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Identification and controllability for neutral retarded differential equation

by

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Identification and controllability for neutral retarded differential equation
(중립 지연 미분 방정식에 대한 동일성과 제어성)

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A dissertation

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요 약

본 논문은 먼저 주어진 Hilbert 상의 지연 미분방정식에서 해석적 반군을 생성하는 주작용소의 고유치에 대한 일반적인 고유공간들의 특성을 조사하였다. 고유 공간상에서 기본해가 해석적 반군에 의한 형태를 구성함을 보여 가제어성을 보이고 이러한 성질들을 이용하여 가제어성과 해의 안정성의 동치관계를 밝혔다.
제3장에서는 주작용소가 타원형 미분연산자에 의해 생성되는 해석적 반군에 대해 보간이론을 정립하고, 생성되는 ζ-볼록성을 가진 보간공간을 설정하여 $W^{-1,p}(\Omega)$상의 지연방정식에 적용하고 해의 유일성을 살펴보았다. 위의 결과를 바탕으로 해석적 반군의 작용소의 고유치에 대한 일반적인 고유공간들의 특성을 이용하여 동일성 문제에 대해 rank condition이 충분조건이 됨을 밝혔다.
제4장에서는 비유계 주작용소를 가진 유리계수를 가진 지연 중립 미분방정식의 해의 형태를 감마함수를 이용하여 기본해 구성을 하고, 비선형 항이 없는 선형포물선방정식의 해를 구성하는 해석적반군의 특성을 통해 비선형항이 섭동된 기본해의 존재성과 L^2-정칙성을 증명한다.
제5장에서는 로컬 립쉬츠 연속성을 만족하는 비선형항을 지닌 중립형 미분방정식에 대해 일반적인 선형 발전방정식의 해석적반군의 특성과 분수승의 작용소의 특성을 통해 주어진 방정식의 해의 존재성을 밝히고, 아울러 제어기가 포함된 준선형 중립형 컨트롤 시스템의 근사적 가제어성을 증명한다.

Chapter 1
Introduction and Preliminaries

This paper is discussed mathematical interpretations including the regularity of solutions and control problems for partial differential equations with time delay based mainly on the results from analytic semigroups generated by differential operators. The main tool of research is to apply some characteristic of interpolation spaces of Banach spaces, and establish some simple properties for interpolation spaces associated with the domain of a generator of an analytic semigroup, which is applicable for the maximal regularity and the existence of solutions of evolution equations of parabolic type with unbounded operators. Based on these theory, sufficient condition for identification condition which is one of the inverse problems is given as the so called rank condition in terms of the initial values and eigenvectors of adjoint operator was obtained, and some results on the control problems for retarded functional differential equations of parabolic type with unbounded principal operators are established.

In Chapter 2, we deal with the control problems for the following semilinear retarded functional differential equation with initial values in a Hilbert space H:

$$\begin{cases} x'(t) = & A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ & +f(t, x(t)) + B_0u(t), \\ x(0) = & g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{RE}$$

Here, the principal operator A_0 generates an analytic semigroup $S(t)$ on H, and $A_iA_0^{-1}(i=1,2)$ are bounded in H. Here, B_0 is a linear bounded operator from U to H and U is some Banach space.

Little is known about the relationship between controllability and stabilizability for solutions of the semilinear equation (RE), which is one of our motivations. We assume that

$$\sigma^+ = \sigma(A) \cap \{\lambda : \mathrm{Re}\,\lambda > 0\}$$

consists entirely of a finite number of eigenvalues of A_0(see [26])

Goal of this chapter is to extend the control theory govern by general semilinear systems to the equations with delays. Based on the semilinear control system with positive isolated spectrum points, we will derive the equivalent relation between controllability and stabilizability of the solution for the control system(RE) with a condition of the completeness of system of the generalized eigenspaces of A_0.

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order in $L^1(\Omega)$:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x).$$

In Chapter 3, we consider the inverse problem for the following retarded functional differential equation defined as $A_0u = -\mathcal{A}(x, D_x)u$:

$$\begin{cases} u'(t) = A_0u(t) + \gamma A_0u(t-h) + \int_{-h}^{0} a(s)A_0u(t+s)ds, \\ u(0) = g^0, \quad u(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{IE}$$

Here A_0, γ, and $a(\cdot)$ are unknown quantities to be identified and the initial condition $g = (g^0, g^1)$ is known.

In [27, 30] the author discussed the control problem for the following retarded system with $L^1(\Omega)$-valued controller:

$$u(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds + \Phi_0 w(t), \qquad (3.1.1)$$

where $A_i(i = 1, 2)$ are second order linear differential operators with real coefficients, and the controller Φ_0 is a bounded linear operator from a control Banach space to $L^1(\Omega)$.

In [16], they established some results concerning identification problems for (IE) of specific form by taking the observation. Furthermore, Yamamoto and Nakagiri [47] studied the identifiability problem for evolution equations in Banach spaces with unknown operators and initial values by means of spectral theory for linear operators.

In view of Sobolev's embedding theorem we may also consider $L^1(\Omega) \subset W^{-1,p}(\Omega)$ if $1 \le p < n/(n-1)$ as is seen in [27]. Hence, we can investigate the system (IE) in the space $W^{-1,p}(\Omega)$ considering Φ_0 as an operator into $W^{-1,p}(\Omega)$. Here, we note that the space $W^{-1,p}(\Omega)$ is ζ-convex(as for the definition and fundamental facts of a ζ-convex see [24, 10]). Consequently, in view of Dore and Venni [18] the maximal regularity for the linear initial value problem:

$$u'(t) = A_0u(t) + f(t), u(0) = u_0$$

in the space $W^{-1,p}(\Omega)$ holds true.

Furthermore, with the aid of a result by Seeley [55] and [27], we can obtain the maximal regularity for solutions of the retarded linear initial value problem (IE) in the space $W^{-1,p}(\Omega)$. In view of these results, we deal with an inverse problem of (IE) in $W^{-1,p}(\Omega)$.

In Chapater 4, we study the existence of solutions and L^2-regularity for the following fractional order retarded neutral functional differential equation:

$$\begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t)+g(t,x_t)] = A_0x(t) + \int_{-h}^0 a_1(s)A_1x(t+s)ds + (Fx)(t) + k(t), \ t>0, \\ x(0)=\phi^0, \quad x(s)=\phi^1(s), \quad -h\le s<0, \end{cases} \tag{NE}$$

where $1/2<\alpha<1$, $h>0$, $a_1(\cdot)$ is Hölder continuous, k is a forcing term, and g, f, are given functions satisfying some assumptions. Moreover, $A_0 : H \to H$ is unbounded but A_1 is bounded. For each $s\in[0,T]$, we define $x_s : [-h,0]\to H$ as $x_s(r)=x(s+r)$ for $r\in[-h,0]$ and $(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$. We propose a different approach of the earlier works used properties of the relative compactness. Our approach is that regularity results of general retarded linear systems of Di Blasio et al. [17] and semilinear systems of [31] remain valid under the above formulation of fractional order retarded neutral differential system (NE) even though the system (NE) contains unbounded principal operators, delay term, and local Lipschitz continuity of the nonlinear term. The methods of the functional analysis concerning an analytic semigroup of operators and some fixed point theorems are applied effectively.

In Chapter 5, we are concerned with the global existence of solution and the approximate controllability for the following abstract neutral functional differential system in a Hilbert space H:

$$\begin{cases} \frac{d}{dt}[(x(t) + (Bx)(t)] = Ax(t) + f(t, x(t)) + (Cu)(t), \quad t \in (0, T], \\ x(0) = x_0, \quad (Bx)(0) = y_0, \end{cases} \tag{CE}$$

where A is an operator associated with a sesquilinear form on $V \times V$ satisfying Gårding's inequality, f is a nonlinear mapping of $[0, T] \times V$ into H satisfying the local Lipschitz continuity, $B : L^2(0, T; V) \to L^2(0, T; H)$ and $C : L^2(0, T; U) \to L^2(0, T; H)$ are appropriate bounded linear mapping.

We propose a different approach of the earlier works (briefly introduced in [41,42],[58-61] about the mild solutions of neutral differential equations). Our approach is that results of the linear cases of Di Blasio et al. [17] and semilinear cases of [31] on the L^2-regularity remain valid under the above formulation of the neutral differential equation (CE). For the basic of our study, the existence of local solutions of (CE) are established in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \hookrightarrow C([0, T]; H)$ for some $T > 0$ by using fractional power of operators and Sadvoskii's fixed point theorem. Thereafter, by showing some variational of constant formula of solutions, we will obtain the global existence of solutions of (CE), and the norm estimate of a solution of (CE) on the solution space. Consequently, in view of the properties of the nonlinear term, we can take advantage of the fact that the solution mapping $u \in L^2(0, T; U) \mapsto x$ is Lipschitz continuous, which is applicable for control problems and the optimal control problem of systems governed by nonlinear properties.

The second purpose of this chapter is to study the approximate controllability for the neutral equation (CE) based on the regularity for (CE), namely that the reachable set of trajectories is a dense subset of H.

Chapter 2
Semilinear retarded control systems

2.1 Introduction

In this paper we deal with the control problems for the following semilinear retarded functional differential equation with initial values in a Hilbert space H:

$$\begin{cases} x'(t) = & A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds \\ & +f(t,x(t)) + B_0u(t), \\ x(0) = & g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{RE}$$

Here, the principal operator A_0 generates an analytic semigroup $S(t)$ on H, and $A_iA_0^{-1}(i = 1, 2)$ are bounded in H. Here, B_0 is a linear bounded operator from U to H and U is some Banach space.

The reachable set for its corresponding linear system of (RE) in case where $f \equiv 0$ is independent of the time T if A_0 generates an analytic semigroup. But it does not hold in general case where A_0 generates C_0-semigroup as seen in Theorem 3.10 and remark 3.4 of [19]. Similar considerations of linear and semilinear systems have dealt with in many references(see the bibliographies of [2-5]). In [36], the approximate controllability for the semilinear system (RE) was established by a condition for the range of the controller B_0 without the inequality condition and see that the necessary assumption is more flexible than one in [3,4,7]. However, little is known about the relationship between controllability and stabilizability for solutions of the semilinear equation (RE), which is one of our motivations. We assume that

$$\sigma^+ = \sigma(A) \cap \{\lambda : \operatorname{Re} \lambda > 0\}$$

consists entirely of a finite number of eigenvalues of A_0(see [26])

Our goal of this paper is to extend the control theory govern by general semilinear systems to the equations with delays. Based on the semilinear control system with positive isolated spectrum points, we will derive the equivalent relation between controllability and stabilizability of the solution for the control system(RE) with a condition of the completeness of system of the generalized eigenspaces of A_0.

2.2 Applications for semilinear retarded systems

Consider the following linear retarded functional differential equation with initial values in a Hilbert space H:

$$\begin{cases} x'(t) = & A_0 x(t) + A_1 x(t-h) + \int_{-h}^{0} a(s) A_2 x(t+s) ds \\ & + f(t, x(t)) + B_0 u(t), \\ x(0) = & g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{2.2.1}$$

Let V be a Hilbert space densely and continously embedded in H. The notations $||\cdot||$ and $|\cdot|$ denote the norms of V and H as usual, respectively. Let $-A_0$ be the operator associated with a bounded sesquilinear form $b(u, v)$ defined on $V \times V$ satisfying Gårding's inequality

$$\operatorname{Re} b(u, u) \geq c||u||, \quad c > 0.$$

It is known that A_0 generates an analytic semigroup $S_0(t)(t>0)$ in both of H and V^*. It is assumed that A_1 and A_2 are bounded linear operators from V to V^* and $A_iA_0^{-1}(i=1,2)$ are bounded in H. Here, B_0 is a linear bounded operator from U to H and U is some Banach space.

Let f be a nonlinear mapping $[0,T]\times V$ into H for given $T>0$. We consider the following cases:

Assumption (F) For any $x_1,\ x_2\in V$ there exists a constant $L>0$ such that

$$|f(t,x_1)-f(t,x_2)|\le L||x_1-x_2||.$$

Using the Maximal regularity for more general retarded parabolic systems as in [9,10], we know the following results.

Proposition 2.2.1. *Let $T>0$, $(g^0,g^1)\in H\times L^2(-h,0;V)$ and Assumption(F) be satisfied. Then there exists a unique solution x of equation (RE) such that*

$$x\in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\subset C([0,T];H).$$

Moreover, there exists a constant c_1 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}\le c_1(|g^0|+||g^1||_{L^2(-h,0;V)}+||u||_{L^2(0,T;U)}).$$

Let $W(\cdot)$ be the fundamental solution of the linear equation associated with (RE) which is the operator valued function satisfying

$$\begin{cases} W(t) &= S_0(t)+\int_0^t S_0(t-s)\{A_1W(s-h) \\ &\quad +\int_{-h}^0 a(\tau)A_2W(s+\tau)d\tau\}ds, \quad t>0 \\ W(0) &= I, \qquad W(t)=0, \quad -h\le t<0, \end{cases}$$

where $S_0(\cdot)$ is the semigroup generated by A_0. Then the solution $x(t; f, u)$ for the equation (RE) can be written by

$$x(t; f, u) = W(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds$$

$$+ \int_0^t W(t-s)\{f(s, x(s; f, u)) + Bu(s)\}ds,$$

$$U_t(s) = W(t-s-h)A_1 + \int_{-h}^{s} W(t-s+\sigma)a(\sigma)A_2 d\sigma.$$

In this section we investigate the spectral properties of the infinitesimal generator A_0 of $S_0(t)$ in the special case where $A_1 = \gamma A_0$ with some constant γ, $A_2 = A_0$ and the embedding $V \subset H$ is compact. Thus, in what follows we consider the equation

$$\begin{cases} \frac{d}{dt}x(t) = & A_0 x(t) + \gamma A_0 x(t-h) + \int_{-h}^{0} a(s)A_0 x(t+s)ds \\ & + f(t, x(t)) + B_0 u(t), \\ x(0) = & g^0, \quad x(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{2.2.2}$$

According to Riesz-Schauder theorem A_0 has discrete spectrum

$$\sigma(A_0) = \{\lambda_j : j = 1, \cdots\}$$

which has no point of accumulation except possibly $\lambda = \infty$. The spectrum of A is denoted by $\sigma(A)$. We assume:

$$\sigma(A) \cap \{\lambda : \mathrm{Re}\,\lambda = 0\} = \emptyset.$$

Set

$$\sigma_+ = \sigma(A) \cap \{\lambda : \mathrm{Re}\,\lambda > 0\},\ \sigma_- = \sigma(A) \cap \{\lambda : \mathrm{Re}\,\lambda < 0\}.$$

We make natural assumption that σ_+ is a finite and $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma_-\} < 0$, that is,

$$\sigma_+ = \{\lambda_1, \ldots, \lambda_N\},$$

$$-\omega_0 = \sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma_-\} < 0$$

and for each $j = 1, \ldots, N$.

Let $Z \equiv H \times L^2(-h, 0; V)$ be the state space of the equation (2.2.2). Z is a product Hilbert space with the norm

$$||g||_Z = (|g^0|^2 + \int_{-h}^{0} ||g^1(s)||^2 ds)^{\frac{1}{2}}, \quad g = (g^0, g^1) \in Z.$$

Let $g \in Z$ and $x(t; g, f, u)$ be a solution of (2.2.2) associated with the non-linear term f and a control u at the time t. The segment x_t is given by $x_t(s; g, f, u) = x(t + s; g, f, u)$, $s \in [-h, 0)$. The solution semigroup $S(t)$ for the equation (2.2.2) is defined by

$$S(t)g = (x(t; g, 0, 0), x_t(\cdot; g, 0, 0)), \tag{2.2.3}$$

where $x(t; g, 0, 0)$ is the solution of the equation (2.2.2) with $f \equiv 0$ and $B \equiv 0$.

Here, we remark that the operator $S(t)$ is a C_0-semigroup on Z and the infinitesimal generator A of $S(t)$ is characterized by

$$D(A) = \{g = (g^0, g^1) : \ g^0 \in H, \ g^1 \in W^{1,2}(-h, 0; V),$$

$$g^1(0) = g^0, \ A_0 g^0 + \gamma A_0 g^1(-h) + \int_{-h}^{0} a(s) A_0 g^1(s) ds \in H\},$$

$$Ag = (A_0 g^0 + A_0 g^1(-h) + \int_{-h}^{0} a(s) A_0 g^1(s) ds, \dot{g}^1).$$

It is also known that if the embedding $V \subset H$ is compact, then

$$\sigma(A) = \{\lambda : m(\lambda) \neq 0, \ \ \lambda/m(\lambda) \in \sigma(A_0)\},$$

where $m(\lambda) = 1 + \gamma e^{-\lambda h} + \int_{-h}^{0} e^{\lambda s} a(s) ds$.

The equation (2.2.2) can be transformed into an abstract equation as follows.

$$\begin{cases} z'(t) = Az(t) + F(t, z(t)) + Bu(t), \\ z(0) = g, \end{cases} \tag{2.2.4}$$

where $z(t) = (x(t; g, f, u), x_t(\cdot; g, f, u)) \in Z$ and $g = (g^0, g^1) \in Z$. The nonlinear operator F on Z is defined by $F(t, z(t)) = (f(t, x(t), 0)$ and the control operator B defined by $Bu = (B_0 u, 0)$. The mild solution of initial value problem (2.2.4) is the following form:

$$z(t; g, f, u) = S(t)g + \int_0^t S(t-s)\{F(s, z(s)) + Bu(s)\} ds.$$

We consider also the adjoint problem

$$\begin{cases} \frac{d}{dt} y(t) & = A_0^* y(t) + \gamma A_0^* y(t-h) + \int_{-h}^{0} a(s) A_0^* y(t+s) ds, \\ y(0) & = \phi^0, \quad y(s) = \phi^1(s) \quad s \in [-h, 0), \end{cases} \tag{2.2.5}$$

where $A_0^* \in B(V, V^*)$ is adjoint operator of A_0 and $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$.

Let A_{T} be the infinitesimal generator of $S_{\mathrm{T}}(t)$ associated with the system (2.2.5). Then the equation (2.2.5) can also be transformed into the following equation:

$$\begin{cases} \hat{z}'(t) = A_{\mathrm{T}} \hat{z}(t), \\ \hat{z}(0) = \phi, \end{cases} \tag{2.2.6}$$

where $\hat{z}(t) = (y(t;\phi, f, 0), (y_t(\cdot;\phi, f, 0)) \in Z$ and $\phi = (\phi^0, \phi^1) \in Z$. The structural operator H is defined by

$$Gg = ([Gg]^0, [Gg]^1), \quad g = (g^0, g^1) \in Z$$

$$[Gg]^0 = g^0, \quad [Gg]^1(s) = \gamma A_0 g^1(-h-s) + \int_{-h}^{s} a(\tau) A_0 g^1(\tau - s) d\tau.$$

The spectral projection

$$P_j = \frac{1}{2\pi i}\int_{\Gamma_j} (\lambda - A)^{-1} d\lambda$$

is an operator of finite rank, where Γ_j is a small circle centered at λ_j such that it surrounds no point of $\sigma(A)$ except λ_j, and set

$$Q_j = \frac{1}{2\pi i}\int_{\Gamma_j} (\lambda - \lambda_j)(\lambda - A)^{-1} d\lambda.$$

Let P_{λ_j} and Z_{λ_j} denote the spectral projection and the generalized eigenspace for λ_j, respectively. Just as in [5] it can be shown that $\overline{\lambda_1}, \cdots, \overline{\lambda_N}$ are eigenvalues of A_T. The spectral projection and the generalized eigenspace for $\overline{\lambda_j}$ are defined by $P^T_{\overline{\lambda_j}}$ and $Z^T_{\overline{\lambda_j}}$, respectively. Thus, we obtain the following theorem in virtue of Theorem 3.1 of [26].

