Thesis for the Degree of
Master of Philosophy

Interpolation spaces governed by analytic semigroups of operators

by

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January 2019

Interpolation spaces governed by analytic semigroups of operators
(작용소의 해석적 반군에 의해 제어되는 보간공간)

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A thesis submitted in partial fulfillment of the requirements
for the degree of

Master of Philosophy

in Department of Applied Mathematics, The Graduate School,
Pukyong National University

January 2019

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February 22, 2019

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요 약

본 논문은 Banach 공간에서 해석적 반군에 의해 생성되는 보간들을 보간 공간의 이론들을 이용하여 공간들의 특성과 수식화를 정리하였다.

주결과의 내용은 첫째로, $0 < \theta < 1$, $0 \le t$, $T(t)$가 A에 의해 생성되는 해석적 반군이라면

$$(1-\theta)\left\{\int_0^\infty (t^{\theta-1} \| (T(t)-I)x \|)^p \frac{dt}{t}\right\}^{\frac{1}{p}} \le \left\{\int_0^\infty (t^{\theta} \| AT(t)x \|)^p \frac{dt}{t}\right\}^{\frac{1}{p}}$$

$$\le \frac{K}{1-2^{-\theta}}\left\{\int_0^\infty (t^{\theta-1} \| (T(t)-I)x \|)^p \frac{dt}{t}\right\}^{\frac{1}{p}}$$

이고, 따라서 $(D(A),X)_{\theta,p} = \left\{x \in X : \int_0^\infty (t^\theta \| AT(t)x \|)^p \frac{dt}{t} < \infty\right\}$.

둘째로, $p = 1/\theta$일 때, $(D(A),X)_{\theta,\frac{1}{\theta}} = \left\{x \in X : \int_0^\infty \| AT(t)x \|^{\frac{1}{\theta}} dt < \infty\right\}$ 임을 알 수 있다. 이것을 이용하여 다음과 같은 일반적인 초기치 문제에 대한 해의 정칙성도 증명하였다.

H 와 V 를 Hilbert 공간으로 하고 V 가 조밀한 공간으로서 그의 공액공간을 V^*로 하자. 다음과 같이 Banach 공간 H 상에서 A 를 포함하는 초기치 문제:

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + f(t), \quad t \in (0,T], \\ x(0) = x_0 \end{cases}$$

에서, $f \in L^2(0,T;\ V^*)$ 그리고 $x_0 \in H$ 로 주어지면 위의 초기치 문제의 해는 유일 하게 존재하며, 아울러

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];\ H).$$

임을 증명하였다.

Interpolation spaces governed by analytic semigroups of operators

October 4, 2018

1 Introduction

In this paper, we consider some characteristic of interpolation spaces of Banach spaces, and establish some simple properties for interpolation spaces associated with the domain of a generator of an analytic semigroup.

Let H be a complex Hilbert space and V be a real separable Hilbert space such that V is a dense subspace of H. Identifying the antidual of V with V^* we may consider $V \subset H \subset V^*$. Consider the following abstract Cauchy problems with initial data x_0:

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + f(t), \quad 0 < t \leq T, \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where $f : [0, T) \to H$ for any $T > 0$. First, we will prove that A generates an analytic semigroup in both H and V^*. We refer to [?]-[?]as for problems of application of various equations governed by semigroups. Blasio *et al.* [?] showed the existence and uniqueness of the solution

$$x \in L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; (D(A), H)_{1/2,2})$$

for $x_0 \in D(A)$ and $f \in L^2(0, T; H)$. By using properties of interpolation spaces, we will show that

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\} = (V, V^*)_{1/2,2},$$

where $|| \cdot ||_*$ is the norm of the element of V^*, and A associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying Gårding's inequality generates an analytic semigroup in both H and V^*. Hence, the equation (1.1) may be considered as an equation in H as well as in V^*. Therefore, we can apply the method of Blasio *et al.* [?] to the problem (1.1) to show the existence and uniqueness of the solution

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

with more general conditions on an initial value $x_0 \in H$ and a forcing term $f \in L^2(0,T;V^*)$. The last inclusion relation on continuity is well known and is an easy consequence of the definition of real interpolation spaces by the trace method.

