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Thesis for the Degree of Doctor of Philosophy

First order semilinear differential equations and their applications

by

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First order semilinear differential equations and their applications
(1계 준선형 미분방정식과 응용)

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요 약

본 논문에서 먼저 Cauchy 문제 중 제어기가 L^1-값을 가진 경우를 다루었다. 주작용소가 타원형 미분연산자가 각 주어지는 것으로 생성되는 해석적 반군에 의한 보간이론을 이용하여 L^1-공간을 포함하는 공간에서 정의됨을 밝혀 아직 까지 밝혀져 있지 않은 L^1-값을 갖는 제어문제를 다루었다. 그 후 Hilbert 공간상에서 정의된 작용소로 구성되어진 지연항을 가진 미분방정식에 대하여 잘 알려져 있는 기본해를 구성하여, 일반화된 고유공간이 완전할 때 근사적인 가제어성이 됨을 고유공간의 성질을 이용하여 밝힐 수 있다. 더 나아가 확대된 State 공간상에서 일반적인 준선형 발전방정식으로 전환하여 근사적인 가제어성을 확인 할 수 있다. 마지막 장은 주어진 지연 미분방정식에서 주작용소가 비유계 작용소인 경우, 기본해 구조를 해석하여 일반적인 선형결과에 대한 최적제어 이론을 확대하는 것이다. 우리는 관측에 의해 정의된 목적함수의 최적 제어의 존재성과 유일성을 증명하였으며, 비선형항의 미분가능의 조건이 없는 선형지연방정식으로 표현되는 수반행렬 상태로 묘사되어진 최적제어의 존재에 대한 필수조건을 유도 할 것이다. 아울러 우리는 목표값의 관측함수에 대한 제어이론과 적분미분형태의 비선형 항이 주어진 방정식에 대한 최적제어의 존재성도 밝힐 것이다.

Chapter 1
Introduction and Preliminaries

This paper is devoted to the functional analytic method for partial differential equations. We intend to present the fundamentals of the theory of abstract parabolic evolution equations and to show to apply to semilinear differential equations and systems arising in science. This kind of evolution differential equations arises in many practical mathematical models, such as, option pricing, population dynamics, physical, biological and engineering problems, etc. Main approach is known to the abstract parabolic evolution equations, namely, the semigroup methods, the variational methods, and the methods of using operational equations. The semigroup methods, which go back to the invention of the analytic semigroups in the middle of the last century, are characterized by precise formulas representing the solutions of the Cauchy problem for evolution equations.

In Chapter 2, we deal with the theory of interpolation spaces between initial Banach functional spaces and the domain of an elliptic differential operator. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order in $L^1(\Omega)$:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x), \qquad \text{(EO)}$$

where $(a_{i,j}(x) : i,j = 1,\cdots,n)$ is a positive definite symmetric matrix for each $x \in \Omega$, $a_{i,j} \in C^1(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$. The operator

$$\mathcal{A}'(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\cdot) + c(x)$$

is the formal adjoint of $\mathcal{A}$.

The object of Chapter 3 is to investigate the quality of reachable set of the following semilinear retarded parabolic type equation

$$\frac{d}{dt}x(t) = A_0x(t) + f(t), \quad t \in (0,T], \tag{CS}$$

where

$$f(t) = A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + f(t, x(t)) + B_0u(t).$$

Then the initial condition of system (CS) is given as follows:

$$x(0) = g^0, \quad x(s) = g^1(s), \quad \text{for} \quad s \in [-h, 0]. \tag{IC}$$

The existence and uniqueness of solution of the above system are proved in [19]. The condition for equivalence between the reachable set of the semilinear system and that of its corresponding linear system was established in [19, 10] and recently, [25, 26]. This paper is dealt with another applicable condition for controller of approximate control problem. Thus, the main result in this paper will show that the system (CS) with some conditions for the operator A_0 satisfies a sufficient condition for approximate controllability obtained in [19].

In Chapter 4, we deal with control problem for semilinear parabolic type equation in Hilbert space H as follows.

$$\begin{cases} \frac{d}{dt}x(t) = & A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds \\ & +f(t,x(t)) + \Phi_0 u(t), \\ x(0) = & g^0, \quad x(s) = g^1(s), \quad s \in [-h,0). \end{cases} \tag{PS}$$

Let $Z = H \times L^2(-h,0;V)$ be the state space of the equation (PS). Z is a product Hilbert space with the norm

$$||g||_Z = (|g^0|^2 + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

The operator A is defined as follows:

$$D(A) = \{g = (g^0, g^1) : \; g^0 \in H, \; g^1 \in L^2(-h,0;V),$$

$$g^1(0) = g^0, \; A_0 g^0 + \int_{-h}^{0} a(s)A_2 g^1(s)ds \in H\},$$

$$Ag = (A_0g^0 + \int_{-h}^{0} a(s)A_2g^1(s)ds, \dot{g}^1).$$

The equation (PS) can be transposed to an following general initial problem

$$\frac{d}{dt}z(t) = Az(t) + F(t,z(t)) + \Phi u(t), \tag{IP}$$

where $\Phi f = (\Phi_0 f, 0)$, $F(t,z(t)) = (f(t,x(t)),0)$

we will show from the approximately controllable of the system (IP) in space Z with the general assumption of nonlinear part. Moreover, we derive the relations between the controllability of the system (PS) and one of (IP).

In the last Chapter is concerned with the optimal control problem of the semilinear functional differential equation with delay in a Hilbert space. Applications of the optimal control problems for two types of cost functions are given; one is the averaging observation control and the other is the observation of terminal value. The principal operator of given equations generates an analytic semigroup and the nonlinear term is uniformly Lipschitz continuous with respect to the second variable. Two applications of the main results are given; one gives a uniqueness of the optimal control of the cost function defined by distributed observation and the other gives a feedback control law for the observation function of terminal value. Here, using techniques for the linear control problems and the properties of solutions of semilinear system as developed in [26, 12, 4], we obtain the existence of optimal controls for the equation, where the nonlinear term is given by the convolution product and give the maximal principle for given cost functions and present the necessary conditions of optimality which are described by the adjoint state corresponding to the linear retarded equation without a condition of differentiability of nonlinear term.

Thus, we give the existence and uniqueness of the optimal control of the cost function defined by distributed observation, and establish the maximal principle represented by the necessary conditions of optimality which are described by the adjoint state corresponding to the linear retarded equation without a condition of differentiability of nonlinear term. Moreover, we give a feedback control law for the observation function of terminal value, and the existence of optimal controls for the equation, where the nonlinear term is given by the convolution product.

Chapter 2
Intermediate spaces related to analytic semigroups generated by elliptic operators

2.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order in $L^1(\Omega)$:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x), \tag{EO}$$

where $(a_{i,j}(x) : i, j = 1, \cdots, n)$ is a positive definite symmetric matrix for each $x \in \Omega$, $a_{i,j} \in C^1(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$. The operator

$$\mathcal{A}'(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)) + c(x)$$

is the formal adjoint of $\mathcal{A}$.

For $1 < p < \infty$, we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

For $p^{'} = p/(p-1)$, we can also define the realization $\mathcal{A}^{'}$ in $L^{p^{'}}(\Omega)$ under Dirichlet boundary condition by $A^{'}_{p^{'}}$:

$$D(A^{'}_{p^{'}}) = W^{2,p^{'}}(\Omega) \cap W^{1,p^{'}}_0(\Omega),$$

$$A^{'}_{p^{'}}u = \mathcal{A}^{'}u \quad \text{for} \quad u \in D(A^{'}_{p^{'}}).$$

It is known that $-A_p$ and $-A^{'}_{p^{'}}$ generate analytic semigroups in $L^p(\Omega)$ and $L^{p^{'}}(\Omega)$, respectively, and $A^*_p = A^{'}_{p^{'}}$.

For brevity, we assume that $0 \in \rho(A_p)$. From the result of Seeley [32] (see also Triebel [9, p. 321]) we obtain that

$$[D(A_p), L^p(\Omega)]_{\frac{1}{2}} = W^{1,p}_0(\Omega),$$

and hence, may consider that

$$D(A_p) \subset W^{1,p}_0(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A^{'}_{p^{'}})^*.$$

Let $(A^{'}_{p^{'}})^{'}$ be the adjoint of $A^{'}_{p^{'}}$ considered as a bounded linear operator from $D(A^{'}_{p^{'}})$ to $L^{p^{'}}(\Omega)$. Let A be the restriction of $(A^{'}_{p^{'}})^{'}$ to $W^{1,p}_0(\Omega)$. Then by the interpolation theory, the operator A is an isomorphism from $W^{1,p}_0(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction $A^{'}$ of $(A_p)^{'}$ belonging to $B(L^{p^{'}}(\Omega), D(A_p)^*)$ to $W^{1,p^{'}}_0(\Omega)$ is an isomorphism from $W^{1,p^{'}}_0(\Omega)$ to $W^{-1,p^{'}}(\Omega)$. For $q \in (1, \infty)$, we set

$$H_{p,q} = (W^{1,p}_0, W^{-1,p})_{1/q,q}.$$

As seen in proposition 3.1 in Jeong [16], the operators $-A$ and $-A'$ generate an analytic semigroup in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively. Furthermore, $-A$ also generates an analytic semigroup in $H_{p,q}$. The spaces $H_{p,q}$ is ζ-convex(as for the definition and fundamental results of a ζ-convex space, see [11, 3]), and the inequality

$$||(A)^{is}||_{B(W^{-1,p}(\Omega))} \leq Ce^{\gamma|s|}, \quad -\infty < s < \infty$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$. Let us consider

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = Bw(t), \quad t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{CP}$$

where the controller B is a bounded linear operator from some Banach space U to $L^1(\Omega)$, and $w \in L^q(0, T; U)$ for $1 < q < \infty$. Noting that if $1 < p < n/(n-1)$ we may consider $L^1(\Omega) \subset W^{-1,p}(\Omega)$, and so, we cannot express $u(t)$ using the solution semigroup since B is a mapping into $W^{-1,p}(\Omega)$ not into $H_{p,q}$. Therefore, based on the theory of the definition and basic properties of Besob spaces, we will show that if $\frac{1}{p'} < 1/n(1 - 2/q')$ then

$$H_{p',q'} \subset C_0(\overline{\Omega}) \subset L^\infty(\Omega).$$

Thus, we may consider

$$H_{p,q} = H^*_{p'.q'} \supset C_0(\overline{\Omega})^* \supset L^1(\Omega)$$

and B is bounded mapping from U to $H_{p,q}$. Hence, it is possible to investigate the control problem for (CP) in $H_{p,q}$. Consequently, in view of the maximal regularity result by Dore and Venni [5], the initial value problem (CP) has a unique solution $u \in L^q\big(0,T;W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\big)\bigcap W^{1,q}(0,T;H_{p,q})$ for any $u_0 \in W_0^{1,p}(\Omega)$.

2.2 Notations

Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ and closure $\overline{\Omega}$. $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω.

$C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω.

$W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p,\Omega} = (\sum_{\alpha\leq m} ||D^\alpha f||_{p,\Omega}^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$||f||_{m,\infty,\Omega} = \max_{\alpha\leq m} ||D^\alpha u||_{\infty,\Omega},$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $||\cdot||_{p,\Omega}$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. For $p = 2$, we denote $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{2,p}(\Omega) = H_0^m(\Omega)$.

Let $p' = p/(p-1)$, $1 < p < \infty$. $W^{-1,p}(\Omega)$ stands for the dual space $W_0^{1,p'}(\Omega)^*$ of $W_0^{1,p'}(\Omega)$ whose norm is denoted by $||\cdot||_{-1,p}$.

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable.

$C^m([0,T];X)$ will denote the set of all m-times continuously differentiable functions from $[0,T]$ into X.

If X and Y are two Banach spaces, $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$.

For an interpolation couple of Banach spaces X_0 and X_1, $(X_0,X_1)_{\theta,p}$ for any $\theta \in (0,1)$ and $1 \le p \le \infty$ and $[X_0,X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively(see [9]).

2.3 Relationship of $H_{p,q} \subset L^1(\Omega)$ as a Besov space

Let A be the operator mentioned in Section 1. Then it was shown that the operators $-A$ generates an analytic semigroup in $W^{-1,p}(\Omega)$ in seen [16].

