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Controllability and regularity for semilinear retarded evolution equations

by

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(준선형 지연 발전 방정식에 대한 제어성과 정칙성)

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요 약

본 논문의 목적은 편미분방정식 중 시간에 의존하는 발전방정식에서 주 작용소가 해석적 반군을 생성하고 과거의 메모리와 데이터를 포함한 지연 함수미분방정식을 수학적으로 해석하고 그 응용으로서 제어이론을 유도하는 데 있다. 이러한 모델은 공학, 경제학, 자연과학에서 현재의 현상이 과거 상태의 파생물 등 과거 정보를 포함하는 유전 시스템에 대하여 비선형계에 대한 독창적인 해석과 정칙성과 가제어성에 대한 이론을 체계적으로 정립하였다. 방정식의 형태와 가제어성의 충분조건을 유도하는 과정은 다음과 같다. 제2장에서는 비연속점을 포함한 준선형 충동적 함수에 대하여 비선형항의 일반적인 립시츠조건으로 해의 정칙성을 유도하였고, 제3장에서는 2계 준선형 충동미분방장식의 주작용소가 cosine족을 생성할 경우 해의 정칙성을 유도하기 위해 기본적인 성질을 규명하고 비선형 항의 조건을 제시하였다. 제4장에서는 p-Laplacian을 포함하는 추상적인 포물선방정식에 대한 해의 규칙성을 찾는다. 포함된 작용소들이 타원형 미분연산자에 의해 생성되는 것으로 해석적 반군을 생성함을 증명하였다. 함수공간에서의 보간이론을 정립하고, ζ-볼록 공간의 성질을 이용하여 작용소의 정의구역과 sovolev공간과의 보간 공간을 해석적 반군에 의한 수식으로 표현하여 해의 존재 가능한 최대영역과 정칙성을 정립하였다. 위의 결과를 바탕으로 해석적 반군의 작용소의 고유치에 대한 일반적인 고유공간들의 특성을 이용하여 공학적으로 중요한 응용으로서 동일성 문제에 대해 충분조건으로 유한차원의 rank condition을 유도하였다. 제5장에서는 Hilbert 공간에서 준선형 적분-미분 제어방정식에 대한 해의 성질을 체계적으로 수식화하여 일반적인 비선형계의 이론인 Degree이론을 이용하여 가제어성의 충분조건을 유도하였다. 제6장에서는 이 논문의 핵심적인 부분으로 비선형 항의 유계성과 homogeneous(동형) 성질을 만족하는 경우 기존의 증명방식을 사용하지 않고 새로운 증명 방법을 사용하여 가제어성을 유도하였고 다른 여러 비선형계에 대한 응용할 수 있는 모티브를 제공하였다.

Chapter 1
Introduction and Preliminaries

The purpose of this paper is to give a systematic presentation of the theory of evolution equation with time delays based on the theory of analytic semigroups of bounded linear operators and its applications to partial functional differential equations. The system with delays means that the future state of given models in engineering, economics and natural sciences depends on only on the present but on the past state and the derivative of the past state. Such models that contain past information are called hereditary systems. In this paper, we obtain a number of criteria for controllability and regularity for various semilinear retarded functional differential control systems with unbounded principal operators and more general conditions of parameters in Hilbert spaces. Throughout this paper, we study a class of abstract retarded equations in some Hilbert spaces.

The paper is organized as follows: In chapter 2, we are concerned with the global existence of solution for the semilinear impulsive system:

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T], \quad t \neq t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0, \end{cases} \tag{2.1.1}$$

where H and V be real Hilbert spaces such that V is a dense subspace in H. Let A be the operator associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying Gårding's inequality:

$$(Au, v) = a(u, v), \quad u, \ v \in V$$

where V is a Hilbert space such that $V \subset H \subset V^*$. Then $-A$ generates an analytic semigroup in both H and V^*(see [76, Theorem 3.6.1]) and so the equation (2.1.1) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0,T] \times V$ to H is assumed to be locally Lipschitz continuous with respect to the second variable, and k is a forcing term. The impulsive condition

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m,$$

is a combination of traditional evolution systems and short term perturbations whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc. We propose a different approach of the earlier works (briefly introduced in [76, 33, 43]) about the mild, strong, and classical solutions of Cauchy problems. Our approach is that results of the linear cases of Di Blasio [16] on the L^2-regularity remain valid under the above formulation of the semilinear problem (1.2). Based on the regularity for (2.1.1), we can apply for the approximate controllability for (2.1.1). Approximate controllability for semilinear control systems can be founded in [4, 10-18]. We note that the contents of this chapter have been published in [39].

In chapter 3, We are concerned with the regularity of the following second-order semilinear impulsive differential system

$$\begin{cases} w''(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w'(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m \end{cases} \tag{3.1.1}$$

in a Banach space X. Here k belongs to $L^2(0,T)$ and $g : [0,T] \times D(A) \to X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuous. In (3.1.1), the principal operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$. The impulsive condition

$$\Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m$$

is combination of traditional evolution systems whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc. We allow implicit arguments about L^2-regularity results for semilinear hyperbolic equations with impulsive condition. These consequences are obtained by showing that results of the linear cases [37, 10] and semilinear case [41] on the L^2-regularity remain valid under the above formulation of (3.1.1). Earlier works prove existence of solution by using Azera Ascoli theorem. But we propose a different approach from that of earlier works to study mild, strong and classical solutions of Cauchy problems by using the properties of the linear equation in the hereditary part, which is seen in [40].

In Chapter 4, we consider the regularity of solutions for an abstract parabolic type equation involving p-Laplacian:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + \mathcal{A}(x,D_x)u(x,t) - \operatorname{div}(|\nabla u(x,t)|^{p-2}\nabla u(x,t)) = f(t), \ (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{cases} \tag{4.1.1}$$

where, Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x,D_x)$ be an elliptic differential operator of second order as follows:

$$\mathcal{A}(x,D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \Omega$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$.

If we put $Au = -\mathcal{A}(x,D_x)u$ then it is known that A generates an analytic semigroup in $L^p(\Omega)(see[1, 67])$. In view of Sobolev's embedding theorem, we remark that $L^p(\Omega) \subset W^{-1,p}(\Omega)$, where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p'}(\Omega)(p' = p/(p-1))$. The space $W^{-1,p}(\Omega)$ is ζ-convex(as for the definition and fundamental facts of a ζ-convex space see [18, 38]). Therefore, from the interpolation theory it is easily seen that the operator A generates an analytic semigroup in both $H_{p,q} \equiv (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q}$ and $W^{-1,p}(\Omega)$. Hence, we can investigate the semilinear form (4.1.1) in the space $W^{-1,p}(\Omega)$ and apply the method of [26] to the system (4.1.1) to show the existence and uniqueness of the solution

$$u \in L^q(0,T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T; W^{-1,p}(\Omega)) \subset C([0,T]; H_{p,q})$$

for any $u_0 \in H_{p,q}$ and $f \in L^q(0,T; W^{-1,p}(\Omega))(p > 2)$. These details are published in [44].

In Chapter 5, we deal with the approximate controllability for semilinear integro-differential functional control equations in the form

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \quad 0 < t \le T, \\ x(0) & = x_0 \end{cases} \tag{5.1.1}$$

in a Hilbert space H, where k belongs to $L^2(0,T)(T > 0)$ and g is a nonlinear mapping as detailed in Section 2. The principal operator A generates an analytic semigroup $(S(t))_{t \ge 0}$ and B is a bounded linear operator from another Hilbert space U to H. We want to use a different method than the previous one. Our used tool is the theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda T(x) - F(x) = y$ in dependence on the real number λ, where T and F are nonlinear operators defined a Banach space X with values in a Banach space Y. In order to obtain the approximate controllability for a class of semilinear integro-differential functional control equations, it is necessary to suppose that T acts as the identity operator while F related to the nonlinear term of (5.1.1) is completely continuous, whose information is detailed in [50].

In Chapter 6, we deal with the approximate controllability for a semilinear control system in the form:

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t,x(t)) + (Bu)(t), \quad 0 < t \le T, \\ x(0) & = x_0. \end{cases} \tag{6.1.1}$$

Let V and H be complex Hilbert spaces forming a Gelgand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

by identifying the antidual of H with H, where V is a Hilbert space densely and continously embedded in H. Here, A is the operator associate with a sesquilinear form satisfying Gårding's inequality as detailed in Section 2. The motivation for the choice of Hilbert spaces setting for System (6.1.1) is the application to L^2-regularity using fact that the principal operator A generates an analytic semigroup $(S(t))_{t\geq 0}$ in both H and V^*(see Jeong,1999; Tanabe, 1979). The controller B is a bounded linear operator from another Hilbert space $L^2(0,T;U)(T>0)$ to $L^2(0,T;U)$. k belongs to $L^2(0,T)$ and f is a nonlinear mapping satisfying Lipschitz continuity.

We want to use a new approach by using the surjectivity theorems similar to the Fredholm alternative for nonlinear operators motivated by the work Kang and Jeong (2019), which is about the solution of nonlinear operator equations $\lambda B(u) - F(u) = f$ provided that $\lambda B(u) - F(u) \neq 0$ for each u. In order to obtain the approximate controllability for System (6.1.1), it is necessary to suppose that B acts as an odd homeomorphism operator while F is odd completely continuous and homogeneous as defined in Section 3. By using this method, the approximate controllability of System (6.1.1) can be given as applicable conditions without restrictions such as the inequality constraints for Lipschitz constant of f or the compactness of $S(t)$. These contents have been dealt with by International Journal of Control [42].

Chapter 2
On semilinear impulsive differential equations with local Lipschitz continuity

2.1 Introduction

In this paper, we are concerned with the global existence of solution and the approximate controllability for the semilinear impulsive control system:

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T], \quad t \neq t_k, \\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{2.1.1}$$

Let H and V be real Hilbert spaces such that V is a dense subspace in H. Let A be the operator associated with a sesquilinear form $a(\cdot, \cdot)$ defined on $V \times V$ satisfying Gårding's inequality:

$$(Au, v) = a(u, v), \quad u, \ v \in V$$

where V is a Hilbert space such that $V \subset H \subset V^*$. Then $-A$ generates an analytic semigroup in both H and V^*(see [76, Theorem 3.6.1]) and so the equation (2.1.1) may be considered as an equation in H as well as in V^*. The nonlinear operator f from $[0, T] \times V$ to H is assumed to be locally Lipschitz continuous with respect to the second variable, and k is a forcing term.

The impulsive condition

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, 2, \cdots, m,$$

is a combination of traditional evolution systems and short term perturbations whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc.

The existence of solutions for a class of semilinear functional differential equations has been studied by many authors. Recently, Kobayashi et el. [51] introduced the notion of semigroups of locally Lipschitz operators which provide us with mild solutions to the Cauchy problem for semilinear evolution equations. The regularity for the semilinear heat equations has been developed as seen in section 4.3.1 of Barbu [10] and [46, 67, 82].

In this paper, we propose a different approach of the earlier works (briefly introduced in [76, 33, 43]) about the mild, strong, and classical solutions of Cauchy problems. Our approach is that results of the linear cases of Di Blasio [16] on the L^2-regularity remain valid under the above formulation of the semilinear problem (1.2). Based on the regularity for (2.1.1), we can apply for the approximate controllability for (2.1.1). Approximate controllability for semilinear control systems can be founded in [4, 10-18].

The paper is organized as follows. In section 2, the results of general linear evolution equations besides notations and assumptions are stated. In section 3, we will obtain that the regularity for parabolic linear equations can also be applicable to (2.1.1) with nonlinear terms satisfying local Lipschitz continuity. The approach used here is similar to that developed in [76, 46] on the general semilnear evolution equations, which is an important role to extend the theory of practical nonlinear partial differential equations. In order to apply control systems, we need some compactness hypothesis. So we make the natural assumption that the embedding $D(A) \subset V$ is compact instead of the compact property of semigroup used in [23, 80]. Then by virtue of the result in Aubin [6], we can take advantage of the fact that the solution mapping $u \in L^2(0, T; U) \mapsto x(T; f, u)$ is compact. Finally we give a simple example to which our main result can be applied.

2.2 Regularity for linear equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l,v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = a(u, v), \quad u,\ v \in V.$$

Then $-A$ is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. From the following inequalities

$$\omega_1 ||u||^2 \le \operatorname{Re} a(u,u) + \omega_2 |u|^2 \le C|Au|\,|u| + \omega_2 |u|^2 \le \max\{C, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.2.3}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.2.4}$$

where each space is dense in the next one which continuous injection.

Lemma 2.2.1. *With the notations (2.2.3), (2.2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* *(Section 1.3.3 of [79]).*

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. For the sake of simplicity we assume that $\omega_2 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \geq 0\}$ is contained in the resolvent set of A.

If X is a Banach space, $L^2(0, T; X)$ is the collection of all strongly measurable square integrable functions from $(0, T)$ into X and $W^{1,2}(0, T; X)$ is the set of all absolutely continuous functions on $[0, T]$ such that their derivative belongs to $L^2(0, T; X)$. $C([0, T]; X)$ will denote the set of all continuously functions from $[0, T]$ into X with the supremum norm. If X and Y are two Banach space, $\mathcal{L}(X, Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X, X)$ is simply written as $\mathcal{L}(X)$. Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}(T) \subset C([0,T];V), \quad \mathcal{W}_1(T) \subset C([0,T];H).$$

Thus, there exists a constant $M_0 > 0$ such that

$$||x||_{C([0,T];V)} \le M_0 ||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \le M_0 ||x||_{\mathcal{W}_1(T)}. \tag{2.2.5}$$

The semigroup generated by $-A$ is denoted by $S(t)$ and there exists a constant M such that

$$|S(t)| \le M, \quad ||s(t)||_* \le M.$$

The following Lemma is from Lemma 3.6.2 of [76].

Lemma 2.2.2. *There exists a constant $M > 0$ such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^*:*

$$|S(t)x| \le Mt^{-1/2}||x||_*, \quad ||S(t)x|| \le Mt^{-1/2}|x|.$$

Lemma 2.2.3. *(a) A^α is a closed operator with its domain dense.*

(b) If $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$.

(c) For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$.

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \le \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(H,V)} \le \frac{C_\alpha}{t^{3\alpha/2}}.$$

Proof. From [1,Lemma 3.6.2] it follows that there exists a positive constant C such that the following inequalities hold for all $t > 0$ and every $x \in H$ or V^* :

$$|AS(t)x| \le \frac{C}{t}|x|, \quad ||AS(t)x|| \le \frac{C}{t^{3/2}}|x|.$$

□

First of all, consider the following linear system

$$\begin{cases} x'(t) + Ax(t) = k(t), \\ x(0) = x_0. \end{cases} \tag{2.2.6}$$

By virtue of Theorem 3.3 of [16](or Theorem 3.1 of [46], [76]), we have the following result on the corresponding linear equation of (2.2.6).