2 Basic results of interpolation spaces

Let X and Y be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}$ such that the embedding mapping of both X and Y in $\mathcal{X}$ is continuous. Let $X \cap Y$ be a dense subspace in both X and Y. Let X and Y be Banach spaces such that the embedding $X \subset Y$ is continuous.

For $1 < p < \infty$, we denote by $L^p_*(X)$ the Banach space of all functions $t \to u(t)$, $t \in (0,\infty)$ and $u(t) \in X$, for which the mapping $t \to u(t)$ is strongly measurable with respect to the measure dt/t and the norm $||u||_{L^p_*(X)}$ is finite, where

$$||u||_{L^p_*(X)} = \{\int_0^\infty ||u(t)||^p_X \frac{dt}{t}\}^{\frac{1}{p}}.$$

For $0 < \theta < 1$, set

$$||t^\theta u||_{L^p_*(X)} = \{\int_0^\infty ||t^\theta u(t)||^p_X \frac{dt}{t}\}^{\frac{1}{p}},$$
$$||t^\theta u^{'}||_{L^p_*(Y)} = \{\int_0^\infty ||t^\theta u^{'}(t)||^p_Y \frac{dt}{t}\}^{\frac{1}{p}}.$$

We now introduce a Banach space

$$V = \{u : ||t^\theta u||_{L^p_*(X)} < \infty, \quad ||t^\theta u'||_{L^p_*(Y)} < \infty\}$$

with norm

$$||u||_V = ||t^\theta u||_{L^p_*(X)} + ||t^\theta u'||_{L^p_*(Y)}.$$

It is easily seen that $u(0) \in \mathcal{X}$. In fact, choose an $q \in C^1_0([0,\infty))$ satisfying $q(t) \geq 0, \quad q(0) = 1$, we know

$$\begin{aligned} u(0) = q(0)u(0) &= -\int_0^\infty \frac{d}{dt}(q(t)u(t))dt \\ &= -\int_0^\infty q^{'}(t)u(t)dt - \int_0^\infty q(t)u^{'}(t)dt. \end{aligned}$$

By the simple calculation, from

$$
\begin{aligned}
&|| \int_0^\infty q^{'}(t)u(t)dt||_X = || \int_0^\infty t^{1-\theta} q^{'}(t)t^{\theta}u(t)\frac{dt}{t}||_X \\
&\le \{\int_0^\infty |t^{1-\theta}q^{'}(t)|^{p^{'}}\frac{dt}{t}\}^{\frac{1}{p^{'}}}\{\int_0^\infty ||t^{\theta}u(t)||_X^p\frac{dt}{t}\}^{\frac{1}{p}} \\
&= \{\int_0^\infty t^{(1-\theta)p^{'}-1}|q^{'}(t)|^{p^{'}}dt\}^{\frac{1}{p^{'}}}||t^{\theta}u||_{L_*^p(X)} < \infty
\end{aligned}
$$

where $p^{'} = p/(p-1)$, it follows $\int_0^\infty q^{'}(t)u(t)dt \in X \subset \mathcal{X}$. By the similar way since

$$
\begin{aligned}
|| \int_0^\infty q(t)u^{'}(t)dt||_Y &= || \int_0^\infty t^{1-\theta}q(t)t^{\theta}u^{'}(t)\frac{dt}{t}||_Y \\
&\le \{\int_0^\infty |t^{1-\theta}q(t)|^{p^{'}}\frac{dt}{t}\}^{\frac{1}{p^{'}}}\{\int_0^\infty ||t^{\theta}u^{'}(t)||_Y^p\frac{dt}{t}\}^{\frac{1}{p}} \\
&= \{\int_0^\infty t^{(1-\theta)p^{'}-1}|q(t)|^{p^{'}}dt\}^{\frac{1}{p^{'}}}||t^{\theta}u^{'}||_{L_*^p(Y)} < \infty
\end{aligned}
$$

it follows $\int_0^\infty q(t)u^{'}(t)dt \in Y$. Thus, $u(0) \in X \cup Y \subset \mathcal{X}$.

