFHPWA}_{\zeta}(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_{1j}\in r_{1j},\gamma_{2j}\in r_{2j},\cdots,\gamma_{mj}\in r_{mj}} \left\{ \frac{\prod_{i=1}^{m}\left(1 + (\zeta - 1)\gamma_{ij}\right)^{\rho_{ij}} - \prod_{i=1}^{m}(1 - \gamma_{ij})^{\rho_{ij}}}{\prod_{i=1}^{m}\left(1 + (\zeta - 1)\gamma_{ij}\right)^{\rho_{ij}} + (\zeta - 1)\prod_{i=1}^{m}(1 - \gamma_{ij})^{\rho_{ij}}} \right\}, \tag{50}$$

or the HFHPWG operator (Eq. (18)):

$$r_j = \mathrm{HFHPWG}_{\zeta}(h_1, h_2, \cdots, h_n)$$

$$= \cup_{\gamma_{1j}\in r_{1j},\gamma_{2j}\in r_{2j},\cdots,\gamma_{mj}\in r_{mj}} \left\{ \frac{\zeta \prod_{i=1}^{m}(\gamma_{ij})^{\rho_{ij}}}{\prod_{i=1}^{m}(1 + (\zeta - 1)(1 - \gamma_{ij}))^{\rho_{ij}} + (\zeta - 1)\prod_{i=1}^{m}(\gamma_{ij})^{\rho_{ij}}} \right\}. \tag{51}$$

to aggregate all the evaluating values $\bar{r}_{ij}$ $(i = 1,2,\cdots,m)$ of the jth column and get the overall rating value $\bar{r}_j$ corresponding to the alternative $x_j(j = 1,2,\cdots,n)$.

$step$ 5: Rank the order of all alternatives. Utilize the method in Definition 3 to rank the overall rating values $r_j(j = 1, 2, \cdots, n)$, rank all the alternatives $x_j(j = 1, 2, \cdots, n)$ in accordance with $r_j(j = 1, 2, \cdots, n)$ in descending order, and finally select the most desirable alternative(s) with the largest overall evaluation value.

$step$ 6: End.

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