Theorem 2.2.1. *Suppose that $\gamma \neq 0$ and the system of the generalized eigenspaces of A_0 is complete. Let us assume the hypotheses (F) and let $f(\cdot,\cdot)$ be uniformly bounded. If for $j = 1, \cdots, N$,*

$$Ker B^* \cap Range\{\sum_{n=0}^{k_j - 1} (Q_j^n)^*\} = \{0\}, \tag{2.2.7}$$

then the system (2.2.6) is Z_+-approximately controllable on $[0, T]$.

Remark 2.2.1. *According to S. Nakagiri [5], we represent the fundamental solution $W(t)$ for (4.1) by*

$$W(t)g^0 = \begin{cases} x(t; (g^0, 0), 0, 0), & t \geq 0 \\ 0 & t < 0 \end{cases}$$

for $g^0 \in H$. Therefore, if for each $p \in L^2(0, T; H_+)$ there exists an element q belonging to the range of B such that

$$\int_0^T W(T-s)p(s)ds = \int_0^T W(T-s)q(s)ds,$$

then we can directly prove the approximate controllability for (2.2.2) as in Theorem 4.1 of [31].

Theorem 2.2.2. *Suppose that $\gamma = 0$. Let us assume the hypotheses (F) and let $f(\cdot, \cdot)$ be uniformly bounded. Then, the following statements are equivalent.*

(a) For any $g \in Z$ there exists an $u \in L^2(0, \infty; U)$ such that the mild solution x of (2.2.2) satisfies $(x, x_t) \in L^2(0, \infty; Z)$, i.e.,

$$\int_0^\infty \{|x(t)|^2 + \int_{-h}^0 ||x(t+s)||^2 ds\} dt < \infty.$$

(b) The system of (2.2.4) is Z_+-approximately controllable.

(c) $\{z^* \in Z_j^T : B_0^*[(A_T - \overline{\lambda_j})^n z^*]^0 = 0,\ 1 \leq j \leq N,\ n = 0, \ldots, k_j - 1\} = \{0\}$.

Proof. As seen in Proposition 4.1 of [29], if $\gamma = 0$, then the solution semigroup $S(t)$ of (2.2.3) is Hölder continuous in $(3h, \infty)$ in operator norm. Thus, by Proposition 3.1 of [29], we know that (a) holds iff

$$\{z^* \in Z_j^* : B^*(A^* - \overline{\lambda_j})^n z^* = 0,\ n = 0, \ldots, k_j - 1\} = \{0\}. \tag{2.2.8}$$

Just as Theorems 4.2 and 8.1 of [5] it can be shown that the structural operator H^* maps $D(A_T)$ to $D(A^*)$ and $A^*G^* = G^*A_T$ on $D(A_T)$, and G^* is an isomorphism from $Z^T_{\overline{\lambda_j}}$ to $Z^*_{\overline{\lambda_j}}$. Hence, (2.2.8) is equivalent to the fact that

$$\{z^* \in Z_j^T : B_0^*[(A_T - \overline{\lambda_j})^n z^*]^0 = 0,\ n = 0, \ldots, k_j - 1\} = \{0\}.$$

□

Chapter 3
Identification problems of retarded differential systems in Hilbert spaces

3.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order in $L^1(\Omega)$:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x).$$

In this paper, we consider the inverse problem for the following retarded functional differential equation defined as $A_0 u = -\mathcal{A}(x, D_x)u$:

$$\begin{cases} u'(t) = A_0 u(t) + \gamma A_0 u(t-h) + \int_{-h}^{0} a(s) A_0 u(t+s) ds, \\ u(0) = g^0, \quad u(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{IE}$$

Here A_0, γ, and $a(\cdot)$ are unknown quantities to be identified and the initial condition $g = (g^0, g^1)$ is known.

In the field of control engineering, the inverse problem, or the parameter estimations of systems has attracted much interest and has been investigated in many references, for example, as for one dimensional heat equation with an unknown spatially-varying conductivity in [11-14], an abstract linear first order evolution equation within the framework of operator theory in [61], and linear retarded functional differential systems in reflexive Banach spaces in [15-17]. In [27, 30] the author discussed the control problem for the following retarded system with $L^1(\Omega)$-valued controller:

$$u^{'}(t) = A_0 u(t) + A_1 u(t-h) + \int_{-h}^{0} a(s) A_2 u(t+s) ds + \Phi_0 w(t), \qquad (3.1.1)$$

where $A_i (i = 1, 2)$ are second order linear differential operators with real coefficients, and the controller Φ_0 is a bounded linear operator from a control Banach space to $L^1(\Omega)$. In [16], they established some results concerning identification problems for (IE) of specific form by taking the observation. Furthermore, Yamamoto and Nakagiri [47] studied the identifiability problem for evolution equations in Banach spaces with unknown operators and initial values by means of spectral theory for linear operators.

In view of Sobolev's embedding theorem we may also consider $L^1(\Omega) \subset W^{-1,p}(\Omega)$ if $1 \le p < n/(n-1)$ as is seen in [27]. Hence, we can investigate the system (IE) in the space $W^{-1,p}(\Omega)$ considering Φ_0 as an operator into $W^{-1,p}(\Omega)$. Here, we note that the space $W^{-1,p}(\Omega)$ is ζ-convex(as for the definition and fundamental facts of a ζ-convex see [24, 10]). Consequently, in view of Dore and Venni [18] the maximal regularity for the linear initial value problem:

$$u'(t) = A_0 u(t) + f(t), u(0) = u_0$$

in the space $W^{-1,p}(\Omega)$ holds true.

Furthermore, with the aid of a result by Seeley [55] and [27], we can obtain the maximal regularity for solutions of the retarded linear initial value problem (IE) in the space $W^{-1,p}(\Omega)$. In view of these results, we deal with an inverse problem of (IE) in $W^{-1,p}(\Omega)$.

The paper is organized as follows. Section 2 presents some notations.

In Section 3 from the definitions of operator A_0 and the interpolation theory as in Theorem 3.5.3 of Butzer and Berens [52], we can apply Theorem 3.2 of Dore and Venni [18] to general linear Cauchy problem in the space $W^{-1,p}(\Omega)$. Thereafter, by using the method of Di Blasio et al. [17] to the system (3.1.1) with the forcing term f in place of the control term $\Phi_0 w$, Section 4 is devoted to studying the wellposedness and regularity for solutions of (IE) by using a solution semigroup $S(t)$ in the initial data space $Z_{p,q} = H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega))$, where $H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p})_{1/q,q}(\Omega)$ for $1 < q < \infty$.

In Section 5, in order to identify the parameters, we investigate the spectrum of the infinitesimal generator $\mathit{\Lambda}$ of $S(t)$. We will give that the spectrum of $\mathit{\Lambda}$ is composed of two parts of cluster points and discrete eigenvalues. Moreover, we are concerned with the representations of spectral projections and the problem of completeness of generalized eigenspaces. Based on this result, we establish a sufficient condition for the inverse problem is given as the so called rank condition in terms of the initial values and eigenvectors of adjoint operator.

Finally we give a simple example to which our main result can be applied.

3.2 Notations

Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ and closure $\overline{\Omega}$.

$C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω.

$C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω.

$W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p,\Omega} = (\sum_{\alpha \leq m} ||D^\alpha f||_{p,\Omega}^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$||f||_{m,\infty,\Omega} = \max_{\alpha \leq m} ||D^\alpha u||_{\infty,\Omega},$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $||\cdot||_{p,\Omega}$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$

For $p = 2$ we denote $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{2,p}(\Omega) = H_0^m(\Omega)$

Let $p' = p/(p-1),\ 1 < p < \infty$. $W^{-1,p}(\Omega)$ stands for the dual space $W_0^{1,p'}(\Omega)^*$ of $W_0^{1,p'}(\Omega)$ whose norm is denoted by $||\cdot||_{-1,p,\infty}$.

If X is a Banach space and $1 < p < \infty$,

$L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable.

$C^m([0,T];X)$ will denote the set of all m-times continuously differentiable functions from $[0,T]$ into X.

If X and Y are two Banach spaces, $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$.

For an interpolation couple of Banach spaces X_0 and X_1, $(X_0,X_1)_{\theta,p}$ for any $\theta \in (0,1)$ and $1 \le p \le \infty$ and $[X_0,X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively(see [22]).

Let A is a closed linear operator in a Banach space. Then

$D(A)$ denotes the domain of (A) and $R(A)$ the range of A.

$\rho(A)$ denotes the resolvent set of A, $\sigma(A)$ the spectrum of A, and $\sigma_p(A)$ the point spectrum of A.

The kernel or null space $\{x \in D(A) : Ax = 0\}$ of A is denoted by $Ker(A)$.

3.3 Cauchy problems on ζ-convex spaces

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider an elliptic differential operator of second order

$$\mathcal{A}(x,D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

where $(a_{i,j}(x) : i, j = 1, \cdots, n)$ is a positive definite symmetric matrix for each $x \in \Omega$, $a_{i,j} \in C^1(\overline{\Omega})$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$. The operator

$$\mathcal{A}'(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\cdot) + c(x)$$

is the formal adjoint of $\mathcal{A}$.

For $1 < p < \infty$ we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

For $p' = p/(p-1)$, we can also define the realization $\mathcal{A}'$ in $L^{p'}(\Omega)$ under Dirichlet boundary condition by $A'_{p'}$:

$$D(A'_{p'}) = W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega),$$

$$A'_{p'} u = \mathcal{A}' u \quad \text{for} \quad u \in D(A'_{p'}).$$

It is known that $-A_p$ and $-A'_{p'}$ generate analytic semigroups in $L^p(\Omega)$ and $L^{p'}(\Omega)$, respectively, and $A_p^* = A'_{p'}$. For brevity, we assume that $0 \in \rho(A_p)$. From the result of Seeley [54] (see also Triebel [22, p. 321]) we obtain that

$$[D(A_p), L^p(\Omega)]_{\frac{1}{2}} = W_0^{1,p}(\Omega),$$

and hence, may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A'_{p'})^*.$$

Let $(A'_{p'})'$ be the adjoint of $A'_{p'}$ considered as a bounded linear operator from $D(A'_{p'})$ to $L^{p'}(\Omega)$. Let $\widetilde{A}$ be the restriction of $(A'_{p'})'$ to $W_0^{1,p}(\Omega)$. Then by the interpolation theory, the operator $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction $\widetilde{A'}$ of $(A_p)' \in B(L^{p'}(\Omega), D(A_p)^*)$ to $W_0^{1,p'}(\Omega)$ is an isomorphism from $W_0^{1,p'}(\Omega)$ to $W^{-1,p'}(\Omega)$. Furthermore, as seen in proposition 3.1 in Jeong [27], we obtain the following result.

Proposition 3.3.1. *The operators $\widetilde{A}$ and $\widetilde{A'}$ generate analytic semigroups in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively, and the inequality*

$$||(\widetilde{A})^{is}||_{B(W^{-1,p}(\Omega))} \leq Ce^{\gamma|s|}, \quad -\infty < s < \infty,$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$.

We set

$$H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{\frac{1}{q},q}, \quad q \in (1, \infty). \tag{3.3.1}$$

Since $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ are ζ-convex spaces, it is easily seen that $H_{p,q}$ is also ζ-convex. From the definitions of operator $\widetilde{A}$ and the interpolation space $H_{p,q}$ as in Theorem 3.5.3 of Butzer and Berens [52], we can apply Theorem 3.2 of Dore and Venni [18] to general linear Cauchy problem as the following result.

Proposition 3.3.2. *Let* $(u_0, f) \in H_{p,q} \times L^q(0,T; W^{-1,p}(\Omega))(1 < q < \infty)$. *Then the Cauchy problem*

$$u'(t) = \widetilde{A}u(t) + f(t), \quad u(0) = u_0$$

has a unique solution

$$u \in L^q(0,T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T; W^{-1,p}(\Omega)) \hookrightarrow C([0,T]; H_{p,q}).$$

The last inclusion relation is well known and is an easy consequence of the definition of real interpolation spaces by the trace method.

3.4 Retarded equations and lemmas

In this section, we apply Propositions 3.3.1 and 3.3.2 to the retarded functional differential equation in the space $W^{-1,p}(\Omega)$. Consider the following retarded equation in $W^{-1,p}(\Omega)$:

$$\begin{cases} u'(t) = A_0u(t) + A_1u(t-h) + \int_{-h}^{0} a(s)A_2u(t+s)ds + f(t), \quad t \in (0,T] \\ u(0) = g^0, \quad u(s) = g^1(s) \quad s \in [-h,0). \end{cases} \tag{3.4.1}$$

Here, $A_0 = -\widetilde{A}$, and $A_\iota u$ $(\iota = 1,\ 2)$ are the restrictions $W_0^{1,p}(\Omega)$ of the linear differential operators $\mathcal{A}_\iota(\iota = 1,\ 2)$ with real coefficients:

$$\mathcal{A}_\iota(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}^\iota(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i^\iota(x)\frac{\partial}{\partial x_i} + c^\iota(x),$$

where

$$a_{i,j}^\iota = a_{j,i}^\iota \in C^1(\overline{\Omega}), \quad b_i^\iota \in C^1(\overline{\Omega}), \quad c^\iota \in L^\infty(\Omega),$$

and $(a^{\iota}_{i,j}), \iota = 1, 2$ are positive definite. The kernel $a(\cdot)$ belongs to $L^{q'}(-h, 0)$. For $q \in (1, \infty)$ we set

$$Z_{p,q} \equiv H_{p,q} \times L^q(-h, 0; W_0^{1,p}(\Omega)).$$

Using Proposition 3.3.2 we can follow the argument as in [17] term by term to deduce the following result(see Proposition 4.1 of [27]).

Lemma 3.4.1. *Given $g = (g^0, g^1) \in Z_{p,q}$ and $f \in L^q(0, T; W^{-1,p}(\Omega))$. Then the problem (3.4.1) has a unique solution*

$$u \in L^q(0, T; W_0^{1,p}(\Omega))) \cap W^{1,q}(0, T; W^{-1,p}(\Omega))) \subset C([0, T]; H_{p,q}).$$

Moreover, we have

$$||u||_{L^q(0,T;W_0^{1,p}(\Omega))\cap W^{1,q}(0,T;W^{-1,p}(\Omega))} \le c(||g^0||_{H_{p,q}} + ||g^1||_{L^q(-h,0;W_0^{1,p}(\Omega))} + ||f||_{L^q(0,T;W^{-1,p}(\Omega))}),$$

where c is a constant.

Let $F \equiv 0$ in (3.4.1) and consider the equation on whole $[0, \infty)$. Then by virtue of Lemma 3.4.1, we can define the solution semigroup $S(t)(t \ge 0)$ for the system (3.4.1) as follows [17, Theorem 4.1](or [61, 63]):

$$S(t) = (u(t; g), u_t(\cdot; g))$$

where $g = (g^0, g^1) \in Z_{p,q}$, $u(t; g)$ is a solution of (3.4.1) and $u_t(\cdot; g)$ is the function $u_t(s; g) = u(t+s; g)$ defined in $[-h, 0]$. It is also known that $S(t)$ is a C_0-semigroup on $Z_{p,q}$. As in Theorem 4.2 of [17], the infinitesimal generator is characterized as follows.

Lemma 3.4.2.

(i) The operator $S(t)$ is a C_0-semigroup on $Z_{p,q}$.

(ii) The infinitesimal generator Λ of $S(t)$ is characterized by

$$D(\Lambda) = \{g = (g^0, g^1):\ g^0 \in W_0^{1,p}(\Omega),\ g^1 \in W^{1,q}(-h, 0; W_0^{1,p}(\Omega)),$$

$$g^1(0) = g^0,\ A_0 g^0 + A_1 g^1(-h) + \int_{-h}^0 a(s) A_2 g^1(s) ds \in H_{p,q}\},$$

$$\Lambda g = (A_0 g^0 + A_1 g^1(-h) + \int_{-h}^0 a(s) A_2 g^1(s) ds, \dot{g}^1).$$

The equation (3.4.1) can be transformed into an abstract equation in $Z_{p,q}$ as follows.

$$z'(t) = \Lambda z(t) + G(t), \quad z(0) = g, \tag{3.4.2}$$

where $G(t) = (f(t), 0)$, $z(t) = (u(t; g), u_t(\cdot; g)) \in Z_{p,q}$ and $g = (g^0, g^1) \in Z_{p,q}$. The mild solution of initial value problem (3.4.2) is the following form:

$$z(t; g) = S(t)g + \int_0^t S(t-s)G(s)ds.$$

We introduce the transposed problem of (3.4.1):

$$\begin{cases} y'(t) = A_0^* y(t) + A_1^* y(t-h) + \int_{-h}^0 a(s) A_2^* y(t+s) ds, & t \in (0, T], \\ y(0) = \phi^0, \quad y(s) = \phi^1(s), & s \in [-h, 0). \end{cases} \tag{3.4.3}$$

Here, we remark that A_0^*, A_1^* $A_2^* \in B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$. We can also define the solution semigroup $S_T(t)$ of (3.4.3) by

$$S_T(t)\phi = (y(t;\phi), y_t(\cdot,\phi)) \quad \forall \phi = (\phi^0, \phi^1) \in Z_{p',q'},$$

where $y(t;\phi)$ is the solution of (3.4.3). Let $\mathit{\Lambda}_{\mathrm{T}}$ be the infinitesimal generator of $S_{\mathrm{T}}(t)$ associated with the system (3.4.3).

For $\lambda \in \mathbb{C}$ we define a densely defined closed linear operator by

$$\Delta(\lambda) = \lambda - A_0 - e^{-\lambda h} A_1 - \int_{-h}^{0} e^{\lambda s} a(s) A_2 ds,$$

$$\Delta_T(\lambda) = \lambda - A_0^* - e^{-\lambda h} A_1^* - \int_{-h}^{0} e^{\lambda s} a(s) A_2^* ds.$$

The operators $\Delta(\lambda)$ and $\Delta_T(\lambda)$ are bounded in $B(W_0^{1,p}(\Omega), W^{-1,p}(\Omega))$ and $B(W_0^{1,p'}(\Omega), W^{-1,p'}(\Omega))$, respectively. Noting that if $\lambda \in \rho(A_0)$

$$\Delta(\lambda) = \{I - (e^{-\lambda h} A_1 + \int_{-h}^{0} e^{\lambda s} A_2 ds)(\lambda - A_0)^{-1}\}(\lambda - A_0).$$

The structural operator F is defined by

$$Fg = ([Fg]^0, [Fg]^1), \tag{3.4.4}$$

$$[Fg]^0 = g^0,$$

$$[Fg]^1(s) = A_1 g^1(-h-s) + \int_{-h}^{s} a(\tau) A_2 g^1(\tau - s) d\tau, \;\; s \in [-h, 0)$$

for $g = (g^0, g^1) \in Z_{p,q}$. It is easy to see that $F \in B(Z_{p,q}, Z^*_{p',q'})$, $F^* \in B(Z_{p',q'}, Z^*_{p,q})$, where

$$[F^*\phi]^0 = \phi^0,$$

$$[F^*\phi]^1(s) = A_1^*\phi^1(-h-s) + \int_{-h}^{s} a(\tau)A_2^*\phi^1(\tau - s)d\tau, \;\; s \in [-h, 0)$$

for $\phi \in Z_{p',q'}$. As in [27, 63] we have that

$$FS(t) = S_T^*(t)F, \quad F^*S_T(t) = S^*(t)F^*. \tag{3.4.5}$$

Let λ be a pole of $(\lambda - \Lambda)^{-1}$ whose order we denote by k_λ and P_λ be the spectral projection associated with λ:

$$P_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\mu - \Lambda)^{-1} d\mu,$$

where Γ_λ is a small circle centered at λ such that it surrounds no point of $\sigma(\Lambda)$ except λ. And we know that $\overline{\lambda} \in \sigma(A_T)$ is a pole of $(\lambda - \Lambda_T)^{-1}$ and the spectral projection is given by

$$P^T_{\overline{\lambda}} = \frac{1}{2\pi i}\int_{\Gamma_{\overline{\lambda}}} (\mu - \Lambda_T)^{-1} d\mu.$$

As is well known λ is an eigenvalue of A and the generalized eigenspace corresponding to λ is given by

$$P_\lambda Z_{p,q} = \{P_\lambda u : u \in Z_{p,q}\} = Ker(\lambda I - \Lambda)^{k_\lambda}.$$

Let us set

$$Q_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (\lambda - \lambda)(\lambda - \Lambda)^{-1} d\lambda.$$

Then we remark that

$$Q^i_{\lambda_j} = \frac{1}{2\pi i}\int_{\Gamma_{\lambda_j}} (\lambda-\lambda_j)^i(\lambda-\Lambda)^{-1}d\lambda.$$

It is also well known that $Q^{k_{\lambda_j}}_{\lambda_j} = 0$ (nilpotent) and $(\Lambda-\lambda)P_{\lambda_j} = Q_{\lambda_j}$(cf. [63, 42]). The following subset of $\sigma(\Lambda)$ are especially of use:

$\sigma_p(\Lambda) =$ the point spectrum of Λ,

$\sigma_d(\Lambda) = \{\lambda \in \sigma(\Lambda) : \lambda$ is isolated and $\dim(P_\lambda Z_{p,q}) = d_\lambda < \infty\}$.