Definition 2.1. *We define $(X,Y)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, to be the space of all elements $u(0)$ where $u \in V$, that is,*

$$(X,Y)_{\theta,p} = \{u(0) : u \in V\}.$$

Lemma 2.1. *(Young's inequality) Let $a > 0$, $b > 0$ and $\frac{1}{p}+\frac{1}{q} = 1$ where $1 < p < \infty$. Then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$*

Proposition 2.1. *For $0 < \theta < 1$ and $1 \le p \le \infty$, the space $(X,Y)_{\theta,p}$ is a Banach space with the norm*

$$||a||_{\theta,p} = \inf\{||u|| : u \in V, \quad u(0) = a\}.$$

Furthermore, there is a constant $C_\theta > 0$ such that

$$||a||_{\theta,p} = C_\theta \inf\{||t^{\theta}u||_{L_*^p(X)}^{1-\theta}||t^{\theta}u'||_{L_*^p(Y)}^{\theta} : u(0) = a, \quad u \in V\}.$$

Proof. We only prove the last equality. For $u \in V$ satisfying $u(0) = a$, we know $||a||_{\theta,p} \le ||u||_V$. Putting

$$u_\lambda(t) = u(\lambda t), \quad \lambda > 0,$$

it holds that

$$u_\lambda \in V, \quad u_\lambda(0) = u(0) = a$$

and

$$||a||_{\theta,p} \leq ||u_\lambda||_V = ||t^\theta u_\lambda||_{L^p_*(X)} + ||t^\theta u_\lambda^{'}||_{L^p_*(Y)}. \tag{2.1}$$

Since

$$\begin{aligned} ||t^\theta u_\lambda||_{L^p_*(X)} &= \{\int_0^\infty ||t^\theta u_\lambda(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}} = \{\int_0^\infty ||t^\theta u(\lambda t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}} \\ &= \{\int_0^\infty ||(\frac{t}{\lambda})^\theta u(t)||_X^p \frac{dt}{t}\}^{\frac{1}{p}} = \lambda^{-\theta}||t^\theta u||_{L^p_*(X)} \end{aligned}$$

and

$$\begin{aligned} ||t^\theta u_\lambda^{'}||_{L^p_*(Y)} &= \{\int_0^\infty ||t^\theta u_\lambda^{'}(t)||_Y^p \frac{dt}{t}\}^{\frac{1}{p}} = \{\int_0^\infty ||t^\theta \lambda u^{'}(\lambda t)||_Y^p \frac{dt}{t}\}^{\frac{1}{p}} \\ &= \lambda\{\int_0^\infty ||(\frac{t}{\lambda})^\theta u^{'}(t)||_Y^p \frac{dt}{t}\}^{\frac{1}{p}} = \lambda^{1-\theta}||t^\theta u^{'}||_{L^p_*(Y)}, \end{aligned}$$

from (2.1) it follows that

$$\begin{aligned} ||a||_{\theta,p} &\leq \lambda^{-\theta}||t^\theta u||_{L^p_*(X)} + \lambda^{1-\theta}||t^\theta u^{'}||_{L^p_*(Y)} \\ &= \lambda^{-\theta}A + \lambda^{1-\theta}B. \end{aligned} \tag{2.2}$$