Lemma 2.3.1. *There exists a positive constant C such that for any $t > 0$*

$$||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))} \le Ct^{-\frac{1}{2}}, \tag{2.3.1}$$

and

$$||(t+A)^{-1}||_{B(L^p(\Omega),W_0^{1,p}(\Omega))} \le Ct^{-\frac{1}{2}}. \tag{2.3.2}$$

Proof. Let A_p be the realization of (EO) in $L^p(\Omega)$ in the distribution sense under the Dirichlet boundary condition. Then $-A_p$ generates an analytic semigroup in $L^p(\Omega)$, and A_p is the restriction of A to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Hence, (2.3.2) follows from the moment inequality

$$||u||_{W^{1,p}(\Omega)} \le C||u||^{\frac{1}{2}}_{W^{2,p}(\Omega)}||u||^{\frac{1}{2}}_{L^p(\Omega)}$$

and the estimate

$$||(t+A)^{-1}||_{B(L^p(\Omega))} \le Ct^{-1}$$

proved in [16, Eq(3.5)]. Replacing p by p' we get

$$||(t+A')^{-1}||_{B(L^{p'}(\Omega), W_0^{1,p'}(\Omega))} \le Ct^{-\frac{1}{2}},$$

where A' is the realization in $W^{-1,p'}(\Omega)$ under the Dirichlet boundary condition. Taking the adjoint we obtain (2.3.1). □

Let Y_0 and Y_1 be two Banach spaces contained in a locally convex linear topological space $\mathcal{Y}$ such that the identity mapping of Y_i $(i = 0,\ 1)$ into $\mathcal{Y}$ is continuous, and their norms will be denoted by $||\cdot||_i$. The algebraic sum Y_0+Y_1 of Y_0 and Y_1 is the space of all elements $a \in \mathcal{Y}$ of the form $a = a_0+a_1$, $a_0 \in Y_0$ and $a_1 \in Y_1$. The intersection $Y_0 \cap Y_1$ and the sum Y_0+Y_1 are Banach spaces with the norms

$$||a||_{Y_0\cap Y_1} = \max\ \{||a||_0, ||a||_1\}$$

and

$$||a||_{Y_0+Y_1} = \inf_a\ \{||a_0||_0 + ||a_1||_1\}, \quad a = a_0 + a_1,\ a_i \in Y_i,$$

respectively.

Definition 2.3.1. *[14] We say that an intermediate space Y of Y_0 and Y_1 belongs to*

(i) *the class $\underline{K}_\theta(Y_0, Y_1)$, $0 < \theta < 1$, if for any $a \in Y_0 \cap Y_1$,*

$$||a||_Y \le c||a||_0^{1-\theta}||a||_1^\theta$$

where c is a constant;

(ii) *the class* $\overline{K}_\theta(Y_0, Y_1),\ 0 < \theta < 1,$ *if for any* $a \in Y$ *and* $t > 0$ *there exist* $a_i \in Y_i\ (i = 1,\ 2)$ *such that* $a = a_0 + a_1$ *and*

$$||a_0||_0 \leq ct^{-\theta}||a||_Y, \quad ||a_1||_1 \leq ct^{1-\theta}||a||_Y$$

where c is a constant;

(iii) *the class* $K_\theta(Y_0, Y_1),\ 0 < \theta < 1,$ *if the space* Y *belongs to both* $\underline{K}_\theta(Y_0, Y_1)$ *and* $\overline{K}_\theta(Y_0, Y_1)$.

Here, we note that by replacing t with t^{-1} the condition in (ii) is rewritten as follows:

$$||a_0||_0 \leq ct^{\theta}||a||_Y, \quad ||a_1||_1 \leq ct^{\theta-1}||a||_Y.$$

The following result is due to Lions-Peetre [14, Theorem 2.3].

Proposition 2.3.1. *For* $0 < \theta_0 < \theta < \theta_1 < 1$, *if the spaces* X_0 *and* X_1 *belong to the class* $K_{\theta_0}(Y_0, Y_1)$ *and the class* $K_{\theta_1}(Y_0, Y_1)$, *respectively, then*

$$(X_0, X_1)_{\frac{\theta-\theta_0}{\theta_1-\theta_0},p} = (Y_0, Y_1)_{\theta,p}.$$

The following corollary is verified following the proof of Proposition 2.3.1.

Corollary 2.3.1. *If the space* X_1 *is of the class* $K_{\theta_1}(Y_0, Y_1)$ *and* $0 < \theta < \theta_1 < 1$, *then*

$$(Y_0, X_1)_{\frac{\theta}{\theta_1},p} = (Y_0, Y_1)_{\theta,p}.$$

If the space X_0 *is of the class* $K_{\theta_0}(Y_0, Y_1)$ *and* $0 < \theta_0 < \theta < 1$, *then*

$$(X_0, Y_1)_{\frac{\theta-\theta_0}{1-\theta_0},p} = (Y_0, Y_1)_{\theta,p}.$$

Proposition 2.3.2. *For* $1 < p < \infty$, $L^p(\Omega)$ *is of the class* $K_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$.

Proof. For any $u \in W_0^{1,p}(\Omega)$ and $t > 0$, from Lemma 2.3.1 and

$$u = A(t+A)^{-1}u + t(t+A)^{-1}u = (t+A)^{-1}Au + t(t+A)^{-1}u,$$

it follows

$$\begin{aligned}||u||_{p,\Omega} \leq &||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))}||Au||_{-1,p,\Omega} \\ &+ t||(t+A)^{-1}||_{B(W^{-1,p}(\Omega),L^p(\Omega))}||u||_{-1,p,\Omega} \\ \leq &Ct^{-\frac{1}{2}}||u||_{1,p,\Omega} + Ct^{\frac{1}{2}}||u||_{-1,p,\Omega}.\end{aligned}$$

By choosing $t > 0$ such that $t^{-1/2}||u||_{1,p,\Omega} = t^{1/2}||u||_{B(W_0^{1,p'}(\Omega))^*} = t^{1/2}||u||_{-1,p,\Omega}$, we obtain

$$||u||_{p,\Omega} \leq C||u||_{1,p,\Omega}^{\frac{1}{2}}||u||_{-1,p,\Omega}^{\frac{1}{2}}.$$

Therefore, $L^p(\Omega)$ belongs to the class $\underline{K}_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$. Put $u_0 = t(t+A)^{-1}u$ and $u_1 = A(t+A)^{-1}u$ for any $u \in L^p(\Omega)$. Then $u = u_0 + u_1$, and we obtain that

$$||u_0||_{1,p,\Omega} \leq t||(t+A)^{-1}u||_{B(L^p(\Omega),W_0^{1,p}(\Omega))}||u||_{p,\Omega} \leq Ct^{\frac{1}{2}}||u||_{p,\Omega}$$

$$||u_1||_{-1,p,\Omega} \leq C||(t+A)^{-1}u||_{1,p,\Omega} \leq Ct^{-\frac{1}{2}}||u||_{p,\Omega}.$$

Therefore, $L^p(\Omega)$ belongs to the class $\overline{K}_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$, and hence, it is of the class $K_{1/2}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$. □

Theorem 2.3.1. *Let $1 < p < \infty$, $1 < q < \infty$ and $0 < \theta < 1$. If $1-2\theta-1/p \neq 0$ and $2\theta - 2 + 1/p \neq 0$ then*

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} \mathring{B}_{p,q}^{1-2\theta}(\Omega) & \theta < \frac{1}{2}(1-\frac{1}{p}), \\ B_{p,q}^{1-2\theta}(\Omega) & \theta > \frac{1}{2}(1-\frac{1}{p}), \end{cases}$$

where $\mathring{B}_{p,q}^{1-2\theta}(\Omega) = \{u \in B_{p,q}^{1-2\theta}(\Omega) : u|_{\partial\Omega} = 0\}$. In particular, we obtain that

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q} = B_{p,q}^{0}(\Omega).$$

Proof. Let $0 < \theta < 1/2$. Then from Corollary 2.3.1, we obtain that

$$\begin{aligned}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} &= (W_0^{1,p}(\Omega), L^p(\Omega))_{2\theta,q} \\ &= (L^p(\Omega), W_0^{1,p}(\Omega))_{1-2\theta,q}.\end{aligned}$$

Therefore, in view of the result of Grisvard [30] (see also Triebel [9][6; p. 321]),

$$(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} = \begin{cases} \mathring{B}_{p,q}^{1-2\theta}(\Omega) & 1-2\theta > \frac{1}{p}, \\ B_{p,q}^{1-2\theta}(\Omega) & 1-2\theta < \frac{1}{p}. \end{cases}$$

Let $1/2 < \theta < 1$. Then from Corollary 2.3.1, it follows

$$\begin{aligned}(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\theta,q} &= (L^p(\Omega), W^{-1,p}(\Omega))_{2\theta-1,q} \\ &= ((L^{p'}(\Omega), W_0^{-1,p'}(\Omega))_{2\theta-1,q'})^*.\end{aligned}$$

In view of Grisvard's theorem, if $2\theta - 1 - 1/p' \neq 0$ then

$$(L^{p'}(\Omega), W_0^{1,p'}(\Omega))_{2\theta-1,q'} = \begin{cases} \mathring{B}_{p',q'}^{2\theta-1}(\Omega) & 2\theta-1 > \frac{1}{p'}, \\ B_{p',q'}^{2\theta-1}(\Omega) & 2\theta-1 < \frac{1}{p'}. \end{cases}$$

From Theorem 4.8.2 in Triebel [9, p. 332], we obtain that

$$(\mathring{B}^{2\theta-1}_{p',q'}(\Omega))^* = B^{1-2\theta}_{p,q}(\Omega) \text{ and } (B^{2\theta-1}_{p',q'}(\Omega))^* = B^{1-2\theta}_{p,q}(\Omega)$$

according as $2\theta - 1 - 1/p' \gtrless 0$. Since $2\theta - 1 - 1/p' \neq 0$ if $1/2 < \theta < 1$ and $2\theta - 2 + 1/p \neq 0$, we get

$$(W^{1,p}_0(\Omega), W^{-1,p}(\Omega))_{\theta,q} = B^{1-2\theta}_{p,q}(\Omega).$$

Consequently, we obtain that

$$(W^{1,p}_0(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q} = B^{\theta}_{p,q}(\Omega), \qquad \text{if} \quad 0 < \theta < \frac{1}{p}$$

and

$$(W^{1,p}_0(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q} = B^{-\theta}_{p,q}(\Omega) \qquad \text{if} \quad 0 < \theta < 1 - \frac{1}{p}.$$

Hence, if $0 < \theta < \min\{1/p, 1 - 1/p\}$, then

$$\begin{aligned}
&(W^{1,p}_0(\Omega), W^{-1,p}(\Omega))_{\frac{1}{2},q} \\
&\quad = ((W^{1,p}_0(\Omega), W^{-1,p}(\Omega))_{\frac{1-\theta}{2},q}, (W^{1,p}_0(\Omega), W^{-1,p}(\Omega))_{\frac{1+\theta}{2},q})_{\frac{1}{2},q} \\
&\quad = (B^{\theta}_{p,q}(\Omega), B^{-\theta}_{p,q}(\Omega))_{\frac{1}{2},q} = B^{0}_{p,q}(\Omega).
\end{aligned}$$

The last equality is obtained from Theorem 1 of section 4.3.1 in Triebel [9]. Hence the proof is complete. □

Theorem 2.3.2. *Let* $1 < p,\ q < \infty$.