Lemma 3.4.3. *Let* $\lambda \in \sigma_p(\Lambda) = \sigma_p(\Delta)$, *where* $\sigma_p(\Delta) = \{\lambda : \Delta(\lambda)$ is not invertival$\}$. *Then*

1) For any $k = 1, 2, \cdots,$

$$Ker(\lambda-\Lambda)^k = \Big\{\Big(\phi^0_0, e^{\lambda s}\sum_{i=0}^{k-1}(-s)^i\phi^0_i/i!\Big) :$$

$$\sum_{i=j-1}^{k-1}(-1)^{i-j}\Delta^{(i-j+1)}(\lambda)\phi^0_i/(i-j+1)! = 0,\ j = 1, \cdots, k\Big\}.$$

2) $\lambda \in \rho(\Lambda) = \rho(\Lambda^*_T)$,

$$F(\lambda-\Lambda)^{-1} = (\lambda-\Lambda^*_T)^{-1}F.$$

In particular, if $\lambda \in \sigma_p(\Lambda)$ *then*

$$FP_\lambda = (P^T_\lambda)^*F.$$

The proof of 1) and 2) is from Proposition 7.2 and Theorem 6.1 of Nakagiri [63, 60], respectively.

Definition 3.4.1. *The system of generalized eigenspaces of Λ is complete if*

$$\mathrm{Cl}(\mathrm{span}\{\bigcup_{k=1}^{\infty}\mathrm{Ker}(\lambda-\Lambda)^k : \lambda\in\sigma_p(\Lambda)\}) = Z_{p,q},$$

where Cl *is denotes the closure in $Z_{p,q}$.*

We know that $\lambda\in\sigma_d(\Lambda)$ if and only if $\overline{\lambda}\in\sigma_d(\Lambda_T)$ and that $P_\lambda Z_{p,q} = d_\lambda = P_{\overline{\lambda}}^T Z_{p',q'} = d_\lambda$. Let $\{\phi_{\lambda 1},...,\phi_{\lambda d_\lambda}\}$ and $\{\psi_{\lambda 1},...,\psi_{\lambda d_\lambda}\}$ be the bases of $P_\lambda Z_{p,q}$ and $P_\lambda^T Z_{p',q'}$, respectively. As is shown by the same method as Proposition 7.4 and Theorem 8.1 of [63], noting that F^* is an isomorphism from $P_\lambda^T Z_{p',q'}$ to $(P_{\overline{\lambda}})^* Z_{p,q}^*$, we can suppose that

$$(F^*\psi_{\lambda i},\phi_{\overline{\lambda} j}) = \delta_{ij},\quad i,j=1,\cdots,d_\lambda. \tag{3.4.6}$$

Here, $(\cdot,\cdot)$ denotes the duality between $Z_{p,q}^*$ and $Z_{p,q}$. The duality between $Z_{p',q'}^*$ and $Z_{p',q'}$ ia s also denoted by $(\cdot,\cdot)$.

Lemma 3.4.4.
(1) Let $\lambda\in\sigma_d(\Lambda_T)$. Then for any $g\in Z_{p',q'}$, the spectral projection has the following representation

$$P_\lambda^T g = \sum_{i=1}^{d_\lambda}(F^*g,\phi_{\overline{\lambda} i})\psi_{\lambda i}.$$

(2) Let $\lambda \in \sigma_d(\Lambda)$*. Then the spectral projection has the following representation*

$$P_\lambda g = \sum_{i=1}^{d_\lambda} (Fg, \psi_{\overline{\lambda} i}) \phi_{\lambda i}$$

for any $g \in Z_{p,q}$.

Proof. We prove only (1) since the proof of (2) is similar. For any $g \in Z_{p',q'}$, $P_\lambda^T g$ is written as $\sum_{i=1}^{m_\lambda} c_i \psi_{\lambda i}$ for $c_i \in \mathbb{C}$ and then by (3.4.6)

$$(F^* P_\lambda^T g, \phi_{\overline{\lambda} j}) = \sum_{i=1}^{m_\lambda} c_i (F^* \psi_{\lambda i}, \phi_{\overline{\lambda} j}) = c_j.$$

From the Laplace transform of the second equality in (3.4.5) we have

$$F^* (\mu - \Lambda_T)^{-1} = (\mu - \Lambda^*)^{-1} F^*$$

and

$$\begin{aligned} F^* P_\lambda^T &= F^* \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \Lambda_T)^{-1} d\mu = \frac{1}{2\pi i} \int_{\Gamma_\lambda} F^* (\mu - \Lambda_T)^{-1} d\mu \\ &= \frac{1}{2\pi i} \int_{\Gamma_\lambda} (\mu - \Lambda^*)^{-1} F^* d\mu = (P_{\overline{\lambda}})^* F^*. \end{aligned}$$

Therefore, we have

$$c_j = (F^* P_\lambda^T g, \phi_{\overline{\lambda} j}) = ((P_\lambda)^* F^* g, \phi_{\overline{\lambda} j}) = (F^* g, P_{\overline{\lambda}} \phi_{\overline{\lambda} j}) = (F^* g, \phi_{\overline{\lambda} j}).$$

The proof of (1) is completed. □

3.5 Identification problem in case $A_1 = \gamma A_0$ & $A_2 = A_0$

In this section we deal with the identification problem in the case where $A_1 = \gamma A_0$ with some constant γ, $A_2 = A_0$ as follows.

$$\begin{cases} u'(t) = A_0 u(t) + \gamma A_0 u(t-h) + \int_{-h}^{0} a(s) A_0 u(t+s) ds, \\ u(0) = g^0, \quad u(s) = g^1(s), \quad s \in [-h, 0). \end{cases} \tag{3.5.1}$$

Here A_0, γ, and $a(\cdot)$ are unknown quantities to be identified and the initial conditions $g_i = (g_i^0, g_i^1) \in Z_{p,q}, i = 1, \ldots, l$ are known.

We denote by the model system $(3.5.1)^m$ by the equation (3.5.1) with A_0, γ, a replaced by A_0^m, γ^m, a^m respectively. The solutions of (3.5.1) and the model system $(3.5.1)^m$ are denoted by $u(t; g)$ and $u^m(t; g)$, respectively, and the solution semigroup for model system by $S^m(t)$. We assume that A_0^m and a^m satisfy the same type of assumptions as A_0 and a.

The identifiability for (3.5.1) is to find conditions such that if

$$u(t; g_i) \equiv u^m(t; g_i), \ \ i = 1, \ldots, l,$$

for $g_i = (g_i^0, g_i^1) \in Z_{p,q}, i = 1, \ldots, l$, is a finite set of initial values, then

$$A_0 = A_0^m, \ \ \gamma = \gamma^m, \ \ a(s) \equiv a^m(s)$$

follows.

At first we investigate the spectral properties of the infinitesimal generator Λ^m of solution semigroup $S^m(t)$ for the equation $(3.5.1)^m$. Since Ω is bounded, the imbedding of $W_0^{1,p}(\Omega)$ to $H_{p,q}$ is compact. From [56, Theorem

3.4], it follows that the system of generalized eigenspaces of A_0 is complete in $H_{p,q}$. According to Riesz-Schauder theorem A_0^m has discrete spectrum

$$\sigma(A_0^m) = \{\mu_j : j = 1, \cdots\}$$

which has no point of accumulation except possibly $\lambda = \infty$.

For $\lambda \in \mathbb{C}$ we have

$$\Delta^m(\lambda) = \lambda - m(\lambda) A_0^m$$

where

$$m(\lambda) = 1 + \gamma^m e^{-\lambda h} + \int_{-h}^{0} e^{\lambda s} a^m(s) ds. \tag{3.5.2}$$

It is clear that m is an entire function and

$$m(\lambda) \to 1 \text{ as } \mathrm{Re}\lambda \to \infty.$$

Just as Theorems 1 and 2 of [28] for A_0^m we can prove the following two Lemmas.

Lemma 3.5.1. *(1) Let $\rho(\Lambda^m)$ be the resolvent set of the infinitesimal generator Λ^m of $S^m(t)$. Then*

$$\begin{aligned} \rho(\Lambda^m) &= \{\lambda : m(\lambda) \neq 0, \ \frac{\lambda}{m(\lambda)} \in \rho(A_0^m)\} \\ &= \{\lambda : \Delta(\lambda) \ \textit{is an isomorphism from} \ W_0^{1,p}(\Omega) \ \textit{onto} \ W^{-1,p}(\Omega)\}. \end{aligned}$$

(2) Let $\sigma(\Lambda^m)$ be the spectrum of Λ^m. Then

$$\sigma(\Lambda^m) = \sigma_e(\Lambda^m) \cup \sigma_p(\Lambda^m),$$

where $\sigma_e(\Lambda^m) = \{\lambda : m(\lambda) = 0\}$ *and* $\sigma_p(\Lambda^m) = \{\lambda : m(\lambda) \neq 0,\ \lambda/m(\lambda) \in \sigma(A_0^m)\}$*. Each nonzero point of* $\sigma_e(\Lambda^m)$ *is not an eigenvalue of* Λ^m *but a cluster point of* $\sigma(\Lambda^m)$*.* $\sigma_p(\Lambda^m)$ *consists only of discrete eigenvalues.*

(3) Suppose $m(0) = 0$*. Then there exists an analytic function* g *on neighborhood at* 0 *such that* $g(0) \neq 0$ *and* $m(\lambda) = \lambda^k g(\lambda)$*, and*

$$0 \in \begin{cases} \sigma_p(\Lambda^m), & if \quad k = 1, \\ \sigma_e(\Lambda^m), & if \quad k > 1. \end{cases}$$

Lemma 3.5.2. *Suppose that* $m(0) \neq 0,\ \gamma^m \neq 0$*. Then the system of generalized eigenspaces of* Λ^m *is complete in* $Z_{p,q}$*.*

The structural operator F defined by (3.4.4) is written as

$$Fg = ([Fg]^0, [Fg]^1),$$

$$[Fg]^0 = g^0,$$

$$[Fg]^1(s) = \gamma A_0 g^1(-h-s) + \int_{-h}^{s} a(\tau) A_0 g^1(\tau - s) d\tau, \ \ s \in [-h, 0).$$

for $g = (g^1, g^1) \in Z_{p,q}$. The m^m and F^m are the structural operators of the model system $(3.5.1)^m$ in place of m in (3.5.2) and F, respectively.

Let $\lambda \in \sigma_p(\Lambda^m)$, and $\{\phi_{\lambda_k} : k = 1, \cdots, d_\lambda\}$ denote the basis of $P_\lambda^m Z_{p,q}$. Let Λ_T^m be the infinitesimal generator of transposed solution semigroup associated with (3.5.1). Then $\overline{\lambda} \in \sigma_p(\Lambda_T^m)$. Let $\{\psi_{\overline{\lambda}_k} : k = 1, \cdots, d_\lambda\}$ be a basis of $(P^m)_{\overline{\lambda}}^T Z_{Z_{p',q'}}$, where $(P^m)_\mu^T$ denotes the projection of Λ_T^m at μ. As shown in [29, Theorem 8.1] the projection $(P^m)^T_\lambda$ has the following eqivalent representation

$$P^m_\lambda g = \sum_{k=1}^{d_\lambda} (F^m g, \psi_{\overline{\lambda}_k})\ \phi_{\lambda_k},\ \forall g \in Z_{p,q}.$$

Throughout this section we shall assume following:

- RANK CONDITION: For set of the initial values $\{g_1, \ldots, g_l\}$ is said to be satisfy the Rank condition for the model system $(3.5.1)^m$ if and only if

$$\text{rank}((F^m g_i, \psi_{\overline{\lambda}_k}) : i \to 1, \ldots, l,\ \ k \downarrow 1, \ldots, d_\lambda) = d_\lambda,\ \ \forall\lambda \in \sigma_p(\Lambda^m) \tag{3.5.3}$$

  for $n = 1, 2, \cdots$ and $j = 1, 2, \cdots$.

The assumption of Rank condition is satisfied if and only if

$$\text{Span}\{P^m_\lambda g_1, \ldots, P^m_\lambda g_l\} = P^m_\lambda Z_{p,q},\ \ \forall\lambda \in \sigma_p(\Lambda^m). \tag{3.5.4}$$

Proposition 3.5.1. *Assume that* $u(t; g_i) \equiv u^m(t; g_i),\ \ i = 1, \ldots, l$ *and the rank condition (3.5.3) for* $\{g_1, \ldots, g_l\}$ *be satisfied. Further assum that* $m(0) \neq 0$. *Then*

$$\sigma_p(\Lambda^m) \subset \sigma_p(\Lambda), \quad \sigma_e(\Lambda^m) \subset \sigma_e(\Lambda), \tag{3.5.5}$$

and

$$\Lambda = \Lambda^m \ \text{ on }\ P^m_\lambda Z_{p,q},\ \ \forall\lambda \in \sigma_p(\Lambda^m). \tag{3.5.6}$$

Proof. By the definitionof semigroups $S(t)$ and $S(t)^m$, we have from the assumption that

$$S^m(t)g_i = S(t)g_i,\ \ \forall t > 0,\ \ i = 1, \cdots, l. \tag{3.5.7}$$

By taking the Laplace transform of (3.5.7) and using the analytic continuation of the resolvent operators, we have

$$(\lambda - \Lambda^m)^{-1}g_i = (\lambda - \Lambda)^{-1}g_i,\ \ \forall\lambda \in \sigma_p(\Lambda^m) \cap \sigma_p(\Lambda),\ \ i = 1, \cdots, l. \tag{3.5.8}$$

Let $\lambda_0 \in \sigma_p(\Lambda^m)$. First we note that $\lambda_0 \neq 0$. Because, if $\lambda_0 = 0$, then $m(\lambda_0) \neq 0$ and hence $0 = \lambda_0/m(\lambda_0) \in \sigma(A_0^m)$ by Lemma 3.5.1, which contradicts the fact that $A_0^m : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ is an isomorphism. We shall show $\lambda_0 \in \sigma(\Lambda)$. Assume contrarily that $\lambda_0 \in \rho(\Lambda)$. Then from Lemma 3.5.1 there exists a sufficiently small number $\epsilon > 0$ such that

$$\{\lambda : 0 < |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(\Lambda^m),\ \ \{\lambda : |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(\Lambda).$$

Thus, by (3.5.8), we have

$$\begin{aligned} P^m_{\lambda_0} g_i &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - \Lambda^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - \Lambda)^{-1} g_i d\lambda = 0,\ \ i = 1, \cdots, l. \end{aligned}$$

This implies by the span condition (3.5.4) for $\lambda = \lambda_0$ that $P^m_{\lambda_0} Z_{p,q} = \{0\}$, which yields the contradiction. Thus, $\lambda_0 \in \sigma(\Lambda)$. Suppose $\lambda_0 \in \sigma_e(\Lambda)$. Since $\lambda_0 \neq 0$ by $m(\lambda_0) = 0$ in Lemma 3.5.1 and $m(0) \neq 0$, there exists a sequence $\{\lambda_n\} \subset \sigma_p(\Lambda)$ such that $\lambda_n (\neq \lambda_0)$ converges to λ_0 as $n \to \infty$. Then we can choose a sufficiently small $\epsilon > 0$ and natural number $N \geq 1$ such that $\{\lambda : |\lambda - \lambda_0| = \epsilon\} \subset \rho(\mathit{\Lambda})$ and

$$\{\lambda_n : n \geq N\} \subset \{\lambda : 0 < |\lambda - \lambda_0| \leq \epsilon\} \subset \rho(\mathit{\Lambda}^m),$$

$$\{\lambda_n : 1 \leq n \leq N-1\} \cap \{\lambda : |\lambda - \lambda_0| \leq \epsilon\} = \emptyset.$$

Since λ_n's are discrete, we can also choose a positive sequence $\{\epsilon_n : n \geq N$ such that $\{\lambda : |\lambda - \lambda_0| \leq \epsilon_n\} \subset \rho(\mathit{\Lambda}^m)$ for all $n \geq N$. Therefore, by the residue theorem, we have

$$\begin{aligned} P^m_{\lambda_0} g_i &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - \mathit{\Lambda}^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - \mathit{\Lambda})^{-1} g_i d\lambda = \sum_{n\geq N} \frac{1}{2\pi i}\int_{|\lambda-\lambda_n|=\epsilon_n} (\lambda - \mathit{\Lambda})^{-1} g_i d\lambda \\ &= \sum_{n\geq N} \frac{1}{2\pi i}\int_{|\lambda-\lambda_n|=\epsilon_n} (\lambda - \mathit{\Lambda}^m)^{-1} g_i d\lambda = \sum_{n\geq N} 0 = 0, \;\; i = 1, \cdots, l, \end{aligned}$$

which also contradicts the rank condition for $\lambda = \lambda_0$. This shows $\lambda_0 \in \sigma_p(\mathit{\Lambda})$. Since λ_0 is a discrete eigenvalue of $\mathit{\Lambda}$ and $\mathit{\Lambda}^m$, we have for sufficiently small $\epsilon > 0$ that

$$\begin{aligned} P^m_{\lambda_0} g_i &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - \mathit{\Lambda}^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} (\lambda - \mathit{\Lambda})^{-1} g_i d\lambda = P_{\lambda_0} g_i, \;\; i = 1, \cdots, l. \end{aligned} \tag{3.5.9}$$

Further, again by (3.5.8) and (3.5.9), for all $i = 1, \cdots, l$ we have

$$\begin{aligned} \Lambda^m P^m_{\lambda_0} g_i &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \Lambda^m(\lambda-\Lambda^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \{\lambda-(\lambda-\Lambda^m)\}(\lambda-\Lambda^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \lambda(\lambda-\Lambda^m)^{-1} g_i d\lambda - \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \lambda(\lambda-\Lambda^m)^{-1} g_i d\lambda \\ &= \frac{1}{2\pi i}\int_{|\lambda-\lambda_0|=\epsilon} \lambda(\lambda-\Lambda)^{-1} g_i d\lambda = \Lambda P_{\lambda_0} g_i = \Lambda P^m_{\lambda_0} g_i. \end{aligned} \tag{3.5.10}$$

