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Argument properties of certain meromorphic functions defined by a linear operator

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Argument properties of certain meromorphic functions defined by a linear operator
(선형 연산자에 의해 정의된 유리형 함수의 편각성질)

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선형 연산자에 의해 정의된 유리형 함수의 편각성질

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요 약

기하함수 이론은 지금까지 많은 학자들에 의하여 연구되어 왔다. 특히 Miller와 Mocanu는 일차 미분 종속 이론을 소개하여 해석함수들의 다양한 기하학적 성질들을 조사하였다.

본 논문에서는 Liu와 Srivastava에 의하여 정의된 일차 연산자를 이용하여 유리형 함수들의 새로운 부분 족들을 소개하고 이 부분 족들에 대하여 포함관계를 조사하였다. 그리고 미분종속이론과 Nunokwa에 의해 소개된 결과를 응용하여 유리형 함수들의 편각성질들과 적분 보존 성질들을 조사하였다. 더욱이 논문에서 소개된 주 결과들을 응용하여 기존의 알려진 여러 결과들을 발전 시켰다.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the punctured open unit disk $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ in $\mathbb{U} = \mathbb{D} \cup \{0\}$, and $f(z) = g(w(z))$. We denote this sunordination by $g \prec h$ or $g(z) \prec h(z)$ $(z \in \mathbb{U})$. In particular, if the function h is univalent in $\mathbb{U}$, the above subordination is equivalent to $g(0) = h(0)$ and $g(\mathbb{U}) \subset h(\mathbb{U})$ (see, e.g., Miller and Mocanu [9]).

Now we define the function $\phi(a, c; z)$ by

$$\phi(a, c; z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(a)_{k+1}}{(c)_{k+1}} z^k \quad (a > 0;\ c \neq 0, -1, -2, \cdots;\ z \in \mathbb{D}), \tag{1.1}$$

where $(\lambda)_k$ is the Pochhammer symbol (or the shifted factorial) defined by

$$(\lambda)_k := \begin{cases} 1 & \text{if } k = 0 \\ \lambda(\lambda+1)\cdots(\lambda+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases}$$

Corresponding to the function $\phi(a, c; z)$, we introduce a linear operator $L(a, c)$ which is defined by means of the following Hadamard product (or convolution):

$$L(a, c)f(z) = \phi(a, c; z) * f(z) \quad (f \in \Sigma), \tag{1.2}$$

It is easily verified from (1.1) and (1.2) that

$$z(L(a, c)f(z))' = aL(a + 1, c)f(z) - (a + 1)L(a, c)f(z). \tag{1.3}$$

The operator $L(a, c)$ was introduced and studied by Liu and Srivastava [7] recently. This operator $L(a, c)$ was motivated essentially by the familiar Carson-Shaffer operator $L(a, c)$ which has been used widely on the space of analytic and univalent functions in $\mathbb{U}$(see, for details [3]; see also [15]).

Let $\mathcal{N}$ be the class of analytic functions h with $h(0) = 1$, which are convex and univalent in $\mathbb{U}$ and $\mathrm{Re}\{h(z)\} > 0$ $(z \in \mathbb{U})$.

Making use of the principle of subordination between analytic functions, we introduce the following new subclasses $\Sigma_s(n; a, c; h)$ and $\Sigma_c(n; a, c; A, B; \alpha)$ of the class Σ.

Let the functions $g_1, \cdots, g_n$ be in the class Σ. Then we say that the functions $g_1, \cdots, g_n$ are in the class $\Sigma_s(n; a, c; h)$ if they satisfy the condition:

$$-\frac{z(L(a,c)g_i(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a,c)g_j(z)} \prec h(z) \quad (z \in \mathbb{U};\ i = 1, \cdots, n;\ h \in \mathcal{N}), \tag{1.4}$$

where $z\sum_{j=1}^{n} L(a,c)g_j(z) \neq 0$ in $\mathbb{U}$.
In particular, we set

$$\Sigma_s\left(n; a, c; \frac{1 + Az}{1 + Bz}\right) := \Sigma_s(n; a, c; A, B) \ (-1 < B < A \leq 1;\ z \in \mathbb{U}).$$

For $n = a = c = 1$, the class $\Sigma_s(n; a, c; h)$ is the well-known class of meromorphic starlike functions in $\mathbb{U}$. Furthermore, we note that the classes $\Sigma_s(n; a, 1; h)$ and $\Sigma_s(1; a, 1; h)$ have been studied by Bharati and Rajagopal [2], and Padmanabhan and Manjini [12], respectively.