(i) *If* $2/q - 2 + 1/p \neq 0$ *then*

$$H_{p,q} = \begin{cases} \overset{\circ}{B}{}^{1-\frac{2}{q}}_{p,q}(\Omega) & \text{if} \quad \frac{1}{q} < \frac{1}{2}(1 - \frac{1}{p}), \\ B^{1-\frac{2}{q}}_{p,q}(\Omega) & \text{if} \quad \frac{1}{q} > \frac{1}{2}(1 - \frac{1}{p}). \end{cases}$$

(ii) *If* $n/p^{'} < 1 - 2/q^{'}$ *then*

$$H_{p^{'},q^{'}} \subset C_0(\overline{\Omega}) \subset L^{\infty}(\Omega).$$

Proof. The relation (i) follows directly from Theorem 2.3.1. Let $1/p^{'} < 1/n(1 - 2/q^{'})$ which implies $2/q^{'} - 2 + 1/p^{'} < -1 - (n-1)/p^{'} < 0$ and $1/q^{'} < 1/2(1 - n/p^{'}) < 1/2(1 - 1/p^{'})$. Then from (i) and the imbedding theorem ([2; Theorem 4.6.1 in p. 327-328]), we obtain

$$H_{p^{'},q^{'}} = \overset{\circ}{B}{}^{1-\frac{2}{q^{'}}}_{p^{'},q^{'}}(\Omega) \subset C_0(\overline{\Omega})$$

Hence, the first inclusion in (ii) follows. □

Example 2.3.1. *Let* U *be a Banach space, and let* $w \in L^q(0, T; U)$ *for* $1 < q < \infty$. *Consider the following control problem:*

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = Bw(t), \quad t \in (0, T], \\ u(0) = u_0, \end{cases} \tag{2.3.3}$$

where the controller B is a bounded linear operator from U to $L^1(\Omega)$. Here, A is an elliptic differential operator of second order in $L^1(\Omega)$ as seen in Section 1. By virtue of Theorem 2.3.2, we may consider

$$H_{p,q} = H^*_{p'.q'} \supset C_0(\overline{\Omega})^* \supset L^1(\Omega),$$

where $\frac{1}{p'} < 1/n(1-2/q')$. Since B is a bounded mapping into $H_{p,q}$, we be able to express $u(t)$ using the solution semigroup $S(t) = e^{At}$. Furthermore, it is possible to investigate the control problem for (2.3.3) in $H_{p,q}$. Consequently, in view of the maximal regularity result by dore and venni [5], the initial value problem (CP) has a unique solution $u \in L^q\big(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\big) \bigcap W^{1,q}(0,T; H_{p,q})$ for any $u_0 \in W_0^{1,p}(\Omega)$. As for the maximal regularity problem of (2.3.3) in Hilbert spaces, We refer to [4, 29]. Moreover, The observability of (2.3.3) is defines as

$$B^* S^*(t) f \equiv 0 \quad \textit{implies} \quad f = 0$$

in a usual sense of [11-14] .

Chapter 3
Sufficient conditions for approximate controllability of semilinear control systems

3.1 Introduction

Let H and V be complex Hilbert spaces such that the imbedding $V \subset H$ is compact. The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$, and those in V are by $((\cdot,\cdot))$ and $||\cdot||$, respectively. Let $-A_0$ be the operator associated with a bounded sesquilinear form $a(u,v)$ defined in $V \times V$ and satisfying Gårding inequality

$$\operatorname{Re} a(u,v) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0 \tag{3.1.1}$$

for any $u \in V$. It is known that A_0 generates an analytic semigroup in both of H and V^* where V^* stands for the dual space of V. The object of this paper is to investigate the quality of reachable set of the following semilinear retarded parabolic type equation

$$\frac{d}{dt}x(t) = A_0x(t) + f(t), \quad t \in (0,T], \tag{CS}$$

where

$$f(t) = A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + f(t,x(t)) + B_0u(t).$$

Then the initial condition of system (CS) is given as follows:

$$x(0) = g^0, \quad x(s) = g^1(s), \quad \text{for} \quad s \in [-h, 0]. \tag{IC}$$

The existence and uniqueness of solution of the above system are proved in [19]. The condition for equivalence between the reachable set of the semilinear system and that of its corresponding linear system was established in [19, 10] and recently, [25, 26]. This paper is dealt with another applicable condition for controller of approximate control problem. Thus the main result in this paper will show that the system (CS) with some conditions for the operator A_0 satisfies a sufficient condition for approximate controllability obtained in [19].

3.2 Main results

Let A_0 be the self adjoint operator associated with a sesquilinear form defined on $V \times V$ such that

$$(A_0 u, v) = -a(u, v), \quad u, \ v \in V$$

where $a(\cdot, \cdot)$ is bounded sesquilinear form satisfying Gårding inequality. It is known that A_0 generates an analytic semigroup in both H and V^*. Let us assume that A_i, $i = 1, 2$, are bounded linear operators from V to V^* and $A_i A_0^{-1}$ are also bounded in H. The real valued function $a(s)$ is assumed to be Hölder continuous in $[-h, 0]$ where h is a fixed positive number. The controller B_0 is a bounded linear operator from a subspace U of H to H. Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. Hence, we assume more general Lipschitz condition: for any $x_1,\ x_2 \in V$ there exists a constant $L > 0$ such that

$$\begin{cases} |f(t,x_1) - f(t,x_2)| \le L||x_1 - x_2||, \\ f(t,0) = 0. \end{cases} \tag{3.2.1}$$

Then as is seen in [4, 16] we can obtain the following result.

Proposition 3.2.1. *Under the assumtions (3.2.1), there exists a unique solution of (CS) and (IC) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $g = (g^0, g^1) \in Z \equiv H \times L^2(-h,0;V)$. *Moreover, there exists a constant* C *such that*

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;U)}),$$

where

$$||\cdot||_{L^2(0,T,V)\cap W^{1,2}(0,T;V^*)} = \max\{||\cdot||_{L^2(0,T;V)}, ||\cdot||_{W^{1,2}(0,T;V^*)}\}.$$

Let $g \in Z$ and $x(T; g, f, u)$ be a solution of the system (CS) and (IC) associated with nonlinear term f and control u at time T. We define reachable sets for the system (CS) and (IC) as follows:

$$L_T(g) = \{x(T; g, 0, u) : u \in L^2(0,T;U)\},$$

$$R_T(g) = \{x(T; g, f, u) : u \in L^2(0,T;U)\}.$$

In virtue of the Riesz-Schauder theorem, if the imbedding $V \subset H$ is compact then the operator A_0 has discrete spectrum

$$\sigma(A_0) = \{\mu_n : n = 1,\ 2,\ \dots\}$$

which has no point of accumulation except possibly $\mu = \infty$. Let μ_n be a pole of the resolvent of A_0 of order k_n and P_n the spectral projection associated with μ_n

$$P_n = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - A_0)^{-1} d\mu,$$

where Γ_n is a small circle centered at μ_n such that it surrounds no point of $\sigma(A_0)$ except μ_n. Then the generalized eigenspace corresponding to μ_n is given by

$$H_n = P_n H = \{P_n u : u \in H\},$$

and we have that from $P_n^2 = P_n$ and $H_n \subset V$ it follows that

$$P_n V = \{P_n u : u \in V\} = H_n.$$

Let us set

$$Q_n = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - \mu_n)(\mu - A_0)^{-1} d\mu.$$

Then we remark that $\dim H_n < \infty$ and

$$Q_n^i = \frac{1}{2\pi i}\int_{\Gamma_n} (\mu - \mu_n)^i (\mu - A_0)^{-1} d\mu.$$

It is also well known that $Q_n^{k_n} = 0$ (nilpotent) and $(A_0 - \mu_n)P_n = Q_n$ (see [33, 6, 24]).

Definition 3.2.1. *The system of the generized eigenspaces of A_0 is complete in H if* $\mathrm{Cl}\{\mathrm{span}\{H_n : n = 1,\ 2,\ ...\ \}\} = H$ *where* Cl *denotes the closure in* H.

Let $G(t)$ be an analytic semigroup generated by A_0. We now define the fundamental solution $W(t)$ of (CS) and (IC) by

$$W(t) = \begin{cases} x(t; (g^0, 0), 0, 0), & t \geq 0 \\ 0, & t < 0. \end{cases}$$

According to the above definition $W(t)$ is a unique solotion of

$$W(t) = G(t) + \int_0^t G(t-s)\{A_1 W(s-h) + \int_{-h}^0 a(\tau) A_2 W(s+\tau) d\tau\} ds$$

for $t \geq 0$ (cf. Nakagiri [35, 7]). We denote the bounded linear operator $\hat{W}$ from $L^2(0,T;H)$ to H by

$$\hat{W} p = \int_0^T W(T-s) p(s) ds$$

for $p \in L^2(0,T;H)$.

Definition 3.2.2. *The system (CS) and (IC) is approximately controllable on* $[0,T]$ *if* $\overline{R_T(g)} = H$, *that is, for any* $\epsilon > 0$ *and* $x \in H$ *there exists a control* $u \in L^2(0,T;U)$ *such that* $|x - W(T)g^0 - \int_{-h}^0 U_T(s) g^1(s) ds - \hat{W} f(\cdot, x_u(\cdot)) - \hat{W} B_0 u| < \epsilon$ *where* $U_T(s) = W(T-s-h)A_1 + \int_{-h}^s W(T-s-\sigma) a(\sigma) A_2 d\sigma$ *and* $x_u(\cdot) = x(\cdot; g, f, u)$.

We need the following hypotheses:

(A) The system of the generalized eigenspaces of A_0 is complete.

(B1) For any $\epsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$|\int_0^t G(t-s)p(s)ds - \int_0^t G(t-s)B_0u(s)ds| < \epsilon, \quad 0 \le t \le T.$$

(B2) $B_0P_nH \subset P_nH$ for $n = 1,\ 2,\ \dots$.

Remark 3.2.1. *We know that the condition (B2) is equivalent to the fact that $P_nB_0P_n = B_0P_n$, thus by the definition of Q_n it is also held that if $f \in P_nH$ then $Q_nB_0f = B_0Q_nf$.*

Proposition 3.2.2. *Under the assumption (B1), we have $\overline{L_T(0)} = H$.*

Theorem 3.2.1. *Let us assume the hypotheses (A), (B1) and (B2). Then we have $\overline{R_T(g)} = \overline{L_T(g)}$ for any $g \in H \times L^2(-h,0;V)$.*

In virtue of Proposition 3.2.2 and Theorem 3.2.1 we have known that the system (CS) and (IC) is approximately controllable in conclusion.

Remark 3.2.2. *For the semilinear equation without delay terms in case where $A_1 = A_2 = 0$ we may assume the condition (B1) at only time T, that is, we can rewite the condition (B1) as follows.*

For any $\epsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$|\int_0^T G(t-s)p(s)ds - \int_0^T G(t-s)B_0u(s)ds| < \epsilon.$$

Remark 3.2.3. *In Naito [22] he proved Theorem 2.2.2 under assumptions (B1) and compact operator $G(t)$ and also Zhou in [10] showed it under assumtion (B1) and another condition of range of controller.*

3.3 Proof of main results

First of all, for the meaning of assumption (B1) we need to show the existence of controller satisfying $\mathrm{Cl}\{B_0u : u \in L^2(0,T;U)\} \neq L^2(0,T;H)$. In fact, Consider about the controler B_0 defined by

$$B_0u(t) = \sum_{n=1}^{\infty} u_n(t),$$

where

$$u_n = \begin{cases} 0, & 0 \le t \le \frac{T}{n} \\ P_n u(t), & \frac{T}{n} < t \le T. \end{cases}$$

Hence we see that $u_1(t) \equiv 0$ and $u_n(t) \in \mathrm{Im}\, P_n$. By completion of generalized eigenspaces of A_0 we may write that $f(t) = \sum_{n=1}^{\infty} P_n f(t)$ for $f \in L^2(0,T;H)$. Let us choose $f \in L^2(0,T;H)$ satisfying

$$\int_0^T ||P_1 f(t)||^2 dt > 0.$$

Then since

$$\int_0^T ||f(t) - B_0u(t)||^2 dt = \int_0^T \sum_{n=1}^{\infty} ||P_n(f(t) - B_0u(t))||^2 dt$$

$$\ge \int_0^T ||P_1(f(t) - B_0u(t))||^2 dt = \int_0^T ||P_1 f(t)||^2 dt > 0,$$

the statement mentioned above is reasonable.

Proof of Proposition 3.2.2. Let $x_0 \in D(A_0)$, Then putting $f(s) = (x_0 + sA_0x_0)/t$ it follows that

$$x_0 = \int_0^t G(t-s)f(s)ds.$$

Thus by the condition (B1) there exists $u \in L^2(0,T;U)$ such that

$$||x_0 - \int_0^t G(t-s)B_0u(s)ds|| < \epsilon.$$

Therefore, the density of the domain $D(A_0)$ in H implies approximate controllablity of (CS) and (IC), the proof of Proposition 3.2.2 is complete. □

From now on we go to proof of the Theorem 3.2.1. In what follows in this section, let us assume that the system of the generalized eigenspaces of A_0 is complete. Then we will prove that the assumptions (B1) and (B2) are a sufficient condition for the following statement (H) in Theorem 1, 2 as in [35]:

(H) For any $\epsilon > 0$ and $p \in L^2(0,T;H)$ there exists a $u \in L^2(0,T;U)$ such that

$$|\int_0^t G(t-s)p(s)ds - \int_0^t G(t-s)B_0u(s)ds| < \epsilon, \quad 0 \le t \le T,$$

$$||B_0u||_{L^2(0,T;H)} \le q||p||_{L^2(0,T;H)}$$

where $G(t)$ is an analytic semigroup with infinitesimal generator A_0 and q is a constant independent of p.