Lemma 2.2.4. *Suppose that the assumptions for the principal operator A stated above are satisfied. Then the following properties hold:*
1) For $x_0 \in V = (D(A), H)_{1/2,2}$(see Lemma 2.2.1) and $k \in L^2(0,T;H)$, $T > 0$, there exists a unique solution x of (2.2.6) belonging to $\mathcal{W}(T) \subset C([0,T];V)$ and satisfying

$$||x||_{\mathcal{W}(T)} \le C_1(||x_0|| + ||k||_{L^2(0,T;H)}), \tag{2.2.7}$$

where C_1 is a constant depending on T.
2) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T > 0$. Then there exists a unique solution x of (2.2.6) belonging to $\mathcal{W}_1(T) \subset C([0,T];H)$ and satisfying

$$||x||_{\mathcal{W}_1(T)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.2.8}$$

where C_1 is a constant depending on T.

Lemma 2.2.5. *Suppose that $k \in L^2(0,T;H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}, \tag{2.2.9}$$

$$||x||_{L^2(0,T;H)} \le C_2T||k||_{L^2(0,T;H)}, \tag{2.2.10}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.2.11}$$

Proof. The assertion (2.2.9) is immediately obtained by (2.2.7). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2 dt \le M\int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M\int_0^T t\int_0^t |k(s)|^2 ds dt \le M\frac{T^2}{2}\int_0^T |k(s)|^2 ds \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M/2}||k||_{L^2(0,T;H)}.$$

From (2.2.3), (2.2.9), and (2.2.10) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1 T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\},$$

the proof is complete. □

2.3 Semilinear differential equations

Let f be a nonlinear mapping from V into H.

Assumption (AF). There exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(r_1) \le L(r_2)$ for $r_1 \le r_2$ and

$$|f(t,x)| \le L(r), \quad |f(t,x) - f(t,y)| \le L(r)||x-y||$$

hold for any $t \in [0,T]$, $||x|| \le r$ and $||y|| \le r$.

Assumption (AI). The functions $I_k : V \to H$ are continuous and there exist positive constants $L(I_k)$ and $\beta \in (1/3, 1]$ such that

$$|A^\beta I_k(x)| \le L(I_k)||x||, \quad |A^\beta I_k(x) - I_k(y)| \le L(I_k)||x-y||, \quad k = 1,2,\cdots,m$$

for each $x, y \in V$, and

$$||x(t_k^-)|| \le K, \quad k = 1,2,\cdots,m.$$

From now on, we establish the following results on the local solvability of the following equation;

$$\begin{cases} x^{'}(t) + Ax(t) = f(t, x(t)) + k(t), \quad t \in (0, T], \quad t \neq t_k, \\ \ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)),\ k = 1, 2, \cdots, m, \\ x(0) = x_0. \end{cases} \tag{2.3.1}$$

Let us rewrite $(Fx)(t) = f(t, x(t))$ for each $x \in L^2(0, T; V)$. Then there is a constant, denoted again by $L(r)$, such that

$$||Fx||_{L^2(0,T;H)} \leq L(r)\sqrt{T}, \quad ||Fx_1 - Fx_2||_{L^2(0,T;H)} \leq L(r)||x_1 - x_2||_{L^2(0,T;V)}$$

hold for $x_1,\ x_2 \in B_r(T) = \{x \in L^2(0, T; V) : ||x||_{L^2(0,T;V)} \leq r\}$.

Here, we note that by using interpolation theory, we have that for any $t > 0$,

$$L^2(0, t; V) \cap W^{1,2}(0, t; V^*) \subset C([0, t]; H).$$

Thus, for any $t > 0$, there exists a constant $c > 0$ such that

$$||x||_{C([0,t];H)} \leq c||x||_{L^2(0,t;V) \cap W^{1,2}(0,t;V^*)}. \tag{2.3.2}$$

Let

$$0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m = T.$$

Then by Assumption (AI) and (2.3.1), it is immediately seen that

$$x \in W^{1,2}(t_i, t_{i+1}; V^*), \quad i = 0, \cdots, m - 1.$$

Thus by virtue of Assumption (AI) and (2.3.2), we may consider that there exists a constant $C_3 > 0$ such that

$$\max_{0 \leq t \leq T} \{|x(t)| : x \text{ is a solution of (2.3.1)}\} \leq C_3||x||_{L^2(0,T:V)}. \tag{2.3.3}$$

From now on, we establish the following results on the solvability of the equation(2.3.1).

Theorem 2.3.1. *Let Assumption (AF) be satisfied. Assume that* $x_0 \in H$, $k \in L^2(0,T;V^*)$. *Then, there exists a time* $T_0 \in (0,T)$ *such that the equation (2.3.1) admits a solution*

$$x \in W_1(T_0) \subset C([0,T_0];H). \tag{2.3.4}$$

Proof. For a solution of (2.3.1) in the wider sense, we are going to find a local solution of the following integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)\{(Fx)(s) + k(s)\}ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)). \tag{2.3.5}$$

To prove a local solution, we will use the successive iteration method. First, put

$$x_0(t) = S(t)x_0 + \int_0^t S(t-s)k(s)ds$$

and define $x_{j+1}(t)$ as

$$x_{j+1}(t) = x_0(t) + \int_0^t S(t-s)(Fx_j)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x_j(t_k^-)). \tag{2.3.6}$$

By virtue of Lemma 2.2.4, we have $x_0(\cdot) \in \mathcal{W}_1(t)$, so that

$$||x_0(\cdot)||_{\mathcal{W}_1(t)} \le C_1(|x_0| + ||k||_{L^2(0,t;V^*)}), \tag{2.3.7}$$

where C_1 is a constant in Lemma 2.2.4. Choose $r > C_1(|x_0| + ||k||_{L^2(0,t;V^*)})$. Putting $p(t) = \int_0^t S(t-s)(Fx_0)(s)ds$, by (2.11) of Lemma 2.2.5, we have

$$||p||_{L^2(0,t;V)} \le C_2\sqrt{t}||Fx_0||_{L^2(0,t;H)} \le C_2L(r)t. \tag{2.3.8}$$

Putting $g(t) := S(t-t_k)I_k(x(t_k^-))$, by Assumption (AI) and Lemma 2.2.3, we have

$$||g(t)||_{L^2(0,t;V)} \le 2(3\beta)^{-1/2}(3\beta-1)^{-1}C_{1-\beta}KL(I_k)t^{3\beta/2}. \tag{2.3.9}$$

Put

$$M_1 := \max\{C_2L(r)t,\ \ 2(3\beta)^{-1/2}(3\beta-1)^{-1}C_{1-\beta}KL(I_k)t^{3\beta/2}\}, \qquad (2.3.10)$$

then for any t satisfying, $M_1 < r$, from (2.3.4) and (2.3.5). so that, from(2.3.7) and (2.3.8) and (2.3.9),

$$||x_1||_{L^2(0,t;V)} \le r + C_2L(r)t + 2(3\beta)^{-1/2}(3\beta-1)^{-1}C_{1-\beta}K\sum_{0<t_k<t}L(I_k)t^{3\beta/2} \le 3r.$$

By induction, it can be shown that for all $j = 1, 2, ...$

$$||x_j||_{L^2(0,t;V)} \le 3r, 0 \le t \le M_1. \qquad (2.3.11)$$

Hence, from the equation

$$\begin{aligned} x_{j+1}(t) - x_j(t) = &\int_0^t S(t-s)\{f(t, x_j(s)) - f(t, x_{j-1}(s))\}ds \\ &+ \sum_{0<t_k<t} S(t-t_k)\{I_k(x_j(t_k^-)) - I_k(x_{j-1}(t_k^-)\}. \end{aligned}$$

Set

$$h(t) := S(t-t_k)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}.$$

Then from (2.3.2) and (2.3.3) it follows that

$$\begin{aligned} ||h||_{L^2(0,T;V)} &= [\int_0^T ||\int_{t_k}^t S'(s-t_k)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}ds||^2dt]^{1/2} \\ &\le [\int_0^T ||\int_{t_k}^t AS(s-t_k)\{I_k(x_1(t_k^-)) - I_k(x_2(t_k^-))\}ds||^2dt]^{1/2} \\ &\le [\int_0^T \{\int_{t_k}^t \frac{C_{1-\beta}}{(s-t_k)^{3(1-\beta)/2}}L(I_k)|(x_1(t_k^-) - x_2(t_k^-))|ds\}^2dt]^{1/2} \\ &\le (3\beta)^{-1/2}2(3\beta-1)^{-1}C_{1-\beta}C_3L(I_k)T^{3\beta/2}||x_1 - x_2||_{L^2(0,T;V)}. \end{aligned}$$

Hence, from the equation

$$x_{j+1}(t) - x_j(t) = \int_0^t S(t-s)\{(Fx_j)(s) - (Fx_{j-1})(s)\}ds$$
$$+ \sum_{0<t_k<t} S(t-t_k)\{I_k(x_j(t_k^-)) - I_k(x_{j-1}(t_k^-))\}.$$

Put

$$M_2 := C_2 L(3r)\sqrt{t} + (3\beta)^{-1/2} 2(3\beta-1)^{-1} C_{1-\beta} C_3 \sum_{0<t_k<t} L(I_k) t^{3\beta/2}. \quad (2.3.12)$$

Then from (2.2.11), (2.3.11) and Assumption (AF), we can observe that the inequality

$$\begin{aligned} ||x_{j+1} - x_j||_{L^2(0,t;V)} &\le C_2 L(3r)\sqrt{t} ||x_j - x_{j-1}||_{L^2(0,t;V)} \\ &+ (3\beta)^{-1/2} 2(3\beta-1)^{-1} C_{1-\beta} C_3 \sum_{0<t_k<t} L(I_k) t^{3\beta/2} ||x_j - x_{j-1}||_{L^2(0,t;V)} \\ &\le M_2 ||x_j - x_{j-1}||_{L^2(0,t;V)} \\ &\le (M_2)^j ||x_1 - x_0||_{L^2(0,t;V)}. \end{aligned}$$

Choose $T_0 > 0$ satisfying $max\{M_1, M_2\} < 1$. Then $\{x_j\}$ is strongly convergent to a function x in $L^2(0, T_0; V)$ uniformly on $0 \le t \le T_0$. By letting $j \to \infty$ in (2.3.6) has a unique solution x in $\mathcal{W}_1(T)$. □

From now on, we give a norm estimation of the solution of (2.3.1) and establish the global existence of solutions with the aid of norm estimations.

Theorem 2.3.2. *Under Assumption (AF) for the nonlinear mapping f, there exists a unique solution x of (2.3.1) such that*

$$x \in \mathcal{W}_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H), \quad T > 0.$$

for any $x_0 \in H$, $k \in L^2(0,T;V^*)$. Moreover, there exists a constant C_4 such that

$$||x||_{\mathcal{W}_1(T)} \le C_4(1 + |x_0| + ||k||_{L^2(0,T;V^*)}), \quad (2.3.13)$$

where C_4 is a constant depending on T.

Proof. Let x be a solution of (2.3.1) on $[0, T_0]$, $T_0 > 0$ satisfies $\max\{M_1, M_2\} < 1$. Here M_1 and M_2 be constants in (2.3.10) and (2.3.12), respectively. Then by virtue of Theorem 2.3.1, the solution x is represented as

$$x(t) = x_0(t) + \int_0^t S(t-s)(Fx)(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)).$$

where

$$x_0(t) = S(t)x_0 + \int_0^t S(t-s)k(s)ds.$$

By (2.3.7), we have $x_0(\cdot) \in \mathcal{W}_1(T_0)$, so that

$$||x_0||_{\mathcal{W}_1(T_0)} \le C_1(|x_0| + ||k||_{L^2(0,T_0;V^*)}),$$

where C_1 is constant in Lemma 2.2.4. Moreover, from (2.3.7)-(2.3.9), it follow that

$$||x||_{\mathcal{W}_1(T_0)} \le C_1(|x_0| + ||k||_{L^2(0,T_0;V^*)}) + \max\{M_1, M_2\}||x||_{\mathcal{W}_1(T_0)}. \quad (2.3.14)$$

Thus, moreover, there exists a constant C_4 such that

$$||x||_{\mathcal{W}_1(T_0)} \le C_4(1 + |x_0| + ||k||_{L^2(0,T_0;V^*)}).$$

Now from

$$|S(t)x_0 + \int_0^t S(t-s)\{(Fx)(s) + k(s)\}ds| \le M|x_0| + MtL(r) + M\sqrt{t}||k||_{L^2(0,t;H)},$$

and

$$|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))| \le MK|A^{-\beta}|\sum_{0<t_k<t} L(I_k).$$

it follow

$$|x| \le M|x_0| + MT_0L(r) + M\sqrt{T_0}||k||_{L^2(0,T_0;H)} + MK|A^{-\beta}|\sum_{0<t_k<T_0} L(I_k) < \infty.$$

Hence, we can solve the equation in $[T_0, 2T_0]$ with the initial value $x(T_0)$ and obtain an analogous estimate to (2.3.14). Since the condition (2.3.10),(2.3.12) is independent of initial value, the solution can be extended to the interval $[0, nT_0]$ for any natural number n, i.e., for the initial $u(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (2.3.14) holds for the solution in $[0, (n+1)T_0]$. □

From the following result, we obtain that the solution mapping is continuous, which is useful for physical application of the given equation.

Theorem 2.3.3. *Let Assumptions (AF) and (AI) be satisfied and* $(x_0, k) \in H \times L^2(0, T; V)$. *Then the solution* x *of the equation (2.3.1) belongs to* $x \in \mathcal{W}_1 \equiv L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ *and the mapping*

$$H \times L^2(0, T; V^*) \ni (x_0, k) \mapsto x \in \mathcal{W}_1(T) \tag{2.3.15}$$

is continuous.

Proof. From Theorem 2.3.2, it follows that if $(x_0, k) \in H \times L^2(0, T; V^*)$ then x belongs to $\mathcal{W}_1(T)$. Let $(x_{0i}, k_i) \in H \times L^2(0, T; V^*)$ and $x_i \in \mathcal{W}_1(T)$ be the solution of (2.3.1) with (x_{0i}, k_i) in place of (x_0, k) for $i = 1,\ 2$. Hence, we assume that x_i belongs to a ball $B_r(T) = \{y \in \mathcal{W}_1(T) : ||y||_{\mathcal{W}_1(T)} \le r\}$.