By the span condition (3.5.4) for $\lambda = \lambda_0$, this implies that

$$\Lambda = \Lambda^m \text{ in } P^m_{\lambda_0} Z_{p,q}, \tag{3.5.11}$$

which proves (3.5.6). Next, let $\lambda_0 \in \sigma_e(\Lambda^m)$, then $m(0) = 0$, so that $\lambda_0 \neq 0$ by the assumption $m(0) \neq 0$. Therefore, there exists a sequence $\{\lambda_n\} \subset \sigma_p(\Lambda^m)$ such that λ_n converges to λ_0. Hence from $\{\lambda_n\} \subset \sigma_p(\Lambda^m) \subset \sigma_p(\Lambda)$ in (3.5.5) it follows that λ_0 is a cluster point of $\sigma_p(\Lambda)$ and hence $\lambda_0 \in \sigma_e(\Lambda)$. □

Theorem 3.5.1. *Suppose that $m(0) \neq 0$ and $\gamma^m \neq 0$. Let the set of initial values $\{g_1, \cdots, g_l\}$ satisfy the rank condition (3.5.3) be satisfied. Then*

$$u(t; g_i) \equiv u^m(t; g_i), \;\; g_i = (g_i^0, g_i^1) \in Z_{p,q}, \;\; i = 1, \ldots, l \tag{3.5.12}$$

implies

$$A_0 = A_0^m, \;\; \gamma = \gamma^m, \;\; a(s) \equiv a^m(s). \tag{3.5.13}$$

Proof. By Proposition 3.5.1, it follows from (3.5.12) and the rank condition (3.5.3) that

$$\Lambda = \Lambda^m \ \text{ in } \ P^m_{\lambda_0} Z_{p,q}, \ \ \lambda \in \sigma_p(\Lambda). \tag{3.5.14}$$

Since $m(0) \neq 0$ and $\gamma^m \neq 0$, by Lemma 3.5.2 the system of generalized eigenspaces of Λ^m is projectively complete, i.e.,

$$\mathrm{Cl}(\mathrm{span}\{P^m_\lambda Z_{p,q} : \lambda \in \sigma_p(\Lambda)\}) = Z_{p,q}. \tag{3.5.15}$$

Then by the same argument as in the proof of Theorem 3 in Yamamoto and Nakagiri [47], we can verify by (3.5.14) and (3.5.15) that $D(\Lambda^m) = D(\Lambda)$ and $\Lambda^m g = \Lambda g$ for any $g \in D(\Lambda^m)$. By Lemma 3.4.2, this implies

$$\begin{aligned} &A_0 g^0 + \gamma A_0 g^1(-h) + \int_{-h}^0 a(s) A_0 g^1(s) ds \\ =&A_0^m g^0 + \gamma^m A_0^m g^1(-h) + \int_{-h}^0 a^m(s) A_0^m g^1(s) ds \end{aligned} \tag{3.5.16}$$

for all $g = (g^0, g^1) \in D(\Lambda^m)$ For any $g^0 \in W_0^{1,p}(\Omega)$ and $\epsilon \in (0, h)$, let $g_\epsilon(s)$ be a function in $W^{1,q}(-h, 0; W_0^{1,p}(\Omega))$ such that

$$g_\epsilon(0) = g^0, \quad g_\epsilon(s) = 0 \quad \text{if } s \in [-h, -\epsilon], \quad \text{and } \int_{-h}^0 ||g_\epsilon(s)||^q_{1,p} ds \leq \epsilon^q. \tag{3.5.17}$$

Then $g_\epsilon(s) \in D(\Lambda^m)$, and we apply this g_ϵ to (3.5.16) to have

$$(A_0^m - A_0) g^0 = \int_{-h}^0 (a(s) A_0 - a^m(s) A_0^m) g_\epsilon(s) ds. \tag{3.5.18}$$

By using Hölder inequality, we have from (3.5.17) and (3.5.18) that

$$\begin{aligned}
&\|(A_0^m - A_0)g_0\|_{-1,p} \\
\le&\Big(\int_{-h}^{0} ||a(s)A_0 - a^m(s)A_0^m||^{q'}_{B(W_0^{1,p}(\Omega),W^{-1,p}(\Omega))} ds\Big)^{1/q'}\Big(\int_{-h}^{0} ||g_\epsilon(s)||^q_{1,p} ds\Big)^{1/q} \\
\le&\epsilon||a(\cdot)A_0 - a^m(\cdot)A_0^m||_{L^{q'}(-h,0;B(W_0^{1,p}(\Omega),W^{-1,p}(\Omega)))} \to 0 \text{ as } \epsilon \to 0,
\end{aligned}$$

so that $A_0^m g^0 = A_0 g^0$ in $W^{-1,p}(\Omega)$ for any $g^0 = W_0^{1,p}(\Omega)$. Hence $A_0^m = A_0$ follows. It follows from this and (3.5.16) that

$$(\gamma^m - \gamma)A_0^m g^1(-h) = \int_{-h}^{0} (a(s) - a^m(s))A_0^m g^1(s) ds, \ \forall g = (g^0, g^1) \in D(\Lambda^m). \tag{3.5.19}$$

For any $f^0 \in W_0^{1,p}(\Omega)$ and $\epsilon \in (0, h)$, let f_ϵ be a function in $W^{1,q}(-h, 0; W_0^{1,p}(\Omega))$ such that

$$f_\epsilon(-h) = f^0, \ f_\epsilon(s) = 0 \text{ if } s \in [-h+\epsilon, 0], \quad \int_{-h}^{0} ||f_\epsilon(s)||^q_{1,p} ds \le \epsilon^q. \tag{3.5.20}$$

Then $(0, f_\epsilon) \in D(\Lambda^m)$, and applying this to (3.5.19) and repeating similar argument as above, we have

$$(\gamma^m - \gamma)A_0^m f^0 = 0, \ \forall f^0 \in W_0^{1,p}(\Omega). \tag{3.5.21}$$

Applying $(A_0^m)^{-1}$ to (3.5.21), we obtain $\gamma = \gamma^m$. Finally, from (3.5.19), applying $(A_0^m)^{-1}$ and using the density argument, we have

$$\int_{-h}^{0} (a(s) - a^m(s))A_0^m g^1(s) ds = 0 \text{ in } W_0^{1,p}(\Omega), \quad \forall g^1 \in L^q(-h, 0; W_0^{1,p}(\Omega)). \tag{3.5.22}$$

This implies $a(s) = a^m(s)$ a.e. $s \in [-h, 0]$. □

Remark 3.5.1. *The rank condition (3.5.3) can be replaced by*

$$\operatorname{rank}((F^m g_i, \psi^0_{\lambda_k}) : i \to 1, \ldots, l, \; k \downarrow 1, \ldots, d^0_\lambda) = d^0_\lambda, \;\; \forall \lambda \in \sigma_p(\Lambda^m),$$

$\{\psi^0_{\lambda_k} : k = 1, \cdots, d^0_\lambda\}$ *is a basis of* $\operatorname{Ker}(\overline{\lambda} - \Lambda^m_T)$ *and* $\dim \operatorname{Ker}(\overline{\lambda} - \Lambda^m_T) = d^0_\lambda$ *(cf. Corollary 1 in [47]).*

3.6 example

We consider the following retarded functional differential equation of parabolic type:

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \alpha \frac{\partial^2 u(t,x)}{\partial x^2} + \beta \frac{\partial^2 u(t-h,x)}{\partial x^2} + \int_{-h}^0 a_1(s) \frac{\partial^2 u(t+s,x)}{\partial x^2} ds, \quad (t,x) \in \mathbb{R}^+ \times (0,\pi), \\ u(t,0) = u(t,\pi) = 0, \quad t > 0 \\ u(0,x) = g^0(x), \quad u(s,x) = g^1(s,x) \quad a.e. \; (s,x) \in [-h,0] \times [0,\pi]. \end{cases} \tag{3.6.1}$$

The initial data $(g^0, g^1) \in H_{p,q} \times L^q(-h, 0; W_0^{1,p}(0,\pi))$, $p, q \neq 2$ are known, where $H_{p,q}$ is defined on the domain $\Omega = (0,\pi)$ Here, $\alpha \neq 0$, β and $a(s)$ are unknown except that $a_1 \in L^2(-h, 0; \mathbb{C})$. Let A_0 be the realization in $H_{p,q}$ of the operator $\alpha \frac{\partial^2}{\partial x^2}$ with Dirichlet boundary condition, that is,

$$A_0 = \alpha \frac{\partial^2}{\partial x^2}, \quad D(A_0) = \{u \in H_{p,q} : u(t,0) = u(t,\pi) = 0\}.$$

Then the eigenvalues and eigenfunctions of A_0 are $\mu_n - an^2$ and $e_n(x) = \sin(nx)$, $n = 1, \cdots$, respectively. Let us define as

$$\gamma = \beta/\alpha, \quad a(s) = a_1(s)/\alpha \quad s \in [-h, 0].$$

Then the system (3.6.1) can be written in the same form as of (3.5.1) on the space $H_{p,q}$. It is well known that $\{e_n :, \ n = 1, \cdots\}$ is an orthogonal base for $H_{p,q}$, and so $\{\sin(nx), \ n = 1, \cdots\}$ is complete In $H_{p,q}$. Thus, we can solve the inverse problem of the system (3.6.1) for parameters α, β, and the function $a_1(\cdot)$ in the terminology of Theorem 3.5.1.

As an additional result in this case, we consider the system of generalized eigenspaces of $\mathit{\Lambda}$ as defined Lemma 3.4.2. The spectrum $\sigma(\mathit{\Lambda})$ of $\mathit{\Lambda}$ is given by

$$\sigma(\mathit{\Lambda}) = \bigcup_{n=1}^{\infty} \sigma_n,$$

where

$$\sigma_n = \{\lambda \in \mathbb{C} : \Delta_n(\lambda) = \lambda - n^2\big(\alpha + \beta e^{\lambda h} + \int_{-h}^{0} e^{\lambda s} a_1(s) ds\big) = 0\}$$

as seen in [62, 58]. Hence, $\sigma(\mathit{\Lambda})$ is a countable set consisting entirely of eigenvalues. Let $\{\lambda_{nj}\}_{j=1}^{\infty}$ be the set of roots of $\Delta_n(\lambda) = 0 (n = 1, 2, \cdots)$ and let k_{nj}(in many cases $e_{nj} = 1$) be the multiplicity of λ_{nj}. The generalized eigenspaces $P_{\lambda_{nj}} H_{p,q}$ corresponding to $\lambda_{nj} \in \sigma(\mathit{\Lambda})$ is given by

$$\mathrm{Span}\{\exp(\lambda_{nj} s) \sin(nx), \cdots, s^{k_{nj}-1} \exp(\lambda_{nj} s) \sin(nx)\}. \tag{3.6.2}$$

Since $\{\sin(nx), \ n = 1, \cdots\}$ is complete In $H_{p,q}$, from (3.6.2) and [2, Theorem 5.4] it follows the system of generalized eigenspaces of $\mathit{\Lambda}$ is complete. In the special case of the finite dimensional space, $\sigma(\mathit{\Lambda})$ is a countable set consisting entirely of eigenvalues. Noting that $\gamma \neq 0$ and $0 \neq \sigma(\Lambda)$, the completeness of the system of generalized eigenspaces of Λ is equivalent to $KerF^* = \{0\}$(see Manitius [2]). If h and $a_1 \neq 0$ are known and the multiplicity $d_{nj} = 1$ for all nj and $g^1 = (g_1^0, 0)$ satisfies $(g_1^0, \sin nx) \neq 0$, then α, β, and a_1 in (3.6.1) are identifiable in terminology of Section 5.

Chapter 4
On fractional order retarded neutral differential equations in Hilbert spaces

4.1 Introduction

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. In this paper, we study the existence of solutions and L^2-regularity for the following fractional order retarded neutral functional differential equation:

$$\begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t) + g(t, x_t)] = A_0x(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + (Fx)(t) + k(t), \ t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0, \end{cases} \tag{NE}$$

where $1/2 < \alpha < 1$, $h > 0$, $a_1(\cdot)$ is Hölder continuous, k is a forcing term, and g, f, are given functions satisfying some assumptions. Moreover, $A_0 : H \to H$ is unbounded but A_1 is bounded. For each $s \in [0, T]$, we define $x_s : [-h, 0] \to H$ as $x_s(r) = x(s + r)$ for $r \in [-h, 0]$ and $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$.

This kind of systems arises in many practical mathematical models arising in dynamic systems, economy, physics, biological and engineering problems, etc. (see [43, 67, 13, 41]). There has been a significant development in fractional differential equations in recent years, see [[53, 1, 66, 23]] and the references therein.

In [68, 11, 12], the authors have discussed the existence of solutions for mild solutions for the neutral differential systems with state-dependence delay. Most studies about the neutral initial value problems governed by retarded semilinear parabolic equation have been devoted to the control problems. As for the retarded differential equations, Jeong et al [25, 31], Sukavanam et al. [49], and Wang [44], have discussed the regularity of solutions and controllability of the semilinear retarded systems, and see [25, 31, 49, 44] and references therein for the linear retarded systems.

Recently, the existence of mild solutions for fractional neutral evolution equations has been studied in [51, 1], the existence of solutions of inhomogeneous fractional diffusion equations with a forcing function in Baeumer et al. [5], and the existence and approximation of solutions to fractional evolution equation in Muslim [46]. In addition, Sukavanam et al.[50] studied approximate controllability of fractional order semilinear delay systems.

In this paper, we propose a different approach of the earlier works used properties of the relative compactness. Our approach is that regularity results of general retarded linear systems of Di Blasio et al. [17] and semilinear systems of [31] remain valid under the above formulation of fractional order retarded neutral differential system (NE) even though the system (NE) contains unbounded principal operators, delay term, and local Lipschitz continuity of the nonlinear term. The methods of the functional analysis concerning an analytic semigroup of operators and some fixed point theorems are applied effectively.

The paper is organized as follows. In Section 2, we deal with properties of the analytic semigroup constructing the strict solution of the corresponding linear systems excluded by the nonlinear term and introduce basic properties. In Section 3, by using properties of the strict solutions in dealt in Section 2, we will obtain the L^2-regularity of solutions of (NE), and a variation of constant formula of solutions of (NE). Finally, we also give an example to illustrate the applications of the abstract results.

4.2 Preliminaries and Lemmas

Let H and V be two Hilbert spaces such that V is a dense subspace of H. The norm of H(resp. V) is denoted by $|\cdot|$ (resp. $||\cdot||$) and the corresponding scalar product by $(\cdot,\cdot)$ (resp.$((\cdot,\cdot))$). Assume that the injection of V into H is continuous. The antidual of V is denoted by V^*, and the norm of V^* by $||\cdot||^*$. Identifying H with its antidual we can assume that H is embedded in V^*. Hence we have $V \subset H \subset V^*$ densely and continuously. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v\in V} \frac{|(l,v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and ,for brevity, we may consider

$$||u||_* \leq |u| \leq ||u||, \quad u \in V. \tag{4.2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re}\, a(u,u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0. \tag{4.2.2}$$

Let A_0 be the operator associated with the sesquilinear form $-a(\cdot,\cdot)$:

$$((c_1 - A_0)u, v) = -a(u,v), \quad u,\ v \in V.$$

It follows from (4.2.2) that for every $u \in V$

$$\text{Re}\,(A_0 u, u) \geq c_0||u||^2.$$

Then A_0 is a bounded linear operator from V to V^* according to the Lax-Milgram theorem, and its realization in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V; A_0 u \in H\}$$

is also denoted by A_0. Then A_0 generates an analytic semigroup $S(t) = e^{tA_0}$ in both H and V^* as in Theorem 3.6.1 of [20]. Moreover, there exists a constant C_0 such that

$$||u|| \leq C_0 ||u||_{D(A_0)}^{1/2} |u|^{1/2}, \tag{4.2.3}$$

for every $u \in D(A_0)$, where

$$||u||_{D(A_0)} = (|A_0 u|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A_0)$. Thus we have the following sequence

$$D(A_0) \subset V \subset H \subset V^* \subset D(A_0)^*,$$

where each space is dense in the next one and continuous injection.

Lemma 4.2.1. *With the notations (4.2.1), (4.2.3), we have*

$$(V, V^*)_{1/2,2} = H,$$

$$(D(A_0), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* *(Section 1.3.3 of [22]).*

If X is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p-th powers of norms are integrable. $\mathcal{L}(X, Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X, X)$ is simply written as $\mathcal{L}(X)$.

For the sake of simplicity we assume that the semigroup $S(t)$ generated by A_0 is uniformly bounded, that is, There exists a constant M_0 such that

$$||S(t)||_{\mathcal{L}(H)} \le M_0, \quad ||A_0 S(t)||_{\mathcal{L}(H)} \le \frac{M_0}{t}. \tag{4.2.4}$$

The following lemma is from [20, Lemma 3.6.2].

Lemma 4.2.2. *There exists a constant* M_0 *such that the following inequalities hold:*

$$||S(t)||_{\mathcal{L}(H,V)} \le t^{-1/2} M_0, \tag{4.2.5}$$

$$||S(t)||_{\mathcal{L}(V^*,V)} \le t^{-1} M_0, \tag{4.2.6}$$

$$||A_0 S(t)||_{\mathcal{L}(H,V)} \le t^{-3/2} M_0. \tag{4.2.7}$$

The following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = A_0 x(t) + \int_{-h}^{0} a_1(s) A_1 x(t+s) ds + k(t), \quad 0 < t \le T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \ s \in [-h, 0). \end{cases} \tag{4.2.8}$$

Then the mild solution $x(t)$ is represented by

$$\begin{aligned} x(t) =& S(t)\phi^0 + \int_0^t S(t-s) \int_{-h}^{0} a_1(\tau) A_1 x(s+\tau) d\tau ds \\ &+ \int_0^t S(t-s) k(s) ds, \\ x(0) =& \phi^0, \quad x(s) = \phi^1(s) \quad s \in [-h, 0). \end{aligned}$$

By virtue of Theorem 2.1 of [29] or [17], we have the following result on the corresponding linear equation of (4.2.8).

Lemma 4.2.3. *(1) For $(\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A_0))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (4.2.8) belonging to*

$$L^2(0, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$

and satisfying

$$||x||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} \le C_1(||\phi^0|| + ||\phi^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}), \tag{4.2.9}$$

where C_1 is a constant depending on T and

$$||x||_{L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)} = \max\{||x||_{L^2(0,T;D(A_0))}, ||x||_{W^{1,2}(0,T;H)}\}$$

(2) Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (4.2.8) belonging to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}), \quad (4.2.10)$$

where C_1 is a constant depending on T.

Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0, T; D(A_0)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $c_1 > 0$ such that

$$||x||_{C([0,T];V)} \leq c_1||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \leq c_1||x||_{\mathcal{W}_1(T)}. \qquad (4.2.11)$$

In what follows in this section, we assume $c_1 = 0$ in (4.2.2) without any loss of generality. So we have that $0 \in \rho(A_0)$ and the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A_0. In this case, it is possible to define the fractional power A_0^α for $\alpha > 0$. The subspace $D(A_0^\alpha)$ is dense in H and the expression

$$||x||_\alpha = ||A_0^\alpha x||, \quad x \in D(A_0^\alpha)$$

defines a norm on $D(A_0^\alpha)$. It is also well known that A_0^α is a closed operator with its domain dense and $D(A_0^\alpha) \supset D(A_0^\beta)$ for $0 < \alpha < \beta$. Due to the well known fact that $A_0^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_{-\alpha} > 0$ such that

$$||A_0^{-\alpha}||_{\mathcal{L}(H)} \leq C_{-\alpha}, \quad ||A_0^{-\alpha}||_{\mathcal{L}(V^*,V)} \leq C_{-\alpha}. \tag{4.2.12}$$

Lemma 4.2.4. *For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$:*

$$||A_0^\alpha S(t)||_{\mathcal{L}(H)} \leq \frac{C_\alpha}{t^\alpha}, \quad ||A_0^\alpha S(t)||_{\mathcal{L}(H,V)} \leq \frac{C_\alpha}{t^{3\alpha/2}}. \tag{4.2.13}$$

Proof. The relation is from the inequalities (4.2.6) and (4.2.7) by properties of fractional power of A_0 and the definition of $S(t)$. □

4.3 Existence of solutions

Consider the following fractional order retarded neutral differential system:

$$\begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t) + g(t, x_t)] = A_0 x(t) + \int_{-h}^0 a_1(s) A_1 x(t+s) ds + (Fx)(t) + k(t), \ t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \leq s < 0, \end{cases} \tag{4.3.1}$$

where $0 < \alpha < 1$ and $A_i (i = 0, 1)$ are the linear operators defined as in Section 2. For each $s \in [0, T]$, we define $x_s : [-h, 0] \to H$ as

$$x_s(r) = x(s + r), \quad -h \leq r \leq 0.$$

We will set

$$\Pi = L^2(-h, 0; V).$$

Definition 4.3.1. *The fractional integral of order $\alpha > 0$ with the lower limit 0 from a function f is defined as*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

provided the right hand side is pointwise defined on $[0, \infty)$, Γ is the Gamma function.