Let $\Sigma_c(n; a, c; A, B; \alpha)$ be the class of functions $f \in \Sigma$ satisfying the argument inequality:

$$\left|\arg\left(-\frac{z(L(a,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a,c)g_j(z)}\right)\right| < \frac{\pi}{2}\alpha \tag{1.5}$$

$$(z \in \mathbb{U};\ 0 < \alpha \leq 1;\ g_j \in \Sigma_s(n; a, c; A, B);\ j = 1, \cdots, n).$$

If we take $n = a = c = \alpha = 1$ in (1.5), $\Sigma_c(n; a, c; A, B; \alpha)$ is the familiar subclass of meromorphic close-to-convex functions in $\mathbb{U}$ introduced by Libera and Robertson [6](also, see [14]).

In the present paper, we give some argument properties of meromorphic functions belonging to Σ which contain the basic inclusion relationship among the classes $\Sigma_s(n; a, c; h)$ and $\Sigma_c(n; a, c; A, B; \alpha)$. The integral preserving properties in connection with the operator $L(a, c)$ defined by (1.2) are also considered. Furthermore, we obtain the previous results of Bajpai [1], Bharati and Rajagopal [2], Goel and Sohi [5], Padmanabhan and Manjini [12] as special cases.

2. Main Results

The following results will be required in our investigation.

Lemma 2.1 [4]. *Let h be convex univalent in $\mathbb{U}$ with $h(0) = 1$ and* Re $(\lambda h(z) + \nu) > 0 (\lambda, \nu \in \mathbb{C})$. *If q is analytic in $\mathbb{U}$ with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\lambda q(z) + \nu} \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.2 [8]. *Let h be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with* Re $\omega(z) \geq 0$. *If q is analytic in $\mathbb{U}$ and $q(0) = h(0)$, then*

$$q(z) + \omega(z) zq'(z) \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.3 [11]. *Let q be analytic in $\mathbb{U}$, $q(0) = 1$, and $q(z) \neq 0$ in $\mathbb{U}$. Suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg\ \{q(z)\}| \quad < \quad \frac{\pi}{2}\alpha \ \ \text{for} \ \ |z| < |z_0| \tag{2.1}$$

and

$$|\arg\ \{q(z_0)\}| \quad = \quad \frac{\pi}{2}\alpha \qquad (\alpha > 0). \tag{2.2}$$

Then

$$\frac{z_0 q'(z_0)}{q(z_0)} \quad = \quad ik\alpha, \tag{2.3}$$

where

$$k \geq \frac{1}{2}\left(b + \frac{1}{b}\right) \ \ \text{when} \ \ \arg\ \{q(z_0)\} = \frac{\pi}{2}\alpha, \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(b + \frac{1}{b}\right) \ \ \text{when} \ \ \arg\ \{q(z_0)\} = -\frac{\pi}{2}\alpha, \tag{2.5}$$

and

$$\{q(z_0)\}^{\frac{1}{\alpha}} = \pm ib \qquad (b > 0). \tag{2.6}$$

First of all, with the help of Lemma 2.1 and Lemma 2.2, we obtain the following.

Proposition 2.1. *Let* $a > 0$ *and* $h \in \mathcal{N}$ *with* $\max_{z\in\mathbb{U}} \operatorname{Re}\{h(z)\} < a+1$. *If* $g_1, \cdots, g_n \in \Sigma_s(n; a+1, c; h)$, *then* $g_1, \cdots, g_n \in \Sigma_s(n; a, c; h)$.

Proof. Let

$$p_i(z) = -\frac{z(L(a,c)g_i(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a,c)g_j(z)} \quad (i = 1, \cdots, n).$$

By using the equation (1.3), we get

$$\frac{1}{n}\sum_{j=1}^{n}(L(a,c)g_j(z))p_i(z) - (a+1)L(a,c)g_i(z) = -aL(a+1,c)g_i(z). \tag{2.7}$$