Let

$$(px_j)(t) = \int_0^t S(t-s)Fx_j(s)ds + \sum_{0<t_k<t} S(t-t_k)I_k(x_j(t_k^-)).$$

Then, by virtue Lemma 2.2.4, we get

$$||x_1 - x_2||_{W_1(T)} = C_1\{|x_1 - x_2| + ||k_1 - k_2||_{L^2(0,T;V^*)} + ||px_1 - px_2||_{L^2(0,T;V^*)}\}. \tag{2.3.16}$$

Set $||\cdot||_{L^2(0,T_0;V)} = ||\cdot||_{L^2}$ for brevity, where $T_0 > 0$ satisfies $max\{M_1, M_2\} < 1$. Then, we have

$$\begin{aligned} &||px_1 - px_2||_{L^2(0,T_0;V^*)} \le ||px_1 - px_2||_{L^2} \\ &= ||\int_0^t S(t-s)\{Fx_1 - Fx_2\}ds||_{L^2} \\ &+ ||\sum_{0<t_k<t} S(t-t_k)\{I_k(x_i(t_k^-)) - I_k(x_2(t_k^-))||_{L^2} \\ &\le M_2||x_1 - x_2||_{L^2}. \end{aligned} \tag{2.3.17}$$

Hence, by (2.3.16), (2.3.17), we see that

$$x_n \mapsto x \in \mathcal{W}_1(T_0) \equiv L^2(0,T_0;V) \cap W^{1,2}(0,T_0;V^*).$$

This implies that $(x_n(T_0), (x_n)_{T_0}) \mapsto (x(T_0), x_{T_0})$ in $H \times L^2(0,T;V^*)$. Hence the same argument show that $x_n \mapsto x$ in

$$L^2(0, min\{2T_0, T\}; V) \cap W^{1,2}(0, min\{2T_0, T\}; V^*).$$

Repeating this process, we conclude that $x_n \mapsto x$ in $\mathcal{W}_1(T)$. □

Example 2.3.1. *Let*

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$

$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = -d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$\begin{cases} \frac{\partial}{\partial t}x(t,y) + Ax(t,y) = f'(|x(t,y)|^2)x(t,y) + k(t), \quad t \in (0,T], \quad t \ne t_k, \\ k = 1,2,\cdots,m, \\ \Delta x(t_k) = x(t_k^+, y) - x(t_k^-, y) = I_k(x(t_k^-)), \quad k = 1,2,\cdots,m, \\ x(t,0) = x(t,\pi) = 0, \quad t > 0 \\ x(0,y) = x_0(y). \end{cases} \tag{2.3.18}$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ *and* $z_n(y) = (2/\pi)^{1/2} \sin ny$, *respectively. Moreover,* $\{z_n : n \in N\}$ *is an orthogonal basis of* H *and*

$$S(t)x = \sum_{n=1}^{\infty} e^{n^2 t}(x, z_n)z_n, \quad \forall x \in H, \ t > 0.$$

Moreover, there exists a constant M_0 *such that* $||S(t)||_{\mathcal{L}(H)} \leq M_0$.

Let $0 < \alpha < 1$. *Then the fractional power* $A^\alpha : D(A^\alpha) \subset H \to H$ *of* A *is given by*

$$A^\alpha x = \sum_{n=1}^{\infty} n^{2\alpha}(x, z_n)z_n, \ D(A^\alpha) := \{x : A^\alpha x \in H\}.$$

In particular,

$$A^{-1/2}x = \sum_{n=1}^{\infty} \frac{1}{n}(x, z_n)z_n, \ \text{and} \ ||A^{-1/2}|| = 1.$$

The nonlinear mapping f *is a real valued function belong to* $C^2([0,\infty))$ *which satisfies the conditions*

(f1) $f(0) = 0$, $f(r) \geq 0$ *for* $r > 0$,

(f2) $|f^{'}(r) \leq c(r+1)$ *and* $|qf''(r)| \leq c$ *for* $r \geq 0$ *and* $c > 0$.

If we present

$$F(t, x(t,y)) = f^{'}(|x(t,y)|^2)x(t,y),$$

Then it is well known that F *is a locally Lipschitz continuous mapping from the whole* V *into* H *by Sobolev's imbedding theorem (see [76, Theorem 6.1.6]). As an example of* q *in the above, we can choose* $q(r) = \mu^2 r + \eta^2 r^2/2$ *(*μ *and* η *is constants). It is well known that Assumption (AF) has been satisfied. Thus, with condition on Assumption (AI) there exists a solution of (2.3.18) belongs to* $\mathcal{W}_1(T) = L^2(0,T;V)) \cap W^{1,2}(0,T;V^*) \hookrightarrow C([0,T];H)$ *for any* $(x_0, k) \in H \times L^2(0,T;V^*)$.

Chapter 3
On solutions of semilinear second-order impulsive functional differential equations

3.1 Introduction

In this paper we are concerned with the regularity of the following second-order semilinear impulsive differential system

$$\begin{cases} w''(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w'(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m \end{cases} \tag{3.1.1}$$

in a Banach space X. Here k belongs to $L^2(0,T)$ and $g : [0,T] \times D(A) \to X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuous. In (3.1.1), the principal operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$. The impulsive condition

$$\Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m$$

is combination of traditional evolution systems whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc.

In recent years the theory of impulsive differential systems has been emerging as an important area of investigation in applied sciences. The reason is that it is richer than the corresponding theory of classical differential equations and it is more adequate to represent some processes arising in various disciplines. The theory of impulsive systems provides a general framework for mathematical modeling of many real world phenomena(see

[54, 72] and references therein). The theory of impulsive differential equations has seen considerable development. Impulsive differential systems have been studied in [3, 2, 68, 87], second-order impulsive integrodifferential systems in [5, 69], and Stochastic differential systems with impulsive conditions in [4, 9, 36].

In this paper, we allow implicit arguments about L^2-regularity results for semilinear hyperbolic equations with impulsive condition. These consequences are obtained by showing that results of the linear cases [37, 10] and semilinear case [41] on the L^2-regularity remain valid under the above formulation of (3.1.1). Earlier works prove existence of solution by using Azera Ascoli theorem. But we propose a different approach from that of earlier works to study mild, strong and classical solutions of Cauchy problems by using the properties of the linear equation in the hereditary part.

This paper is organized as follows. In Section 2, we give some definition, notation and the regularity for the corresponding linear equations. In Section 3, by using properties of the strict solutions of linear equations in dealt in Section 2, we will obtain the L^2-regularity of solutions of (3.1.1), and a variation of constant formula of solutions of (3.1.1). Finally, we also give an example to illustrate the applications of the abstract results.

3.2 Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas. Let X be a Banach space with norm denoted by $||\cdot||$.

Definition 3.2.1. *[77] A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in X is called a strongly continuous cosine family if*

c(1) $C(s+t)+C(s-t) = 2C(s)C(t)$, *for all* $s, \ t \in \mathbb{R}$,

c(2) $C(0) = I$,

c(3) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$.

If $C(t),\ \ t \in \mathbb{R}$ is a strongly continuous cosine family in X , then $S(t),\ \ t \in \mathbb{R}$ is the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)xds,\ \ x \in X,\ \ t \in \mathbb{R}. \tag{3.2.1}$$

The infinitesimal generator of a strongly continuous cosine family $C(t),\ \ t \in \mathbb{R}$ is the operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(0)x.$$

We endow with the domain $D(A) = \{x \in X : C(t)x$ is a twice continuously differentiable function of $t\}$ with norm

$$||x||_{D(A)} = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\} + ||Ax||.$$

We shall also make use of the set

$$E = \{x \in X : C(t)x \text{ is a once continuously differentiable function of } t\}$$

with norm

$$||x||_E = ||x|| + \sup\{||\frac{d}{dt}C(t)x|| : t \in \mathbb{R}\}.$$

It is not difficult to show that $D(A)$ and E with given norms are Banach spaces.

The following Lemma is from Proposition 2.1 and Proposition 2.2 of [54].

Lemma 3.2.1. *Let $C(t)(t \in \mathbb{R})$ be a strongly continuous cosine family in X. The following are true :*

c(4) $\quad$ *$C(t) = C(-t)$ for all $t \in \mathbb{R}$,*

c(5) $\quad$ *$C(s), S(s), C(t)$ and $S(t)$ commute for all $s, t \in \mathbb{R}$,*

c(6) $\quad$ *$S(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$,* *c(7)* *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that*

$$||C(t)|| \leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R},$$

$$||S(t_1) - S(t_2)|| \leq K\Big|\int_{t_2}^{t_1} e^{\omega|s|}ds\Big| \text{ for all } t_1, t_2 \in \mathbb{R},$$

c(8) *if* $x \in E$, *then* $S(t)x \in D(A)$ *and*

$$\frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

c(9) *if* $x \in D(A)$, *then* $C(t)x \in D(A)$ *and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

c(10) *if* $x \in X$ *and* $r, s \in \mathbb{R}$, *then*

$$\int_r^s S(\tau)xd\tau \in D(A) \quad \text{and} \quad A(\int_r^s S(\tau)xd\tau) = C(s)x - C(r)x,$$

c(11) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$,

c(12) $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in \mathbb{R}$,

c(13) $C(s+t) = C(t)C(s) - S(t)S(s)$ *for all* $s, t \in \mathbb{R}$,

c(14) $C(s+t) - C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$.

The following Lemma is from Proposition 2.4 of [77].

Lemma 3.2.2. *Let* $C(t)(t \in \mathbb{R})$ *be a strongly continuous cosine family in* X *with infinitesimal generator* A. *If* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$, $y_0 \in E$, *and*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ t \in \mathbb{R},$$

then $w(t) \in D(A)$ *for* $t \in \mathbb{R}$, *w is twice continuously differentiable, and w satisfies*

$$w^{''}(t) = Aw(t) + f(t), \;\; t \in R, \;\; w(0) = x_0, \;\; w^{'}(0) = y_0. \tag{3.2.2}$$

Conversely, if $f : \mathbb{R} \to X$ *is continuous,* $w(t) : \mathbb{R} \to X$ *is twice continuously differentiable,* $w(t) \in D(A)$ *for* $t \in \mathbb{R}$, *and w satisfies* (3.2.2), *then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \;\; t \in \mathbb{R}.$$

Proposition 3.2.1. *Let* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$, $y_0 \in E$. *Then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \;\; t \in \mathbb{R}$$

is a solution of (3.2.2) belonging to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$. *Moreover, we have that there exists a positive constant* C_1 *such that for any* $T > 0$,

$$||w||_{L^2(0,T;D(A))} \le C_1(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{3.2.3}$$

3.3 Nonlinear equations

This section is to investigate the regularity of solutions of a second-order nonlinear impulsive differential system

$$\begin{cases} w^{''}(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w^{'}(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m \end{cases} \tag{3.3.1}$$

in a Banach space X.

Assumption (BG) Let $g : [0,T] \times D(A) \to X$ be a nonlinear mapping such that $t \mapsto g(t,w)$ is measurable and

(bg1) $||g(t, w_1) - g(t, w_2)||_{D(A)} \leq L||w_1 - w_2||$,

for a positive constant L.

Assumption (BI) Let $I_k^1 : D(A) \to X$, $I_k^2 : E \to X$ be continuous and there exist positive constants $L(I_k^1)$, $L(I_k^2)$ such that

(bi1) $||I_k^1(w_1) - I_k^1(w_2)|| \leq L(I_k^1)||w_1 - w_2||_{D(A)}$, for each $w_1, w_2 \in D(A)$
$||I_k^1(w)|| \leq L(I_k^1)$, for $w \in D(A)$

(bi2) $||I_k^2(w_1') - I_k^2(w_2')|| \leq L(I_k^2)||w_1' - w_2'||_E$, for each $w_1', w_2' \in E$
$||I_k^2(w')|| \leq L(I_k^2)||$, for $w' \in E$.

For $w \in L^2(0, T : D(A))$, we set

$$F(t, w) = \int_0^t k(t-s)g(s, w(s))ds$$

where k belongs to $L^2(0, T)$. Then we will seek a mild solution of (3.3.1), that is, a solution of the integral equation

$$\begin{aligned} w(t) =& C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s, w) + f(s)\}ds \\ &+ \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)), \quad t \in \mathbb{R}. \end{aligned} \tag{3.3.2}$$

Remark 3.3.1. *If $g : [0, T] \times X \to X$ is a nonlinear mapping satisfying*

$$||g(t, w_1) - g(t, w_2)|| \leq L||w_1 - w_2||$$

for a positive constant L, then our results can be obtained immediately.

Lemma 3.3.1. *Let $w \in L^2(0,T;D(A))$, $T>0$. Then $F(\cdot,w) \in L^2(0,T;X)$ and*

$$||F(\cdot,w)||_{L^2(0,T;X)} \leq L||k||_{L^2(0,T)}\sqrt{T}||w||_{L^2(0,T;D(A))}.$$

Moreover if $w_1,\ w_2 \in L^2(0,T;D(A))$, then

$$||F(\cdot,w_1)-F(\cdot,w_2)||_{L^2(0,T;X)} \leq L||k||_{L^2(0,T)}\sqrt{T}||w_1-w_2||_{L^2(0,T;D(A))}.$$

Lemma 3.3.2. *If $k \in W^{1,2}(0,T)$, $T>0$, then*

$$\begin{aligned} A\int_0^t S(t-s)F(s,w)ds = &-F(t,w) \\ &+\int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds \\ &+\int_0^t (C(t-s)-I)k(0)g(s,w(s))ds. \end{aligned} \tag{3.3.3}$$

Theorem 3.3.1. *Suppose that the Assumptions (BG) and Assumption (BI) are satisfied. If $f:\mathbb{R}\to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$, and $k \in W^{1,2}(0,T)$, $T>0$, then there exists a time $T \geq T_0 > 0$ such that the functional differential equation (3.3.1) admits a unique solution w in $L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)$.*

Proof. Let us fix $T_0>0$ so that

$$\begin{aligned} C_2 \equiv &\omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||||k||_{W^{1,2}(0,T_0)} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||||k(0)|| \\ &+\{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t}L(I_k^1)Ke^{wT_0} \\ &+\{2w^{-1}K(e^{wT_0}-1)+1\}\sum_{0<t_k<t}L(I_k^2)<1 \end{aligned} \tag{3.3.4}$$

where $K,\ L,\ L(I_k^1)$ and $L(I_k^2)$ are constants in c(7), (bg1) and Assumption (BI) respectively. Invoking Proposition 3.2.1, for any $v \in L^2(0,T_0;D(A))$ we obtain the equation

$$\begin{cases} w''(t) = Aw(t) + F(t,v) + f(t), \quad 0 < t \le T_0, \\ w(0) = x_0, \quad w'(0) = y_0 \\ \Delta w(t_k) = I_k^1(v(t_k)), \quad \Delta w'(t_k) = I_k^2(v'(t_k^+)), \quad k = 1,2,...,m \end{cases} \tag{3.3.5}$$

has a unique solution $w \in L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)$. Let $w_1,\ w_2$ be the solutions of (3.3.5) with v replaced by $v_1,\ v_2 \in L^2(0,T_0;D(A))$, respectively. Put

$$J(w)(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s,v) + f(s)\}ds + \sum_{0<t_k<t} C(t-t_k)I_k^1(v(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(v'(t_k^+)).$$

Then

$$\begin{aligned} J(w_1)(t) - J(w_2)(t) = &\int_0^t S(t-s)\{F(s,v_1) - F(s,v_2)\}ds \\ &+ \sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1(t_k)) - I_k^1(v_2(t_k))\} \\ &+ \sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1'(t_k^+)) - I_k^1(v_2'(t_k^+))\}, \\ =&I_1 + I_2 + I_3. \end{aligned}$$