The fractional derivative of order $\alpha > 0$ in the Caputo sense with the lower limit 0 from a function $f \in C^n[0, \infty)$ is defined as

$$\frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 < \alpha < n.$$

For the basic results about fractional integrals and fractional derivative, one can refer to [23].

The mild solution of the system (4.3.1) is represented as (see [51, 71]):

$$x(t) = S(t)[\phi^0 + g(0, \phi^1)] - g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s) g(s, x_s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s) \{ \int_{-h}^0 a_1(\tau) A_1 x(s+\tau) d\tau + (Fx)(s) + k(s) \} ds. \tag{4.3.2}$$

To establish our results, we introduce the following assumptions on system (4.3.1).

Assumption (A). We assume that $a_1(\cdot)$ is Hölder continuous of order ρ:

$$|a_1(0)| \le H_1, \quad |a_1(s) - a_1(\tau)| \le H_1 (s-\tau)^{\rho}.$$

Assumption (F1). F is a nonlinear mapping of $L^2(0,T;V)$ into $L^2(0,T;H)$ satisfying following:

(i) There exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$||Fx - Fy||_{L^2(0,T;H)} \le L_f(r)||x-y||_{L^2(0,T;V)}, \quad t \in [0,T]$$

hold for $||x||_{L^2(0,T;V)} \le r$ and $||y||_{L^2(0,T;V)} \le r$.

(ii) The inequality

$$||Fx||_{L^2(0,T;H)} \le L_f(r)(||x||_{L^2(0,T;V)} + 1)$$

holds for every $t \in [0,T]$ and $||x||_{L^2(0,T;V)} \le r$.

Assumption (G). Let $g : [0,T] \times \Pi \to H$ be a nonlinear mapping such that there exists a constant L_g satisfying the following conditions hold:

(i) For any $x \in \Pi$, the mapping $g(\cdot, x)$ is strongly measurable;

(ii) There exists a positive constant $\beta > 1 - 2\alpha/3$ such that

$$|A^\beta g(t,0)| \le L_g, \quad |A^\beta g(t,x) - A^\beta g(t,\hat{x})| \le L_g||x-\hat{x}||_\Pi,$$

for all $t \in [0,T]$, and $x, \hat{x} \in \Pi$.

Lemma 4.3.1. *Let $x \in L^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$, and*

$$||x_t||_\Pi \le ||x||_{L^2(-h,t;V)} (t > 0), \tag{4.3.3}$$

$$||x_\cdot||_{L^2(0,T;\Pi)} \le \sqrt{T}||x||_{L^2(-h,T;V)}. \tag{4.3.4}$$

Proof. The first paragraph is easy to verify. Moreover, we have

$$||x_t||_\Pi = \Big[\int_{-h}^{0} ||x(s+\tau)||^2 d\tau\Big]^{1/2} \le \Big[\int_{-h}^{t} ||x(\tau)||^2 d\tau\Big]^{1/2} \le ||x||_{L^2(-h,t;V)}, \ t > 0,$$

and

$$\begin{aligned} ||x.||^2_{L^2(0,T;\Pi)} &\le \int_0^T ||x_s||^2_\Pi ds \le \int_0^T \int_{-h}^0 ||x(s+r)||^2 dr ds \\ &\le \int_0^T ds \int_{-h}^T ||x(r)||^2 dr \le T||x||^2_{L^2(-h,T;V)}. \end{aligned}$$

□

One of the main useful tools in the proof of existence theorems for nonlinear functional equations is the following fixed point theorem.

Lemma 4.3.2. *(See [45]) Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*

(i) $(K_1 + K_2)(\Sigma) \subset \Sigma$,

(ii) K_1 *is a completely continuous mapping,*

(iii) K_2 *is a contraction mapping.*

Then the operator $K_1 + K_2$ has a fixed point in Σ.

From now on, we establish the following results on the solvability of the equation (4.3.1).

Theorem 4.3.1. *Let Assumptions (A), (F1) and (G) be satisfied. Assume that* $(\phi^0, \phi^1) \in H \times \Pi$ *and* $k \in L^2(0, T; V^*)$ *for* $T > 0$*. Then, there exists a solution* x *of the system (4.3.1) such that*

$$x \in \mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \hookrightarrow C([0, T]; H).$$

Moreover, there is a constant C_2 *independent of the initial data* (ϕ^0, ϕ^1) *and the forcing term* k *such that*

$$||x||_{L^2(-h,T;V)} \le C_2(1 + |\phi^0| + ||\phi^1||_\Pi + ||k||_{L^2(0,T;V^*)}). \tag{4.3.5}$$

Proof. Let

$$r := 2\big[C_1|\phi^0| + C_1 C_{-\beta} L_g(||\phi^1|| + 1)\big],$$

and

$$\begin{aligned} N := & C_{-\beta} L_g\big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1\big) \\ & + \frac{C_1 (2\alpha)^{-1/2} (2\alpha - 1)^{-1/2}}{\Gamma(\alpha)} \\ & \times \big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V^*)}\big) \\ & + \frac{C_{1-\beta} L_g}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha + 3\beta - 2\big)^{1/2} \Gamma(\alpha)} (||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1), \end{aligned}$$

where C_1 is the constants in Lemma 4.2.3 and $\beta > 1 - 2\alpha/3$ in Assumption (G). Let

$$T_1^\gamma := \max\{T_1^{1/2}, T_1^{(2\alpha+3\beta-2)/2}\}$$

and choose $0 < T_1 < T$ such that

$$T_1^\gamma N \le \frac{r}{2} = \big[C_1|\phi^0| + C_1 C_{-\beta} L_g(||\phi^1|| + 1)\big], \tag{4.3.6}$$

and

$$\hat{N} := T_1^{\gamma}\{C_{-\beta}L_g + \frac{C_{1-\beta}L_g}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha + 3\beta - 2\big)^{1/2}\Gamma(\alpha)} \tag{4.3.7}$$

$$+ \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}L_f(r)}{\Gamma(\alpha)}\}$$

$$< 1.$$

Let J be the operator on $L^2(0, T_1; V)$ defined by

$$(Jx)(t) = S(t)[\phi^0 + g(0, \phi^1)] - g(t, x_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s, x_s)ds$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} S(t-s)\{\int_{-h}^0 a_1(\tau)A_1 x(s+\tau)d\tau + (Fx)(s) + k(s)\}ds.$$

Let

$$\Sigma = \{x \in L^2(-h, T_1; V) : x(0) = \phi^0, \text{ and } x(s) = \phi^1(s)(s \in [-h, 0))\}.$$

and

$$\Sigma_r = \{x \in \Sigma : ||x||_{L^2(0,T_1;V)} \le r\},$$

which is a bounded closed subset of $L^2(0, T_1; V)$.

Now, in order to show that the operator J has a fixed point in $\Sigma_r \subset L^2(0, T_1; V)$, we take the following steps according to the process of Lemma 4.3.2.

Step 1. J maps Σ_r into Σ_r.

By (4.2.10), (4.2.12) and Assumption (G), and noting $x_0 = \phi^1$, we know

$$||S(\cdot)g(0, x_0)||_{L^2(0,T_1;V)} = C_1|g(0, \phi^1)| \tag{4.3.8}$$

$$= C_1||A^{-\beta}||_{\mathcal{L}(H)}\big(|A^{\beta}g(0, \phi^1) - A^{\beta}g(0, 0)| + |A^{\beta}g(0, 0)|\big)$$

$$\le C_1 C_{-\beta} L_g(||\phi^1||_{\Pi} + 1).$$

From (4.2.10) of Lemma 4.2.3 it follows

$$||S(t)\phi^0||_{L^2(0,T_1;V)} \le C_1|\phi^0|, \tag{4.3.9}$$

and by using Hölder inquality

$$\int_0^t (t-s)^{\alpha-1}\big\|S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + (Fx)(s) + k(s)\big\}\big\|ds \tag{4.3.10}$$

$$\le (2\alpha-1)^{-1/2}t^{(2\alpha-1)/2}C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||Fx||_{L^2(0,t;V^*)} + ||k||_{L^2(0,t;V^*)}).$$

Define the operator I_1 from $L^2(0,T_1;V)$ to itself by

$$(I_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + (Fx)(s) + k(s)\big\}ds.$$

Then according to (4.3.10) we obtain the following inequality

$$||I_1||_{L^2(0,T_1;V)} \le \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}T_1^\alpha}{\Gamma(\alpha)}\big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} \tag{4.3.11}$$

$$+ L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V^*)}\big).$$

By using Assumption (G) and Lemma 4.3.1, we have

$$||g(\cdot,x_\cdot)||_{L^2(0,T_1;V)} = \big(\int_0^{T_1} \big\|A^{-\beta}A^\beta g(t,x_t)\big\|^2 dt\big)^{1/2} \tag{4.3.12}$$

$$\le C_{-\beta}\big(\int_0^{T_1} \big\|A^\beta g(t,x_t)\big\|^2 dt\big)^{1/2} \le C_{-\beta}L_g\sqrt{T_1}\big(||x_t||_\Pi + 1\big)$$

$$\le C_{-\beta}L_g\sqrt{T_1}\big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1\big).$$

Here, we note

$$||x_t||_\Pi \le ||x||_{L^2(-h,T_1;V)} \le ||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}. \tag{4.3.13}$$

Again we define the operator I_2 from $L^2(0,T_1;V)$ to itself by

$$(I_2x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}AS(t-s)g(s,x_s)ds.$$

From Lemma 4.2.4 and Assumption (G) we have

$$\begin{aligned}||(t-s)^{(\alpha-1)}AS(t-s)g(s,x_s)|| &= (t-s)^{(\alpha-1)}||A^{1-\beta}S(t-s)||_{\mathcal{L}(H,V)}|A^\beta g(s,x_s)| \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\alpha+3(1-\beta)/2}}|A^\beta(g(s,x_s)| \\ &\le \frac{C_{1-\beta}}{(t-s)^{1-\alpha+3(1-\beta)/2}}L_g(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1),\end{aligned}$$

and hence, by using Hólder inequality and Assumption (G),

$$||I_2x||_{L^2(0,T_1;V)} = \Big[\int_0^{T_1}||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}AS(t-s)g(s,x_s)ds\Big\|^2 dt\Big]^{1/2} \tag{4.3.14}$$

$$\le \frac{1}{\Gamma(\alpha)}C_{1-\beta}L_g(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1)\Big[\int_0^{T_1}\Big(\int_0^t \frac{1}{(t-s)^{1-\alpha+3(1-\beta)/2}}ds\Big)^2dt\Big]^{1/2}$$

$$\le \frac{C_{1-\beta}L_gT_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1).$$

Thus, from (4.3.8)-(4.3.14) it follows that

$$\begin{aligned}
||Jx||_{L^2(0,T_1;V)} \le & C_1|\phi^0| + C_1C_{-\beta}L_g(||\phi^1||+1) \\
& + C_{-\beta}L_g\sqrt{T_1}\big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1\big) \\
& + \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}T_1^\alpha}{\Gamma(\alpha)} \\
& \times \big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)(||x||_{L^2(0,T_1;V)}+1) + ||k||_{L^2(0,T_1;V)}\big) \\
& + \frac{C_{1-\beta}L_gT_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1), \\
\le & C_1|\phi^0| + C_1C_{-\beta}L_g(||\phi^1||+1) + T_1^\gamma N \le \frac{r}{2} + \frac{r}{2} \le r.
\end{aligned}$$

Therefore, J maps Σ_r into Σ_r.

Define mapping $K_1 + K_2$ on $L^2(0,T_1;V)$ by the formula

$$(Jx)(t) = (K_1x)(t) + (K_2x)(t),$$

where

$$(K_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}S(t-s)\int_0^s a_1(\tau-s)A_1x(\tau)d\tau ds,$$

and

$$(K_2x)(t) = S(t)[\phi^0 + g(0,x_0)] - g(t,x_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}AS(t-s)g(s,x_s)ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}S(t-s)\{\int_{s-h}^0 a_1(\tau-s)A_1\phi^1(\tau)d\tau + F(x)(s) + k(s)\}ds.$$

Step 2. K_1 is a completely continuous mapping.

We can now employ Lemma 4.3.2 with Σ_r. Assume that a sequence $\{x_n\}$ of $L^2(0,T_1;V)$ converges weakly to an element $x_\infty \in L^2(0,T_1;V)$, i.e., $w-\lim_{n\to\infty} x_n = x_\infty$. Then we will show that

$$\lim_{n\to\infty} ||K_1x_n - K_1x_\infty||_{L^2(0,T_1;V)} = 0, \tag{4.3.15}$$

which is equivalent to the completely continuity of K_1 since $L^2(0,T_1;V)$ is reflexive. For a fixed $t \in [0,T_1]$, let $x_t^*(x) = (K_1x)(t)$ for every $x \in L^2(0,T_1;V)$. Then $x_t^* \in L^2(0,T_1;V^*)$ and we have $\lim_{n\to\infty} x_t^*(x_n) = x_t^*(x_\infty)$ since $w-\lim_{n\to\infty} x_n = x_\infty$. Hence,

$$\lim_{n\to\infty}(K_1x_n)(t) = (K_1x_\infty)(t), \quad t \in [0,T_1].$$

By using Hölder inequality, we obtain easily the following inequality:

$$|\int_0^s a_1(\tau-s)A_1x(\tau)d\tau| = \Big|\int_0^s (a_1(\tau-s)-a_1(0)+a_1(0))A_1x(\tau)d\tau\Big| \tag{4.3.16}$$

$$\le \big\{\big((2\rho+1)^{-1}s^{2\rho+1}\big)^{1/2} + \sqrt{s}\big\}H_1||A_1||_{\mathcal{L}(H)}\big(\int_0^s ||x(\tau)||^2 d\tau\big)^{1/2}.$$

Thus, by (4.2.5) and (4.3.16) it holds

$$||(K_1x)(t)|| = \Big\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}S(t-s)\int_0^s a_1(\tau-s)A_1x(\tau)d\tau ds\Big\|$$

$$\le \frac{H_1||A_1||_{\mathcal{L}(H)}||x||_{L^2(0,t;V)}}{\Gamma(\alpha)}\Big\|\int_0^t \frac{1}{(t-s)^{1/2-\alpha}}\big\{((2\rho+1)^{-1}s^{(2\rho+1)/2}+\sqrt{s}\big\}ds\Big\|$$

$$\le \frac{H_1||A_1||_{\mathcal{L}(H)}||x||_{L^2(0,t;V)}}{\Gamma(\alpha)}\big\{(2\rho+1)^{-1}B(1/2+\alpha,(2\rho+3)/2)t^{\rho+1} + B(1/2+\alpha,3/2)t\big\}.$$

$$:= c_2||x||_{L^2(0,t;V)},$$

where c_2 is a constant and $B(\cdot,\cdot)$ is the Beta function, that is,

$$B(1/2+\alpha,(2\rho+3)/2)t^{\rho+1} = \int_0^t (t-s)^{\alpha-1/2} s^{(2\rho+1)/2} ds.$$

And we know

$$\sup_{0\le t\le T_1} ||(K_1 x)(t)|| \le c_2 ||x||_{L^2(0,T_1;V)} \le \infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty} \Big(\int_0^{T_1} ||(K_1 x_n)(t)||^2 dt\Big) = \Big(\int_0^{T_1} ||(K_1 x_\infty)(t)||^2 dt\Big),$$

i.e., $\lim_{n\to\infty} ||K_1 x_n||_{L^2(0,T_1;V)} = ||K_1 x_\infty||_{L^2(0,T_1;V)}$. Since $L^2(0,T_1;V)$ is a reflexive space, it holds (4.3.15).

Step 3. K_2 is a contraction mapping.

For every x_1 and $x_2 \in \Sigma_r$, we have

$$\begin{aligned}(K_2x_1)(t) - (K_2x_2)(t) =& g(t,x_{2_t}) - g(t,x_{1_t}) \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} AS(t-s)\big(g(t,x_{1_s}) - g(t,x_{2_s})\big)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} S(t-s)\{F(x_1)(s) - F(x_2)(s)\}ds.\end{aligned}$$

By the similar way to (4.3.8)-(4.3.14), we have

$$\begin{aligned}||K_2x_1 - K_2x_2||_{L^2(0,T_1;V)} \le& \Big\{C_{-\beta}L_g\sqrt{T_1} + \frac{C_{1-\beta}L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)} \\ &+ \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}L_f(r)T_1^\alpha}{\Gamma(\alpha)}\Big\}||x_1-x_2||_{L^2(0,T_1;V)} \\ \le& \hat{N}||x_1-x_2||_{L^2(0,T_1;V)}.\end{aligned}$$

So by virtue of the condition (4.3.7) the contraction mapping principle gives that the solution of (4.3.1) exists uniquely in $L^2(0,T_1;V)$. This has proved the local existence and uniqueness of the solution of (4.3.1).

Step 4. We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula (4.3.5) of solution of (4.3.1). Let x be a solution of (4.3.1) and $\phi^0 \in H$. Then we have that from (4.3.8)-(4.3.14) it follows that

$$\begin{aligned}
||x||_{L^2(0,T_1;V)} \le & C_1|\phi^0| + C_1 C_{-\beta} L_g(||\phi^1||_\Pi + 1) \\
& + C_{-\beta} L_g \sqrt{T_1}\big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1\big) \\
& + \frac{C_1 (2\alpha)^{-1/2}(2\alpha-1)^{-1/2} T_1^\alpha}{\Gamma(\alpha)} \\
& \times \big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V)}\big) \\
& + \frac{C_{1-\beta} L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)} (||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1), \\
= & \hat{N}||x||_{L^2(0,T_1;V)} + \hat{N}_1,
\end{aligned}$$

where $\hat{N}$ is the constant of (4.3.7) and

$$\begin{aligned}
\hat{N}_1 = & C_1|\phi^0| + C_1 C_{-\beta} L_g(||\phi^1||_\Pi + 1) + C_{-\beta} L_g \sqrt{T_1}\big(||\phi^1||_\Pi + 1\big) \\
& + \frac{C_1 (2\alpha)^{-1/2}(2\alpha-1)^{-1/2} T_1^\alpha}{\Gamma(\alpha)} \big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)) + ||k||_{L^2(0,T_1;V^*)}\big) \\
& + \frac{C_{1-\beta} L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)} (||\phi^1||_\Pi + 1).
\end{aligned}$$

Taking into account (4.3.7) there exists a constant C_2 such that

$$||x||_{L^2(0,T_1;V)} \leq (1-\hat{N})^{-1}\hat{N}_1 \tag{4.3.17}$$
$$\leq C_2(1+|\phi^0|+||\phi^1||_\Pi+||k||_{L^2(0,T_1;V^*)}),$$

which obtain the inequality (4.3.5).