Choosing

$$\lambda = \theta A/(1-\theta)B,$$

(2.2) implies that

$$\begin{aligned} ||a||_{\theta,p} &\leq (\frac{\theta A}{(1-\theta)B})^{-\theta}A + (\frac{\theta A}{(1-\theta)B})^{1-\theta}B \\ &= (\frac{\theta}{1-\theta})^{-\theta}A^{1-\theta}B^\theta + (\frac{\theta}{1-\theta})^{1-\theta}A^{1-\theta}B^\theta \\ &= (1+\frac{\theta}{1-\theta})(\frac{\theta}{1-\theta})^{-\theta}A^{1-\theta}B^\theta \\ &= \frac{1}{1-\theta}(\frac{\theta}{1-\theta})^{-\theta}A^{1-\theta}B^\theta \\ &= \frac{A^{1-\theta}B^\theta}{(1-\theta)^{1-\theta}\theta^\theta} = (\frac{A}{1-\theta})^{1-\theta}(\frac{B}{\theta})^\theta. \end{aligned} \tag{2.3}$$

By regarding as

$$a = (\frac{A}{1-\theta})^{1-\theta}, \quad b = (\frac{B}{\theta})^\theta, \quad p = \frac{1}{1-\theta}, \quad \text{and} \quad q = \frac{1}{\theta}$$

in Young's Lemma 2.1, from (2.3) we have

$$||a||_{\theta,p} \leq \frac{A^{1-\theta}B^{\theta}}{(1-\theta)^{1-\theta}\theta^{\theta}} \leq A + B,$$

that is,

$$\begin{aligned} ||a||_{\theta,p} &\leq \frac{1}{(1-\theta)^{1-\theta}\theta^{\theta}} ||t^{\theta}u||^{1-\theta}_{L^p_*(X)} ||t^{\theta}u^{'}||^{\theta}_{L^p_*(Y)} \\ &\leq ||t^{\theta}u||_{L^p_*(X)} + ||t^{\theta}u^{'}||_{L^p_*(Y)}. \end{aligned}$$

For every $u \in V$ satisfying $u(0) = a$, it holds

$$||a||_{\theta,p} \leq C_{\theta} ||t^{\theta}u||^{1-\theta}_{L^p_*(X)} ||t^{\theta}u^{'}||^{\theta}_{L^p_*(Y)} \leq ||u||_V$$

where $C_{\theta} = 1/(1-\theta)^{1-\theta}\theta^{\theta}$. Therefore

$$||a||_{\theta,p} = C_{\theta} \inf\{||t^{\theta}u||^{1-\theta}_{L^p_*(X)} ||t^{\theta}u^{'}||^{\theta}_{L^p_*(Y)} : u(0) = a, \quad u \in V\}.$$

□

Proposition 2.2. *For* $0 < \theta < 1$ *and* $1 \leq p \leq \infty$, *we have* $(X, X)_{\theta,p} = X$.

Proof. We only proof the relation $(X, X)_{\theta,p} \supset X$. Let $x \in X$ and $q \in C^1_0([0, \infty))$ satisfying $q(0) = 1$. Putting $u(t) = q(t)x$, we have $u(0) = x$. By simple calculation, since

$$\begin{aligned} \int_0^{\infty} ||t^{\theta}u(t)||^p_X \frac{dt}{t} &= \int_0^{\infty} t^{\theta p - 1} |q(t)|^p ||x||^p_X dt < \infty, \\ \int_0^{\infty} ||t^{\theta}u^{'}(t)||^p_X \frac{dt}{t} &= \int_0^{\infty} t^{\theta p - 1} |q^{'}(t)|^p ||x||^p_X dt < \infty \end{aligned}$$

we have $x \in (X, X)_{\theta,p}$. □

Proposition 2.3. *Let* $X \subset Y$ *satisfying that there exists a constant* $c > 0$ *such that* $||u||_Y \leq c||u||_X$. *If* $0 < \theta < \theta^{'} < 1$ *then we have*

$$(X, Y)_{\theta,p} \subset (X, Y)_{\theta^{'},p}.$$

Proof. Let $a \in (X, Y)_{\theta,p}$. Then there exists $u \in V$ such that $u(0) = a$ and

$$||t^{\theta}u||_{L^p_*(X)} \leq \infty, \quad ||t^{\theta}u^{'}||_{L^p_*(Y)} \leq \infty.$$