So, from Lemmas 3.3.1, 3.3.2, it follows that for $0 \le t \le T_0$,

$$\begin{aligned} &|| \int_0^t S(t-s)\{F(s,v_1) - F(s,v_2)\}ds|| \\ &\le \omega^{-1} KLT_0(e^{\omega T_0} - 1)||k||_{L^2(0,T_0)}||v_1 - v_2||_{L^2(0,T_0;D(A))}, \end{aligned}$$

$$
\begin{aligned}
&||\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \\
&\le ||AS(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \\
&= ||S(t)A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||,
\end{aligned}
$$

and

$$
\begin{aligned}
&||A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds|| \\
&\le ||\int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)(g(\tau,v_1(\tau))-g(\tau,v_2(\tau)))d\tau\, ds|| \\
&\quad + ||\int_0^t (C(t-s)-I)k(0)(g(s,v_1(s))-g(s,v_2(s)))ds|| \\
&\le tL||Ke^{\omega t}+1||||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\
&\quad + \sqrt{t}L||Ke^{\omega t}+1||||k(0)||||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
||I_1||_{L^2(0,T_0;D(A))} &\le \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \qquad (3.3.6)\\
&+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\
&+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}
$$

By Assumption (bi1), we obtain

$$
||\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| \le \sum_{0<t_k<T_0} Ke^{wT_0}L(I_k^1)||v_1-v_2||_{D(A)},
$$

$$||\frac{d}{dt}C(t)\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||$$
$$\leq ||AS(t)\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||$$
$$= ||S(t)A\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||,$$

and

$$||A\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| = ||\sum_{0<t_k<t}C(t-t_k)A\{I_k^1(v_1)-I_k^1(v_2)\}||$$
$$\leq \sum_{0<t_k<t}Ke^{wt}||I_k^1(v_1)-I_k^1(v_2)||_{D(A)}$$
$$\leq \sum_{0<t_k<t}Ke^{wt}L(I_k^1)||v_1-v_2||_{D(A)}.$$

Therefore, we have

$$||I_2||_{L^2(0,T_0;D(A))} \leq \{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t}L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}. \tag{3.3.7}$$

We also obtain from Assumption (bi2),

$$||\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1'(t_k^+))-I_k^1(v_2'(t_k^+))\}|| \leq \sum_{0<t_k<T_0}Kw^{-1}(e^{wT_0}-1)L(I_k^2)||v_1-v_2||_{D(A)},$$

$$||\frac{d}{dt}C(t)\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||$$
$$\leq ||AS(t)\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||$$
$$= ||S(t)A\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1')-I_k^1(v_2')\}||,$$

and

$$||A\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1'(t_k^+)) - I_k^1(v_2'(t_k^+))\}|| = ||\sum_{0<t_k<t} \frac{d}{dt}C(t)\{I_k^2(v_1') - I_k^1(v_2')\}||$$
$$\leq \sum_{0<t_k<t} ||I_k^2(v_1') - I_k^1(v_2')||_E$$
$$\leq \sum_{0<t_k<t} L(I_k^2)||v_1' - v_2'||_E.$$

Therefore, we have

$$||I_3||_{L^2(0,T_0;D(A))} \leq \{w^{-1}K(e^{wT_0}-1)+2\} \sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}. \tag{3.3.8}$$

Thus, from (3.3.6),(3.3.7), and (3.3.8), we conclude that

$$\begin{aligned}
&||J(w_1) - J(w_2)||_{L^2(0,T_0;D(A))} \qquad (3.3.9)\\
&\leq \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}L||k||_{L^2(0,T_0)}\sqrt{T_0}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+\{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+\{2w^{-1}K(e^{wT_0}-1)+1\}\sum_{0<t_k<t} L(I_k^2)||v_1-v_2||_{W^{1,2}(0,T_0;D(A))}.
\end{aligned}$$

Moreover, it is easily seen that

$$||J(w_1) - J(w_2)||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \leq C_2||v_1-v_2||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}.$$

So by virtue of the condition (3.3.4) the contraction mapping principle gives that the solution of (3.3.1) exists uniquely in $[0, T_0]$. □

Theorem 3.3.2. *Suppose that the Assumptions (BG) and (BI) are satisfied. If* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A), y_0 \in E$, *and* $k \in W^{1,2}(0,T)$, $T > 0$, *then the solution* w *of (3.3.1) exists and is unique in* $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$, *and there exists a constant* C_3 *depending on* T *such that*

$$||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \leq C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{3.3.10}$$

Proof. Let $w(\cdot)$ be the solution of (3.3.1) in the interval $[0, T_0]$ where T_0 is a constant in (3.3.4) and $v(\cdot)$ be the solution of the following equation

$$v''(t) = Av(t) + f(t), \ \ 0 < t,$$
$$v(0) = x_0, \ \ v'(0) = y_0.$$

Then

$$(w-v)(t) = \int_0^t S(t-s)F(s,w)ds + \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)),$$

and in view of (3.3.9)

$$||w - v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \leq C_2||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}, \tag{3.3.11}$$

that is, combining (3.3.11) with Proposition 3.2.1 we have

$$\begin{aligned} ||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} &\leq \frac{1}{1-C_2}||v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \quad (3.3.12) \\ &\leq \frac{C_1}{1-C_2}(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T_0;X)}). \end{aligned}$$

Now from

$$A\int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds$$

$$= C(T_0)f(0) - f(T_0) + \int_0^{T_0}(C(T_0-s)-I)f^{'}(s)ds$$

$$- F(T_0,w) + \int_0^{T_0}(C(T_0-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds$$

$$+ \int_0^{T_0}(C(T_0-s)-I)k(0)g(s,w(s))ds,$$

$$||A\sum_{0<t_k<t} C(t-t_k)I_k^1(w_1)|| \le Kw^{-1}(e^{wT_0-1})Ke^{wT_0}\sum_{0<t_k<t} L(I_k^1)||w(t_k)||_{D(A)},$$

$$||\sum_{0<t_k<t} S(t-t_k)I_k^2(v_1')|| \le \sum_{0<t_k<t} L(I_k^2)||w'(t_k^+)||_E,$$

and since

$$\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,w)+f(s)\}ds = S(t)A\int_0^t S(t-s)\{F(s,w)+f(s)\}ds,$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t} C(t-t_k)I_k^1(w) \le S(t)A\sum_{0<t_k<t} C(t-t_k)I_k^1(w).$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t} S(t-t_k)I_k^2(w') \le S(t)A\sum_{0<t_k<t} S(t-t_k)I_k^2(w').$$

We have

$$||w(T_0)||_{D(A)} = ||C(T_0)x_0 + S(T_0)y_0 + \int_0^{T_0} S(T_0-s)\{F(s,w)+f(s)\}ds$$
$$+ \sum_{0<t_k<t} C(t-t_k)I_k^1(w) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w')||_{D(A)}$$

$$
\begin{aligned}
&\leq (\omega^{-1}K(e^{\omega T_0}-1)+1)\{Ke^{\omega T_0}||x_0||_{D(A)}+||y_0||_E+T_0L||k||_{L^2(0,T_0)}||w||_{L^2(0,T_0;D(A))}\\
&\quad +||Ke^{\omega T_0}f(0)||+||f(0)||+||K(e^{\omega T_0}+1)\sqrt{T_0}||f||_{W^{1,2}(0,T;X)}\\
&\quad +tL||Ke^{\omega t}+1||\,||k||_{W^{1,2}(0,T_0)}||w||_{L^2(0,T_0;D(A))}\\
&\quad +\sqrt{t}L||Ke^{\omega t}+1||\,||k(0)||||w||_{L^2(0,T_0;D(A))}\}\\
&\quad +\{2+Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t}Ke^{wT_0}L(I_k^1)\\
&\quad +\{1+2Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t}L(I_k^2).
\end{aligned}
$$

Hence, from (3.3.12), there exists a positive constant $C>0$ such that

$$||w(T_0)||_{D(A)}\leq C(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T_0;X)}).$$

Since the condition (3.3.4) is independent of initial values, the solution of (3.3.1) can be extended to the interval $[0,nT_0]$ for every natural number n. An analogous estimate to (3.3.12) holds for the solution in $[0,nT_0]$, and hence for the initial value $(w(nT_0),w^{'}(nT_0))\in D(A)\times E$ in the interval $[nT_0,(n+1)T_0]$. □

Example. We consider the following partial differential equation

$$
\begin{cases}
w^{''}(t,x)=Aw(t,x)+F(t,w)+f(t), \quad 0<t, \quad 0<x<\pi,\\
w(t,0)=w(t,\pi)=0, \quad t\in\mathbb{R}\\
w(0,x)=x_0(x), \quad w^{'}(0,x)=y_0(x), \quad 0<x<\pi\\
\Delta w(t_k,x)=I_k^1(w(t_k))=(\gamma_k||w''(t_k,x)||+t_k), \quad 1\leq k\leq m,\\
\Delta w'(t_k,x)=I_k^2(w'(t_k))=\delta_k||w'(t_k,x)||,
\end{cases}
\tag{BE}
$$

where constants γ_k and $\delta_k (k=1,\cdots,m)$ are small.

Let $X=L^2([0,\pi];\mathbb{R})$, and let $e_n(x)=\sqrt{\frac{2}{\pi}}\sin nx$. Then $\{e_n:n=1,\cdots\}$ is an orthonormal base for X. Let $A:X\to X$ be defined by

$$Aw(x)=w''(x),$$

where $D(A) = \{w \in X : w,\ w'$ are absolutely continuous, $w'' \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2(w, e_n)e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, e_n)e_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(w, e_n)e_n, \quad w \in X.$$

Let $g_1(t, x, w, p), \quad p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, \delta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \le \delta$ such that

(bf1) $g_1(t, x, 0, 0) = 0$,

(bf2) $|g_1(t, x, w, p) - g_1(t, x, w, q)| \le \rho(t, |w|)|p - q|$,

(bf3) $|g_1(t, x, w_1, p) - g_1(t, x, w_2, p)| \le \rho(t, |w_1| + |w_2|)|w_1 - w_2|$.

Let

$$g(t, w)x = g_1(t, x, w, Dw, D^2w).$$

Then noting that

$$\begin{aligned}||g(t, w_1) - g(t, w_2)||_{0,2}^2 \le\ & 2\int_\Omega |g_1(t, x, w_1, p) - g_1(t, x, w_2, q)|^2 dx \\ & + 2\int_\Omega |g_1(t, x, w_1, q) - g_1(t, x, w_2, q)|^2 dx\end{aligned}$$

where $p = (Dw_1, D^2w_1)$ and $q = (Dw_2, D^2w_2)$, it follows from (bf1), (bf2) and (bf3) that

$$||g(t, w_1) - g(t, w_2)||_{0,2}^2 \le L(||w_1||_{D(A)}, ||w_2||_{D(A)})||w_1 - w_2||_{D(A)}$$

where $L(||w_1||_{D(A)}, ||w_2||_{D(A)})$ is a constant depending on $||w_1||_{D(A)}$ and $||w_2||_{D(A)}$. We set

$$F(t, w) = \int_0^t k(t - s)g(s, w(s))ds$$

where k belongs to $L^2(0, T)$. Then, from the results in section 3, the solution w of (BE) exists and is unique in $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$, and there exists a constant C_3 depending on T such that

$$||w||_{L^2(0,T;D(A))} \le C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}).$$

Chapter 4
Regularity for semilinear differential equations with p-Laplacian

4.1 Introduction

This paper is concerned with the regularity of solutions for an abstract parabolic type equation involving p-Laplacian:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) + \mathcal{A}(x,D_x)u(x,t) - \operatorname{div}(|\nabla u(x,t)|^{p-2}\nabla u(x,t)) = f(t), \ (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{4.1.1}$$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x,D_x)$ be an elliptic differential operator of second order as follows:

$$\mathcal{A}(x,D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \Omega$, $b_i \in C^1(\overline{\Omega})$ and $c \in L^\infty(\Omega)$.

If $-A$ is the infinitesimal generator of an analytic semigroup in a complex Banach space X, we find that in general it is false that the following abstract linear problem

$$\begin{cases} du(t)/dt + Au(t) = f(t), \quad t \in (0,T] \\ u(0) = u_0 \end{cases} \tag{4.1.2}$$

has a solution $u \in W^{1,q}(0,T;X) \cap L^q(0,T;D(A))$ in case $f \in L^q(0,T;X)$ for any $1 < q < \infty$. As in Prato and Grisvard [21](also see [56, 16]), we can obtain L^2- regularity for the strong solutions, while in the Hilbert space setting. Moreover as the better result in Dore and Venni [26], if X is ζ-convex, we also obtain L^q-regularity results for solution of (4.1.2).

The background of these variational problems are physics, especially in solid mechanics, where nonconvex and multi-valued constitutive laws lead to differential inclusions. We refer to [66, 65] to see the applications of differential inclusions. Recently, much research has many researches have been devoted to the study of a class of semilinear differential equations [52, 31, 20]. Especially, [14, 15, 32] showed the existence of infinitely many solutions for fractional p-Laplacian equations, and [17] discussed upper semicontinuity of attractors and continuity of equilibrium sets for parabolic problems with degenerate p-Laplacian. Most of them considered the existence of weak solutions for differential inclusions of various forms by using the Faedo-Galerkin approximation method. In Yang et al.[84] proved that the existence of global attractors in $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$ for the following p-Laplacian equation:

$$\begin{cases} du(t)/dt - \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(u(t)) = g(t), \quad 0 < t, \\ u(0) = u_0. \end{cases}$$

If we put $Au = -\mathcal{A}(x, D_x)u$ then it is known that A generates an analytic semigroup in $L^p(\Omega)(see[1, 67])$. In view of Sobolev's embedding theorem, we remark that $L^p(\Omega) \subset W^{-1,p}(\Omega)$, where $W^{-1,p}(\Omega)$ is the dual space of $W_0^{1,p'}(\Omega)(p' = p/(p-1))$. The space $W^{-1,p}(\Omega)$ is ζ-convex(as for the definition and fundamental facts of a ζ-convex space see [18, 38]). Therefore, from the interpolation theory it is easily seen that the operator A generates an analytic semigroup in both $H_{p,q} \equiv (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q}$ and $W^{-1,p}(\Omega)$. Hence, we can investigate the semilinear form (4.1.1) in the space $W^{-1,p}(\Omega)$ and apply the method of [26] to the system (4.1.1) to show the existence and uniqueness of the solution

$$u \in L^q(0,T;W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T;W^{-1,p}(\Omega)) \subset C([0,T];H_{p,q})$$

for any $u_0 \in H_{p,q}$ and $f \in L^q(0,T;W^{-1,p}(\Omega))(p > 2)$.

For the basic of our study, some variational of constant formula of solutions are established considering as an equation in $L^p(\Omega)$ as well as in $W^{-1,p}(\Omega)$. Thereafter, by showing that the nonlinear mapping of p-Laplacian term is Lipschitz continuous, we will obtain the existence for solutions of semilinear equation (4.1.1) by converting the problem into the contraction mapping principle and the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;W_0^{1,p}(\Omega)) \cap W^{1,2}(0,T;W^{-1,p}(\Omega)) \cap C([0,T];H_{p,q})$ as seen in [4]. Consequently, in view of the properties of p-Laplacian term, we show that the mapping

$$H_{p,q} \times L^q(0,T;W^{-1,p}(\Omega)) \ni (x_0,f) \mapsto u \in L^q(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];H_{p,q})$$

is continuous.