Now we will prove that $|x(T_1)| < \infty$ in order that the solution can be extended to the interval $[T_1, 2T_1]$. From (4.2.11) and Lemma 4.2.3 it follows that

$$|S(T_1)[\phi^0+g(0,x_0)]| \leq c_1||S(\cdot)[\phi^0+g(0,x_0)||_{\mathcal{W}_1(T_1)} \tag{4.3.18}$$
$$\leq c_1C_1|\phi^0+g(0,\phi^1)|$$
$$\leq c_1C_1\{|\phi^0|+C_{-\beta}L_g(||\phi^1||_\Pi+1)\} := I,$$

and by using Assumption (G) we have

$$|g(T_1,x_{T_1})| \leq \|A^{-\beta}A^\beta g(t,x_{T_1})\|, \tag{4.3.19}$$
$$\leq C_{-\beta}L_g\big(||x_{T_1}||_\Pi+1\big)$$
$$\leq C_{-\beta}L_g\big(||\phi^1||_\Pi+||x||_{L^2(0,T_1;V)}+1\big) := II.$$

By (4.3.10), we have

$$|(I_1x)(T_1)| \tag{4.3.20}$$
$$\leq \frac{1}{\Gamma(\alpha)}\Big\|\int_0^{T_1}(T_1-s)^{\alpha-1}S(T_1-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+(Fx)(s)+k(s)\big\}ds\Big\|$$
$$\leq (2\alpha-1)^{-1/2}\Gamma(\alpha)^{-1}{T_1}^{(2\alpha-1)/2}$$
$$\times C_1(|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+L_f(r)(||x||_{L^2(0,T_1;V)}+1)+||k||_{L^2(0,T_1;V^*)}) := III$$

From Lemma 4.2.4 and Assumption (G) we have

$$
\begin{aligned}
&|(T_1-s)^{(\alpha-1)}AS(T_1-s)g(s,x_s)| \le \\
&= (T_1-s)^{(\alpha-1)}|A^{1-\beta}S(T_1-s)|_{\mathcal{L}(H)}|A^{\beta}(g(s,x_s)| \\
&\le \frac{C_{1-\beta}}{(T_1-s)^{1-\alpha+(1-\beta)}}|A^{\beta}(g(s,x_s)| \\
&\le \frac{C_{1-\beta}}{(T_1-s)^{2-\alpha-\beta}}L_g(||\phi^1||_{\Pi}+||x||_{L^2(0,T_1;V)}+1),
\end{aligned}
$$

and so

$$
|(I_2x)(T_1)| = \Big|\frac{1}{\Gamma(\alpha)}\int_0^{T_1}(T_1-s)^{(\alpha-1)}AS(T_1-s)g(s,x_s)ds\Big| \tag{4.3.21}
$$

$$
\le C_{1-\beta}\big(\alpha+\beta-1\big)^{-1}T_1^{\alpha+\beta-1}L_g(||\phi^1||_{\Pi}+||x||_{L^2(0,T_1;V)}+1) := IV.
$$

Thus,by (4.3.17)-(4.3.21) we have

$$
\begin{aligned}
|x(T_1)| &= \big|S(T_1)[\phi^0+g(0,x_0)]-g(T_1,x_{T_1})+(I_1x)(T_1)+(I_2x)(T_1)\big| \\
&\le I+II+III+IV<\infty.
\end{aligned}
$$

Hence we can solve the equation in $[T_1, 2T_1]$ with the initial $(x(T_1), x_{T_1})$ and an analogous estimate to (4.3.4). Since the condition (4.3.6) is independent of initial values, the solution can be extended to the interval $[0, nT_1]$ for any natural number n, and so the proof is complete. □

Remark 4.3.1. *Thanks for Lemma 4.2.3, we note that the solution of (4.3.1) under conditions of Theorem 4.3.1 with* $(\phi^0,\phi^1)\in V\times L^2(0,T;D(A))$ *and* $k\in L^2(0,T;H)$ *for* $T>0$ *belongs to*

$$
\mathcal{W}(T) = L^2(0,T;D(A)))\cap W^{1,2}(0,T;H)\hookrightarrow C([0,T];V).
$$

Moreover, there is a constant C_2 *independent of the initial data* (ϕ^0, ϕ^1) *and the forcing term* k *such that*

$$||x||_{L^2(-h,T;D(A))} \leq C_2(1 + ||\phi^0|| + ||\phi^1||_{L^2(0,T;D(A))} + ||k||_{L^2(0,T;H)}).$$

Now, we obtain that the solution mapping is Lipschitz continuous in the following result, which is useful for the control problem and physical applications of the given equation.

Theorem 4.3.2. *Let Assumptions (A), (F1) and (G) be satisfied. Assuming that the initial data* $(\phi^0, \phi^1) \in H \times \Pi$ *and the forcing term* $k \in M^2(0, T; V^*)$*. Then the solution* x *of the equation (4.3.1) belongs to* $x \in L^2(0, T; V)$ *and the mapping*

$$H \times \Pi \times L^2(0, T; V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in L^2(0, T; V) \tag{4.3.22}$$

is Lipschitz continuous.

Proof. From Theorem 4.3.1, it follows that if $(\phi^0, \phi^1, k) \in L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*)$ then x belongs to $M^2(0, T; V)$. Let $(\phi^0_i, \phi^1_i, k_i)$ and x^i be the solution of (4.3.1) with $(\phi^0_i, \phi^1_i, k_i)$ in place of (ϕ^0, ϕ^1, k) for $i = 1, 2$. Let

$x_i (i=1,2) \in \Sigma_r$. Then it holds

$$x^1(t) - x^2(t) = S(t)[(\phi_1^0 - \phi_2^0) + (g(0, x_0^1) - g(0, x_0^2))]$$

$$- (g(t, x_t^1) - g(t, x_t^2)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s)(g(s, x_s^1) - g(t, x_s^2)) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s) \{ \int_{-h}^0 a_1(\tau) A_1 (x^1(s+\tau) - x^2(s+\tau)) d\tau ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s) \{ ((Fx^1)(s) - (Fx^2)(s)) + (k_1(s) - k_2(s)) \} ds.$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s)(k_1(s) - k_2(s)) ds$$

Hence, by applying the same argument as in the proof of Theorem 4.3.1, we have

$$||x_1 - x_2||_{L^2(0,T_1;V)} \leq \hat{N} ||x_1 - x_2||_{L^2(0,T_1;V)} + \hat{N}_2,$$

where

$$\begin{aligned} \hat{N}_2 =& C_1 |\phi_1^0 - \phi_2^0| + C_1 C_{-\alpha} L_g (||\phi_1^1 - \phi_2^1||_\Pi) + C_{-\alpha} L_g \sqrt{T_1} ||\phi_1^1 - \phi_2^1||_\Pi \\ &+ \frac{C_1 (2\alpha)^{-1/2} (2\alpha - 1)^{-1/2} T_1^\alpha}{\Gamma(\alpha)} \\ &\times \big(|\phi_1^0 - \phi_2^0| + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0;V)} + ||k_1 - k_2||_{L^2(0,T_1;V^*)} \big) \\ &+ \frac{C_{1-\beta} L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha + 3\beta - 2\big)^{1/2} \Gamma(\alpha)} ||\phi_1^1 - \phi_2^1||_\Pi \end{aligned}$$

which implies

$$||x||_{M^2(0,T_1;V)} \leq \hat{N}_2 (1 - \hat{N})^{-1}.$$

Therefore, it implies the inequality (4.3.22). □

Corollary 4.3.1. *For a forcing term $k \in L^2(0,T;V^*)$ let x_k be the solution of equation (4.3.1). Let us assume that the embedding $V \subset H$ is compact. Then the mapping $k \mapsto x_k$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$.*

Proof. If $k \in L^2(0,T;V^*)$, then in view of Theorem 4.3.1

$$||x_k||_{\mathcal{W}_1(T)} \le C_3(1 + |g^0| + ||g^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}).$$

Hence if k is bounded in $L^2(0,T;V^*)$, then so is x_k in $L^2(0,T;V)) \cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow L^2(0,T;H)$$

is compact in view of Theorem 2 of J. P. Aubin [32]. □

4.4 example

Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi).$$

Consider the following retarded neutral stochastic differential system in Hilbert space H:

$$\begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t,y) + g(t,x_t(t,y))] = Ax(t,y) + \int_{-h}^0 a_1(s)A_1x(t+s,y)ds \\ \qquad\qquad + f'(|x(t,y)|^2)x(t,y) + k(t,y), \quad (t,y) \in [0,T] \times [0,\pi], \\ x(0,y) = \phi^0(y), \quad x(s,y) = \phi^1(s,y), \quad (s,y) \in [-h,0) \times [0,\pi], \end{cases} \tag{4.4.1}$$

where $h > 0$, $a_1(\cdot)$ is Hölder continuous, and $A_1 \in \mathcal{L}(H)$. Let

$$a(u,v) = \int_0^\pi \frac{du(y)}{dy}\overline{\frac{dv(y)}{dy}}dy.$$

Then

$$A = \partial^2/\partial y^2 \quad \text{with} \quad D(A) = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $z_n(y) = (2/\pi)^{1/2}\sin ny$, respectively. Moreover,

(a1) $\{z_n : n \in N\}$ is an orthogonal basis of H and

$$S(t)x = \sum_{n=1}^{\infty} e^{n^2 t}(x, z_n)z_n, \quad \forall x \in H, \ t > 0.$$

Moreover, there exists a constant M_0 such that $||S(t)||_{\mathcal{L}(H)} \le M_0$.

(a2) Let $0 < \alpha < 1$. Then the fractional power $A^\alpha : D(A^\alpha) \subset H \to H$ of A is given by

$$A^\alpha x = \sum_{n=1}^{\infty} n^{2\alpha}(x, z_n)z_n, \ D(A^\alpha) := \{x : A^\alpha x \in H\}.$$

In particular,

$$A^{-1/2}x = \sum_{n=1}^{\infty} \frac{1}{n}(x, z_n)z_n, \ \text{and} \ ||A^{-1/2}|| = 1.$$

The nonlinear mapping f is a real valued function belong to $C^2([0,\infty))$ which satisfies the conditions

(f1) $f(0) = 0$, $f(r) \ge 0$ for $r > 0$,

(f2) $|f^{'}(r) \le c(r+1)$ and $|qf''(r)| \le c$ for $r \ge 0$ and $c > 0$.

If we present

$$F(t,x(t,y)) = f^{'}(|x(t,y)|^2)x(t,y),$$

Then it is well known that F is a locally Lipschitz continuous mapping from the whole V into H by Sobolev's imbedding theorem (see [20, Theorem 6.1.6]). As an example of q in the above, we can choose $q(r) = \mu^2 r + \eta^2 r^2/2$ (μ and η is constants).

Define $g : [0,T] \times \Pi \to H$ as

$$g(t,x_t) = \sum_{n=1}^{\infty} \int_0^t e^{n^2 t} (\int_{-h}^0 a_2(s)x(t+s)ds, z_n)z_n, \quad ,t > 0.$$

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t,x_t) = (S(t) - I)\int_{-h}^0 a_2(s)x(t+s)ds,$$

where I is the identity operator form H to itself and

$$|a_2(0)| \le H_2, \quad |a_2(s) - a_2(\tau)| \le H_2(s-\tau)^\kappa, \quad s,\ \tau \in [-h,0]$$

for a constant $\kappa > 0$. Hence we have

$$\begin{aligned}|Ag(t,x_t)| \le &(M_0+1)\big\{\big|\int_{-h}^0 (a_2(s) - a_2(0))x(t+s)d\tau\big| + \big|\int_{-h}^0 a_2(0)x(t+s)d\tau\big|\big\} \\ \le &(M_0+1)H_2\big\{(2\kappa+1)^{-1}h^{2\rho+1} + h\big\}||x_t||_\Pi.\end{aligned}$$

It is immediately seen that Assumption (G) has been satisfied. Thus, all the conditions stated in Theorem 4.3.1 have been satisfied for the equation (4.4.1), and so there exists a solution of (4.4.1) belongs to $\mathcal{W}_1(T) = L^2(0,T;V)) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H)$.

Chapter 5
Control problems for semilinear neutral differential equations in Hilbert spaces

5.1 Introduction

Let H and V be real Hilbert spaces such that V is a dense subspace in H. Let U be a Banach space of control variables. In this paper, we are concerned with the global existence of solution and the approximate controllability for the following abstract neutral functional differential system in a Hilbert space H:

$$\begin{cases} \frac{d}{dt}[(x(t) + (Bx)(t)] = Ax(t) + f(t, x(t)) + (Cu)(t), \quad t \in (0, T], \\ x(0) = x_0, \quad (Bx)(0) = y_0, \end{cases} \tag{CE}$$

where A is an operator associated with a sesquilinear form on $V \times V$ satisfying Gårding's inequality, f is a nonlinear mapping of $[0, T] \times V$ into H satisfying the local Lipschitz continuity, $B : L^2(0, T; V) \to L^2(0, T; H)$ and $C : L^2(0, T; U) \to L^2(0, T; H)$ are appropriate bounded linear mapping.

This kind of equations arises in population dynamics, in heat conduction in material with memory and in control systems with hereditary feed back control governed by an integro-differential law.

Recently, the existence of solutions for mild solutions for neutral differential equations with state-dependence delay has been studied in the literature in [11, 12]. As for partial neutral integro-differential equations, we refer to [58-61]. The controllability for neutral equations has been studied by many authors, for example, local controllability of neutral functional differential systems with unbounded delay in [68], neutral evolution integrodifferential systems with state dependent delay in [69, 7] , impulsive neutral functional evolution integrodifferential systems with infinite delay in [6], and second order neutral impulsive integrodifferential systems in [8, 14]. However there are few papers treating the regularity and controllability for the systems with local Lipschipz continuity, we can just find a recent article Wang [44] in case semilinear systems. Similar considerations of semilinear systems have been dealt with in many references [40],[67-69] .

In this paper, we propose a different approach of the earlier works (briefly introduced in [41,42],[58-61] about the mild solutions of neutral differential equations). Our approach is that results of the linear cases of Di Blasio et al. [17] and semilinear cases of [31] on the L^2-regularity remain valid under the above formulation of the neutral differential equation (CE). For the basic of our study, the existence of local solutions of (CE) are established in $L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H)$ for some $T > 0$ by using fractional power of operators and Sadvoskii's fixed point theorem. Thereafter, by showing some variational of constant formula of solutions, we will obtain the global existence of solutions of (CE), and the norm estimate of a solution of (CE) on the solution space. Consequently, in view of the properties of the nonlinear term, we can take advantage of the fact that the solution mapping $u \in L^2(0,T;U) \mapsto x$ is Lipschitz continuous, which is applicable for control problems and the optimal control problem of systems governed by nonlinear properties.

The second purpose of this paper is to study the approximate controllability for the neutral equation (CE) based on the regularity for (CE), namely that the reachable set of trajectories is a dense subset of H. This kind of equations arise naturally in biology, in physics, control engineering problem, etc.

The paper is organized as follows. In section 2, we introduce some notations. In section 3, the regularity results of general linear evolution equations besides fractional power of operators and some relations of operator spaces are stated. In section 4, we will obtain the regularity for neutral functional differential (CE) with nonlinear terms satisfying local Lipschitz continuity. The approach used here is similar to that developed in [31, 44] on the general semilnear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations. Thereafter, we investigate the approximate controllability for the problem (CE) in Section 5. Our purpose in this paper is to obtain the existence of solutions and the approximate controllability for neutral functional differential control systems without using many of the strong restrictions considering in the previous literature.

Finally, we give a simple example to which our main result can be applied.

5.2 Regularity for linear equations

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality with $c_1 = 0$ in (4.2.2)

$$\text{Re } a(u,u) \geq c_0||u||^2, \quad c_0 > 0. \tag{5.2.1}$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u,\ v \in V.$$

Then the operator A is mentioned in Section 2 of Chapter 4.

Lemma 5.2.1. *Let $S(t)$ be the semigroup generated by $-A$. Then there exists a constant M such that*

$$|S(t)| \leq M, \quad ||s(t)||_* \leq M.$$

For all $t > 0$ and every $x \in H$ or V^* there exists a constant $M > 0$ such that the following inequalities hold:

$$|S(t)x| \leq Mt^{-1/2}||x||_*, \quad ||S(t)x|| \leq Mt^{-1/2}|x|.$$

By virtue of (5.2.1), we have that $0 \in \rho(A)$ and the closed half plane $\{\lambda : \text{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A. In this case, there exists a neighborhood U of 0 such that

$$\rho(A) \supset \{\lambda : |\arg\lambda| > \omega\} \cup U.$$

Hence, we can choose a the path Γ runs in the resolvent set of A from $\infty e^{i\theta}$ to $\infty e^{-i\theta}$, $\omega < \theta < \pi$, avoiding the negative axis. For each $\alpha > 0$, we put

$$A^{-\alpha} = \frac{1}{2\pi i} \int_\Gamma \lambda^{-\alpha}(A - \lambda)^{-1} d\lambda,$$

where $\lambda^{-\alpha}$ is chosen to be for $\lambda > 0$. By assumption, $A^{-\alpha}$ is a bounded operator. So we can assume that there is a constant $M_0 > 0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)} \leq M_0, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)} \leq M_0. \tag{5.2.2}$$

For each $\alpha \geq 0$, we define an operator A^α as follows:

$$A^\alpha = \begin{cases} (A^{-\alpha})^{-1} & \text{for } \alpha > 0, \\ I & \text{for } \alpha = 0. \end{cases}$$

The subspace $D(A^\alpha)$ is dense in H and the expression

$$||x||_\alpha = ||A^\alpha x||, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$.

Lemma 5.2.2. *(a) A^α is a closed operator with its domain dense.*

(b) If $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$.

(c) For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$:

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \leq \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(H,V)} \leq \frac{C_\alpha}{t^{3\alpha/2}}. \tag{5.2.3}$$

Proof. From [20, Lemma 3.6.2] it follows that there exists a positive constant C such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:

$$|AS(t)x| \le \frac{C}{t}|x|, \quad ||AS(t)x|| \le \frac{C}{t^{3/2}}\,|x|,$$

which implies (5.2.3) by properties of fractional power of A. For more details about the above lemma, we refer to [20, 3]. □

First of all, consider the following linear system

$$\begin{cases} x'(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{5.2.4}$$

By virtue of Theorem 3.3 of [38](or Theorem 3.1 of [33], [20]), we have the following result on the corresponding linear equation of (5.2.4).

Lemma 5.2.3. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*

1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 5.2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (5.2.4) belonging to $\mathcal{W}(T) \subset C([0,T];V)$ and satisfying

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{5.2.5}$$

where C_1 is a constant depending on T.

2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (5.2.4) belonging to $\mathcal{W}_1(T) \subset C([0,T];H)$ and satisfying

$$||x||_{\mathcal{W}_1(T)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{5.2.6}$$

where C_1 is a constant depending on T.