Let $q \in C_0^1([0,\infty))$ satisfying $q(0) = 1, \quad 0 \le q(t) \le 1$ for $t \in (0,1)$ and $q(t) = 0$ for $1 \le t$. Putting $v(t) = q(t)u(t)$, we have

$$\begin{aligned}
||t^{\theta'} v||_{L^p_*(X)} &= \{\int_0^\infty (t^{\theta'} ||v(t)||_X)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \{\int_0^1 (t^{\theta'} q(t)||u(t)||_X)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \{\int_0^1 (t^{\theta'} ||u(t)||_X)^p \frac{dt}{t}\}^{\frac{1}{p}} < \infty,
\end{aligned}$$

and

$$\begin{aligned}
||t^{\theta'} v'||_{L^p_*(Y)} &= \{\int_0^\infty (t^{\theta'} ||q(t)u'(t) + q'(t)u(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \{\int_0^\infty (t^{\theta'} q(t)||u'(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \{\int_0^\infty (t^{\theta'} q'(t)||u(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \{\int_0^\infty (t^{\theta'} ||u'(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \max|q'(t)|\{\int_0^\infty (t^{\theta'} ||u(t)||_Y)^p \frac{dt}{t}\}^{\frac{1}{p}} < \infty,
\end{aligned}$$

hence we obtain that $a = v(0) \in (X,Y)_{\theta',p}$. □

From now on we will deal with the complex interpolation methods introduced by [5], which the complex methods will yield new interpolation spaces. Let

$$\begin{aligned}
F = \{u : u \text{ is } & \text{ continuous function in } 0 \le \text{Re}z \le 1 \text{ with } u(z) \in \mathcal{X}, \\
& \text{and analytic in } 0 < \text{Re}z < 1. \text{ For } -\infty < y < \infty,\ u(iy) \in X, \\
& u(1+iy) \in Y,\ u(iy) \text{ and } u(1+iy) \text{ are strongly continous and} \\
& \text{bounded on } X \text{ and } Y, \text{ respectively}\}.
\end{aligned}$$

As is seen in Triebel [**?**], we know that F is a Banach space equipped with the norm

$$||u||_F = \max\{\sup_{-\infty<y<\infty} ||u(iy)||_X, \quad \sup_{-\infty<y<\infty} ||u(1+iy)||_Y\}.$$

Definition 2.2. *Let $\{X,Y\}$ be an interpolation couple. We now introduce a Banach space. For $0 < \theta < 1$, put*

$$[X,Y]_\theta = \{u(\theta) : u \in F\}$$

with norm

$$||a||_{[X,Y]_\theta} = \inf_{u(\theta)=a} ||u||_F.$$

It is well known result that if H_1 and H_2 are Hilbert spaces satisfying that H_1 is a dense space of H_2 and the embedding is continuous then

$$[H_1, H_2]_\theta = (H_1, H_2)_{\theta,2} \quad 0 < \theta < 1.$$

3 Interpolation spaces by analytic semigroup generator

In this section we consider an interpolation method between the initial Banach space and the domain of the infinitesimal generator A of the analytic semigroup $T(t)$. We will verify the fact that

$$(D(A), X)_{\theta,p} = \{x \in X : \int_0^\infty (t^\theta ||AT(t)x||)^p \frac{dt}{t} < \infty\},$$

for $0 < \theta < 1$, $1 \leq p \leq \infty$,

which is mainly on the role of interpolation spaces in the study of analytic semigroup of operators.

Let X be a Banach space with norm $||\cdot||$ and $T(t)$ be an analytic semigroup with infinitesimal generator A. We may assume that

$$||T(t)|| \leq M, \quad ||AT(t)|| \leq \frac{K}{t}$$

for some positive constants M, K and $t \geq 0$.