4.2 Notations

Let Ω be a region in an n-dimensional Euclidean space $\mathbb{R}^n$ and closure $\overline{\Omega}$. For an integer $m \geq 0$, $C^m(\Omega)$ is the set of all m-times continuously differential functions on Ω. $C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of these functions which have compact support in Ω. For $1 \leq p \leq \infty$, $W^{m,p}(\Omega)$ is the set of all functions $f = f(x)$ whose derivative $D^\alpha f$ up to degree m in distribution sense belong to $L^p(\Omega)$. As usual, the norm is then given by

$$||f||_{m,p} = (\sum_{\alpha \leq m} ||D^\alpha f||_p^p)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

$$||f||_{m,\infty} = \max_{\alpha \leq m} ||D^\alpha f||_\infty,$$

where $D^0 f = f$. In particular, $W^{0,p}(\Omega) = L^p(\Omega)$ with the norm $||\cdot||_p$. Let $p' = p/(p-1),\ 1 < p < \infty$. $W^{-1,p}(\Omega)$ stands for the dual space $W_0^{1,p'}(\Omega)^*$ of $W_0^{1,p'}(\Omega)$ whose norm is denoted by $||\cdot||_{-1,p}$.

For a closed linear operator of A in some Banach space, $\rho(A)$ denotes the resolvent set of A.

If X is a Banach space and The notation $(\cdot,\cdot)_{X^*,X}$ is the duality pairing between X^* and X.

$L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable. $C^m([0,T];X)$ will denote the set of all m-times continuously differentiable functions from $[0,T]$ into X.

If X and Y are two Banach spaces, $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$. The intersection $X \cap Y$ is a Banach spaces with the norms

$$||a||_{X\cap Y} = \max\{||a||_X, ||a||_Y\}, \quad \forall a \in X \cap Y.$$

For an interpolation couple of Banach spaces X_0 and X_1, $(X_0, X_1)_{\theta,p}$ and $[X_0, X_1]_\theta$ denote the real and complex interpolation spaces between X_0 and X_1, respectively.

4.3 Elliptic boundary value problem in $W^{-1,p}(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the following elliptic differential operator of second order with real and smooth coefficients:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

where $\{a_{i,j}(x)\}$ is a positive definite symmetric matrix for each $x \in \overline{\Omega}$. The operator

$$\mathcal{A}^{'}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(b_i(x)\cdot) + c(x)$$

is the formal adjoint of $\mathcal{A}$.

For $1 < p < \infty$, we denote the realization of $\mathcal{A}$ in $L^p(\Omega)$ under the Dirichlet boundary condition by A_p:

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \tag{4.3.1}$$

$$A_p u = \mathcal{A}u \quad \text{for} \quad u \in D(A_p).$$

For $p^{'} = p/(p-1)$, we can also define the realization $\mathcal{A}^{'}$ in $L^{p^{'}}(\Omega)$ under Dirichlet boundary condition by $A^{'}_{p'}$:

$$D(A^{'}_{p'}) = W^{2,p^{'}}(\Omega) \cap W_0^{1,p^{'}}(\Omega),$$

$$A^{'}_{p'} v = \mathcal{A}^{'} v \quad \text{for} \quad v \in D(A^{'}_{p'}).$$

It is known that the adjoint operator of A_p considered as a closed linear operator in $L^p(\Omega)$ coincide with $A^{'}_{p'}$:

$$A_p^* = A^{'}_{p'}$$

and $-A_p$ and $-A^{'}_{p'}$ generate analytic semigroups in $L^p(\Omega)$ and $L^{p^{'}}(\Omega)$, respectively[[67], section 7.3]. For the sake of simplicity we assume that the closed half plane $\{\lambda : \mathrm{Re}\lambda \le 0\}$ is contained in $\rho(A_p) \cap \rho(A^{'}_{p'})$, hence in particular $0 \in \rho(A_p) \cap \rho(A^{'}_{p'})$, by adding some positive constant to $\mathcal{A}$ if necessary.

In what follows, we make $D(A_p)$ and $D(A^{'}_{p'})$ Banach space endowing them with graph norm of A_p and $A^{'}_{p'}$, respectively. Since $D(A^{'}_{p'})$ and $W_0^{1,p^{'}}(\Omega)$ are dense subspaces of $W_0^{1,p^{'}}(\Omega)$ and $L^{p^{'}}(\Omega)$, respectively, we may consider that

$$D(A_p) \subset W_0^{1,p}(\Omega) \subset L^p(\Omega) \subset W^{-1,p}(\Omega) \subset D(A^{'}_{p'})^*.$$

Lemma 4.3.1. *Let $(A'_{p'})'$ be the adjoint operator $A'_{p'}$. Then $(A'_{p'})'$ is an isomorphism from $L^p(\Omega)$ to $D(A'_{p'})^*$, and the restriction of $(A'_{p'})'$ to $D(A_p)$ coincides with A_p.*

Proof. For any $f \in L^p(\Omega)$ and $v \in D(A'_{p'})$, we have

$$((A'_{p'})'f, v)_{(D(A'_{p'})^*, D(A'_{p'}))} = (f, A'_{p'}v)_{(L^p(\Omega), L^{p'}(\Omega))}.$$

So, due to $0 \in \rho(A'_{p'})$, we have that $(A'_{p'})'$ is an isomorphism from $L^p(\Omega)$ to $D(A'_{p'})^*$. If $f \in L^p(\Omega)$ and $v \in D(A'_{p'})$, then

$$((A'_{p'})'f, v)_{(D(A'_{p'})^*, D(A'_{p'}))} = (f, A'_{p'}v)_{(L^p(\Omega), L^{p'}(\Omega))} = (A_p f, v)_{(D(A'_{p'})^*, D(A'_{p'}))}.$$

This implies that the restriction of $(A'_{p'})'$ to $D(A_p)$ coincides with A_p. □

Lemma 4.3.2. *Let $\widetilde{A}$ be the restriction of $(A'_{p'})'$ to $W_0^{1,p}(\Omega)$. Then the operator $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$. Similarly, we consider that the restriction $\widetilde{A'}$ of $(A_p)' \in B(L^{p'}(\Omega), D(A_p)^*)$ to $W_0^{1,p'}(\Omega)$ is an isomorphism from $W_0^{1,p'}(\Omega)$ to $W^{-1,p'}(\Omega)$.*

Proof. From the result of Seeley [74] (see also Triebel [[79], p. 321], [55]), we obtain that

$$[D(A_p), L^p(\Omega)]_{1/2} = W_0^{1,p}(\Omega), \tag{4.3.2}$$

$$[D(A'_{p'}), L^{p'}(\Omega)]_{1/2} = W_0^{1,p'}(\Omega). \tag{4.3.3}$$

Regarding the dual spaces, from (4.3.3) it follows that

$$[L^p(\Omega), D(A'_{p'})^*]_{1/2} = [D(A'_{p'}), L^{p'}(\Omega)]^*_{1/2} = W^{-1,p}(\Omega).$$

This, together with $0 \in \rho(A'_{p'})$, implies that the operator $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ to $W^{-1,p}(\Omega)$ by the interpolation theory. □

It is not difficult to see that, for $u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,p'}(\Omega)$, $\widetilde{A}u = \mathcal{A}u$ and $\widetilde{A'}v = \mathcal{A}'v$, both understood in the distribution sense, and

$$(\widetilde{A}u, v) = a(u, v) = (u, \widetilde{A'}v), \tag{4.3.4}$$

where $a(u, v)$ is the associated sesquilinear form:

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^{n} (a_{i,j}(x) \frac{\partial u}{\partial x_i} \overline{\frac{\partial v}{\partial x_j}} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \overline{v} + c(x) u \overline{v} \right\} dx.$$

The following results are from Section 3 in Jeong [38].

Lemma 4.3.3. *The operators $-\widetilde{A}$ and $-\widetilde{A'}$ generate analytic semigroups in $W^{-1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$, respectively. Furthermore, the inequality*

$$||(\widetilde{A})^{is}||_{B(W^{-1,p}(\Omega))} \le Ce^{\gamma|s|}, \quad -\infty < s < \infty \tag{4.3.5}$$

holds for some constants $C > 0$ and $\gamma \in (0, \pi/2)$.

For any $q \in (1, \infty)$, we set

$$Z_{p,q} = (D(A_p), L^p(\Omega))_{1/q,q}, \quad H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q}. \tag{4.3.6}$$

Remark 4.3.1. *Concerning ζ-convex Banach space, we recall that every Hilbert space is ζ-convex. Cartesian products and quotients of ζ-convex spaces are ζ-convex. If (X, Y) is an interpolation couple spaces of ζ-convex spaces, $(X, Y)_{\theta,p}$ with $1 < p < \infty$, and $[X, Y]_\theta$ are ζ-convex. Moreover, if X is ζ-convex and $1 < p < \infty$ then every L^p space of X-valued functions is ζ-convex(see [21, 18] and the bibliography therein). Since $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ and $W^{-1,p}(\Omega)$ are ζ-convex spaces. From the interpolation theory and definitions of the operator $\widetilde{A}$, it is easily seen that $H_{p,q}$ and $Z_{p,q}$ are also ζ-convex.*

Proposition 4.3.1. *The operators* $-\widetilde{A}$ *and* $-\widetilde{A'}$ *generate analytic semigroups in* $H_{p,q}$ *and* $H_{p',q'}$*, respectively.*

Proof. By lemma 4.3.3, since $-A_p$ and $-\widetilde{A}$ generate analytic semigroup in $L^p(\Omega)$ and $W^{-1,p}(\Omega)$, respectively, there exists an angle $\gamma \in (0, \frac{\pi}{2})$ such that

$$\Sigma = \{\lambda : \gamma \leq \arg \lambda \leq 2\pi - \gamma\} \subset \rho(A_p) \cap \rho(\widetilde{A}), \tag{4.3.7}$$

$$||(\lambda - A_p)^{-1}||_{B(L^p(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma, \tag{4.3.8}$$

$$||(\lambda - \widetilde{A})^{-1}||_{B(W^{-1,p}(\Omega))} \leq C/|\lambda|, \quad \lambda \in \Sigma. \tag{4.3.9}$$

In view of (4.3.8)

$$||A_p(\lambda - A_p)^{-1}u||_p = ||(\lambda - A_p)^{-1}A_p u||_p \leq \frac{C}{|\lambda|}||A_p u||_p,$$

for any $u \in D(A_p)$, we have

$$||(\lambda - A_p)^{-1}||_{B(D(A_p))} \leq \frac{C}{|\lambda|}. \tag{4.3.10}$$

From (4.3.8) and (4.3.10) it follows that

$$||(\lambda - \widetilde{A})^{-1}||_{B(W_0^{1,p}(\Omega))} \leq \frac{C}{|\lambda|} \tag{4.3.11}$$

and hence, from (4.3.10), (4.3.11) and the definition of the space $H_{p,q}$, we have that

$$||(\lambda - \widetilde{A})^{-1}||_{B(H_{p,q})} \leq \frac{C}{|\lambda|}.$$

Therefore, we have shown that $-\widetilde{A}$ generates an analytic semigroup in $H_{p,q}$. □

Proposition 4.3.2. *There exists a constant $C > 0$ such that*

$$||\widetilde{A}^{is}||_{B(H_{p,q})} \leq Ce^{\gamma|\lambda|}, s \in \mathbb{R},$$

where γ is the constant in (4.3.7).

Proof. From Theorem 1 of Seeley [73] and Proposition 3.2 of Jeong [38] there exists a constant $C > 0$ such that

$$||(A_p)^{\epsilon+is}||_{B(L^p(\Omega))} \leq Ce^{\gamma|s|}, \tag{4.3.12}$$

$$||\widetilde{A}^{\epsilon+is}||_{B(W^{-1,p}(\Omega))} \leq Ce^{\gamma|s|} \tag{4.3.13}$$

for any $s \in \mathbb{R}$ and $\epsilon > 0$. From (4.3.12) it follows

$$||(A_p)^{\epsilon+is}||_{B(D(A_p))} \leq Ce^{\gamma|s|}, \tag{4.3.14}$$

and hence, from (4.3.12) and (4.3.14) we obtain

$$||\widetilde{A}^{\epsilon+is}||_{B(W_0^{1,p}(\Omega))} \leq Ce^{\gamma|s|}. \tag{4.3.15}$$

Hence from (4.3.5), (4.3.14) and (4.3.15) we have shown that

$$||\widetilde{A}^{\epsilon+is}||_{B(H_{p,q})} \leq Ce^{\gamma|s|}.$$

So the proof is complete. □

Remark 4.3.2. *Propositions 4.3.1, 4.3.2 say that $-\widetilde{A}$ generates analytic semigroup $\{e^{t\widetilde{A}} : t \geq 0\}$ in $H_{p,q}$ as well as in $W^{-1,p}(\Omega)$. Hence, we may assume that there is a constant $M_0 > 0$ such that*

$$||e^{t\widetilde{A}}||_{B(L^p(\Omega))} \leq M_0, \quad ||e^{t\widetilde{A}}||_{B(H_{p,q})} \leq M_0, \quad ||e^{t\widetilde{A}}||_{B(W^{-1,p}(\Omega))} \leq M_0.$$

From now on, in virtue of Proposition 4.3.1, 4.3.2, we study such a simple initial value problem in $W^{-1,p}(\Omega)$ or in $H_{p,q}$ as

$$\begin{cases} u'(t) + \widetilde{A}u(t) = f(t), \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{LE}$$

Remark 4.3.3. *If $-A$ is the infinitesimal generator of an analytic semigroup in a complex Banach space X, we find that in general it is false that problem (LE) has a solution $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ in case $f \in L^p(0,T;X)$. As in Da Prato and Grisvard [21](also see [56, 16], section 5.5 of [76]), we can obtain L^2- regularity for the strong solutions, while in the Hilbert space setting. Moreover, as the better result in [26], if X is ζ-convex, we also obtain $L^p(p > 1)$-regularity results for solution of (LE) mentioned above.*

From Theorem 3.5.3 of Butzer and Berens [19] we obtain the following result.

Lemma 4.3.4. *For any $1 < p$ and $q \in (0, \infty)$, we have*

$$Z_{p,q} = (D(A_p), L^p(\Omega))_{1/q,q} = \{x \in L^p(\Omega) : \int_0^T ||\widetilde{A}e^{t\widetilde{A}}x||_p^q dt < \infty\},$$

and

$$H_{p,q} = (W_0^{1,p}(\Omega), W^{-1,p}(\Omega))_{1/q,q} = \{x \in W^{-1,p}(\Omega) : \int_0^T ||\widetilde{A}e^{t\widetilde{A}}x||_{-1,p}^q dt < \infty\}.$$

In order to prove the solvability of the initial equation (LE), we establish necessary estimates applying the result of [26] to (LE) considered as an equation in $H_{p,q}$ as well as in $W^{-1,p}(\Omega)$.