Lemma 5.2.4. *For every $k \in L^2(0,T;H)$, let $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{5.2.7}$$

Proof. By (5.2.5) we have

$$||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}. \tag{5.2.8}$$

Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \le M\int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M\int_0^T t\int_0^t |k(s)|^2 ds dt \le M\frac{T^2}{2}\int_0^T |k(s)|^2 ds \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}. \tag{5.2.9}$$

From (4.2.3), (5.2.8), and (5.2.9) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, the proof is completed. □

5.3 Semilinear differential equations

Consider the following abstract neutral functional differential system:

$$\begin{cases} \frac{d}{dt}[(x(t)+(Bx)(t)] = Ax(t) + f(t, x(t)) + k(t), \quad t \in (0, T], \\ x(0) = x_0, \quad (Bx)(0) = y_0. \end{cases} \tag{5.3.1}$$

Then we will show that the initial value problem (5.3.1) has a solution by solving the integral equation:

$$\begin{aligned} x(t) =& S(t)[x_0 + y_0] - (Bx)(t) \\ &+ \int_0^t AS(t-s)Bx(s)ds + \int_0^t S(t-s)\{f(s, x(s)) + k(s)\}ds. \end{aligned} \tag{5.3.2}$$

Now we give the basic assumptions on the system (5.3.1)

Assumption (B). Let $B : L^2(0, T; V) \to L^2(0, T; H)$ be a bounded linear mapping such that there exist constants $\beta > 1/3$, $L > 0$, and a continuous nondecreasing function $b(t) : [0, T] \to \mathbb{R}$ with $b(0) = 0$ such that

$$||A^\beta Bx||_{L^2(0,t;H)} \leq b(t)||x||_{L^2(0,t;V)}, \quad \forall (t, x) \in (0, T] \times L^2(0, T; V).$$

Assumption (F2). f is a nonlinear mapping of $[0, T] \times V$ into H satisfying following:

(i) There exists a function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|f(t, x) - f(t, y)| \leq L_1(r)||x - y||, \quad t \in [0, T]$$

hold for $||x|| \leq r$ and $||y|| \leq r$.

(ii) The inequality

$$|f(t,x)| \le L_1(r)(||x||+1)$$

holds for every $t \in [0,T]$ and $x \in V$.

Let us rewrite $(Fx)(t) = f(t,x(t))$ for each $x \in L^2(0,T;V)$. Then there is a constant, denoted again by $L_1(r)$ such that

$$||Fx||_{L^2(0,T;H)} \le L_1(r)(||x||_{L^2(0,T;V)}+1),$$
$$||Fx_1 - Fx_2||_{L^2(0,T;H)} \le L_1(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x \in L^2(0,T;V)$ and $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0,T;V) : ||x||_{L^2(0,T;V)} \le r\}$.

One of the main useful tools in the proof of existence theorems for functional equations is Sadvoskii's fixed point theorem of Lemma 4.3.2 of Chapter 4.

From now on, we establish the following results on the solvability of the equation (5.3.1).

Theorem 5.3.1. *Let Assumptions (B) and (F2) be satisfied. Assume that* $x_0 \in H$, $k \in L^2(0,T;V^*)$ *for* $T > 0$. *Then, there exists a solution* x *of the equation (5.3.1) such that*

$$x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, there is a constant C_3 *independent of* x_0 *and the forcing term* k *such that*

$$||x||_{\mathcal{W}_1(T)} \le C_3(1+|x_0|+||k||_{L^2(0,T;V^*)}). \tag{5.3.3}$$

Proof. Let

$$r_0 = 2C_1|x_0 + y_0|,$$

where C_1 is constant in Lemma 5.2.4. Let $\beta > 1/3$, choose $0 < T_1 < T$ such that

$$T_1^{3\beta/2}\big[\{C_2L_1(r_0)(r_0+1) + C_2||k||_{L^2(0,T_1;V)}\} + 2r_0b(T_1)C_{1-\beta}(3\beta)^{-1/2}(3\beta-2)^{-1}\big] \tag{5.3.4}$$

$$+ r_0M_0b(T_1) \le C_1|x_0 + y_0|,$$

where C_2 is constant in Lemma 5.2.5. Let

$$\hat{M} \equiv T_1^{3\beta/2}\{C_2L_1(r_0) + 2(3\beta)^{-1/2}(3\beta-2)^{-1}C_{1-\beta}b(T_1)\} + M_0b(T_1) < 1. \tag{5.3.5}$$

Define a mapping $J : L^2(0,T_1;V) \to L^2(0,T_1;V)$ as

$$(Jx)(t) = S(t)(x_0 + y_0) - (Bx)(t)$$
$$+ \int_0^t AS(t-s)(Bx)(s)ds + \int_0^t S(t-s)\{f(s,x(s)) + k(s)\}ds.$$

It will be shown that the operator J has a fixed point in the space $L^2(0,T_1;V)$. By assumptions (B) and (F2), it is easily seen that J is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma = \{x \in L^2(0,T_1;V) : ||x||_{L^2(0,T_1;V)} \le r_0,\ x(0) = x_0\},$$

which is a bounded closed subset of $L^2(0,T_1;V)$. From (5.2.5) it follows

$$||S(\cdot)(x_0 + y_0)||_{L^2(0,T_1;V)} \le C_1|x_0 + y_0|. \tag{5.3.6}$$

By (5.2.2), (4.2.11) and Assumption (B) we have

$$||Bx||_{L^2(0,T_1;V)} = ||A^{-\beta}A^{\beta}Bx||_{L^2(0,T_1;V)} \tag{5.3.7}$$

$$\leq ||A^{-\beta}||_{\mathcal{L}(H,V)}||A^{\beta}Bx||_{L^2(0,T_1;H)} \leq r_0M_0b(T_1).$$

By virtue of (5.2.7) in Lemma 5.2.4, for $0 < t < T_1$, it holds

$$||\int_0^t S(t-s)\{f(s,x(s)) + k(s)\}ds||_{L^2(0,T_1;V)} \leq C_2\sqrt{T_1}||Fx + k||_{L^2(0,T_1;H)} \tag{5.3.8}$$

$$\leq C_2\sqrt{T_1}\{L_1(r_0)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V)}\}$$

$$\leq C_2\sqrt{T_1}\{L_1(r_0)(r_0 + 1) + ||k||_{L^2(0,T_1;V)}\}.$$

Since (5.2.3) and Assumption (B) the following inequality holds:

$$||AS(t-s)Bx(s)|| = ||A^{1-\beta}S(t-s)A^{\beta}Bx(s)|| \leq \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}r_0b(T_1).$$

Let

$$(Wx)(t) = \int_0^t AS(t-s)Bx(s)ds.$$

Then there holds

$$||Wx||_{L^2(0,T_1;V)} = \big[\int_0^{T_1} ||\int_0^t AS(t-s)Bx(s)ds||^2 dt\big]^{1/2} \tag{5.3.9}$$

$$\leq \big[\int_0^{T_1} \big(\int_0^t \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}r_0b(T_1)ds\big)^2 dt\big]^{1/2}$$

$$\leq 2r_0b(T_1)C_{1-\beta}(3\beta-2)^{-1}\big(\int_0^{T_1} t^{3\beta-1}dt\big)^{1/2}$$

$$= 2r_0b(T_1)C_{1-\beta}(3\beta)^{-1/2}(3\beta-2)^{-1}T_1^{3\beta/2}.$$

Therefore, from (5.3.4), (5.3.6)-(5.3.9) it follows that

$$||Jx||_{L^2(0,T_1;V)} \leq C_1|x_0 + y_0| + r_0 M_0 b(T_1)$$

$$+ T_1^{3\beta/2}\big[\{C_2 L_1(r_0)(r_0 + 1) + C_2||k||_{L^2(0,T_1;V)}\} + 2(3\beta)^{-1/2}(3\beta - 2)^{-1} r_0 b(T_1) C_{1-\beta}\big]$$

$$\leq r_0,$$

and hence J maps Σ into Σ.

Define mapping $K_1 + K_2$ on $L^2(0, T_1; V)$ by the formula

$$(Jx)(t) = (K_1 x)(t) + (K_2 x)(t),$$

$$(K_1 x)(t) = -(Bx)(t)$$

$$(K_2 x)(t) = S(t)(x_0 + y_0) + \int_0^t AS(t-s)(Bx)(s)ds$$

$$+ \int_0^t S(t-s)\{f(s, x(s)) + k(s)\}ds.$$

We can now employ Lemma 5.3.1 with Σ. Assume that a sequence $\{x_n\}$ of $L^2(0, T_1; V)$ converges weakly to an element $x_\infty \in L^2(0, T_1; V)$, i.e., $w - \lim_{n\to\infty} x_n = x_\infty$. Then we will show that

$$\lim_{n\to\infty} ||K_1 x_n - K_1 x_\infty|| = 0, \tag{5.3.10}$$

which is equivalent to the completely continuity of K_1 since $L^2(0, T_1; V)$ is reflexive. For a fixed $t \in [0, T_1]$, let $x_t^*(x) = (K_1 x)(t)$ for every $x \in L^2(0, T_1; V)$. Then $x_t^* \in L^2(0, T_1; V^*)$ and we have $\lim_{n\to\infty} x_t^*(x_n) = x_t^*(x_\infty)$ since $w - \lim_{n\to\infty} x_n = x_\infty$. Hence,

$$\lim_{n\to\infty} (K_1 x_n)(t) = (K_1 x_\infty)(t), \quad t \in [0, T_1]. \tag{5.3.11}$$

By (5.2.3), (4.2.11) and Assumption (B) we have

$$||(K_1x)(t)|| = ||(Bx)(t)|| = ||A^{-\beta}A^{\beta}Bx(t)|| \le ||A^{-\beta}||_{\mathcal{L}(H,V)}||A^{\beta}Bx||_{L^2(0,T_1;H)} \le \infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty}\int_0^{T_1}||(K_1x_n)(t)||^2dt = \int_0^{T_1}||(K_1x_\infty)(t)||^2dt,$$

i.e., $\lim_{n\to\infty}||K_1x_n||_{L^2(0,T_1;V)} = ||K_1x_\infty||_{L^2(0,T_1;V)}$. Since $L^2(0,T_1;V)$ is a Hilbert space, it holds (5.3.10). Next, we prove that K_2 is a contraction mapping on Σ. Indeed, for every x_1 and $x_2 \in \Sigma$, we have

$$\begin{aligned}(K_2x_1)(t) - (K_2x_2)(t) = &\int_0^t AS(t-s)\{(Bx_1)(s) - (Bx_2)(t)\}ds \\ &+ \int_0^t S(t-s)\{f(s,x_1(s)) - f(s,x_2(s))\}ds.\end{aligned}$$

By similar to (5.3.8) and (5.3.9), we have

$$\begin{aligned}&||K_2x_1 - K_2x_2||_{L^2(0,T_1;V)} \\ &\le T_1^{3\beta/2}\big\{C_2L_1(r_0) + 2(3\beta)^{-1/2}(3\beta-2)^{-1}C_{1-\beta}b(T_1)\big\}||x_1 - x_2||_{L^2(0,T_1;V)}.\end{aligned}$$

So by virtue of the condition (5.3.5) the contraction mapping principle gives that the solution of (5.3.1) exists uniquely in $[0,T_1]$.

So by virtue of the condition (5.3.5), K_2 is contractive. Thus, Lemma 5.3.1 gives that the equation of (5.3.1) has a solution in $\mathcal{W}_1(T_1)$.

From now on we establish a variation of constant formula (5.3.3) of solution of (5.3.1). Let x be a solution of (5.3.1) and $x_0 \in H$. Then we have that from (5.3.6)-(5.3.9) it follows that

$$
\begin{aligned}
||x||_{L^2(0,T_1;V)} \leq & C_1|x_0+y_0| + M_0 b(T_1)||x||_{L^2(0,T_1;V)} \\
& + T_1^{3\beta/2}\big[\{C_2 L_1(r_0)(||x||_{L^2(0,T_1;V^*)}+1) + C_2||k||_{L^2(0,T_1;V^*)}\} \\
& + 2(3\beta)^{-1/2}(3\beta-2)^{-1}C_{1-\beta}b(T_1)||x||_{L^2(0,T_1;V)}\big].
\end{aligned}
$$

Taking into account (5.3.5) there exists a constant C_3 such that

$$
\begin{aligned}
||x||_{L^2(0,T_1;V)} \leq & (1-\hat{M})^{-1}\big[C_1|x_0+y_0| + r_0 M_0 b(T_1) \\
& + T_1^{3\beta/2}\{C_2 L_1(r_0) + C_2||k||_{L^2(0,T_1;V^*)}\}\big] \\
\leq & C_3(1+|x_0|+||k||_{L^2(0,T_1;V^*)})
\end{aligned}
$$

which obtain the inequality (5.3.3). Since the conditions (5.3.4) and (5.3.5) are independent of initial value and by (4.2.11)

$$
|x(T_1)| \leq ||x||_{C([0,T_1;H])} \leq M_1||x||_{\mathcal{W}_1(T)},
$$

by repeating the above process, the solution can be extended to the interval $[0,T]$. □

Corollary 5.3.1. *If $M_0 b(T_1) < 1$ then the uniqueness of the solution solution of (5.3.1) in $\mathcal{W}_1(T)$ is obtained.*

Proof. Let $M_0 L < 1$. Then instead of the condition (5.3.5), we can choose T_1 such that

$$
M_0 b(T_1) + T_1^{3\beta/2}\big\{C_2 L_1(r_0) + 2(3\beta)^{-1/2}(3\beta-2)^{-1}C_{1-\beta}b(T_1)\big\} < 1. \quad (5.3.12)
$$

For every x_1 and $x_2 \in \Sigma$, we have

$$(Jx_1)(t) - (Jx_2)(t) = (Bx_2)(t) - (Bx_1)(t) + \int_0^t AS(t-s)\{Bx_1(s) - Bx_2(t)\}ds$$

$$+ \int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s))\}ds.$$

By similar to (5.3.8) and (5.3.9), we have

$||Jx_1 - Jx_2||_{L^2(0,T_1;V)}$

$\leq \big[M_0 b(T_1) + T_1^{3\beta/2}\{C_2 L_1(r_0) + 2(3\beta)^{-1/2}(3\beta-2)^{-1}C_{1-\beta}b(T_1)\}\big]||x_1 - x_2||_{L^2(0,T_1;V)}.$

So by virtue of the condition (5.3.12) the contraction mapping principle gives that the solution of (5.3.1) exists uniquely in $[0, T_1]$. □

Remark 5.3.1. *Let Assumptions (B) and (F2) be satisfied and* $(x_0, k) \in D(A) \times L^2(0,T;H)$. *Then by the argument of the proof of Theorem 5.3.1 term by term, we also obtain that there exists a solution* x *of (5.3.1) such that*

$$x \in \mathcal{W}(T) \equiv L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V).$$

Moreover, there exists a constant C_3 *such that*

$$||x||_{\mathcal{W}(T)} \leq C_3(1 + ||x_0|| + ||k||_{L^2(0,T;H)}),$$

where C_3 *is a constant depending on* T.

The following inequality is refereed to as the Young inequality.

Lemma 5.3.1. *(Young inequality) Let* $a > 0,\ b > 0$ *and* $1/p + 1/q = 1$ *where* $1 \le p < \infty$ *and* $1 < q < \infty$. *Then for every* $\lambda > 0$ *one has*

$$ab \le \frac{\lambda^p a^p}{p} + \frac{b^q}{\lambda^q q}\ .$$

From the following result, we obtain that the solution mapping is continuous, which is useful for physical applications of the given equation.

Theorem 5.3.2. *Let Assumptions (B) and (F2) be satisfied and* $(x_0, y_0, k) \in H \times H \times L^2(0, T; V^*)$. *Then the solution* x *of the equation (5.3.1) belongs to* $x \in \mathcal{W}_1(T) \equiv L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ *and the mapping*

$$H \times H \times L^2(0, T; V^*) \ni (x_0, y_0, k) \mapsto x \in \mathcal{W}_1(T)$$

is continuous.

Proof. From Theorem 5.3.1, it follows that if $(x_0, k) \in H \times L^2(0, T; V^*)$ then x belongs to $\mathcal{W}_1(T)$. Let $(x_{0i}, y_{0i}, k_i) \in H \times H \times L^2(0, T; V^*)$ and $x_i \in \mathcal{W}_1(T)$ be the solution of (5.3.1) with (x_{0i}, y_{0i}, k_i) in place of (x_0, y_0, k) for $i = 1,\ 2$. Let $x_i (i = 1, 2) \in \varSigma$. Then as seen in Theorem 5.3.1, it holds

$$\begin{cases} \frac{d}{dt}[x_1(t) - x_2(t) + (Bx_1)(t) - (Bx_2)(t)] = A(x_1(t) - x_2(t)) \\ +f(t, x_1(t)) - f(t, x_2(t)) + k_1(t) - k_2(t), \\ x_1(0) - x_2(0) = x_{01} - x_{02}. \end{cases}$$

So the solution of the above equation is represented by

$$
\begin{aligned}
x_1(t) - x_2(t) =& S(t)\{(x_{01} - x_{02}) + (y_{01} - y_{02})\} + (Bx_2)(t) - (Bx_1)(t) \\
&+ \int_0^t AS(t-s)\{(Bx_1)(t) - (Bx_2)(t)\} ds \\
&+ \int_0^t S(t-s)\{f(s, x_1(t)) - f(s, x_2(s) + k_1(s) - k_2(s)\} ds,
\end{aligned}
$$

and hence, it holds

$$
\begin{aligned}
||x_1 - x_2||_{L^2(0,T_1;V)} \le& C_1(|x_{01} - x_{02}| + |y_{01} - y_{02}|) + C_2 T_1^{3\beta/2} ||k_1 - k_2||_{L^2(0,T_1;V^*)} \\
&+ T_1^{3\beta/2}\big\{M_0 L + C_2 L_1(r) + 2(3\beta)^{-1/2}(3\beta-2)^{-1} b(T_1) C_{1-\beta}\big\}\big] ||x_1 - x_2||_{L^2(0,T_1;V)}.
\end{aligned}
$$

From (5.3.4), we have

$$
\begin{aligned}
|||x_1 - x_2||_{L^2(0,T_1;V)} \le& (1 - \hat{M})^{-1}\big(C_1(|x_{01} - x_{02}| + |y_{01} - y_{02}|) \\
&+ C_2 T_1^{3\beta/2} ||k_1 - k_2||_{L^2(0,T_1;V^*)}\big).
\end{aligned}
$$

Hence, repeating this process as seen in Theorem 5.3.1, we conclude that the solution mapping is continuous. □

For $k \in L^2(0,T;V^*)$ let x_k be the solution of equation (5.3.1) with k instead of Bu.

Theorem 5.3.3. *Let us assume that the embedding $V \subset H$ is compact. For $k \in L^2(0,T;V^*)$ let x_k be the solution of equation (5.3.1). Then the mapping $k \mapsto x_k$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$. Moreover, if we define the operator $\mathcal{F}$ by*

$$\mathcal{F}(k) = f(\cdot, x_k),$$

then $\mathcal{F}$ is also a compact mapping from $L^2(0,T;V^*)$ to $L^2(0,T;H)$.