If $\mu_n \in \sigma(A_0)$ then we have the Laurent expansion for $R(\mu - A_0) \equiv (\mu - A_0)^{-1}$ at $\mu = \mu_n$ whose principal part (the part consisting of all the negative power of $(\mu - \mu_n)$) is a finite series:

$$R(\mu - A_0) = \frac{P_n}{\mu - \mu_n} + \sum_{i=1}^{k_n - 1} \frac{Q_n^i}{(\mu - \mu_n)^{i+1}} + R_0(\mu),$$

where $R_0(\mu)$ is a holomorphic part of $R(\mu - A_0)$ at $\mu = \mu_n$.

Since the system of generalized eigenspaces of A_0 is complete, it holds that for any $\epsilon > 0$

$$|f(s) - \sum_{n=1}^{\infty} P_n f(s)| < \frac{\epsilon}{2M\sqrt{T}} \tag{3.3.1}$$

for $f \in L^2(0, T; H)$, where M is a constant such that $|G(t)| \leq M$ for the sake of simplicity. Here, in what follows we put $u_n = P_n B_0 u$.

Since A_0^{-1} is compact we note that there exists an arc C_n which joints μ_n and some z_0 with $\operatorname{Re} z_0 < \inf\{\operatorname{Re} \mu_n : \mu_n \in \sigma(A_0)\}$ and $C_n - \{\mu_n\} \subset \rho(A_0)$ where $\rho(A_0)$ is the resolvent set of A_0.

Lemma 3.3.1. *Let $G(t)$ be the semigroup generated by A_0. Then we give an expression of the semigroup that*

$$G(t)f = e^{\mu_n t} \sum_{i=1}^{k_n - 1} \frac{t^i}{i!} Q_n^i f, \quad t \geq 0$$

for any $f \in P_n H$.

Proof. From the well known fact that

$$A_0 P_n = A_0 \frac{1}{2\pi i} \int_{\Gamma_n} (\mu - A_0)^{-1} d\mu$$

$$= \frac{1}{2\pi i} \int_{\Gamma_n} \mu(\mu - A_0)^{-1} d\mu$$

we have

$$G(t)P_n = \frac{1}{2\pi i} \int_{\Gamma_n} e^{\mu t}(\mu - A_0)^{-1} d\mu.$$

If $f \in P_n H$ then $f = P_n f$ and hence

$$\begin{aligned} G(t)f = G(t)P_n f &= \frac{1}{2\pi i} \int_{\Gamma_n} e^{\mu t}(\mu - A_0)^{-1} f d\mu \\ &= e^{\mu_n t} \frac{1}{2\pi i} \int_{\Gamma_n} e^{(\mu - \mu_n)t}(\mu - A_0)^{-1} f d\mu \\ &= e^{\mu_n t} \{ \sum_{i=0}^{\infty} \frac{t^i}{i!} (\frac{1}{2\pi i} \int_{\Gamma_n} (\mu - \mu_n)^i (\mu - A_0)^{-1} f d\mu) \} \\ &= e^{\mu_n t} \sum_{i=0}^{k_n - 1} \frac{t^i}{i!} Q_n^i f. \end{aligned}$$

Hear, we used the nilpotent property of the operator Q_n in the last equality. The proof of Lemma is complete. □

Remark 3.3.1. *Let $f \in P_n H$. Then in virtue of Lemma 3.3.1 it holds that $B_0 G(t) f = G(t) B_0 f$ for every $t \geq 0$.*

Let $f \in L^2(0, t; H)$. Then by the assumption (B1) for any $\epsilon > 0$ there exists a control $v \in L^2(0, t; U)$ such that

$$| \int_0^t G(t-s) f(s) ds - \int_0^t G(t-s) B_0 v(s) ds | < \frac{\epsilon}{2}, \quad 0 \leq t \leq T, \qquad (3.3.2)$$

and

$$|v(s) - \sum_{n=1}^{\infty} P_n v(s)| < \frac{\epsilon}{2M||B_0||\sqrt{T}}. \tag{3.3.3}$$

Let us define $h \in H$ by

$$h = \sum_{n=1}^{\infty} \int_0^t G(t-s)P_n v(s)ds$$

$$= \sum_{n=1}^{\infty} h_n.$$

Here, we put $h_n = \int_0^t G(t-s)P_n v(s)ds$. Since $P_n v(s) \in P_n H$, in terms of Lemma 3.3.1 we have that

$$h_n = \int_0^t G(t-s)P_n v(s)ds \tag{3.3.4}$$

$$= \sum_{i=1}^{k_n-1} \int_0^t e^{\mu_n(t-s)} \frac{(t-s)^i}{i!} Q_n^i P_n v(s)ds.$$

Define

$$u(s) = \sum_{n=1}^{\infty} u_n(s), \quad u_n(s) = (\sum_{i=1}^{k_n-1} \frac{t^{i+1}}{(i+1)!})^{-1} e^{-\mu_n(t-s)} Q_n^{k_n-1} h_n.$$

Then $u_n(s) \in P_nH$ and from remark 3.2.1 it follows

$$\int_0^t G(t-s)B_0u(s)ds = \sum_{n=1}^{\infty}\int_0^t G(t-s)B_0u_n(s)ds$$

$$= \sum_{n=1}^{\infty}\int_0^t e^{\mu_n(t-s)}\sum_{i=1}^{k_n-1}\frac{(t-s)^i}{i!}Q_n^iB_0u_n(s)ds$$

$$= B_0\sum_{n=1}^{\infty}h_n = B_0h.$$

Thus from (3.3.2), (3.3.3) it follows that

$$|\int_0^t G(t-s)B_0u(s)ds - \int_0^t G(t-s)f(s)ds|$$

$$\leq |\int_0^t G(t-s)B_0u(s)ds - B_0h| +$$

$$|B_0h - \int_0^t G(t-s)B_0v(s)ds| +$$

$$|\int_0^t G(t-s)B_0v(s)ds - \int_0^t G(t-s)f(s)ds|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon.$$

Moreover, by Hölder inequality we also have

$$||B_0u||_{L^2(0,t;H)} \leq \int_0^t |\sum_{n=1}^{\infty}B_0u_n(s)|^2ds$$

$$\leq \int_0^t |\sum_{n=1}^{\infty}B_0(\sum_{i=1}^{k_n-1}\frac{t^{i+1}}{(i+1)!})^{-1}e^{-\mu_n(t-s)}Q_n^{k_n-1}h_n|^2ds$$

$$\leq c\int_0^t |\sum_{n=1}^{\infty}B_0h_n|^2ds,$$

where c is a constant. From remark 3.3.1 we also note that

$$B_0 h_n = B_0 \int_0^t G(t-s) P_n v(s) ds$$

$$= \int_0^t G(t-s) B_0 P_n v(s) ds,$$

and, hence from (3.3.2) and (3.3.3) it holds

$$|\sum_{n=1}^{\infty} B_0 h_n| = |\sum_{n=1}^{\infty} \int_0^t G(t-s) B_0 P_n v(s) ds|$$

$$\leq |\int_0^t G(t-s) B_0 \sum_{n=1}^{\infty} P_n v(s) ds - \int_0^t G(t-s) B_0 v(s) ds| +$$

$$|\int_0^t G(t-s) B_0 v(s) ds - \int_0^t G(t-s) B_0 f(s) ds| +$$

$$|\int_0^t G(t-s) f(s) ds|$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} + |\int_0^t G(t-s) f(s) ds|$$

$$\leq q||f||_{L^2(0,t;H)} + \epsilon$$

where q is a constant. Thus, from the above equality we can conclude that

$$||B_0 u||^2_{L^2(0,t;H)} \leq q||f||_{L^2(0,t;H)} + \epsilon.$$

Here, we note the constant q is independent of f. Since ϵ is arbitrary we have proof that the assumption of Theorem 3.2.1 implies the condition (H). In virtue of Theorem 4.2 of [19] the proof of Theorem 3.2.1 is complete. □

3.4 Examples of controller

Example 1. Define the controller B_0 by

$$B_0 u(t) = \sum_{n=1}^{\infty} u_n(t)$$

where

$$u_n(t) = \begin{cases} 0, & 0 \le t \le \frac{T}{n}, \\ P_n u(t), & \frac{T}{n} \le t \le T. \end{cases}$$

Then as is seen in section 3 we define h, u by

$$h = \sum_{n=1}^{\infty} h_n, \quad u(s) = \sum_{n=1}^{\infty} u_n(s)$$

where $h_n(s)$ is defined by as (3.3.4),

$$u(s) = \sum_{n=1}^{\infty} u_n(s), \quad u_n(s) = \sum_{i=1}^{k_n-1} (T - \frac{T}{n})^{i+1} i! e^{-\mu_n(T-s)} Q_n^{k_n-i} h_n,$$

respectively. Then $u_n(s) \in P_n H$ and $\int_0^T G(T-s)B_0 u(s)ds = \sum_{n=1}^{\infty} h_n$, and this controller is satisfied the conditions in Thereom 3.2.1.

Example 2. We consider the heat control system studied by Zhou [14, Example 1] and Naito [17, Example 1]. Let $H = L^2(0,\pi)$ and $A_0 = -d^2/dx^2$ $H = L^2(0,\pi)$ and $A_0 = -d^2/dx^2$ with

$$D(A_0) = \{y \in H : d^2y/dx^2 \in H \ \text{ and } \ y(0) = y(\pi) = 0\}.$$

Then $\{e_n = (2/\pi)^{1/2} \sin nx : 0 \leq x \leq \pi, \ \ n = 1, \ ... \}$ is orthonomal base for H. Define an infinite dimensional space U by

$$U = \{\sum_{n=2}^{\infty} u_n e_n : \sum_{n=2}^{\infty} u_n^2 < \infty\}$$

with norm defined by $||u||_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. Define a continuous linear operator B_0 from U to H as follows:

$$B_0 u = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n e_n \in U.$$

It is directly seen that the above controller B_0 satisfies the conditions (B1) and (B2). We can also check breifly by using the assumption (H). In fact, let $f \in L^2(0, T; H)$ and $f = \sum_{n=1}^{\infty} f_n(s) e_n$. Then we choose a function $u \in L^2(0, t; U)$ for $0 \leq t \leq T$ such that $u_2 = \frac{1}{2} f_1 + f_2$ and $u_n = f_n$ for $n = 2, \ 3, \ ....$ Hence, choosing a constant in condition (H) such that $q > \frac{7}{2}$, not only the system (CS) and (IC) with the operator A_0 mentioned above but also the general semilinear case is approximate controllable.

Chapter 4
Controllability for semilinear systems of parabolic type with delays

4.1 Introduction

In this paper, we deal with control problem for semilinear parabolic type equation in Hilbert space H as follows.

$$\begin{cases} \frac{d}{dt}x(t) = & A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds \\ & +f(t,x(t)) + \Phi_0 u(t), \\ x(0) = & g^0, \quad x(s) = g^1(s), \quad s \in [-h,0). \end{cases} \tag{PS}$$

Let A_0 be the operator associated with a sesquilinear form defined on $V \times V$ satisfying Gårding inequality:

$$(A_0u, v) = -a(u,v), \quad u,\ v \in V$$

where V is a Hilbert space such that $V \subset H \subset V^*$. Then it is known that A_0 generates an analytic semigroup in both H and V^*. Let $Z = H \times L^2(-h,0;V)$ be the state space of the equation (PS) . Z is a product Hilbert space with the norm

$$||g||_Z = (|g^0|^2 + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

The operator A is defined as follows:

$$D(A) = \{g = (g^0, g^1) : \ g^0 \in H, \ g^1 \in L^2(-h, 0; V),$$

$$g^1(0) = g^0, \ A_0 g^0 + \int_{-h}^{0} a(s) A_2 g^1(s) ds \in H\},$$

$$Ag = (A_0 g^0 + \int_{-h}^{0} a(s) A_2 g^1(s) ds, \dot{g}^1).$$

The equation (PS) can be transposed to an following general initial problem

$$\frac{d}{dt} z(t) = Az(t) + F(t, z(t)) + \Phi u(t), \qquad \text{(IP)}$$

where $\Phi f = (\Phi_0 f, 0)$, $F(t, z(t)) = (f(t, x(t)), 0)$ Recently, Approximate controllability for semilinear control systems can be founded in [17, 26] with a range condition of the control action operator. In [22, 23], Naito showed approximately controllability of the system (PS) by using the assumption that the semigroup generated by A is compact operator, also Nakagiri and Yamamoto [36] showed it in case where the operator A generator an analytic semigroup. We note that in our case the semigoup generated by A is not compact operator but only C_0-semigroup. So, we show from the approximately controllable of the system (IP) in space Z with the general assumption of nonlinear part. Moreover, we derive the relations between the controllability of the system (PS) and one of (IP).