Lemma 3.1. *Let $0 < \theta < 1$, $1 < p < \infty$ and $\phi(t) \geq 0$ almost everywhere. Then*

$$\{\int_0^\infty (t^{\theta-1} \int_0^t \phi(s)ds)^p \frac{dt}{t}\}^{\frac{1}{p}} \leq \frac{1}{1-\theta} \{\int_0^\infty (t^\theta \phi(t))^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

Proof. Let $0 < \epsilon < N < \infty$. Then

$$\int_\epsilon^N (t^{\theta-1} \int_0^t \phi(s)ds)^p \frac{dt}{t} = \int_\epsilon^N t^{(\theta-1)p-1} (\int_0^t \phi(s)ds)^p dt$$

$$= [\frac{t^{(\theta-1)p}}{(\theta-1)p} (\int_0^t \phi(s)ds)^p]_\epsilon^N - \int_\epsilon^N \frac{t^{(\theta-1)p}}{(\theta-1)p} p\phi(t) (\int_0^t \phi(s)ds)^{p-1} dt$$

$$\leq \frac{\epsilon^{(\theta-1)p}}{(1-\theta)p} (\int_0^\epsilon \phi(s)ds)^p + \frac{1}{1-\theta} \int_\epsilon^N t^{(\theta-1)p} \phi(t) (\int_0^t \phi(s)ds)^{p-1} dt.$$

Since

$$\int_0^\epsilon \phi(s)ds = \int_0^\epsilon s^{1-\theta}s^\theta\phi(s)\frac{ds}{s}$$
$$\leq \{\int_0^\epsilon s^{(1-\theta)p'}\frac{ds}{s}\}^{\frac{1}{p'}}\{\int_0^\epsilon (s^\theta\phi(s))^p\frac{ds}{s}\}^{\frac{1}{p}}$$
$$= \{\frac{\epsilon^{(1-\theta)p'}}{(1-\theta)p'}\}^{\frac{1}{p'}}\{\int_0^\epsilon (s^\theta\phi(s))^p\frac{ds}{s}\}^{\frac{1}{p}},$$

we see that

$$\epsilon^{\theta-1}(\int_0^\epsilon \phi(s)ds) \leq (\frac{1}{(1-\theta)p'})^{\frac{1}{p'}}\{\int_0^\epsilon (s^\theta\phi(s))^p\frac{ds}{s}\}^{\frac{1}{p}}$$

tends to zero as ϵ tends to zero. If $\epsilon \to 0$ and $N \to \infty$, then we have

$$\int_0^\infty (t^{\theta-1}\int_0^t \phi(s)ds)^p\frac{dt}{t} \leq \frac{1}{1-\theta}\{\int_0^\infty t^{(\theta-1)p}\phi(t)(\int_0^t \phi(s)ds)^{p-1}dt\}$$
$$= \frac{1}{1-\theta}\int_0^\infty t^{(\theta-1)(p-1)+\theta}\phi(t)(\int_0^t \phi(s)ds)^{p-1}\frac{dt}{t}$$
$$= \frac{1}{1-\theta}\int_0^\infty t^\theta\phi(t)(t^{\theta-1}\int_0^t \phi(s)ds)^{p-1}\frac{dt}{t}$$
$$\leq \frac{1}{1-\theta}\{\int_0^\infty (t^\theta\phi(t))^p\frac{dt}{t}\}^{\frac{1}{p}}\{\int_0^\infty (t^{\theta-1}\int_0^t \phi(s)ds)^p\frac{dt}{t}\}^{1-\frac{1}{p}}.$$

Hence the proof is complete. □

The main results in this paper is the following.

Theorem 3.1. *Let* $0 < \theta < 1$, $0 \leq t$. *Then*

$$(1-\theta)\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p\frac{dt}{t}\}^{\frac{1}{p}}$$
$$\leq \{\int_0^\infty (t^\theta||AT(t)x||)^p\frac{dt}{t}\}^{\frac{1}{p}}$$
$$\leq \frac{K}{1-2^{-\theta}}\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p\frac{dt}{t}\}^{\frac{1}{p}}.$$

Therefore, we have

$$(D(A),X)_{\theta,p} = \{x \in X : \int_0^\infty (t^\theta||AT(t)x||)^p\frac{dt}{t} < \infty\}.$$