By differentiating both sides of (2.7) with respect to z, and simplifying, we have

$$p_i(z) + \frac{zp_i'(z)}{-\frac{1}{n}\sum_{i=1}^{n} p_i(z) + a + 1} = -\frac{z(L(a+1,c)g_i(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a+1,c)g_j(z)} \prec h(z). \tag{2.8}$$

$$(z \in \mathbb{U};\ i = 1, \cdots, n),$$

since $g_1, \cdots, g_n \in \Sigma_s(n; a+1, c; h)$. Since h is convex, for any $z_0 \in \mathbb{U}$, there exists a point $\zeta_0 \in \mathbb{U}$ such that

$$q(z_0) + \frac{z_0 q'(z_0)}{-q(z_0) + a + 1} = h(\zeta_0),$$

where

$$q(z) = \frac{1}{n}\sum_{i=1}^{n} p_i(z).$$

Then we obtain from Lemma 2.1 that $q \prec h$. Applying Lemma 2.2 with

$$\omega(z) = \frac{1}{-q(z) + a + 1}$$

to (2.8) again, it follows that $p_i \prec h$ for all i $(i = 1, \cdots, n)$, which implies $g_1, \cdots, g_n \in \Sigma_s(n; a, c; h)$.

Next, we prove that

$$z \sum_{j=1}^{n} L(a,c) g_j(z) \neq 0 \ \ (z \in \mathbb{U}).$$

Since $g_1, \cdots, g_n \in \Sigma_s(n; a+1, c; h)$ and h is convex, we find that there exists a point $\zeta_0 \in \mathbb{U}$ such that for any $z_0 \in \mathbb{U}$,

$$r(z_0) := -\frac{z_0(\sum_{j=1}^{n} L(a+1,c) g_j(z_0))'}{\sum_{j=1}^{n} L(a+1,c) g_j(z_0)} = h(\zeta_0),$$

and hence $r \prec h$. Also, we note that

$$\sum_{j=1}^{n} L(a,c) g_j(z) = \frac{a}{z^{a+1}} \int_0^z t^a \sum_{j=1}^{n} L(a+1,c) g_j(t) dt.$$

Thus, by applying Theorem 1 of [10], we conclude that

$$z \sum_{j=1}^{n} L(a,c) g_j(z) \neq 0 \ \ (z \in \mathbb{U}).$$

This evidently completes the proof of Proposition 2.1.

If we take $h(z) = \frac{1+Az}{1+Bz}$ $(-1 < B < A \leq 1)$ in Proposition 2.1, we have the following.

Corollary 2.1. *Let* $1 + A < (a+1)(1+B)$ $(a > 0;\ -1 < B < A \leq 1)$. *Then the inclusion relation:*

$$\Sigma_s(n; a+1, c; A, B) \subset \Sigma_s(n; a, c; A, B)$$

holds true.

Remark 2.1. If we let $c = 1$ in Proposition 2.1, then we have the result of Bharati and Rajagopal [2], which includes the results given by Padmanabhan and Manjini [12] as a special case.

Proposition 2.2. *Let $\mu > 0$ and $h \in \mathcal{N}$ with $\max_{z\in\mathbb{U}} \operatorname{Re}\{h(z)\} < \mu + 1$. If $g_1, \cdots, g_n \in \Sigma_s(n; a, c; h)$, then $F_\mu(g_1), \cdots, F_\mu(g_n) \in \Sigma_s(n; a, c; h)$, where F_μ is the integral operator defined by*

$$F_\mu(g_i) := F_\mu(g_i)(z) = \frac{\mu}{z^{\mu+1}} \int_0^z t^\mu g_i(t)dt \quad ((1, \cdots, n;\ \mu \geq 0). \tag{2.9}$$

Proof. Let

$$p_i(z) = -\frac{z(L(a,c)F_\mu(g_i)(z))'}{\frac{1}{n}\sum_{j=1}^n L(a,c)F_\mu(g_j)(z)} \quad (i = 1, \cdots, n).$$

From (2.9), we have

$$z(L(a,c)F_\mu(g_i)(z))' = \mu L(a,c)g_i(z) - (\mu+1)L(a,c)F_\mu(g_i)(z). \tag{2.10}$$

Then, by using (2.10), we get

$$\frac{1}{n}\sum_{j=1}^n (L(a,c)F_\mu(g_j)(z))p_i(z) - (\mu+1)L(a,c)F_\mu(g_i)(z) = -\mu L(a,c)g_i(z). \tag{2.11}$$

Differentiating the both sides of (2.11) with respect to z and simplifying, we have

$$p_i(z) + \frac{zp_i'(z)}{-\frac{1}{n}\sum_{j=1}^n p_i(z) + \mu + 1} = -\frac{z(L(a,c)g_i(z))'}{-\frac{1}{n}\sum_{j=1}^n L(a,c)g_j(z)}.$$

Then, by the same arguments as in the proof of Proposition 2.1, it follows that Proposition 2.2 holds true as stated.