4.2 Preliminaries

Let V and H be Hilbert spaces forming a Gelfand triple $V \subset H \subset V^*$ with pivot space H. and the operator A_0 is the operater mention in section 3.1. Moreover, there exists a constant C_1 such that

$$||u|| \leq C_1 ||u||_{D(A_0)}^{1/2} |u|^{1/2}, \tag{4.2.1}$$

for every $u \in D(A_0)$, where

$$||u||_{D(A_0)} = (|A_0 u|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A_0) \subset V \subset H \subset V^* \subset D(A_0)^*,$$

where each space is dense in the next one and continuous injection.

Lemma 4.2.1. *With the notations (4.2.1), we have*

$$(V, V^*)_{1/2,2} = H,$$

$$(D(A_0), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* *([29],Section 1.3.3 of [9]).*

The operators A_1 and A_2 in the system (PS) are bounded linear operators from V to V^* such that they map $D(A_0)$ into H. The function $a(\cdot)$ is assume to be a real valued Hölder continous in $[-h, 0]$ and the controller operator Φ_0 is a bounded linear operator from a Banach space U to H, where U is called a control set.

Let f be a nonlinear mapping from $\mathbb{R} \times V$ into H. We assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that

$$\begin{cases} |f(t,x_1) - f(f,x_2)| \le L||x_1 - x_2|| \\ f(t,0) = 0. \end{cases} \tag{4.2.2}$$

Assume that (3.1.1) holds for $c_1 = 0$. Noting that $A_0 + c_1$ is an isomorphism from V to V^* if $c_1 \neq 0$. Corresponding the linear system [4, 14, 32], we have the following result of semilinear equation (PS) as is seen in [16, 20].

Proposition 4.2.1. *Under the assumption (4.2.2), then there exists a unique solution x of (PS) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

for any $g = (g^0, g^1) \in Z = H \times L^2(-h;V)$. Moreover, there exists a constant C such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C(|g^0| + ||g^1||_{L^2(-h,0;V)} + ||u||_{L^2(0,T;U)}).$$

4.3 Controllability

Let $Z \equiv H \times L^2(-h,0;V)$ be the state space of the equation (PS). Z is a product Hilbert space with the norm

$$||g|| = (|g^0|^2 + \int_{-h}^0 ||g^1(s)||_V^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

Let $g \in Z$ and $x(t; g, f, \Phi_0 u)$ be a solution of the equation (PS) associated with nonlinear term f and control $\Phi_0 u$ at time t. In view of the result of

Proposition 4.2.1 considered as an equation in V^*, we can define the solution semigroup for the problem (PS) as follows:

$$S(t)g = (x(t; g, 0, 0), x_t(\cdot; g, 0, 0)) \tag{4.3.1}$$

where $g = (g^0, g^1) \in Z$, $x(t; g, 0, 0)$ is the solution of (PS) and (IP) with $f(t, x) = 0$ and $\Phi_0 = 0$ and $x_t(s; g, 0, 0) = x(t + s; g, 0, 0)$ defined in $[-h, 0]$. It is known that the operator $S(t)$ is a C_0-semigroup on Z (see [35]). and the infinitesimal generator A of $S(t)$ is characterized by

$$\begin{cases} D(A) = & \{g = (g^0, g^1) : g^0 \in H,\ g^1 \in L^2(-h, 0; V), \\ & g^1(0) = g^0,\ A_0 g^0 + A_1 g^1(-h) + \int_{-h}^0 a(s) A_2 g^1(s) ds \in H, \\ Ag = & (A_0 g^0 + A_1 g^1(-h) + \int_{-h}^0 a(s) A_2 g^1(s) ds, \dot{g}^1). \end{cases}$$

Note that $a(\cdot)$ need not be Hólder continuous for the above results to hold. It has only to belong to $L^2(-h, 0)$.

For the sake of simplicity, we assume that $S(t)$ is uniformly bounded, that is, there exists a constant $M \geq 1$ such that

$$||S(t)||_Z \leq M. \tag{4.3.2}$$

as is seen in [36], the equation (PS) can be transformed into an abstract equation

$$\begin{cases} z(t) = Az(t) + F(z(t)) + Bu(t), \\ z(0) = g, \end{cases} \tag{4.3.3}$$

where $z(t) = (x(t), x_t(\cdot))$ belongs to the Hilbert space Z and $g = (g^0, g^1) \in Z$. The operator A is the infinitesimal generator of C_0-semigroup $S(t)$, $F(z(t)) =$

$(f(t,x(t)),0)$ and $Bu=(B_0u,0)$. The mild solution of initial problem (4.3.3) is the following form:

$$z(t;g,f,Bu)=S(t)g+\int_0^t S(t-s)F(z(s))ds+\int_0^t S(t-s)Bu(s)ds.$$

For $T>0$, $g\in Z$ and $u\in L^2(0,T;U)$ we set

$$L_T(g)=\{z(T;g,0,Bu):u\in L^2(0,T;U)\},$$

$$R_T(g)=\{z(T;g,f,Bu):u\in L^2(0,T;U)\},$$

$$L(g)=\bigcup_{T>0}L_T(g),\quad R(g)=\bigcup_{T>0}R_T(g),$$

$$L_T^K(g)=\{z(T;g,0,Bu):||u||_{L^2(0,T;U)}\le K\},$$

$$R_T^K(g)=\{z(T;g,f,Bu):||u||_{L^2(0,T;U)}\le K\}.$$

Definition 4.3.1. *The system (4.3.3) is said to be approximately controllable on* $[0,T]$ *if* $\overline{R(T)}=Z$. *If* $\overline{L(T)}=Z$, *the linear system (4.3.3) is said to be approximately controllable on* $[0,T]$.

Here, we remark that if the system (4.3.3) is said to be approximately controllable on $[0,T]$ if $\overline{R(T)}=Z$, so is the system (PS). In view of H. Tanabe[11; Lemma 7.4.1] $L_T(0)$ is independent of time T.

We need the following hypothesis:

(H) $|f(t,x)|\le M,\ x\in H(t\ge 0)$.

Theorem 4.3.1. *Let us assume the hypothesis (H) and* $\overline{L_T^K(g)}$ *have interior points. Then we have that for any* $g \in Z$ *there exists a constant* c *such that*

$$\overline{R_T^K(g)} \subset c\overline{L_T^K(g)}.$$

Proof. Since $\overline{L_T^K(g)} = S(T)g + \overline{L_T(0)}$ and $\overline{L_T^K(0)}$ is a balanced closed subspace, there exists a $z_0 \in Z$ such that

$$\inf\{||z_0 - S(T)g - z|| : z \in \overline{L_T(0)}\} > 2M||g||_Z + M^2T. \tag{4.3.4}$$

Then $z_0 \notin \overline{R_T(g)}$. In fact, from (4.3.4) we obtain that

$$\begin{cases} || & z(t;0,f,Bu) - z_0 + S(T)g||_Z \\ & \geq || \int_0^t S(t-s)Bu(s)ds - z_0||_Z - 2||S(T)g||_Z - || \int_0^t S(t-s)F(z(s))ds||_Z \\ & > 0. \end{cases}$$

Hence there exists a constant a c such that

$$\overline{R_T^K(g)} \subset c\overline{L_T^K(g)}.$$

□

We also assume that

(H1) $z(t;g,f,0) \in \overline{L_t(0)}$ and $z(t;g,0,0) \in \overline{L_t(0)}$ $(t > 0)$.

With the aid of the hypothesis (H1) it holds that $\overline{L_t(0)} = \overline{L_t(g)}$ for every $g \in Z$.

Theorem 4.3.2. *Under the hypotheses (H) and (H1), we have that*

$$\overline{L_T(g)} \subset \overline{R_T(g)}.$$

Proof. Let $\epsilon > 0$ and $z_0 \in \overline{L_T(g)}$ and let $\delta \leq \frac{1}{2}\epsilon M^2$. Put $z_0(s) = z(s; g, f, 0)$ and $z_1 = z(T-\delta; g, f, 0)$, where $z(T-\delta; g, f, 0) = S(T-\delta)g + \int_0^{T-\delta} S(T-\delta-s)F(z_0(s))ds$. Consider the following problem:

$$\begin{cases} \frac{d}{dt}y(t) = Ay(t) + Bu(t), \quad T-\delta < s \leq T, \\ y(T-\delta) = z_1. \end{cases}$$

Then since the form of a solution of above equation is $y_u(T) = S(\delta)z_1 + \int_{T-\delta}^{t} S(T-s)\Phi u(s)ds$.

and $L_t(0)$ is independent of time t, we have $y_u(T) \in \overline{L^T(0)}$. Here, we used the fact that $S(t)\overline{L_{t'}(0)} \subset \overline{L_{t+t'}(0)}$. From the hypothesis (H1) there exists $u_1 \in L^2(T-\delta, T; U)$ such that

$$||y_{u_1}(T) - z_0|| < \frac{\epsilon}{2} \tag{4.3.5}$$

where $y_{u_1}(T) = S(\delta)z_1 + \int_{T-\delta}^{T} S(T-s)Bu_1(s)ds$. Now we set

$$v(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq T-\delta, \\ u_1(s) & \text{if } T-\delta < s \leq T. \end{cases}$$

Then $v \in L^2(0, T; U)$ and from (4.3.5) we obtain that

$$\begin{aligned} ||z(T; g, f, Bv) - z_0||_Z &\leq ||S(\delta)z_1 + \int_{T-\delta}^{T} S(T-s)Bu_1(s)ds - z_0||_Z \\ &\quad + ||\int_{T-\delta}^{T} S(T-s)F(z_{u_1}(s))ds||_Z \\ &< \frac{\epsilon}{2} + M^2\delta \leq \epsilon. \end{aligned}$$

Hence the proof is complete. □

Corollary 4.3.1. *Let us assume the hypotheses (H) and (H1). Then* $\overline{L_t(g)} = Z$ *if and only if* $\overline{R_t(g)} = Z$. *therefore, if the linear system (4.3.3) is said to be approximately controllable on* $[0, T]$, *so is semilinear system (4.3.3)*

The proof of this Corollary holds from Theorems 4.3.1 and 4.3.2.

Chapter 5
Optimal Control Problems for Semilinear Retarded Functional Differential Equations

5.1 Introduction

This paper is concerned with the optimal control problem of the semilinear functional differential equation with delay in a Hilbert space. Applications of the optimal control problems for two types of cost functions are given; one is the averaging observation control and the other is the observation of terminal value. The principal operator of given equations generates an analytic semigroup and the nonlinear term is uniformly Lipschitz continuous with respect to the second variable. This is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [21].

The optimal control problems of linear systems have been so extensively studied by [13, 37, 8] and the references cited there. In [37], Nakagiri obtained some standard optimal control problems, namely, the fixed time integral convex cost problem and the time optimal control problem for general linear retarded systems in reflexive Banach spaces. In [27], Papageorgiou established the existence of the optimal control for a broad class of nonlinear evolution control systems and in [28], the author obtained necessary conditions for optimality using the penalty method first introduced in Balakrishnan [2]. Actually, the work of [2] is to introduce a computational procedure for optimal nonlinear control problems with condition of the Gâteaux differentiability of the nonmonotone terms. Indeed, the optimal control problems on semilinear partial differential equations with delay terms are not so many.

The purpose of this paper is to extend the optimal control theory for the general linear results as in [37, 17, 15] to practical semilinear retarded systems using the construction of the fundamental solution in case where the principal operators are unbounded operators. Two applications of the main results are given; one gives a uniqueness of the optimal control of the cost function defined by distributed observation and the other gives a feedback control law for the observation function of terminal value. Here, using techniques for the linear control problems and the properties of solutions of semilinear system as developed in [26, 12, 4], we obtain the existence of optimal controls for the equation, where the nonlinear term is given by the convolution product and give the maximal principle for given cost functions and present the necessary conditions of optimality which are described by the adjoint state corresponding to the linear retarded equation without a condition of differentiability of nonlinear term.