Proof. From

$$
\begin{aligned}
AT(t)x &= -\sum_{k=0}^{n}(AT(2^{k+1}t) - AT(2^k t))x + AT(2^{n+1}t)x \\
&= -\sum_{k=0}^{n}(AT(2^k t)(T(2^k t) - I)x + AT(2^{n+1}t)x
\end{aligned}
$$

it follows that

$$
\begin{aligned}
||AT(t)x|| &\leq \sum_{k=0}^{n}\frac{K}{2^k t}||(T(2^k t) - I)x|| + \frac{K}{2^{n+1}t}||x||, \\
t^{\theta}||AT(t)x|| &\leq K\sum_{k=0}^{n}2^{-k\theta}(2^k t)^{\theta-1}||(T(2^k t) - I)x|| \\
&\quad + \frac{K}{2^{n+1}}||x||t^{\theta-1},
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\{\int_0^{\infty}(t^{\theta}||AT(t)x||)^p\frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq K\sum_{k=0}^{n}2^{-k\theta}\{\int_{\epsilon}^{\infty}((2^k t)^{\theta-1}||(T(2^k t) - I)x||)^p\frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \frac{K}{2^{n+1}}||x||\{\int_{\epsilon}^{\infty}t^{(\theta-1)p}\frac{dt}{t}\}^{\frac{1}{p}} \\
&= K\sum_{k=0}^{n}2^{-k\theta}\{\int_{2^k\epsilon}^{\infty}(t^{\theta-1}||(T(t) - I)x||)^p\frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \frac{K}{2^{n+1}}||x||(\frac{\epsilon^{(\theta-1)p}}{(1-\theta)p})^{\frac{1}{p}}, \quad (2^k t \to t),
\end{aligned}
$$

for every $\epsilon > 0$. Thus,

$$
\begin{aligned}
&\{\int_{\epsilon}^{\infty}(t^{\theta}||AT(t)x||)^p\frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq K\sum_{k=0}^{\infty}2^{-k\theta}\{\int_{2^k\epsilon}^{\infty}(t^{\theta-1}||(T(t) - I)x||)^p\frac{dt}{t}\}^{\frac{1}{p}}
\end{aligned}
$$

as $n \to \infty$. Therefore, passing $\epsilon \to 0$, we obtain

$$\begin{aligned}
&\{\int_0^\infty (t^\theta ||AT(t)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le K \sum_{k=0}^{\infty} 2^{-k\theta} \{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \frac{K}{1-2^{-\theta}} \{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}}.
\end{aligned}$$

On the other hand, since

$$||(T(t)-I)x|| = ||\int_0^t AT(s)xds|| \le \int_0^t ||AT(s)x||ds,$$

we have

$$\begin{aligned}
&\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \{\int_0^\infty (t^{\theta-1}\int_0^t ||AT(s)x||ds)^p \frac{dt}{t}\}^{\frac{1}{p}},
\end{aligned}$$

from Lemma 4.1, it follows

$$\begin{aligned}
&\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\le \frac{1}{1-\theta} \{\int_0^\infty (t^\theta ||AT(t)x||)^p \frac{dt}{t}\}^{\frac{1}{p}},
\end{aligned}$$

hence, the proof is complete. □

Corollary 3.1. *Let $T(t)$ be an analytic semigroup with generator A in X. Then*

$$(D(A), X)_{\theta,\frac{1}{\theta}} = \{x \in X : \int_0^\infty ||AT(t)x||^{\frac{1}{\theta}} dt < \infty\}.$$

In particular, if $\theta = \frac{1}{2}$ then

$$L^2(0,\infty; D(A)) \cap W^{1,2}(0,\infty; X) \subset C([0,\infty); (D(A), X)_{\frac{1}{2},2})$$

where

$$(D(A), X)_{\frac{1}{2},2} = \{x \in X : \int_0^\infty ||AT(t)x||^2 dt < \infty\}.$$

4 Applications for Initial value problem

We consider the regular problem for the following functional differential equation of parabolic type