Proposition 4.3.3. *Suppose that $\widetilde{A}$ is defined as in Lemma 4.3.2. Then the following results hold:*
1) Let $1 < p,\ q < \infty$, Then for any $u_0 \in H_{p,q}$ and $f \in L^q(0,T;W^{-1,p}(\Omega))$, there exists a unique solution u of (LE) belonging to

$$\mathcal{W} \equiv L^q(0,T;W_0^{1,p}(\Omega))\bigcap W^{1,q}(0,T;W^{-1,p}(\Omega)) \subset C([0,T];H_{p,q}), \quad (4.3.16)$$

and satisfying

$$||u||_{\mathcal{W}} \le C_1(||u_0||_{p,q} + ||f||_{L^q(0,T;W^{-1,p}(\Omega))}), \qquad (4.3.17)$$

where C_1 is a constant depending on T.
2) Let $u_0 \equiv 0$ and $f \in L^q(0,T;H_{p,q})$, $T > 0$. Then there exists a unique solution u of (LE) belonging to

$$\mathcal{W}_0 \equiv L^q(0,T;W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))\bigcap W^{1,q}(0,T;H_{p,q}),$$

and satisfying

$$||u||_{\mathcal{W}_0} \le C_1||f||_{L^q(0,T;H_{p,q})},$$

where C_1 is a constant depending on T.

Proof. In virtue of Remark 4.3.2 the mild solution of (LE) is represented by

$$u(t) = e^{-t\widetilde{A}}u_0 + \int_0^t e^{-(t-s)\widetilde{A}}f(s)ds, \quad t \ge 0.$$

If $t \mapsto f(t)$ belongs to $L^q(0,T;X)$ we set $||f(t)||_{L_t^q(0,T;X)} = ||f||_{L^q(0,T;X)}$. Analogous notations, we are used when $L^q(0,T;X)$ is replaced by another Banach space of functions. For the sake of simplicity, we may consider

$$||v||_{-1,p} \le ||v||_{p,q}, \quad v \in H_{p,q}.$$

Now, to prove that

$$||e^{-t\widetilde{A}}u_0||_{L_t^q(0,T;W_0^{1,p}(\Omega))} \le c_0||u_0||_{p,q}$$

for some $c_0 > 0$, it is sufficient to observe that by Lemma 4.3.4 and Remark 4.3.2,

$$||e^{-t\widetilde{A}}u_0||_{L^q_t(0,T;W^{1,p}_0(\Omega))} \leq \left(\int_0^T ||e^{t\widetilde{A}}u_0||^q dt\right)^{1/q} + \left(\int_0^T ||\widetilde{A}e^{t\widetilde{A}}u_0||^q_{-1,p} dt\right)^{1/q}$$
$$\leq ||e^{t\widetilde{A}}u_0||_{W^{1,q}_t(0,T;W^{-1,p}(\Omega))} \leq c_0||u_0||_{p,q}.$$

For any $f \in L^q(0,T;W^{-1,p}(\Omega))$, set

$$(e^{-\widetilde{A}} * f)(t) = \int_0^t e^{(t-s)\widetilde{A}} f(s)ds, \quad 0 \leq t \leq T.$$

Since $-\widetilde{A}$ generates an analytic semigroup $\{e^{-t\widetilde{A}} : 0 \leq t < \infty\}$ in $W^{-1,p}(\Omega)$ and applying Theorem 4.3.2 of [26] to the equation (LE), we have (4.3.17)(see Theorem 2.3 of [16]) and

$$e^{-\widetilde{A}} * f \in L^q(0,T;W^{1,p}_0(\Omega)) \bigcap W^{1,q}(0,T;W^{-1,p}(\Omega)).$$

The last inclusion relation of (4.3.16) is well known and is an easy consequence of the definition of real interpolation space by the trace method.

The proof of 2) is obtained by applying the argument of 1) term by term to the equation (LE) due to (4.3.1) in the space $H_{p,q}$. □

Remark 4.3.4. *By terms of Proposition 4.3.3, the result of [[26], Theorem 2.1] implies that if $u_0 \in (D(A), L^p(\Omega))_{1/q,q} \equiv Z_{p,q}$ and $f \in L^q(0,T;L^p(\Omega))$, then there exists a unique solution u of (LE) belonging to*

$$\mathcal{W}_1 \equiv L^q(0,T;D(A)) \bigcap W^{1,q}(0,T;L^p(\Omega)) \subset C([0,T];Z_{p,q}), \qquad (4.3.18)$$

and satisfying

$$||u||_{\mathcal{W}_1} \leq C_1(||u_0||_{Z_{p,q}} + ||f||_{L^q(0,T;L^p(\Omega))}), \qquad (4.3.19)$$

where C_1 is a constant depending on T.

4.4 Existence of solutions in the strong sense

This section is to investigate the regularity of solutions for an abstract parabolic type equation (4.1.1) involving p-Laplacian in the strong sense in case for any $u_0 \in H_{p,q}(2 \le p,\ 1 < q < \infty)$ and $f \in L^q(0,T;W^{-1,p}(\Omega))$. Now, we put that

$$Au = \mathcal{A}(x, D_x)u \quad \text{i.e.,} \quad A = \widetilde{A} \tag{4.4.1}$$

which was defined in the previous section, and $\mathcal{A}(x, D_x)$ is restriction to $W_0^{1,p}(\Omega)$ with real coefficients:

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i}) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

where $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$ and $\{a_{ij}(x)\}$ is positive definite uniformly in Ω, i.e., there exists a positive number c_1 such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge c_1|\xi|^2 \tag{4.4.2}$$

for all $x \in \bar{\Omega}$ and all real vectors ξ, $b_i \in C^1(\Omega)$, and $c \in L^\infty(\Omega)$. On the other hand, by this hypothesis, there exists a certain K such that $|b_i(x)| \le K$ and $|c(x)| \le K$ hold almost everywhere.

We denote the pairings between $L^{p'}(\Omega)$ and $L^p(\Omega)$, $W^{-1,p}(\Omega)$ and $W_0^{1,p'}(\Omega)$, and $D(A'_{p'})^*$ and $D(A'_{p'})$ all by $(\cdot,\cdot)$ with no fear of confusion. In what follows this section, the norms on $L^p(\Omega)$, $W_0^{1,p}(\Omega)$, and $W^{-1,p}(\Omega)$ will be denoted by $||\cdot||$, $||\cdot||_1$, and $||\cdot||_{-1}$, respectively.

We may consider that there exists a constant C_0 such that for any $u \in W_0^{1,p}(\Omega)$

$$||u||_1 \le C_0||u||_{D(A)}^{1/2}||u||^{1/2}. \tag{4.4.3}$$

For $u \in L^q(0,T;W_0^{1,p}(\Omega))$, we set

$$B(u(t)) = -\mathrm{div}(|\nabla u(t)|^{p-2}\nabla u(t)). \tag{4.4.4}$$

Now we recall that the operator B is hemicontinuous,

$$(B(u(t)), u(t)) = ||\nabla u(t)||^p,$$

and monotone, i.e., when $p \geq 2$, there exists a positive constant δ such that

$$(B(u_1(t)) - B(u_2(t)), u_1(t) - u_2(t)) \geq \delta ||u_1(t) - u_2(t)||_1^p$$

(cf. [56]). Moreover, we have obtain the following results.

Lemma 4.4.1. *The operator B defined as (4.4.4) is locally Lipschitz continuous, i.e., for $p > 2$ and $r > 0$, there exists a number $L(r) > 0$ such that*

$$||(Bu_1)(t) - (Bu_2)(t)||_{-1} \leq L(r)||u_1(t) - u_2(t)||_1,$$

holds with $||u_1(t)||_1 < r$, $||u_2(t)||_1 < r$. Let $u_1,\ u_2 \in L^q(0,T;W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ with $||u_1|| < r$, $||u_2|| < r$. Then

$$||Bu_1 - Bu_2||_{L^q(0,T_0;L^p(\Omega))} \leq L(r)||u_1 - u_2||_{L^q(0,T_0;W_0^{1,p}(\Omega))}. \tag{4.4.5}$$

Proof. Let $u_1(t),\ u_2(t) \in W_0^{1,p}(\Omega)$. For any $z(t) \in W_0^{1,p}(\Omega)$, considering the boundary value condition, we have

$$\begin{aligned}
&|((Bu_1)(t) - (Bu_2)(t), z(t))| \\
&= -\big(\mathrm{div}(|\nabla u_1(t)|^{p-2}\nabla u_1(t)) - \mathrm{div}(|\nabla u_2(t)|^{p-2}\nabla u_2(t)), z(t)\big) \\
&\leq \big((|\nabla u_1(t)|^{p-2} - |\nabla u_2(t)|^{p-2})\nabla u_1(t), \nabla z(t)\big) \\
&\quad + |\nabla u_2(t)|^{p-2}(\nabla u_1(t) - \nabla u_2(t)), \nabla z(t)).
\end{aligned}$$

So, if $p > 2$, there exists a function $L : \mathbb{R}_+ \to \mathbb{R}$ such that $L(r_1) \leq L(r_2)$ for $r_1 \leq r_2$ and

$$||(Bu_1)(t) - (Bu_2)(t)||_{-1} \leq L(r)||u_1(t) - u_2(t)||_1$$

holds with $||u_1(t)||_1 < r$, $||u_2(t)||_1 < r$. The proof of the second paragraph (4.4.5) is similar. □

From now on, we establish the following results on the local solvability of the following equation;

$$\begin{cases} du(t)/dt + Au(t) + B(u(t)) = f(t), & t \in (0, T] \\ u(0) = u_0. \end{cases} \tag{4.4.6}$$

Theorem 4.4.1. *For any $p > 2$, and $q > 1$, assume that $u_0 \in H_{p,q}$, $f \in L^q(0, T; W^{-1,p}(\Omega))$. Then, there exists a time $T_0 \in (0, T)$ such that the equation (4.4.6) admits a unique solution*

$$u \in L^q(0, T_0; W_0^{1,p}(\Omega)) \cap W^{1,q}(0, T_0; W^{-1,p}(\Omega)) \subset C([0, T_0]; H_{p,q}). \tag{4.4.7}$$

Proof. Let $\alpha = (\frac{p}{p+q})^{1/q}$ and let the constant T_0 satisfy the following inequality:

$$\alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}} < 1, \tag{4.4.8}$$

where C_0, C_1, and $L(r)$ are given by (4.4.3), Proposition 4.3.3, and (4.4.5), respectively. Let B_r be the ball of radius r centered at zero of $L^q(0, T_0; W_0^{1,p}(\Omega))$, i.e., $B_r = \{v \in L^q(0, T_0; W_0^{1,p}(\Omega)) : ||v|| \leq r\}$. Invoking Proposition 4.2.1, for a given $w \in B_1$, the problem

$$\begin{cases} du(t)/dt + Au(t) + B(w(t)) = f(t), & t \in (0, T_0] \\ u(0) = u_0 \end{cases} \tag{4.4.9}$$

has a unique solution $u \in L^q(0, T; W_0^{1,p}(\Omega)) \cap W^{1,q}(0, T_0; W^{-1,p}(\Omega))$. To prove the existence and uniqueness of solutions of semilinear type (4.4.6), by virtue of Proposition 4.3.3, we are going to show that the mapping defined by $w \mapsto u$ maps is strictly contractive from the ball B_r into itself if the condition

(4.4.8) is satisfied. Let $u_1,\ u_2$ be the solutions of (4.4.9) with w replaced by $w_1,\ w_2 \in B_r$. Then from Remark 4.3.4, it follows

$$
\begin{aligned}
||u_1 - u_2||_{L^q(0,T_0;D(A))\cap W^{1,q}(0,T_0;L^p(\Omega))} &\le C_1||Bw_1 - Bw_2||_{L^q(0,T_0;L^p(\Omega))} \\
&\le C_1 L(r)||w_1 - w_2||_{L^q(0,T_0;W_0^{1,p}(\Omega))}.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
||u_1 - u_2||_{L^q(0,T_0;L^p(\Omega))} &= \left\{ \int_0^{T_0} ||u_1(t) - u_2(t)||^q dt \right\}^{1/q} \qquad (4.4.10) \\
&\le \left\{ \int_0^{T_0} || \int_0^t (\dot{u}_1(s) - \dot{u}_2(s)) ds||^q dt \right\}^{1/q} \\
&\le \left\{ \int_0^{T_0} t^{q/p} \int_0^t ||\dot{u}_1(s) - \dot{u}_2(s)||^q ds dt \right\}^{1/q} \\
&\le \alpha T_0^{\frac{p+q}{pq}} ||u_1 - u_2||_{W^{1,q}(0,T_0;L^p(\Omega)))},
\end{aligned}
$$

and in view of (4.4.3), we have

$$
\begin{aligned}
&||u_1 - u_2||_{L^q(0,T_0;W_0^{1,p}(\Omega))} \le C_0||u_1 - u_2||^{1/2}_{L^q(0,T_0;D(A))}||u_1 - u_2||^{1/2}_{L^q(0,T_0;L^p(\Omega))} \\
&\le C_0||u_1 - u_2||^{1/2}_{L^q(0,T_0;D(A))}(\frac{p}{p+q})^{1/q} T_0^{\frac{p+q}{pq}} ||u_1 - u_2||^{1/2}_{W^{1,q}(0,T_0;L^p(\Omega))} \\
&\le \alpha C_0 T_0^{\frac{p+q}{pq}} ||u_1 - u_2||_{L^q(0,T_0;D(A))\cap W^{1,q}(0,T_0;L^p(\Omega))} \\
&\le \alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}} ||w_1 - w_2||_{L^q(0,T_0;W_0^{1,p}(\Omega))}. \qquad (4.4.11)
\end{aligned}
$$

So by virtue of (4.4.11), the mapping defined by $w \mapsto u$ maps is strictly contractive from B_r into itself. Therefore, the contraction mapping principle gives that the equation (4.4.6) has a unique solution in $[0, T_0]$. Since $\widetilde{A}$ is an isomorphism from $W_0^{1,p}(\Omega)$ onto $W^{-1,p}(\Omega)$ by Lemma 4.3.1, the solution of (4.4.6) belongs to $W^{1,q}(0, T_0; W^{-1,p}(\Omega))$. The last inclusion relation (4.4.7) is well known and is an easy consequence of the definition of real interpolation spaces by the trace method. □

Theorem 4.4.2. *For any $p > 2$, $q > 1$, and $T > 0$, assume that $u_0 \in H_{p,q}$, $f \in L^q(0,T;W^{-1,p}(\Omega))$. Then, the solution u of (4.4.6) exists and is unique in*

$$\mathcal{W} \equiv L^q(0,T;W_0^{1,p}(\Omega)) \bigcap W^{1,q}(0,T;W^{-1,p}(\Omega)) \subset C([0,T];H_{p,q}), \quad (4.4.12)$$

and satisfying

$$||u||_{\mathcal{W}} \le C_2(||u_0||_{p,q} + ||f||_{L^q(0,T;W^{-1,p}(\Omega))}), \quad (4.4.13)$$

where $||\cdot||_{p,q}$ is the norm as an element of $H_{p,q}$, and C_2 is a constant depending on T and $\mathcal{W}$.