5.2 Preliminaries and Local Solutions

Let V and H be Hilbert spaces forming a Gelfand triple $V \subset H \subset V^*$ with pivot space H. and the operator A_0 is the operater mention in section 3.1.

The control space will be modeled by a Banach space Y. Let the controller B is a bounded linear operator from Y to H.

We denote by $W^{m,p}(0,T;V^*)$ the sobolev space of V^*-valued functions on $[0,T]$ whose distributional derivatives up to m belong to $L^p(0,T;V^*)$.

First, we introduce the following linear retarded functional differential equation:

$$\begin{cases} x'(t) = A_0x(t) + \int_{-h}^{0} a(s)A_1x(t+s)ds + q(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0. \end{cases} \tag{RE}$$

The operator A_1 is a bounded linear operator from V to V^* such that its restriction to $D(A_0)$ is a bounded linear operator from $D(A_0)$ to H. The function $a(\cdot)$ is assumed to be real valued and Hölder continuous in the interval $[-h, 0]$:

$$|a(s) - a(\tau)| \le K(s-\tau)^{\rho} \tag{5.2.1}$$

for constants K and $0 < \rho < 1$.

Let $W(\cdot)$ be the fundamental solution of the linear equation associated with (RE) which is the operator valued function satisfying

$$\begin{cases} W(t) &= S(t) + \int_0^t S(t-s)\int_{-h}^{0} a(\tau)A_1W(s+\tau)d\tau ds, \quad t > 0, \\ W(0) &= I, \qquad W(s) = 0, \quad -h \le s < 0, \end{cases} \tag{5.2.2}$$

where $S(\cdot)$ is the semigroup generated by A_0. Then

$$x(t) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-s)q(s)ds,$$

$$U_t(s) = \int_{-h}^{s} W(t-s+\sigma)a(\sigma)A_1 d\sigma.$$

Recalling the formulation of mild solutions, we know that the mild solution of (RE) is represented by

$$x(t) = \begin{cases} S(t)\phi^0 + \int_0^t S(t-s)\{\int_{-h}^0 a(\tau)A_1 x(s+\tau)d\tau + q(s)\}ds, \\ \phi^1(s), \quad -h \le s < 0. \end{cases}$$

From Proposition 4.2 in [4] it follows the following results.

Proposition 5.2.1. *The fundamental solution* $W(t)$ *to* (RE) *exists uniquely. For any natural number* n *and* $i = 0, 1$, *there exists a constant* C_n *such that*

$$||W(t)|| \le C_n, \tag{5.2.3}$$

$$||A_i W(t)|| \le C_n/t, \tag{5.2.4}$$

$$||A_i W(t) A_0^{-1}|| \le C_n \tag{5.2.5}$$

on $[0, nh]$ *and*

$$|| \int_t^{t'} A_i W(\tau)d\tau|| \le C_n, \quad 0 \le t < t' \le nh. \tag{5.2.6}$$

Here, $|| \cdot ||$ *stands for the operator norm simply.*

Considering as an equation in V^* we also obtain the same norm estimates of (5.2.3)-(5.2.6) in the space $\mathcal{L}(V^*)$ which is the collection of all bounded linear operators from V^* to itself.

Proposition 5.2.2. *There exists a constant* C *such that the following inequalities hold for all* $t > 0$ *and every* $x \in H$ *or* V^*:

$$|W(t)x| \le Ct^{-1/2}||x||_*, \tag{5.2.7}$$

$$||W(t)x|| \le Ct^{-1/2}|x| \tag{5.2.8}$$

Proof. As in Lemma 3.6.2 of [8], if $\{S(t)\}$ represents the semigroup generated by A_0, then there exists a constant C such that

$$|S(t)x| \le Ct^{-1/2}||x||_*, \quad ||S(t)x|| \le Ct^{-1/2}|x|, \tag{5.2.9}$$

for $t > 0$ and $x \in H$ or V^*. With the aid of the change of the variable and noting $W(t) = 0$ for $t < 0$, we get

$$W(t)x = S(t)x + \int_0^t S(t-s)\int_0^s a(\tau - s)A_1 W(\tau)xd\tau ds$$

on $[0, h]$ and

$$W(t)x = S(t)x + \int_0^t S(t-s)\int_{s-h}^s a(\tau - s)A_1 W(\tau)xd\tau ds$$

on $[h, \infty]$. We write the second term of the representation $W(t)$ in $[h, \infty]$ as

$$\begin{aligned}
&\int_0^t S(t-s)\int_{s-h}^s a(\tau - s)A_1 W(\tau)xd\tau ds\\
&= \int_0^t S(t-s)\int_{s-h}^s (a(\tau - s) - a(-s))A_1 W(\tau)xd\tau ds\\
&\quad + \int_0^t S(t-s)a(-s)\int_{s-h}^s A_1 W(\tau)xd\tau ds\\
&= I + II.
\end{aligned}$$

Combining (5.2.1), (5.2.7)(or (5.2.8)), and (5.2.9), we get

$$\begin{aligned}
||I|| &\le \int_0^t C(t-s)^{-1/2}\int_{s-h}^s KC_n\tau^{\rho-1}|x|d\tau ds\\
&\le 2CC_nK\rho^{-1}B(\frac{1}{2}, \rho + 1)|x|,
\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function. From (5.2.6), it follows that II is also bounded. We can also obtain the similar inequalities in $[0,h]$. Thus, the proof is easily obtained from (5.2.9) and using the representation (5.2.2) of $W(t)$. □

By virtue of Theorem 3.3 of [4] we have the following result on the linear equation (RE).

Proposition 5.2.3. *1) Let $F=(D(A_0),H)_{\frac{1}{2},2}$, where $(D(A_0),H)_{1/2,2}$ denote the real interpolation space between $D(A_0)$ and H. Let*

$$(\phi^0,\phi^1)\in F\times L^2(-h,0;D(A_0)) \ textand \ q\in L^2(0,T;H), T>0.$$

Then there exists a unique solution x of (RE) belonging to

$$\mathcal{W}_0(T)\equiv L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)\subset C([0,T];F)$$

and satisfying

$$||x||_{\mathcal{W}_0(T)}\le C_1'(||\phi^0||_F+||\phi^1||_{L^2(-h,0;D(A_0))}+||q||_{L^2(0,T;H)}), \tag{5.2.10}$$

where C_1' is a constant depending on T.
2) Let $(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$ and $q\in L^2(0,T;V^*)$, $T>0$. Then there exists a unique solution x of (RE) belonging to

$$\mathcal{W}_1(T)\equiv L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H)$$

and satisfying

$$||x||_{\mathcal{W}_1(T)}\le C_1'(|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+||q||_{L^2(0,T;V^*)}). \tag{5.2.11}$$

In what follows we assume that $W(t)$ is uniformly bounded: There is a constant $M > 0$ such that

$$||W(t)|| \leq M, \quad t > 0 \tag{5.2.12}$$

for the sake of simplicity. Let $g : [0,T] \times V \to H$ be a nonlinear mapping. We assume that there exists a constant $L > 0$ such that $t \mapsto g(t,x)$ is measurable and

$$|g(t,x) - g(t,y)| \leq L||x - y||, \quad g(t,0) = 0 \tag{5.2.13}$$

for all $x_1, x_2 \in V$.

For $x \in L^2(0,T;V)$ and $k \in L^2(0,T)$ we set

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds. \tag{5.2.14}$$

Now, we consider the following semilinear retarded functional differential equation

$$\begin{cases} x'(t) = A_0 x(t) + \int_{-h}^0 a(s)A_1 x(t+s)ds + f(t,x(t)) + q(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \leq s < 0. \end{cases} \tag{5.2.15}$$

Lemma 5.2.1. *Let $x \in L^2(0,T;V)$, $T > 0$. Then $f(\cdot,x) \in L^2(0,T;H)$. and*

$$||f(\cdot,x)||_{L^2(0,T;H)} \leq L||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)}.$$

Moreover, if $x_1,\ x_2 \in L^2(0,T;V)$, then

$$||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} \leq L||k||_{L^2(0,T)}\sqrt{T}||x_1 - x_2||_{L^2(0,T;V)}.$$

Proof. From (5.2.13) and using the Hölder inequality it is easily seen that

$$||f(\cdot,x)||^2_{L^2(0,T;H)} \le \int_0^T |\int_0^t k(t-s)g(s,x(s))ds|^2 dt$$

$$\le ||k||^2_{L^2(0,T)} \int_0^T \int_0^t L^2 ||x(s)||^2 ds dt$$

$$\le TL^2 ||k||^2_{L^2(0,T)} ||x||^2_{L^2(0,T;V)}.$$

The proof of the second paragraph is similar. □

In virtue of Lemma 5.2.1, using the maximal regularity for more general retarded parabolic system from Theorem 3.1 in [31], we establish the following result on the solvability of (5.2.15).

Proposition 5.2.4. *Suppose that the assumptions* (5.2.13) *is satisfied. Then for any*

$$(\phi^0, \phi^1) \in H \times L^2(-h,0;V) \text{ and } q \in L^2(0,T;V^*), \quad T > 0,$$

the solution x of (5.2.15) *exists and is unique in* $L^2(-h,T;V) \cap W^{1,2}(0,T;V^*)$, *and there exists a constant C_2' depending on T such that*

$$||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2'(1 + |\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||q||_{L^2(0,T;V^*)}). \tag{5.2.16}$$

Let $I = [0,T]$, $T > 0$ be a finite interval. We introduce the transposed system which is exactly the same as in Nakagiri [37]. Let $q_0^* \in H$, $q_1^* \in L^1(I;H)$. The retarded transposed system in H is defined by

$$\begin{cases} \frac{dz(t)}{dt} + A_0^* z(t) + \int_{-h}^0 a(s) A_1^* z(t-s)ds - q_1^*(t) = 0 \text{ a.e. } t \in I, \\ z(T) = q_0^*, \quad z(s) = 0 \text{ a.e. } s \in]T, T+h]. \end{cases} \tag{5.2.17}$$

Let $W^*(t)$ denote the adjoint of $W(t)$. Then the mild solution of (5.2.17) is defined as follows:

$$z(t) = W^*(T-t)q_0^* + \int_t^T W^*(\xi - t)q_1^*(\xi)d\xi,$$

for $t \in I$ in the weak sense. The transposed system will be used to describe a formulation of the optimality conditions for optimization problems.

5.3 Optimal Control for the Distributed Observation

In this section we assume that the embedding $D(A_0) \subset V$ is compact. Choose a bounded subset U of Y and call it a control set. Suppose that an admissible control $u \in L^2(0,T;Y)$ is a strongly measurable function satisfying $u(t) \in U$ for almost all t, and let $x(t; f, u)$ be a solution of (5.2.15) associated with the nonlinear term f and a control u at time t. The solution $x(t; f, u)$ of (5.2.15) for each admissible control u is called a trajectory corresponding to u.

Let $\mathcal{F}$ and $\mathcal{B}$ be the Nemitsky operators corresponding to the map f and B, which are defined by

$$(\mathcal{F}u)(\cdot) = f(\cdot, x_u(\cdot)) \quad \text{and} \quad (\mathcal{B}u)(\cdot) = Bu(\cdot),$$

respectively. Then,

$$\begin{aligned} x(t; f, u) &= x(t;\phi) + \int_0^t W(t-s)\{f(s, x(s) + Bu(s)\}ds \\ &= x(t;\phi) + \int_0^t W(t-s)((\mathcal{F}+\mathcal{B})u)(s)ds, \end{aligned}$$

where

$$x(t;\phi) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds.$$

Let Z be a real Hilbert space and let $C(t)$ be bounded from H to Z for each t and be continuous in $t \in [0,T]$. Let $y \in L^2(0,T;Z)$. Suppose that there exists no admissible control which satisfies $C(t)x(t;f,u) = y(t)$ for almost all t. Then, we consider a cost function given by

$$J(u) = \frac{1}{2}\int_0^T |C(t)x(t;f,u) - y(t)|^2 dt. \tag{5.3.1}$$

Let $u \in L^1(0,T;Y)$. Then it is well known that

$$\lim_{h\to 0} h^{-1}\int_0^h ||u(t+s) - u(t)||_Y ds = 0 \tag{5.3.2}$$

for almost all points of $t \in [0,T]$.