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + f(t), \quad 0 < t \le T, \\ x(0) = x_0 \end{cases} \tag{4.1}$$

in a Hilbert space H. Let V be another Hilbert space such that $V \subset H$. Identifying the antidual of V with V^* we may consider $V \subset H \subset V^*$. Therefore, for the sake of simplicity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad v \in V$$

where the notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of H, V and V^*, respectively as usual. Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{4.2}$$

Let A be the operator associated with a sesquilinear form

$$(Au, v) = -a(u, v), \quad u, \ v \in V.$$

Then the operator A is a bounded linear from V to V^*. We may assume that $(D(A), H)_{1/2,2} = V$ satisfying

$$||u|| \le C_1||u||^{1/2}_{D(A)}|u|^{1/2} \tag{4.3}$$

for some a constant $C_1 > 0$ where $(D(A), H)_{\theta,p}$ denotes the real interpolation space between $D(A)$ and H as is seen in Section 2.

Lemma 4.1. *With the notations (4.2), (4.3), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^**(Section 1.3.3 of [?, ?]).*

We may assume that (4.2) holds for $c_1 = 0$ as noting that $A_0 + c_1$ is an isomorphism from V to V^* if $c_1 \ne 0$.

By virtue of Theorem 3.3 of [?] we have the following result on the corresponding linear equation of (4.1).

Proposition 4.1. *Suppose that the assumptions mentioned above are satisfied. For $x_0 \in (D(A), H)_{\frac{1}{2},2}$ and $f \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (4.1) belonging to*

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];(D(A),H)_{\frac{1}{2},2})$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \leq C_2(||x_0||_{(D(A),H)_{\frac{1}{2},2}} + ||f||_{L^2(0,T;H)}),$$

where C_2 is a constant depending on T and

$$||\cdot||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} = \max\{||\cdot||_{L^2(0,T;D(A))}, ||\cdot||_{W^{1,2}(0,T;H)}\}.$$

Lemma 4.2. *Let $T > 0$. Then*

$$H = \{x \in V^* : \int_0^T ||Ae^{tA}x||_*^2 dt < \infty\},$$

where $||\cdot||_*$ is the norm of the element of V^*.

Proof. Put $u(t) = e^{tA}x$ for $x \in H$. From

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u(t)|^2 &= \text{Re }(\dot{u}(t), u(t)) = \text{Re }(Au(t), u(t))\\ &= -\text{Re }\ a(u(t), u(t)) \leq -c_0||u(t)||^2,\end{aligned}$$

it follows

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + c_0||u(t)||^2 \leq 0.$$

By integrating over t, it yields

$$\frac{1}{2}|u(t)|^2 + c_0\int_0^t ||u(s)||^2 ds \leq \frac{1}{2}|x|^2.$$

Hence, we obtain that

$$\int_0^T ||Ae^{tA}x||_*^2 dt \leq \int_0^T ||A||_{B(V,V^*)}||u(s)||^2 ds < \infty.$$

Conversely, suppose that $x \in V^*$ and $\int_0^T ||Ae^{tA}x||_*^2 dt < \infty$. Put $u(t) = e^{tA}x$. Then since A is an isomorphism operator from V to V^* there exists a constant $c > 0$ such that

$$\int_0^T ||u(t)||^2 dt \leq c\int_0^T ||Au(t)||_*^2 dt = c\int_0^T ||Ae^{tA}x||_*^2 dt.$$

From the assumptions and $\dot{u}(t) = Ae^{tA}x$ it follows

$$u \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Therefore, $x = u(0) \in H$. □

The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. It is known that A generates an analytic semigroup in both H and V^*(see [**?**, **?**, **?**]). Replaying the interpolation space F in Blasio *et al.* [**?**] with the space H, we can derive the results of [**?**] regarding term by term to deduce the following result.

Theorem 4.1. *Let $x_0 \in H$ and $f \in L^2(0,T;V^*)$. Then for each $T > 0$, a solution x of the equation (4.1) belongs to*

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, for some constant C_2 we have

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_2(|x_0| + ||f||_{L^2(0,T;V^*)}).$$

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