Proof. Let x be a solution of (4.4.6) and let w be the solution of the following linear functional differential equation parabolic type;

$$\begin{cases} dw(t)/dt + Aw(t) = f(t), \quad t \in (0,T_0]. \\ w(0) = u_0. \end{cases}$$

Then we have

$$\begin{cases} d(u-w)(t)/dt + A((u-w)(t)) = -B(u(t)), \quad t \in (0,T_0]. \\ (u-w)(0) = 0. \end{cases}$$

Suppose that x and y belong to B_r. In view of (4.3.1) and (4.3.19), we have

$$\begin{aligned} &||u-w||_{L^q(0,T_0;D(A))\cap W^{1,q}(0,T_0;L^p(\Omega))} \le C_1||Bu||_{L^q(0,T_0;L^p(\Omega))} \\ &\le C_1L(r)||u||_{L^q(0,T_0:W_0^{1,p}(\Omega))} \\ &\le C_1L(r)(||u-w||_{L^q(0,T_0:W_0^{1,p}(\Omega))} + ||w||_{L^q(0,T_0:W_0^{1,p}(\Omega))}). \end{aligned}$$

Thus arguing as in the proof of (4.4.11)

$$\begin{aligned} &||u-w||_{L^q(0,T_0;W_0^{1,p}(\Omega))} \le \alpha C_0 T_0^{\frac{p+q}{pq}} ||u-w||_{L^q(0,T_0;D(A))\cap W^{1,q}(0,T_0;L^p(\Omega))} \\ &\le \alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}} (||u-w||_{L^q(0,T_0;W_0^{1,p}(\Omega))} + ||w||_{L^q(0,T_0;W_0^{1,p}(\Omega))}). \end{aligned}$$

Therefore, we have

$$||u-w||_{L^q(0,T_0;W_0^{1,p}(\Omega))} \leq \frac{\alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}}}{1-\alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}}} ||w||_{L^q(0,T_0;W_0^{1,p}(\Omega))}\},$$

and hence, with the aid of 1) of Proposition 4.3.3

$$\begin{aligned} ||u||_{L^q(0,T_0;W_0^{1,p}(\Omega))} &\leq \frac{1}{1-\alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}}} ||w||_{L^q(0,T_0;W_0^{1,p}(\Omega))} \\ &\leq \frac{C_1}{1-\alpha C_0 C_1 L(r) T_0^{\frac{p+q}{pq}}} (||u_0||_{p,q} + ||f||_{L^q(0,T_0;W^{-1,p}(\Omega))}). \end{aligned} \tag{4.4.14}$$

We know that there exists a positive constant M_0 such that for any $v \in L^p(\Omega)$

$$||v||_{-1,p} \leq M_0 ||v||_p. \tag{4.4.15}$$

On the other hand, using Proposition 4.3.3, Remark 4.3.4, and (4.4.15) we get

$$\begin{aligned} &||u||_{L^q(0,T_0;W_0^{1,p}(\Omega))\cap W^{1,p}(0,T_0;W^{-1,p}(\Omega))} \\ &\leq C_1(||u_0||_{p,q} + ||Bu+f||_{L^q(0,T_0;W^{-1,p}(\Omega))}) \\ &\leq C_1(||u_0||_{p,q} + M_0||Bu||_{L^q(0,T_0;L^p(\Omega))} + ||f||_{L^q(0,T_0;W^{-1,p}(\Omega))}) \\ &\leq C_1(||u_0||_{p,q} + ||f||_{L^q(0,T;W^{-1,p}(\Omega))} + M_0 L(r)||u||_{L^q(0,T_0;W_0^{1,p}(\Omega))}). \end{aligned} \tag{4.4.16}$$

Combining (4.4.14) and (4.4.16) we obtain

$$||u||_{L^q(0,T_0;W_0^{1,p}(\Omega))\cap W^{1,q}(0,T_0;W^{-1,p}(\Omega))} \leq C(||u_0||_{p,q} + ||f||_{L^q(0,T;W^{-1,p}(\Omega))}) \tag{4.4.17}$$

for some constant C. Now from Theorem 4.4.1 it follows that

$$||u(T_0)||_{p,q} \leq ||u||_{C([0,T_0];H_{p,q})} \leq C_2(||u_0||_{p,q} + ||f||_{L^2(0,T_0;W^{-1,p}(\Omega))}). \tag{4.4.18}$$

So, we can solve the equation in $[T_0, 2T_0]$ and obtain an analogous estimate to (4.4.17). Since the condition (4.4.8) is independent of initial values, the solution of (4.4.6) can be extended the internal $[0, nT_0]$ for a natural number n, i.e., for the initial $u(nT_0)$ in the interval $[nT_0, (n+1)T_0]$, as analogous estimate (4.4.17) holds for the solution in $[0, (n+1)T_0]$. Furthermore, the estimate (4.4.13) is easily obtained from (4.4.17) and (4.4.18). □

The following inequality is refereed to as the Young inequality.

Lemma 4.4.2. *(Young inequality) Let* $a > 0,\ b > 0$ *and* $1/p + 1/q = 1$ *where* $1 \leq p, \infty$ *and* $1 < q < \infty$. *Then for every* $\lambda > 0$ *one has*

$$ab \leq \frac{\lambda^p a^p}{p} + \frac{b^q}{\lambda^q q} \ .$$

Theorem 4.4.3. *For any* $p \geq 2,\ q > 1,$ *and* $T > 0$, *assume that* $(u_0, f) \in Z_{p,q} \times L^q(0,T; L^p(\Omega))$ *where* $Z_{p,q} \equiv (D(A), L^p(\Omega))_{1/q,q}$. *Then the solution* u *of the equation (4.4.6) belongs to* $u \in L^q(0,T; D(A)) \cap W^{1,q}((0,T); L^p(\Omega)) \subset C([0,T]; Z_{p,q})$ *and the mapping*

$$Z_{p,q} \times L^q(0,T; L^p(\Omega)) \ni (u_0, f) \mapsto u \in L^q(0,T; D(A)) \cap W^{1,q}((0,T); L^p(\Omega))$$

is Lipschitz continuous.

Proof. It is easy to show that if $u_0 \in Z_{p,q}$ and $f \in L^q(0,T; L^p(\Omega))$, then u belongs to $L^q(0,T; D(A)) \cap W^{1,q}(0,T; L^p(\Omega))$. Let $(u_{0i}, f_i) \in Z_{p,q} \times L^q(0,T; L^p(\Omega))$ and $u_i \in B_r \subset L^q(0,T; D(A))$ be the solution of (4.4.6) with (u_{0i}, f_i) in place of (u_0, f) for $i = 1,\ 2$. Then in view of Proposition 4.3.3, we have

$$||u_1 - u_2||_{L^q(0,T;D(A)) \cap W^{1,q}(0,T;L^p(\Omega))} \tag{4.4.19}$$
$$\leq C_1\{||u_{01} - u_{02}||_{Z_{p,q}} + ||Bu_1 - Bu_2||_{L^q(0,T;L^p(\Omega))} + ||f_1 - f_2||_{L^q(0,T;L^p(\Omega))}\}$$
$$\leq C_1\{||u_{01} - u_{02}||_{Z_{p,q}} + L(r)||u_1 - u_2||_{L^q(0,T:W_0^{1,p}(\Omega))} + ||f_1 - f_2||_{L^q(0,T;L^p(\Omega))}\}.$$

Since

$$u_1(t) - u_2(t) = u_{01} - u_{02} + \int_0^t (\dot{u}_1(s) - \dot{u}_2(s))ds,$$

and, by (4.4.10) we get

$$||u_1 - u_2||_{L^q(0,T;L^p(\Omega))} \le \sqrt[q]{T}||u_{01} - u_{02}||_p + \alpha T^{\frac{p+q}{pq}}||u_1 - u_2||_{W^{1,q}(0,T;L^p(\Omega))}.$$

Hence, by Lemma 4.4.2 and regarding as $||\cdot||_p \le ||\cdot||_{Z_{p,q}}$, we get

$$||u_1 - u_2||_{L^q(0,T;W_0^{1,p}(\Omega))} \le C_0||u_1 - u_2||^{1/2}_{L^q(0,T;D(A))}||u_1 - u_2||^{1/2}_{L^q(0,T;L^p(\Omega))} \tag{4.4.20}$$

$$\le C_0||u_1 - u_2||^{1/2}_{L^q(0,T;D(A))}\{T^{1/(2q)}||u_{01} - u_{02}||^{1/2}_{Z_{p,q}} + (\alpha T^{\frac{p+q}{pq}})^{1/2}||u_1 - u_2||^{1/2}_{W^{1,q}(0,T;L^p(\Omega))}\}$$

$$\le C_0T^{1/(2q)}||u_{01} - u_{02}||^{1/2}_{Z_{p,q}}||u_1 - u_2||^{1/2}_{L^q(0,T;D(A))}$$

$$+ C_0\sqrt{\alpha}T^{\frac{p+q}{2pq}}||u_1 - u_2||_{L^q(0,T;D(A))\cap W^{1,q}(0,T;L^p(\Omega))}$$

$$\le \frac{1}{4}\alpha^{-1/2}C_0T^{\frac{p-q}{2pq}}||u_{01} - u_{02}||_{Z_{p,q}} + 2C_0\sqrt{\alpha}T^{\frac{p+q}{2pq}}||u_1 - u_2||_{L^q(0,T;D(A))\cap W^{1,q}(0,T;L^p(\Omega))}.$$

Combining (4.4.19) with (4.4.20) we obtain

$$||u_1 - u_2||_{L^q(0,T;D(A))\cap W^{1,q}(0,T;L^p(\Omega))} \tag{4.4.21}$$

$$\le C_1\{||u_{01} - u_{02}||_{Z_{p,q}} + ||f_1 - f_2||_{L^q(0,T;L^p(\Omega))}\} + C_0C_1\frac{1}{4}\alpha^{-1/2}T^{\frac{p-q}{2pq}}||u_{01} - u_{02}||_{Z_{p,q}}$$

$$+ 2C_0C_1\sqrt{\alpha}T^{\frac{p+q}{2pq}}L(r)||u_1 - u_2||_{L^q(0,T;D(A))\cap W^{1,q}(0,T;L^p(\Omega))}.$$

Suppose that $(u_{0n}, f_n) \to (u_0, f)$ in $Z_{p,q} \times L^2(0,T;H)$, and let u_n and u be the solutions (4.4.6) with (u_{0n}, f_n) and (u_0, f) respectively. Let $0 < T_1 \le T$ be such that

$$2C_0C_1\sqrt{\alpha}T^{\frac{p+q}{2pq}}L(r) < 1.$$

Then by virtue of (4.4.21) with T replaced by T_1 we see that

$$u_n \to u \quad \text{in} \;\; L^2(0,T_1;D(A)) \cap W^{1,2}(0,T_1;L^p(\Omega)).$$

This implies that $u_n(T_1) \mapsto u(T_1)$ in $Z_{p,q}$. Hence the same argument shows that $u_n \to u$ in

$$L^q(T_1, \min\{2T_1, T\}; D(A)) \cap W^{1,q}(T_1, \min\{2T_1, T\}; L^p(\Omega)).$$

Repeating this process we conclude that $u_n \to u$ in $L^q(0,T;D(A)) \cap W^{1,q}(0,T;L^p(\Omega))$. $\square$

Remark 4.4.1. *The result of Theorem 4.4.3 is important to apply for the control problems and the optimal control theory for technologically given cost functions. In particular, by the similar way to Theorem 4.4.3 we have that if* $(u_0, f) \in H_{p,q} \times L^q(0,T;W^{-1,p}(\Omega))$ *for any* $p > 2$, $q > 1$, *and* $T > 0$. *Then the solution* u *of the equation (4.4.6) belongs to* $u \in L^q(0,T;V) \cap W^{1,q}(0,T;W^{-1,p}(\Omega)) \subset C([0,T];H)$, *and the mapping*

$$H_{p,q} \times L^q(0,T;W^{-1,p}(\Omega)) \ni (u_0, f) \mapsto u \in L^q(0,T;W_0^{1,p}(\Omega)) \cap W^{1,q}(0,T;W^{-1,p}(\Omega))$$

is continuous.

Chapter 5
Approximate controllability for semilinear integro-differential control equations in Hilbert spaces

5.1 Introduction

In this paper, we deal with the approximate controllability for semilinear integro-differential functional control equations in the form

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \quad 0 < t \le T, \\ x(0) & = x_0 \end{cases} \tag{5.1.1}$$

in a Hilbert space H, where k belongs to $L^2(0,T)(T > 0)$ and g is a nonlinear mapping as detailed in Section 2. The principal operator A generates an analytic semigroup $(S(t))_{t\ge 0}$ and B is a bounded linear operator from another Hilbert space U to H.

The controllability problem is a question of whether is possible to steer a dynamic system from an initial state to an arbitrary final state using the set of admissible controls. Naito [64] was the first to deal with the range condition argument of controller in order to obtain the approximate controllability of a semilinear control system. In [22, 46, 81, 85], they have studied continuously about controllability of semilinear systems dominated by linear parts(in case $g \equiv 0$) by assuming that $S(t)$ is compact operator for each $t > 0$ as matters connected with [64]. Another approach used to obtain sufficient conditions for approximate solvability of nonlinear equations is a fixed point theorem combined with technique of operator transformations by configuring the resolvent as seen in [8]

The controllability for various nonlinear equations has been studied by many authors, for example, see [28, 29, 61] for local controllability of neutral functional differential systems with unbounded delay, [53, 70] for neutral evolution integrodifferential systems with state dependent delay, and [68] for impulsive neutral functional evolution integrodifferential systems with infinite delay. Moreover, the approximate controllability for semi-linear retarded stochastic systems has been studied by [60, 62, 63].

Sukavanam and Tomar [75] studied the approximate controllability for the general retarded initial value problem by assuming that the Lipschitz constant of the nonlinear term is less then 1, and Wang [81] for general retarded semilinear equations assuming the growth condition of the nonlinear term and the compactness of the semigroup.

In this paper, authors want to use a different method than the previous one. Our used tool is the theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda T(x) - F(x) = y$ in dependence on the real number λ, where T and F are nonlinear operators defined a Banach space X with values in a Banach space Y. In order to obtain the approximate controllability for a class of semilinear integro-differential functional control equations, it is necessary to suppose that T acts as the identity operator while F related to the nonlinear term of (5.1.1) is completely continuous.

In Section 2, we introduce regularity properties for (5.1.1). Since we apply the Fredholm theory in the proof of the main theorem, we assume some compactness of the embedding between intermediate spaces. Then by virtue of Aubin [6], we can show that the solution mapping of a control space to the terminal state space is completely continuous. Based on Section 2, it is shown the sufficient conditions on the controller and nonlinear terms for approximate controllability for (5.1.1) by using the Fredholm theory. Finally, a simple example to which our main result can be applied is given.

5.2 Semilinear functional equations

Throughout this Chapter, as seen in Section 2.2, V, H and V^* are complex Hilbert spaces forming a Gelgand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*.$$

Moreover, A in System (6.1.1) is also the operator in place of $-A$ in Section 2.2. It is known that A is a bounded linear operator from V to V^*, and A generates an analytic semigroup $S(t)(t \geq 0)$ in both of H and V^*(see [76]).

Consider the following initial value problem for the abstract semilinear parabolic equation

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + \int_0^t k(t-s)g(s, x(s), u(s))ds + Bu(t), \\ x(0) & = x_0. \end{cases} \tag{5.2.1}$$

Let U be a Hilbert space and the controller operator B be a bounded linear operator from U to H.