Definition 5.3.1. *The point t, which permits* (5.3.2) *to hold, is called a Lebesgue point of u.*

Lemma 5.3.1. *Let x_u be the solution of* (5.2.15) *corresponding to u. Then the mapping $u \mapsto x_u$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;V)$.*

Proof. We define the solution mapping S from $L^2(0,T;Y)$ to $L^2(0,T;V)$ by

$$(Su)(t) = x_u(t), \quad u \in L^2(0,T;Y).$$

In virtue of (5.2.10), (5.2.16), and Lemma 5.2.1

$$
\begin{aligned}
&||Su||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} = ||x_u||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \\
&\leq C_1'(||\phi^0||_F + ||\phi^1||_{L^2(-h,0;D(A_0))} + ||(\mathcal{F}+\mathcal{B})u||_{L^2(0,T;H)}) \\
&\leq C_1'(||\phi^0||_F + ||\phi^1||_{L^2(-h,0;D(A_0))} \\
&\quad + L||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)} + ||B||||u||_{L^2(0,T;Y)}) \\
&\leq C_1'\{||\phi^0||_F + ||\phi^1||_{L^2(-h,0;D(A_0))} + L||k||_{L^2(0,T)}\sqrt{T}(C_2'(1+|\phi^0| \\
&\quad + ||\phi^1||_{L^2(-h,0;V)} + ||B||||u||_{L^2(0,T;Y)}) + ||B||||u||_{L^2(0,T;Y)}\}.
\end{aligned}
$$

Hence if u is bounded in $L^2(0,T;Y)$, then so is x_u in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$. Noting that $D(A_0)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H) \subset L^2(0,T;V))$$

is also compact in view of Theorem 2 of J. P. Aubin [18]. Hence, the mapping $u \mapsto Su = x_u$ is compact from $L^2(0,T;Y)$ to $L^2(0,T;V)$. □

Theorem 5.3.1. *Let U be a bounded closed convex subset of Y. Then, there exists an optimal control for the cost function* (5.3.1).

Proof. Let $\{u_n\}$ be a minimizing sequence of J such that

$$\inf_{u\in U} J(u) = \lim_{n\to\infty} J(u_n).$$

Since U is bounded and weakly closed, there exist a subsequence, which we write again by $\{u_n\}$, of $\{u_n\}$ and a $\hat{u}\in U$ such that

$$u_n \to \hat{u} \quad \text{weakly in } L^2(0,T;Y).$$

Now we show that $\hat{u}$ is admissible as follows. Since U is a closed convex set of Y, by Mazur's theorem as an important consequence of the Hahn-Banach theorem, there exists an $f_0 \in Y^*$ and $c \in [-\infty, \infty]$ be such that $f_0(u) \le c$ for all $u \in U$. Let s be a Lebesgue point of $\hat{u}$. and put

$$w_{\epsilon,n} = \frac{1}{\epsilon}\int_s^{s+\epsilon} u_n(t)dt$$

for each $\epsilon > 0$ and n. Then, $f_0(w_{\epsilon,n}) \le c$ and we have

$$w_{\epsilon,n \to} w_\epsilon = \frac{1}{\epsilon}\int_s^{s+\epsilon} \hat{u}(t)dt \quad \text{weakly as } n \to \infty.$$

By letting $\epsilon \to 0$, it holds that $w_\epsilon \to \hat{u}(s)$ and $f_0(\hat{u}) \le c$, so that $\hat{u}(s) \in U$. Noting that

$$x_n(t) = x(t;\phi) + \int_0^t W(t-s)((\mathcal{F}+\mathcal{B})u_n)(s)ds, \tag{5.3.3}$$

where

$$x(t;\phi) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds,$$

it follows from Proposition 5.2.4 that $\{x_n(t)\}$ is bounded and hence weakly sequentially compact. From (5.2.13) and Lemma 5.3.1, we see that $\mathcal{F}$ is a compact operator from $L^2(0,T;Y)$ to $L^2(0,T;H)$ and hence, it holds $\mathcal{F}u_n \to \mathcal{F}u_0$ strongly in $L^2(0,T;H)$. Hence, since $\mathcal{B}$ and $W(t)$ are strongly continuous on $[0,T]$, we have

$$x_n(t) \to x(t;f,\hat{u}) \ \text{ weakly in } \ H,$$

where

$$x(t; f, \hat{u}) = x(t; \phi) + \int_0^t W(t-s)\{(\mathcal{F} + \mathcal{B})\hat{u}(s)\}(s)ds.$$

Therefore, we have

$$\inf J(u) \le J(\hat{u}) \le \liminf J(u_n) = \inf J(u).$$

Thus, this $\hat{u}$ is an optimal control. $\square$

The maximum principle is derived from the optimal condition as follows.

Theorem 5.3.2. *Let* (5.2.13) *be satisfied and let* $\hat{u}$ *be an optimal control. Then the equality*

$$\max_{v \in U}(v, B^* z(s)) = (\hat{u}(s), B^* z(s))$$

holds, where

$$z(s) = \int_s^T W^*(t-s) C^*(t)(y(t) - C(t)x(t; f, \hat{u}))dt.$$

Here, $z(s)$ satisfies the following transposed system:

$$\begin{cases} z'(t) + A_0^* z(t) \int_{-h}^0 a(s) A_1^* z(t-s)ds - C^*(t)(y(t) - C(t)x(t; f, \hat{u})) = 0 \;\; a.e. \;\; t \in I, \\ z(T) = 0, \quad z(s) = 0 \quad a.e. \; s \in [T, T+h]. \end{cases} \tag{5.3.4}$$

in the weak sense.

Proof. Let $\hat{u}$ be an optimal control and $\hat{x}(t) = x(t; f, \hat{u})$. For $\epsilon > 0$, choose $v \in L^2(0, T; U)$ so that $||\hat{u} - v||_{L^2(0,T;U)} < \epsilon$. Let t_0 be a Lebesgue point of $\hat{u}, v$. For $t_0 < t_0 + \epsilon < T$, put

$$u(t) = \begin{cases} v, & \text{if } t_0 < t < t_0 + \epsilon, \\ \hat{u}(t), & \text{otherwise.} \end{cases} \tag{5.3.5}$$

Then u is an admissible control. Let $x(t) = x(t; f, u)$. Then $x(t) - \hat{x}(t) = 0$ for $0 \leq t \leq t_0$ and

$$x(t) - \hat{x}(t) = \int_{t_0}^{t_0+\epsilon} W(t-s)\{f(s, x(s)) - f(s, \hat{x}(s)) + B(v - \hat{u}(s))\}ds, \tag{5.3.6}$$

for $t_0 + \epsilon \leq t \leq T$. If $t_0 < t < t_0 + \epsilon$ then

$$x(t) - \hat{x}(t) = \int_{t_0}^{t} W(t-s)\{f(s, x(s)) - f(s, \hat{x}(s)) + B(v - \hat{u}(s))\}ds. \tag{5.3.7}$$

Let $t_0 + \epsilon \leq t \leq T$ and let us put

$$w(t) = \int_{t_0}^{t_0+\epsilon} W(t-s)B(v - \hat{u}(s))ds.$$

Noting that $v - \hat{u}$ is admissible and t_0 is a Lebesgue point of $v - \hat{u}$ and for $0 < \alpha < 1$

$$\log(1+\epsilon) = \int_1^{1+\epsilon} \frac{1}{y} dy \leq \int_1^{1+\epsilon} \frac{1}{y^{1-\alpha}} dy = \frac{1}{\alpha}((1+\epsilon)^\alpha - 1) \leq \frac{\epsilon^\alpha}{\alpha},$$

we have that from (5.2.8) of proposition 5.2.2 and Hölder inequality, it follows

$$
\begin{aligned}
||w(t)|| &\le C\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}|B(v-\hat{u}(s))|ds \qquad (5.3.8)\\
&\le C||B||\{\log\frac{t-t_0}{t-t_0-\epsilon}\}^{\frac{1}{2}}||\hat{u}-v||_{L^2(0,T;U)}\\
&\le \alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||.
\end{aligned}
$$

From (5.3.6) and (5.3.8), it follows that

$$
\begin{aligned}
&||x(t)-\hat{x}(t)|| \qquad (5.3.9)\\
&\le ||\int_{t_0}^{t_0+\epsilon}W(t-s)\{f(s,x(s))-f(s,\hat{x}(s))\}ds||+||w(t)||\\
&\le C\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}|f(s,x(s))-f(s,\hat{x}(s))|ds+||w(t)||\\
&\le CL\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}||x(s)-\hat{x}(s)||ds+\alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||.
\end{aligned}
$$

Therefore, if we put $X(t) = ||x(t) - \hat{x}(t)||$, then

$$X(t) \le CL\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}X(s)ds + \alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||$$

$$\le CL\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}\{CL\int_0^s(s-\tau)^{-1/2}X(\tau)d\tau$$

$$+\alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||\}ds + \alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||$$

$$=(CL)^2\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}\int_0^s(s-\tau)^{-1/2}X(\tau)d\tau ds$$

$$+\alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||\int_{t_0}^{t_0+\epsilon}(t-s)^{-1/2}ds$$

$$+\alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||$$

$$=(CL)^2\int_{t_0}^{t_0+\epsilon}\int_\tau^t(t-s)^{-1/2}(s-\tau)^{-1/2}dsX(\tau)d\tau$$

$$+\frac{1}{2}\alpha^{-1/2}\epsilon^{3/2}(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B|| + \alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||$$

$$=(CL)^2B(\frac{1}{2},\frac{1}{2})\int_{t_0}^{t_0+\epsilon}X(\tau)d\tau + \frac{1}{2}\alpha^{-1/2}\epsilon^{3/2}(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||$$

$$+\alpha^{-1/2}\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2}C||B||,$$

where $B(\cdot,\cdot)$ is the Beta function. Gronwal's inequality implies

$$|x(t)-\hat{x}(t)| \le ||x(t)-\hat{x}(t)|| \le c_2\epsilon(\frac{\epsilon}{t-t_0-\epsilon})^{\alpha/2} \tag{5.3.10}$$

for some constant $c_2 > 0$. It holds the inequality (5.3.10) in case where $0 \leq t < t_0 + \epsilon$ naturally. Since $\hat{u}$ is optimal, we have

$$0 \leq \frac{1}{\epsilon}(J(u) - J(\hat{u})) \tag{5.3.11}$$

$$= \frac{1}{\epsilon}\int_0^T (C(t)(x(t) - \hat{x}(t)), C(t)\hat{x}(t) - y(t))dt$$

$$+ \frac{1}{2\epsilon}\int_0^T |C(t)(x(t) - \hat{x}(t))|^2 dt$$

$$= I + II.$$

From (5.3.10) it follows that

$$\lim_{\epsilon \downarrow 0} II = 0. \tag{5.3.12}$$

The first term of (5.3.11) can be represented as

$$I = \frac{1}{\epsilon}\int_{t_0}^T (C(t)(x(t) - \hat{x}(t)), C(t)\hat{x}(t) - y(t))dt = \frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} + \frac{1}{\epsilon}\int_{t_0+\epsilon}^T = I_1 + I_2.$$

On account of (5.3.10), it holds that

$$\lim_{\epsilon \downarrow 0} I_1 = 0. \tag{5.3.13}$$

Noting

$$|f(s, x(s)) - f(s, \hat{x}(s))| \leq L||k||_{L^2(0,T)}(\int_0^s |x(\tau) - \hat{x}(\tau)|^2 d\tau)^{1/2}$$

$$\leq \epsilon c_2 \sqrt{T} L ||k||_{L^2(0,T)},$$

we have that

$$|\int_{t_0}^{t_0+\epsilon} W(t-s)(f(s,x(s))-f(s,\hat{x}(s)))ds| \le \epsilon^2 c_2\sqrt{T}CL||k||_{L^2(0,T)}. \quad (5.3.14)$$

Hence, we obtain

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}(x(t)-\hat{x}(t))$$

$$= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} W(t-s)\{f(s,x(s))-f(s,\hat{x}(s))+B(v-\hat{u})(s)\}ds$$

$$= W(t-t_0)B(v-\hat{u})(t_0).$$

Thus, as in (5.3.11), we have

$$\lim_{\epsilon\downarrow 0} I_2 = \int_{t_0}^{T}(C(t)(W(t-t_0)B(v-\hat{u})(t_0), C(t)\hat{x}(t)-y(t))dt,$$

that is, from (5.3.11)-(5.3.14) it follows

$$\int_s^T (C(t)W(t-s)B(v-\hat{u})(s), C(t)\hat{x}(t)-y(t))dt \ge 0$$

holds for every $v\in U$ and for all Lebesgue points s of $\hat{u}$. Hence, we have

$$(v-\hat{u}(s), B^*z(s))dt \le 0,$$

where

$$z(s) = \int_s^T W^*(t-s)C^*(t)(y(t)-C(t)\hat{x}(t))dt.$$

Here, $z(s)$ is a solution in a weak sense of the equation (5.3.4). □

The optimality condition J is often used to derive the uniqueness of optimal control. To give such an application we need the following lemma, which is well known for the fundamental solution $W(t)$[[37], Lemma 5.1].