Proof. If $(x_0, k) \in H \times L^2(0,T;V^*)$, then in view of Theorem 5.3.1

$$||y_k||_{\mathcal{W}_1(T)} \leq C_2(|x_0| + ||k||_{L^2(0,T;V^*)}).$$

Since $x_k \in L^2(0,T;V)$, we have $f(\cdot, x_k) \in L^2(0,T;H)$. Consequently, by (4.2.11), we know $x_k \in \mathcal{W}_1(T) \subset C([0,T];H)$. With aid of 1) of Lemma 5.2.3, noting that $||x_k||_{L^2(0,T;V)} \leq ||x_k||_{\mathcal{W}_1(T)}$, we have

$$||x_k||_{\mathcal{W}_1(T)} \leq C_3(1 + |x_0| + ||k||_{L^2(0,T;V^*)}\}$$

Hence if k is bounded in $L^2(0,T;V^*)$, then so is x_k in $\mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding

$$\mathcal{W}_1(T) \subset L^2(0,T;H)$$

is compact in view of Theorem 2 of Aubin [32]. Hence $k \mapsto x_k$ is compact from $L^2(0,T;V^*)$. Moreover, it is immediately that $\mathcal{F}$ is a compact mapping of

$$L^2(0,T;V^*) \hookrightarrow \mathcal{W}_1(T) \hookrightarrow L^2(0,T;H),$$

which is of $L^2(0,T;V^*)$ to $L^2(0,T;H)$. □

5.4 Approximate Controllability

In this section, we show that the controllability of the corresponding linear equation is extended to the nonlinear differential equation. Let U be a Banach space of control variables. Here C is a linear bounded operator from

$L^2(0,T;U)$ to $L^2(0,T;H)$, which is called a controller. For $x \in L^2(0,T;H)$ we set

$$(Bx)(t) = \int_0^t N(t-s)x(s)ds,$$

where $N : [0,\infty) \to \mathcal{L}(H,V)$ is strongly continuous. Then it is immediately seen that $Bx \in C([0,T];V)$ and hence $AS(s)(Bx)(s) = AS(s)(Bx)(s)$ for $0 \le s \le T$ because $D(A) = V$. Since $t \to N(t)$ is strong continuous, by the uniform boundedness principle there exists a constant M_N such that for any $T > 0$,

$$\sup_{t\in[0,T]} ||AN(t)||_{\mathcal{L}(H,V^*)} \le M_N.$$

Consider the following neutral control equation

$$\begin{cases} \frac{d}{dt}[x(t) + (Bx)(t)] = Ax(t) + f(t,x(t)) + (Cu)(t), \quad t \in (0,T], \\ x(0) = x_0, \quad (Bx)(0) = y_0. \end{cases} \tag{5.4.1}$$

Let $x(T;B,f,u)$ be a state value of the system (5.4.1) at time T corresponding to the operator B, the nonlinear term f and the control u. We note that $S(\cdot)$ is the analytic semigroup generated by $-A$. Then the solution $x(t;B,f,u)$ can be written as

$$\begin{aligned} x(t;B,f,u) =& S(t)(x_0+y_0) - (Bx)(t) \qquad (5.4.2)\\ &+ \int_0^t S(t-s)\{A(Bx)(s)ds + f(s,x(s)) + (Cu)(s)\}ds. \end{aligned}$$

and in view of Theorem 5.3.1,

$$||x(\cdot;B,f,u)||_{\mathcal{W}_1(T)} \le C_3(|x_0| + ||C||_{\mathcal{L}(U,H)}||u||_{L^2(0,T;U)}). \tag{5.4.3}$$

We define the reachable sets for the system (5.3.1) as follows:

$$R(T) = \{x(T; B, f, u) : u \in L^2(0, T; U)\},$$

$$L(T) = \{x(T; 0, 0, u) : u \in L^2(0, T; U)\}.$$

Definition 5.4.1. *The system (5.4.1) is said to be approximately controllable on* $[0, T]$ *if for every desired final state* $z_T \in H$ *and* $\epsilon > 0$ *there exists a control function* $u \in L^2(0, T; U)$ *such that the solution* $x(T; B, f, u)$ *of (5.4.1) satisfies* $|x(T; f, u) - z_T| < \epsilon$*, that is,* $\overline{R_T(f)} = H$ *where* $\overline{R(T)}$ *is the closure of* $R(T)$ *in* H*.*

We define the linear operator $\hat{S}$ from $L^2(0, T; H)$ to H by

$$\hat{S}p = \int_0^T S(T - s)p(s)ds$$

for $p \in L^2(0, T; H)$.

We need the following hypothesis:

Assumption (S).

(i) For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$\begin{cases} |\hat{S}p - \hat{S}Cu| < \varepsilon, \\ ||Cu||_{L^2(0,t;H)} \leq q_1 ||p||_{L^2(0,t;H)}, \quad 0 \leq t \leq T. \end{cases}$$

where q_1 is a constant independent of p.

(ii) f is a nonlinear mapping of $[0, T] \times H$ into H satisfying following:

There exists a function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|f(t, x) - f(t, y)| \le L_1(r)|x - y|, \quad t \in [0, T]$$

hold for $|x| \le r$ and $|y| \le r$.

By virtue of the condition (i) of Assumption (S) we note that $AS(t - s)Bx = S(t - s)ABx$ for each $x \in V$. Therefore, the system (5.4.1) is approximately controllable on $[0, T]$ if for any $\varepsilon > 0$ and $z_T \in H$ there exists a control $u \in L^2(0, T; U)$ such that

$$||S(T)(x_0 + y_0) - (Bx)(T) + \hat{S}\{ABx + Fx + Cu\} - z_T|| < \varepsilon,$$

where $(Fx)(t) = f(t, x(t))$ for $t \ge 0$. Throughout this section, Invoking (5.4.3), we can choose a constant r_1 such that

$$r_1 > C_3(|x_0| + ||C||_{\mathcal{L}(U,H)}||u||_{L^2(0,T;U)}), \tag{5.4.4}$$

and set

$$G(s, x) = A(Bx)(s) + f(s, x(s)).$$

Lemma 5.4.1. *Let u_1 and u_2 be in $L^2(0, T; U)$. Then under the assumption (S), we have that for $0 \le t \le T$,*

$$|x(t; B, f, u_1) - x(t; B, f, u_2)| \le Me^{M_2}\sqrt{t}||Cu_1 - Cu_2||_{L^2(0,T;H)},$$

where $M_2 = e^{M(M_N T + L_1(r_1))}$

Proof. Let $x_(t) = x(t; B, f, u_1)$ and $x_2(t) = x(t; B, f, u_2)$. Then for $0 \le t \le T$, we have

$$x_1(t) - x_2(t) =(Bx_2)(t) - (Bx_1)(t) + \int_0^t S(t-s)\{G(s, x_1) - G(s, x_2)\}ds \tag{5.4.5}$$

$$+ \int_0^t S(t-s)C(u_1(s) - u_2(s))ds.$$

So we immediately obtain

$$|A(Bx_2)(t) - A(Bx_1)(t)| \le M_N \int_0^t |x_2(s) - x_1(s)|ds,$$

and so it holds

$$|\int_0^t S(t-s)A\{(Bx_2)(s) - (Bx_1)(s)\}ds| \le MM_NT \int_0^t |x_2(s) - x_1(s)|ds$$

Moreover, we have

$$|\int_0^t S(t-s)\{f(s, x_1(s)) - f(s, x_2(s)\}ds| \le ML_1(r_1) \int_0^t |x_2(s) - x_1(s)|ds,$$

$$|\int_0^t S(t-s)\{Cu_1(s) - Cu_2(s)\}ds| \le M\sqrt{t}||Cu_1 - Cu_2||_{L^2(0,T_1;V)}.$$

Thus, from (5.4.5) it follows that

$$|x(t; B, f, u_1) - x(t; B, f, u_2)| \le M\sqrt{t}||Cu_1 - Cu_2||_{L^2(0,T;H)} \tag{5.4.6}$$

$$+ \{MM_NT + ML_1(r_1)\} \int_0^t |x_2(s) - x_1(s)|ds.$$

Therefore, by using Gronwall's inequality this lemma follows. □

Theorem 5.4.1. *Under the assumptions (S), the system (5.4.1) is approximately controllable on* $[0, T]$.

Proof. We will show that $D(A) \subset \overline{R_T(g)}$, i.e., for given $\varepsilon > 0$ and $z_T \in D(A)$ there exists $u \in L^2(0, T; U)$ such that

$$|z_T - x(T; B, f, u)| < \varepsilon,$$

where

$$x(T; B, f, u) = S(T)(x_0+y_0)-(Bx)(T)+\int_0^T S(T-s)\{G(s, x(\cdot; B, f, u))+Cu(s)\}ds.$$

As $z_T \in D(A)$ there exists a $p \in L^2(0, T; Z)$ such that

$$\hat{S}p = z_T + (Bx)(T) - S(T)(x_0 + y_0),$$

for instance, take $p(s) = \{(z_T + (Bx)(T)) - sA(z_T + (Bx)(T))\} - S(s)(x_0 + y_0)/T$. Let $u_1 \in L^2(0, T; U)$ be arbitrary fixed. Since by the assumption (S) there exists $u_2 \in L^2(0, T; U)$ such that

$$|\hat{S}(p - G(\,\cdot\,, x(\,\cdot\,, B, f, u_1))) - \hat{S}Cu_2| < \frac{\varepsilon}{4},$$

it follows

$$|z_T + (Bx)(T) - S(T)(x_0 + y_0) - \hat{S}G(\,\cdot\,, x(\,\cdot\,B, f, u_1)) - \hat{S}Cu_2| < \frac{\varepsilon}{4}. \quad (5.4.7)$$

We can also choose $w_2 \in L^2(0, T; U)$ by the assumption (S) such that

$$|\hat{S}(G(\,\cdot\,x(\,\cdot\,; B, f, u_2)) - G(\,\cdot\,x(\,\cdot\,; B, f, u_1)) - \hat{S}Cw_2| < \frac{\varepsilon}{8} \quad (5.4.8)$$

and by the assumption (S)

$$\|Cw_2||_{L^2(0,t;H)} \le q_1||G(\,\cdot\,, x(\,\cdot\,; B, f, u_1)) - G(\,\cdot\,, x(\,\cdot\,; B, f, u_2))||_{L^2(0,t;H)}$$

for $0 \le t \le T$. Therefore, in view of Lemma 5.4.1 and the assumption (S)

$$\begin{aligned}
||Cw_2||_{L^2(0,t;H)} &\le q_1\{\int_0^t |G(\tau, x(\tau; B, f, u_2)) - G(\tau, x(\tau; B, f, u_1))|^2 d\tau\}^{\frac{1}{2}} \\
&\le q_1(M_N + L(r_1))\{\int_0^t |x(\tau; B, f, u_2) - x(\tau; B, f, u_1)|^2 d\tau\}^{\frac{1}{2}} \\
&\le q_1(M_N + L(r_1))\{\int_0^t (Me^{M_2})^2 \tau ||Cu_2 - Cu_1||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}} \\
&\le q_1(M_N + L(r_1))Me^{M_2}(\int_0^t \tau d\tau)^{\frac{1}{2}} ||Cu_2 - Cu_1||_{L^2(0,t;H)} \\
&= q_1(M_N + L(r_1))Me^{M_2}(\frac{t^2}{2})^{\frac{1}{2}} ||Cu_2 - Cu_1||_{L^2(0,t;H)}.
\end{aligned}$$

Put $u_3 = u_2 - w_2$. We determine w_3 such that

$$|\hat{S}(G(\,\cdot\,, x(\cdot\,; B, f, u_3)) - G(\,\cdot\,, x(\cdot\,; B, f, u_2))) - \hat{S}Cw_3| < \frac{\varepsilon}{8},$$

$$||Cw_3||_{L^2(0,t;H)} \le q_1||G(\,\cdot\,, x(\,\cdot\,; B, f, u_3)) - G(\,\cdot\,, x(\,\cdot\,; B, f, u_2))||_{L^2(0,t;H)}$$

for $0 \le t \le T$. Hence, we have

$$||Cw_3||_{L^2(0,t;H)}$$

$$\le q_1\{\int_0^t |G(\tau, x(\tau; B, f, u_3)) - G(\tau, x(\tau; B, f, u_2))|^2 d\tau\}^{\frac{1}{2}}$$

$$\le q_1(M_N + L(r_1))\{\int_0^t |x(\tau; B, f, u_3) - x(\tau; B, f, u_2)|^2 d\tau\}^{\frac{1}{2}}$$

$$\le q_1(M_N + L(r_1))Me^{M_2}\{\int_0^t \tau||Cu_3 - Cu_2||^2_{L^2(0,\tau:H)} d\tau\}^{\frac{1}{2}}$$

$$\le q_1(M_N + L(r_1))Me^{M_2}\{\int_0^t \tau||Cw_2||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}}$$

$$\le q_1(M_N + L(r_1))Me^{M_2}\{\int_0^t \tau(q_1(M_N + L(r_1))Me^{M_2})^2\frac{\tau^2}{2}||Cu_2 - Cu_1||^2_{L^2(0,\tau;H)} d\tau\}^{\frac{1}{2}}$$

$$\le (q_1(M_N + L(r_1))Me^{M_2})^2(\int_0^t \frac{\tau^3}{2} d\tau)^{\frac{1}{2}}||Cu_2 - Cu_1||_{L^2(0,t;H)}$$

$$= (q_1(M_N + L(r_1))Me^{M_2})^2(\frac{t^4}{2\cdot 4})^{\frac{1}{2}}||Cu_2 - Cu_1||_{L^2(0,t;H)}.$$

By proceeding with this process, and from that

$$||C(u_n - u_{n+1})||_{L^2(0,t;H)} = ||Cw_n||_{L^2(0,t;H)}$$

$$\le (q_1(M_N + L(r_1))Me^{M_2})^{n-1}(\frac{t^{2n-2}}{2\cdot 4\cdots(2n-2)})^{\frac{1}{2}}||Cu_2 - Cu_1||_{L^2(0,t;H)}$$

$$= (\frac{q_1(M_N + L(r_1))Me^{M_2}t}{\sqrt{2}})^{n-1}\frac{1}{\sqrt{(n-1)!}}||Cu_2 - Cu_1||_{L^2(0,t;H)},$$

it follows that

$$\sum_{n=1}^{\infty} ||Cu_{n+1} - Cu_n||_{L^2(0,T;H)}$$

$$\leq \sum_{n=0}^{\infty} (\frac{q_1 T(M_N + L(r_1))Me^{M_2}}{\sqrt{2}})^n \frac{1}{\sqrt{n!}} ||Cu_2 - Cu_1||_{L^2(0,T;H)} < \infty.$$

Therefore, there exists $u^* \in L^2(0,T;H)$ such that

$$\lim_{n\to\infty} Cu_n = u^* \quad \in \quad L^2(0,T;H).$$

From (5.4.7), (5.4.8) it follows that

$$|z_T + (Bx)(T) - S(T)(x_0 + y_0) - \hat{S}G(\cdot, x(\cdot; B, f, u_2)) - \hat{S}Cu_3|$$

$$= \big|z_T + (Bx)(T) - S(T)(x_0 + y_0) - \hat{S}G(\cdot, x(\cdot; B, f, u_1)) - \hat{S}Cu_2 + \hat{S}Cw_2$$

$$- \hat{S}[G(\cdot, x(\cdot; B, f, u_2)) - G(\cdot, x(\cdot; B, f, u_1))]\big|$$

$$< (\frac{1}{2^2} + \frac{1}{2^3})\varepsilon.$$

By choosing $w_n \in L^2(0,T;U)$ by the assumption (B) such that

$$|\hat{S}(G(\,\cdot\, x(\,\cdot\,; B, f, u_n)) - G(\,\cdot\, x(\,\cdot\,; B, f, u_{n-1})) - \hat{S}Cw_n| < \frac{\varepsilon}{2^{n+1}},$$

putting $u_{n+1} = u_n - w_n$, we have

$$|z_T + (Bx)(T) - S(T)(x_0 + y_0) - \hat{S}G(\cdot, x(\cdot; B, f, u_n)) - \hat{S}Cu_{n+1}|$$

$$< (\frac{1}{2^2} + \cdots + \frac{1}{2^{n+1}})\varepsilon, \quad n = 1\ 2,\ \cdots.$$

Therefore, for $\varepsilon > 0$ there exists integer N such that

$$|\hat{S}Cu_{N+1} - \hat{S}Cu_N| < \frac{\varepsilon}{2}$$

and

$$\begin{aligned}
&|z_T + (Bx)(T) - S(T)(x_0 + y_0) - \hat{S}G(\cdot, x(\cdot; B, f, u_N)) - \hat{S}Cu_N| \\
&\leq |z_T + (Bx)(T) - S(T)(x_0 + y_0) - \hat{S}G(\cdot, x(\cdot; B, f, u_N)) - \hat{S}Cu_{N+1}| \\
&\ + |\hat{S}Cu_{N+1} - \hat{S}Cu_N| \\
&< (\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}})\varepsilon + \frac{\varepsilon}{2} \leq \varepsilon.
\end{aligned}$$

Thus the system (5.4.1) is approximately controllable on $[0, T]$ as N tends to infinity. □

5.5 example

Let

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi),$$

$$a(u, v) = \int_0^\pi \frac{du(y)}{dy} \overline{\frac{dv(y)}{dy}} dy$$

and

$$A = \partial^2/\partial y^2 \quad \text{with} \quad D(A) = \{x \in H^2(0, \pi) : x(0) = x(\pi) = 0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(y) = (2/\pi)^{1/2} \sin ny$, respectively. Moreover,

(a) $\{\phi_n : n \in N\}$ is an orthogonal basis of H and

(b) $S(t)x = \sum_{n=1}^{\infty} e^{n^2 t}(x, \phi_n)\phi_n, \quad \forall x \in H,\ t > 0.$

(c) Let $0 < \alpha < 1$. Then the fractional power $A^\alpha : D(A^\alpha) \subset H \to H$ of A is given by

$$A^\alpha x = \sum_{n=1}^{\infty} n^{2\alpha}(x, \phi_n)\phi_n,\ D(A^\alpha) := \{x : A^\alpha x \in H\}.$$

In particular, $A^{-1/2}x = \sum_{n=1}^{\infty} \frac{1}{n}(x, \phi_n)\phi_n$ and $||A^{-1/2}|| = 1$.

Consider the following neutral differential control system:

$$\begin{cases} \partial/\partial t\big[x(t,y) + \int_0^t \int_0^\pi b(t-s,z,y)x(s,z)dzds\big] \\ = Ax(t,y) + g^{'}(|x(t,y)|^2)x(t,y) + (Cu)(t), \quad t \in (0,T], \\ x(t,0) = x(t,\pi_0) = 0, \end{cases} \tag{5.5.1}$$

where g is a real valued function belong to $C^2([0,\infty))$ which satisfies the conditions

(i) $g(0) = 0$, $g(r) \geq 0$ for $r > 0$,

(ii) $|g^{'}(r) \leq c(r+1)$ and $|g^{''}(r)| \leq c$ for $r \geq 0$ and $c > 0$.

If we present

$$f(x(t,y)) = g^{'}(|x(t,y)|^2)x(t,y),$$

f is a mapping from the whole V into H by Sobolev's imbedding theorem (see [20, Theorem 6.1.6]). As an example of g in the above, we can choose $g(r) = \mu^2 r + \eta^2 r^2/2$ (μ and η is constants).

In addition, we need to impose the following conditions(see [68, 70]).

(iii) The function b is measurable and

$$\int_0^\pi \int_0^t \int_0^\pi b^2(t-s,z,y)dzdsdy < \infty.$$

(iv) The function $(\partial^2/\partial z^2)b$ is measurable, $b(0,y,\pi) = b(0,y,0)$, and

$$M_b := \int_0^\pi \int_0^t \int_0^\pi \big(\frac{\partial}{\partial z} b(t-s,z,y)\big)^2 dzdsdy < \infty.$$

(v) $C : L^2(0,T;U) \to L^2(0,T;H)$ is a bounded linear operator.

We define $B : L^2(0,T;V) \to L^2(0,T;H)$ by

$$(Bx)(t) = \int_0^t \int_0^\pi b(t-s,z,y)x(s,z)dyds.$$

From (ii) it follows that B is bounded linear and

$$\begin{aligned} A^{1/2}(Bx)(t) &= \frac{1}{n}\frac{2}{\pi}\big((Bx)(t), \sin ny\big)\phi_n \\ &= \frac{2}{\pi}\big(\int_0^t \int_0^\pi \frac{\partial}{\partial y} b(t-s,z,y)dyds, \cos ny\big)\phi_n \\ &= \frac{2}{\pi}\big((B_1x)(t), \cos ny\big)\phi_n. \end{aligned}$$

where

$$(B_1x)(t) = \int_0^t \int_0^\pi \frac{\partial}{\partial y} b(t-s,z,y)dyds.$$

Thus, by (iv) the operator B_1 is bounded linear with $||B_1|| \le \sqrt{M_b}$ and we have that $B \in D(A^{1/2})$ and $||A^{1/2}Bx|| = ||B_1x||$. Therefore from Theorem 5.3.1, there exists a solution x of the equation (5.5.1) such that

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).$$

Moreover, from Theorem 5.4.1 the neutral system (5.5.1) is approximately controllable on $[0,T]$.

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