From Proposition 2.2, we have immediately the following.

Corollary 2.2. *Let* $1 + A < (\mu + 1)(1 + B)$ $(\mu > 0;\ -1 < B < A \leq 1)$. *If* $g_1, \cdots, g_n \in \Sigma_s(n; a, c; A, B)$, *then* $F_\mu(g_1), \cdots, F_\mu(g_n) \in \Sigma_s(n; a, c; A, B)$, *where* F_μ *is the integral operator defined by* (2.9).

Remark 2.2. If we take $n = a = c = 1$ and $B \to A$ in Corollary 2.2, then we have the corresponding results of Goel and Sohi [5]. In particular, for $n = a = c = \mu = 1$ and $B \to A$, Corollary 2.2 yields the result of Bajpai [1].

Now, we derive

Theorem 2.1. *Let* $0 < \delta \leq 1$ *and* $1 + A < (a + 1)(1 + B)$ $(a > 0;\ -1 < B < A \leq 1)$. *If a function* $f \in \Sigma$ *is satisfies the condition:*

$$\left| \arg \left(-\frac{z(L(a+1, c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a+1, c)g_j(z)} \right) \right| < \frac{\pi}{2}\delta,$$

where $g_1, \cdots, g_n \in \Sigma_s(n; a+1, c; A, B)$, *then*

$$\left| \arg \left(-\frac{z(L(a, c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a, c)g_j(z)} \right) \right| < \frac{\pi}{2}\alpha,$$

where $\alpha (0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha \cos \frac{\pi}{2} t_1}{\frac{A-1}{1-B} + a + 1 + \alpha \sin \frac{\pi}{2} t_1} \right) \tag{2.12}$$

and

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{A - B}{(a+1)(1 - B^2) - (1 - AB)} \right). \tag{2.13}$$

Proof. Let

$$p(z) = -\frac{z(L(a,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a,c)g_j(z)} \text{ and } q(z) = \frac{1}{n}\sum_{i=1}^{n} q_i(z),$$

where

$$q_i(z) = -\frac{z(L(a,c)g_i(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a,c)g_j(z)} \quad (i = 1, \cdots, n).$$

Using (1.3), we have

$$\frac{1}{n}\sum_{j=1}^{n}(L(a,c)g_j(z))p(z) - (a+1)L(a,c)f(z) = -aL(a+1,c)f(z). \tag{2.14}$$

Differentiating (2.14) with respect to z and simplifying, we obtain

$$-\frac{z(L(a+1,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a+1,c)g_j(z)} = p(z) + \frac{zp'(z)}{-q(z)+a+1}.$$

Since $g_1, \cdots, g_n \in \Sigma_s(n; a+1, c; A, B)$, by Corollary 2.1, we know that $g_1, \cdots, g_n \in \Sigma_s(n; a, c; A, B)$ and so

$$q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U};\ -1 \leq B < A \leq 1).$$

Hence we observe [13] that

$$\left| q(z) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \qquad (z \in \mathbb{U};\ -1 < B < A \leq 1) \tag{2.15}$$

Then, by using (2.15), we have

$$-q(z) + a + 1 = \rho e^{i\frac{\pi\phi}{2}},$$

where

$$\begin{cases} a+1-\frac{1+A}{1+B} \ < \ \rho \ < \ a+1+\frac{A-1}{1-B} \\ -t_1 \ < \ \phi \ < \ t_1. \end{cases}$$

when t_1 is given by (2.13).

We note that p is analytic in $\mathbb{U}$ with $p(0) = 1$. Let h be the function which maps onto the angular domain $\{w : |\arg\{w\}| < \frac{\pi}{2}\delta\}$ with $h(0) = 1$. Applying Lemma 2.2 for this h with $\omega(z) = 1/(-q(z) + a + 1)$, we see that Re $p(z) > 0$ and hence $p(z) \neq 0$ in $\mathbb{U}$.