Lemma 5.3.2. *Given an interval $I \subset \mathbb{R}$ and a Banach space X, let $f \in L^p(I;X)$ for $1 \le p \le \infty$. If*

$$\int_0^t W(t-s)f(s)ds = 0, \quad \textit{for all } t \in I,$$

then $f(t) = 0$ almost everywhere $t \in I$.

Now, we give the conditions for the uniqueness of optimal control as follows.

Theorem 5.3.3. *Let $\mathcal{F} + \mathcal{B}$ and $C(t)(t \ge 0)$ be one to one mappings. Then, the optimal control for the cost function* (5.3.1) *is unique.*

Proof. Let u be an admissible control defined by (5.3.5) and let t_0 be a Lebesgue point of $\hat{u}, v$ and $\mathcal{F}(v - \hat{u})$. Putting that $\hat{x}(t) = x(t; f, \hat{u})$ and $x(t) = x(t; f, u)$, we obtain the estimate of $x(t) - \hat{x}(t)$ in H by using simple calculations and known results, which is also obtained from (5.3.10) directly. Using the Hölder inequality it is easily seen that

$$\begin{aligned}||f(\cdot,x) - f(\cdot,x)||^2_{L^2(t_0,t_0+\epsilon;H)} &\le \int_{t_0}^{t_0+\epsilon} |\int_0^s k(s-\tau)(g(\tau,x(\tau)) - g(\tau,\hat{x}(\tau)))d\tau|^2 ds \\ &\le ||k||^2_{L^2(0,T)} \int_{t_0}^{t_0+\epsilon} \int_0^s L^2 ||x(\tau) - \hat{x}(\tau)||^2 d\tau ds \\ &\le \epsilon L^2 ||k||^2_{L^2(0,T)} ||x - \hat{x}||^2_{L^2(t_0,t_0+\epsilon;V)}\end{aligned}$$

and hence, with the aid of Hölder inequality

$$|\int_{t_0}^{t_0+\epsilon} W(t-s)(f(s,x(s)) - f(s,\hat{x}(s)))ds| \tag{5.3.15}$$

$$\leq \epsilon ML||k||_{L^2(0,T)}||x-\hat{x}||_{L^2(t_0,t_0+\epsilon;V)}$$

for $t_0 + \epsilon \leq t \leq T$. Since the control set U is bounded, noting that $v - \hat{u}$ is admissible and t_0 is Lebesgue point of $v - \hat{u}$, there exists a constant $c_1 > 0$ such that

$$|B(v - \hat{u}(t))| \leq c_1, \quad \text{for } 0 \leq t \leq T.$$

Thus, we obtain

$$|x(t) - \hat{x}(t)| \leq |\int_{t_0}^{t_0+\epsilon} W(t-s)\{f(s,x(s)) - f(s,\hat{x}(s)) + B(v - \hat{u}(s))\}ds|$$

$$\leq \epsilon ML||k||_{L^2(0,T)}||x-\hat{x}||_{L^2(t_0,t_0+\epsilon;V)} + \epsilon c_1 M.$$

Hence, in virtue of Proposition 5.2.4, there exists a constant c_3 such that

$$|x(t) - \hat{x}(t)| \leq c_3\epsilon \tag{5.3.16}$$

holds for any $0 \leq t \leq T$. In case where $0 \leq t < t_0 + \epsilon$, it also holds (5.3.16). Let us consider the optimal relation (5.3.11), it follows from (5.3.16) that

$$\lim_{\epsilon \downarrow 0} II = 0. \tag{5.3.17}$$

On account of (5.3.16), as in (5.3.13), it holds that

$$\lim_{\epsilon \downarrow 0} I_1 = 0. \tag{5.3.18}$$

Let $t > t_0$ and $\epsilon \downarrow 0$. Then, we obtain

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}(x(t) - \hat{x}(t)) \\
&= \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} W(t-s)(\mathcal{F}+\mathcal{B})(v-\hat{u})(s)ds \\
&= W(t-t_0)(\mathcal{F}+\mathcal{B})(v-\hat{u})(t_0).
\end{aligned}$$

Hence,

$$\lim_{\epsilon\downarrow 0} I_2 = \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \int_{t_0+\epsilon}^{T} (C(t)(x(t)-\hat{x}(t)), C(t)\hat{x}(t) - y(t))dt \tag{5.3.19}$$

$$= \int_{t_0}^{T} (C(t)W(t-t_0)(\mathcal{F}+\mathcal{B})(v-\hat{u})(t_0), C(t)\hat{x}(t) - y(t))dt.$$

By (5.3.17)-(5.3.19), the inequality

$$\int_{s}^{T} (C(t)W(t-s)(\mathcal{F}+\mathcal{B})(v-\hat{u})(s), C(t)\hat{x}(t) - y(t))dt \geq 0$$

holds for every $v \in U$ and for all Lebesgue points s of $\hat{u}$. Let us denote two optimal controls by u_1 and u_2 and their corresponding by x_1 and x_2. Then, by the similar procedure mentioned above, the inequalities

$$\int_{s}^{T} (C(t)W(t-s)(\mathcal{F}+\mathcal{B})(u_2-u_1)(s), C(t)\hat{x}_1(t) - y(t))dt \geq 0$$

and

$$\int_{s}^{T} (C(t)W(t-s)(\mathcal{F}+\mathcal{B})(u_1-u_2)(s), C(t)\hat{x}_2(t) - y(t)dt \geq 0$$

hold. Add both inequalities and integrate the resultant inequality from 0 to T with respect to s. Then, since

$$x_2(t) - x_1(t) = \int_0^t W(t-s)(\mathcal{F}+\mathcal{B})(u_1 - u_2)(s)ds,$$

it holds

$$\int_0^T |C(t)(x_2(t) - x_1(t))|^2 \leq 0.$$

Since $C(t)$ is one to one, we have that $x_2(t) - x_1(t) \equiv 0$. Hence, by Lemma 5.3.2, it holds that $(\mathcal{F}+\mathcal{B})(u_1 - u_2)(t) = 0$ almost everywhere. From that $\mathcal{F}+\mathcal{B}$ is one to one, $u_1(t) = u_2(t)$ holds for almost all t. □

5.4 Observation of Terminal Value

Let y be an element of H and suppose there exists no admissible control which satisfies

$$x(T; f, u) = y.$$

We assume a cost function given by

$$J_1 = \frac{1}{2}|x(T; f, u) - y|. \tag{5.4.1}$$

Theorem 5.4.1. *Let U be a bounded closed convex subset of Y and let* (5.2.13) *be satisfied. Then, there exists an optimal control for the cost function* (5.4.1). *Moreover, if $\hat{u}$ is an optimal control for* (5.4.1) *then*

$$\max_{v \in U}(v, B^* z(t)) = (\hat{u}(t), B^* z(t)) \tag{5.4.2}$$

almost everywhere in $0 \le t \le T$, *where* $z(t) = W^*(T-t)(x(T;f,u)-y)$ *satisfies the terminal value problem*

$$\begin{cases} z'(t) + A_0^* z(t) + \int_{-h}^0 a(s) A_1^* z(t-s) ds = 0, \\ z(T) = x(T;f,u) - y \end{cases}$$

in the weak sense.

Proof. Let $v \in U$. Let u be an admissible control defined by (5.3.5) and t_0 be a Lebesgue point of $\hat{u}, v \in U$. Put $x(t) = x(t;f,u)$ and $\hat{x}(t) = x(t;f,\hat{u})$, then

$$x(T) - \hat{x}(T) = \int_{t_0}^{t_0+\epsilon} W(T-s)\{f(s,x(s)) - f(s,\hat{x}(s)) + B(v - \hat{u}(s))\}ds.$$

Since $\hat{u}$ is an optimal control, we have

$$0 \le \frac{1}{\epsilon}(J_1(u)^2 - J_1(\hat{u})^2) \tag{5.4.3}$$

$$= \frac{1}{\epsilon}(x(T) - \hat{x}(T), \hat{x}(T) - y) + \frac{1}{2\epsilon}|x(T) - \hat{x}(T)|^2$$

$$= I + II.$$

From (5.3.10) or (5.3.15), we have $II \downarrow 0$ as $\epsilon \downarrow 0$. From (5.3.14) it follows that

$$\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} W(T-s)(f(s,x(s)) - f(s,\hat{x}(s)))ds \to 0,$$

thus,

$$I = (\frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} W(T-s)\{f(s,x(s)) - f(s,\hat{x}(s)) + B(v-\hat{u})(s)\}ds, \hat{x}(T) - y)$$

$$\to (W(T-t_0)B(v-\hat{u})(t_0), \hat{x}(T) - y).$$

Therefore, from (5.4.3) we have

$$0 \leq ((v-\hat{u})(t_0), B^*W^*(T-t_0)(\hat{x}(T)-y)),$$

which implies that (5.4.2) holds at each Lebesgue point $\hat{u}$. $\square$

Definition 5.4.1. *Let $z(t) = W(T-t)^* z_0$ be a solution of the equation*

$$z'(t) + A_0^* z(t) + \int_{-h}^{0} a(s) A_1^* z(t-s)ds = 0, \quad z(T) = z_0. \tag{5.4.4}$$

We say the adjoint system (5.4.4) *is weakly regular if $z_0 = 0$ follows from the existence of a set $E \subset [0,T]$ such that the measure of E is positive and $z(t) = W(T-t)^* z_0 = 0$ for all $t \in E$.*

The examples for which the system (5.5.4) is weakly regular are given in [[1], p. 41] or Section 7.3 of [37].

Theorem 5.4.2. *Let the cost J_1 be given in* (5.4.1)*. Assume that the adjoint system* (5.4.4) *is weakly regular and B^* is one to one, then the optimal control $\hat{u}(t)$ is the bang-bang control, i.e., $\hat{u}(t)$ satisfies*

$$\hat{u}(t) \in \partial U \quad \textit{for} \quad \textit{almost} \quad \textit{everywhere } t \in [0,T], \tag{5.4.5}$$

Proof. For the cost function J_1, the maximal principle is written by

$$\max_{v \in U}(v, B^*z(t)) = (\hat{u}(t), B^*z(t)) \quad \text{a.e. } t \in [0,T],$$

where $z(t) = W(T-t)^* z_0$. It is sufficient to show (5.4.5) that $B^*z(t) \neq 0$ a.e. $t \in [0,T]$. Suppose the contrary that there exists a set E such that the measure of E is positive and $B^*z(t) = 0$ $t \in E$. Since B^* is one to one and (5.4.4) is weakly regular, we have that $z_0 = 0$, which is a contraction. $\square$

The unique problem of the optimal control for the terminal value cost function J_1 is an open problem. One of the difficulties is that we do not obtain the convexity property of nonlinear term.

5.5 Conclusions

The purpose of this paper is to extend the optimal control theory for the general linear results to practical semilinear retarded systems using the construction of the fundamental solution in case where the principal operators are unbounded operators. We give the existence and uniqueness of the optimal control of the cost function defined by distributed observation, and establish the maximal principle represented by the necessary conditions of optimality which are described by the adjoint state corresponding to the linear retarded equation without a condition of differentiability of nonlinear term. Moreover, we give a feedback control law for the observation function of terminal value, and the existence of optimal controls for the equation, where the nonlinear term is given by the convolution product.

The unique problem of the optimal control for the observation function of terminal value cost function J_1 is an open problem. One of the difficulties is that we do not obtain the convexity property of nonlinear term. Further, we intend to show how to apply control problems and optimal control to various nonlinear differential equations and systems arising in science.

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