If there exist a point $z_0 \in \mathbb{U}$ such that the conditions (2.1) and (2.2) are satisfied, then (by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, we suppose that

$$\{p(z_0)\}^{\frac{1}{\alpha}} = ia \ \ (a > 0).$$

Then we obtain

$$\begin{aligned} \arg \quad & \left(p(z_0) + \frac{z_0 p'(z_0)}{-q(z_0) + a + 1} \right) \\ &= \frac{\pi}{2}\alpha + \arg \ \left(1 + i\alpha k (\rho e^{i\frac{\pi\phi}{2}})^{-1} \right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1} \left(\frac{\alpha k \sin\frac{\pi}{2}(1-\phi)}{\rho + \alpha k \cos\frac{\pi}{2}(1-\phi)} \right) \\ &\geq \frac{\pi}{2}\alpha + \tan^{-1} \left(\frac{\alpha \cos\frac{\pi}{2}t_1}{\left(\frac{1+A}{1+B} + a + 1\right) + \alpha \sin\frac{\pi}{2}t_1} \right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ and t_1 are given by (2.12) and (2.13), respectively. This evidently contradicts the assumption of Theorem 2.1.

Next, we suppose that

$$\{p(z_0)\}^{\frac{1}{\alpha}} = -ia \ \ (a > 0).$$

Then we have

$$\begin{aligned} \arg \quad & \left(p(z_0) + \frac{z_0 p'(z_0)}{-q(z_0) + a + 1} \right) \\ & \leq -\frac{\pi}{2}\alpha - \tan^{-1}\left(\frac{\alpha \cos \frac{\pi}{2} t_1}{\left(\frac{1+A}{1+B} + a + 1\right) + \alpha \sin \frac{\pi}{2} t_1} \right) \\ & = -\frac{\pi}{2}\delta, \end{aligned}$$

where δ and t_1 are given by (2.12) and (2.13), respectively. This also is a contradiction to the assumption of Theorem 2.1. Therefore we complete the proof of our theorem.

From Theorem 2.1, we obtain immediately the following.

Corollary 2.3. *Let* $1 + A < (a+1)(1+B)$ $(a > 0; \ -1 < B < A \leq 1)$*. Then the inclusion relation:*

$$\Sigma_c(n; a+1, c; A, B; \alpha) \subset \Sigma_c(n; a, c; A, B; \alpha)$$

holds true.

Next, we prove the following theorem.

Theorem 2.2. *Let* $0 < \delta \leq 1$ *and* $1 + A < (\mu+1)(1+B)$ $(\mu > 0; \ -1 < B < A \leq 1)$*. If a function* $f \in \Sigma$ *is satisfies the condition:*

$$\left| \arg \left(-\frac{z(L(a,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n} L(a,c) g_j(z)} \right) \right| < \frac{\pi}{2}\delta,$$

where $g_1, \cdots, g_n \in \Sigma_s(n; a, c; A, B)$*, then*

$$\left|\arg\left(-\frac{z(L(a,c)F_\mu(f)(z))'}{\frac{1}{n}\sum_{j=1}^{n}L(a,c)F_\mu(g_j)(z)}\right)\right| < \frac{\pi}{2}\alpha,$$

where F_μ is the integral operator defined by (2.9) *and* $\alpha(0<\alpha\le 1)$ *is the solution of the equation* (2.12) *with* $a=\mu$.

Proof. Let

$$p(z) = -\frac{z(L(a,c)F_c(f)(z))'(z)}{\frac{1}{n}\sum_{j=1}^{n}L(a,c)F_c(g_j)(z)} \text{ and } q(z)=\frac{1}{n}\sum_{i=1}^{n}q_i(z),$$

where

$$q_i(z) = -\frac{z(L(a,c)F_c(g_i))'(z)}{\frac{1}{n}\sum_{j=1}^{n}L(a,c)F_c(g_j)(z)} \quad (i=1,\cdots,n).$$

Using the equation (2.7), we obtain

$$\frac{1}{n}\sum_{j=1}^{n}(L(a,c)F_\mu g_j(z))p(z)-(\mu+1)L(a,c)F_\mu f(z) = -\mu L(a,c)f(z). \tag{2.16}$$

Differentiating (2.16) with respect to z and simplifying, we obtain

$$-\frac{z(L(a,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n}L(a,c)g_j(z)} = p(z)+\frac{zp'(z)}{-q(z)+\mu+1}.$$

Since $g_1,\cdots,g_n\in\Sigma_s(n;a,c;A,B)$, by Proposition 2.2, we know that $g_1,\cdots,g_n\in\Sigma_s(n;a,c;A,B)$ Hence, we find that

$$q(z) \prec \frac{1+Az}{1+Bz} \quad (z\in\mathbb{U};\ -1\le B<A\le 1).$$

The remaining part of the proof is similar to that in the proof of Theorem 2.1 and so we omit the details involved.

From Theorem 2.2, we obtain the following.

Corollary 2.4. *Let* $1+A<(\mu+1)(1+B)$ $(\mu>0;\ -1<B<A\leq 1)$. *If* $f\in\Sigma_c(n;a,c;A,B;\alpha)$, *then* $F_\mu(f)\in\Sigma_c(n;a,c;A,B;\alpha)$, *where* F_μ *is the integral operator defined by* (2.6).

Remark 2.3. From Theorem 2.2 or Corollary 2.4, we see that every function in $\Sigma_c(n;a,c;A,B;\alpha)$ preserves the angles under the integral operator defined by (2.6). If we put $n=a=c=\alpha=1$ and $B\to A$ in Corollary 2.4, we obtain the result given earlier by Goel and Sohi [5].

Finally, we state Theorem 2.3 below. The proof is much akin to that of Theorem 2.1 and so the details may be omitted.

Theorem 2.3. *Let* $0<\delta\leq 1$, $\gamma\geq 0$ *and* $1+A<(a+1)(1+B)$ $(a>0;\ -1<B<A\leq 1)$. *If a function* $f\in\Sigma$ *satisfies the condition:*

$$\left|\arg\left(-\left[\gamma\frac{z(L(a+1,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n}L(a+1,c)g_j(z)}+(1-\gamma)\frac{z(L(a,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n}L(a,c)g_j(z)}\right]\right)\right|<\frac{\pi}{2}\delta,$$

where $g_1,\cdots,g_n\in\Sigma_s(n;a+1,c;A,B)$, *then*

$$\left|\arg\left(-\frac{z(L(a,c)f(z))'}{\frac{1}{n}\sum_{j=1}^{n}L(a,c)g_j(z)}\right)\right|<\frac{\pi}{2}\alpha,$$

where α $(0<\alpha\leq 1)$ *is the solution of the equation* :

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\gamma\cos\frac{\pi}{2}t_1}{\left(\frac{1+A}{1+B}+a+1\right)+\alpha\sin\frac{\pi}{2}t_1}\right)$$

when t_1 *is given by* (2.13).

References

1. S. K. Bajpai, *A note on a class of meromorphic univalent functions*, Rev. Roumaine Math. Pures Appl. **22**(1977), 295-297.

2. R. Bharati and R. Rajagopal, Meromorphic functions and differential sunordination, in *New Trends in Geometric function Theory and Applications* (S. Ponnusamy, Editor), World Scientific, Singapore, 1990, pp. 10-17.

3. B. C. Carlson and D. B. Shaffer, *Starlike and prestarlike hypergeometric functions,* SIAM J. Math. Anal. **159**(1984), 737-745.

4. P. Enigenberg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities, **3**(1983)(Birkhauser Verlag-Basel), 339-348.

5. R. M. Goel and N. S. Sohi, *On a class of meromorphic functions*, Glas. Mat. **17(37)**((1981), 19-28.

6. R. J. Libera and M. S. Robertson, *Meromorphic close-to-convex functions*, Michigan Math. J. **8**(1961), 167-176.

7. J. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. **259**(2001), 566-581.

8. S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28**(1981), 157-171.

9. S. S. Miller and P. T. Mocanu, Differential Subordinations, Theory and Applications, Marcel Dekker, Inc., New York, Basel, 2000.

10. P. T. Mocanu and G. S. Salagean, *Integral operators and meromorphic starlike functions*, Mathematica **32(55)**(1990), 147-152.

11. M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci., **69**(1993), 234-237.

12. K. S. Padmanabhan and R. Mangini, *Differential subordination and meromorphic functions*, Ann. Polon. Math. **49**(1998), 21-30.

13. H. Silverman and E. M. Silvia, *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math. **37**(1985), 48-61.

14. R. Singh, *Meromorphic close-to-convex functions*, J. Indian. Math. Soc. **33**(1969), 13-20.

15. H. M. Srivastava and S. Owa, *bSome characterizations and distortions theorems involving fractional calculus, generalized hypergeometric functions, Hadamard products, linear operators, and certain subclasses of analytic functions*, Nagoya Math. J. **106**(1987), 1-28.