Let $g : \mathbb{R}^+ \times V \times U \to H$ be a nonlinear mapping satisfying the following:

Assumption (DF).

(i) For any $x \in V$, $u \in U$ the mapping $g(\cdot, x, u)$ is strongly measurable;

(ii) There exist positive constants L_0, L_1, L_2 such that

(a) $u \mapsto g(t, x, u)$ is an odd mapping $(g(\cdot, x, -u) = -g(\cdot, x, u))$;

(b) for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $u, \hat{u} \in U$,

$$|g(t, x, u) - g(t, \hat{x}, \hat{u})| \leq L_1||x - \hat{x}|| + L_2||u - \hat{u}||_U,$$

$$|g(t, 0, 0)| \leq L_0.$$

For $x \in L^2(0,T;V)$, we set

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds$$

where k belongs to $L^2(0,T)$.

Lemma 5.2.1. *Let Assumption (DF) be satisfied. Assume that $x \in L^2(0,T;V)$ for any $T > 0$. Then $f(\cdot,x,u) \in L^2(0,T;H)$ and*

$$\begin{aligned} &||f(\cdot,x,u))||_{L^2(0,T;H)} \le L_0||k||_{L^2(0,T)}T/\sqrt{2} \\ &\quad + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}). \end{aligned} \tag{5.2.2}$$

Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, then

$$\begin{aligned} &||f(\cdot,x,u) - f(\cdot,\hat{x},\hat{u})||_{L^2(0,T;H)} \\ &\le ||k||_{L^2(0,T)}\sqrt{T}(L_1||x-\hat{x}||_{L^2(0,T;V)} + L_2||u-\hat{u}||_{L^2(0,T;U)}). \end{aligned} \tag{5.2.3}$$

Proof. From Assumption (DF), and using the Hölder inequality, it is easily seen that

$$\begin{aligned} &||f(\cdot,x,u)||_{L^2(0,T;H)} \le ||f(\cdot,0,0)|| + ||f(\cdot,x,u) - f(\cdot,0,0)|| \\ &\le \Big(\int_0^T |\int_0^t k(t-s)g(s,0,0)ds|^2 dt\Big)^{1/2} \\ &\quad + \Big(\int_0^T |\int_0^t k(t-s)\{g(s,x(s),u(s)) - g(s,0,0)\}ds|^2 dt\Big)^{1/2} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}||g(\cdot,x,u) - g(\cdot,0,0)||_{L^2(0,T;H)} \\ &\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}). \end{aligned}$$

The proof of (5.2.3) is similar. □

By virtue of Theorem 2.1 of [45], we have the following result on (5.2.1).

Proposition 5.2.1. *Let Assumption (DF) be satisfied. Then there exists a unique solution x of (5.2.1) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

for any $x_0 \in H$. Moreover, there exists a constant C_3 such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_3(|x_0| + ||u||_{L^2(0,T;U)}). \tag{5.2.4}$$

Corollary 5.2.1. *Assume that the embedding $D(A) \subset V$ is completely continuous. Let Assumption (DF) be satisfied, and x_u be the solution of equation (5.2.1) associated with $u \in L^2(0,T;U)$. Then the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$.*

Proof. If $u \in L^2(0,T;U)$, then in view of Lemma 2.2.4

$$||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_3(|x_0| + ||B||\,||u||_{L^2(0,T;U)}). \tag{5.2.5}$$

Since $x_u \in L^2(0,T;V)$, we have $f(\cdot, x_u, u) \in L^2(0,T;H)$. Consequently

$$x_u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H).$$

Hence, with aid of Lemma 2.2.4, (5.2.2) and (5.2.4),

$$\begin{aligned}
&||x_u||_{L^2(0,T;D(A)\cap W^{1,2}(0,T;H)} \leq C_1(||x_0|| + ||f(\cdot,x_u,u) + Bu||_{L^2(0,T;H)}) \\
&\leq C_1\{||x_0|| + L_0||k||_{L^2(0,T)}T/\sqrt{2} \\
&\quad + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}) + ||Bu||_{L^2(0,T;H)}\} \\
&\leq C_1\big[||x_0|| + L_0||k||_{L^2(0,T)}T/\sqrt{2} \\
&\quad + ||k||_{L^2(0,T)}\sqrt{T}\big\{L_1C_3(|x_0| + ||u||_{L^2(0,T;U)}) + L_2||u||_{L^2(0,T;U)}\big\} + ||Bu||_{L^2(0,T;H)}\big].
\end{aligned}$$

Thus, if u is bounded in $L^2(0,T;U)$, then so is x_u in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is completely continuous in view of Theorem 2 of [6], the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

5.3 Approximate controllability

Throughout this section, we assume that $D(A)$ is compactly embedded in V. Let $x(T; f, u)$ be a state value of the system (5.2.1) at time T corresponding to the nonlinear term f and the control u. We define the reachable sets for the system (5.2.1) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

Definition 5.3.1. *The system (5.2.1) is said to be approximately controllable in the time interval* $[0, T]$ *if for every desired final state* $x_1 \in H$ *and* $\epsilon > 0$ *there exists a control function* $u \in L^2(0, T; U)$ *such that the solution* $x(T; f, u)$ *of (5.2.1) satisfies* $|x(T; f, u) - x_1| < \epsilon$*, that is, if* $\overline{R_T(f)} = H$ *where* $\overline{R_T(f)}$ *is the closure of* $R_T(f)$ *in* H*, then the system (5.2.1) is called approximately controllable at time* T*.*

Let us introduce the theory of the degree for completely continuous perturbations of the identity operator, which is the infinite dimensional version of Borsuk's theorem. Let $0 \in D$ be a bounded open set in a Banach space X, $\overline{D}$ its closure and ∂D its boundary. The number $d[I - T; D, 0]$ is the degree of the mapping $I - T$ with respect to the set D and the point 0 (see Fučik et al. [27] or Lloid [57]).

Theorem 5.3.1. *(Borsuk's theorem) Let* D *be a bounded open symmetric set in a Banach space* X*,* $0 \in D$*. Suppose that* $T : \overline{D} \to X$ *be odd completely continuous operator satisfying* $T(x) \neq x$ *for* $x \in \partial D$*. Then* $d[I - T; D, 0]$ *is odd integer. That is, there exists at least one point* $x_0 \in D$ *such that* $(I - T)(x_0) = 0$*.*

Definition 5.3.2. *Let* T *be a mapping defined by on a Banach space* X *with value in a real Banach space* Y*. The mapping* T *is said to be a* (K, L, α)*-homeomorphism of* X *onto* Y *if* *(i)* *T is a homeomorphism of X onto Y;*

(ii) *there exist real numbers $K > 0$, $L > 0$, and $\alpha > 0$ such that*

$$L||x||_X^\alpha \leq ||T(x)||_Y \leq K||x||_X^\alpha, \quad \forall x \in X.$$

Lemma 5.3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ a continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty.$$

Proof. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\} \subset X$ such that

$$||\lambda T(x_n) - F(x_n)||_Y \leq M \tag{5.3.1}$$

as $x_n \to \infty$. From (5.3.1) it follows that

$$\frac{\lambda T(x_n)}{||x_n||_X^\alpha} - \frac{F(x_n)}{||x_n||_X^\alpha} \to 0.$$

Hence, we have

$$\limsup_{n \to \infty} \frac{|\lambda| ||T(x_n)||_Y}{||x_n||_X^\alpha} = N,$$

and so, $|\lambda| K \geq N \geq |\lambda| L$. It is a contradiction with $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}]$. □

Proposition 5.3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator. Suppose that for $\lambda \neq 0$,*

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \tag{5.3.2}$$

Then $\lambda T - F$ maps X onto Y.

Proof. We follow the proof Theorem 1.1 in Chapter II of Fučik et al. [27]. Suppose that there exists $y \in Y$ such that $\lambda T(x) = y$. Then from (5.3.2) it follows that $FT^{-1} : Y \to Y$ is an odd completely continuous operator and

$$\lim_{||y||_Y \to \infty} ||y - FT^{-1}(\frac{y}{\lambda})||_Y = \infty.$$

Let $y_0 \in Y$. There exists $r > 0$ such that

$$||y - FT^{-1}(\frac{y}{\lambda})||_Y > ||y_0||_Y \geq 0$$

for each $y \in Y$ satisfying $||y||_Y = r$. Let $Y_r = \{y \in Y : ||y||_Y < r\}$ be a open ball. Then by view of Theorem 5.3.1, we have $d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0]$ is an odd number. For each $y \in Y$ satisfying $||y||_Y = r$ and $t \in [0, 1]$, there is

$$||y - FT^{-1}(\frac{y}{\lambda}) - ty_0||_Y \geq ||y - FT^{-1}(\frac{y}{\lambda})||_Y - ||y_0||_Y > 0$$

and hence, by the homotopic property of degree, we have

$$d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, y_0] = d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0] \neq 0.$$

Hence, by the existence theory of the Leray-Schauder degree, there exists a $y_1 \in Y_r$ such that

$$y_1 - FT^{-1}(\frac{y_1}{\lambda}) = y_0.$$

We can choose $x_0 \in X$ satisfying $\lambda T(x_0) = y_1$, and so, $\lambda T(x_0) - F(x_0) = y_0$. Thus, it implies that $\lambda T - F$ is a mapping of X onto Y. □

Combining Lemma 5.3.1 and Proposition 5.3.1, we have the following results.

Corollary 5.3.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then $\lambda T - F$ maps X onto Y. Therefore, if $N = 0$, then for all $\lambda \neq 0$ the operator $\lambda T - F$ maps X onto Y.

First we consider the approximate controllability of the system (5.2.1) in case where the controller B is the identity operator on H under Assumption (DF) on the nonlinear operator f in Section 5.2. Hence, noting that $H = U$, we consider the linear system given by

$$\begin{cases} \frac{d}{dt} y(t) &= Ay(t) + u(t), \\ y(0) &= x_0, \end{cases} \tag{5.3.3}$$

and the following semilinear control system

$$\begin{cases} \frac{d}{dt} x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \\ x(0) &= x_0. \end{cases} \tag{5.3.4}$$

Theorem 5.3.2. *Assume that*

$$\limsup_{||u|| \to \infty} \frac{||f(\cdot, x_u, u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} < 1. \tag{5.3.5}$$

Under the Assumption (DF) we have

$$R_T(0) \subset R_T(f).$$

Therefore, if the linear system (5.3.3) with $f = 0$ is approximately controllable, then so is the semilinear system (5.3.4).

Proof. Let $x(t)$ be solution of (5.3.4) corresponding to a control u. First, we show that there exist a $v \in L^2(0,T;H)$ such that

$$\begin{cases} v(t) &= u(t) - f(t, x(t), v(t)), \quad 0 < t \le T, \\ v(0) &= u(0). \end{cases}$$

Let us define an operator $F : L^2(0,T;H) \to L^2(0,T;H)$ as

$$Fv = -f(\cdot, x_v, v).$$

Then by Corollary 5.2.1, F is a compact mapping from $L^2(0,T;H)$ to itself, and we have

$$\lim_{||v|| \to \infty} ||\lambda I(v) - F(v)||_{L^2(0,T;H)} = \infty,$$

where the identity operator I on $L^2(0,T;H)$ is an odd $(1,1,1)$-homeomorphism. Thus, from (5.3.5) and Corollary 5.3.1, if $\lambda \ge 1$ then $\lambda I - F$ maps $L^2(0,T;H)$ onto itself. Hence, we have showed that there exists a $v \in L^2(0,T;H)$ such that $v(t) = u(t) - f(t, y(t), v(t))$. Let y and x be solutions of (5.3.3) and (5.3.4) corresponding to controls u and v, respectively. Then, equation (5.3.4) is rewritten as

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + v(t), \ 0 < t \le T \\ &= Ax(t) + f(t, x(t), v(t)) + u(t) - f(t, y(t), v(t)) \\ &= Ax(t) + u(t) \end{aligned}$$

with $x(0) = x_0$, which means

$$\begin{aligned} x(t) =& S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s), v(s)) + v(s)\}ds \\ =& S(t)x_0 + \int_0^t S(t-s)u(s)ds = y(t), \end{aligned}$$

where y be solution of (5.3.3) corresponding to a control u. Therefore, we have proved that $R_T(0) \subset R_T(f)$. $\square$

Corollary 5.3.2. *Let us assume that*

$$||k||_{L^2(0,T)}\sqrt{T}\big(L_1C_3 + L_2\big) < 1,$$

where C_3 is the constant in Proposition 5.2.1. Under the Assumption (DF), we have

$$R_T(0) \subset R_T(f)$$

in case where $B \equiv I$.

Proof. By Lemma 5.2.1 and Proposition 5.2.1, we have

$$\begin{aligned}
&||Fu||_{L^2(0,T;H)} = ||f(\cdot, x_u, u)||_{L^2(0,T;H)} \\
&\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}(L_1||x||_{L^2(0,T;V)} + L_2||u||_{L^2(0,T;U)}) \\
&\le L_0||k||_{L^2(0,T)}T/\sqrt{2} + ||k||_{L^2(0,T)}\sqrt{T}\big\{L_1C_3\big(|x_0| + ||u||_{L^2(0,T;U)}\big) + L_2||u||_{L^2(0,T;U)}\big\}.
\end{aligned}$$

Hence, we have

$$\limsup_{||u||\to\infty} \frac{||F(u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;U)}} \le ||k||_{L^2(0,T)}\sqrt{T}\big(L_1C_3 + L_2\big).$$

Thus, from Theorem 5.3.2, it follows that if $\lambda \ge 1$ then $\lambda I - F$ maps $L^2(0,T;H)$ onto itself, and so, by the same argument as in the proof of theorem it holds that $R_T(0) \subset R_T(f)$. □

From now on, we consider the initial value problem for the semilinear parabolic equation (5.2.1). Let U be some Hilbert space and the controller operator B be a bounded linear operator from U to H.

Assumption (DB) There exists a constant $\beta > 0$ such that $\mathrm{R}(f) \subset \mathrm{R}(B)$ and

$$||Bu|| \ge \beta||u||, \quad \forall u \in L^2(0,T;U).$$

Consider the linear system given by

$$\begin{cases} \frac{d}{dt}y(t) &= Ay(t) + Bu(t), \\ y(0) &= x_0. \end{cases} \tag{5.3.6}$$

Theorem 5.3.3. *Under the Assumptions (5.3.5), (DB) and (DF), we have*

$$R_T(0) \subset R_T(f).$$

Therefore, if the linear system (5.3.6) with $f = 0$ is approximately controllable, then so is the semilinear system (5.2.1).

Proof. Let y be a solution of the linear system (5.3.6) with $f = 0$ corresponding to a control u, and let x be a solutions of the semilinear system (5.3.4) corresponding to a control v. Set $v(t) = u(t) - B^{-1}f(t, x(t), v(t))$. Then, system (5.2.1) is rewritten as

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + f(t, x(t), v(t)) + Bv(t),\ 0 < t \leq T \\ &= Ax(t) + f(t, x(t), v(t)) + Bu(t) - f(t, x(t), v(t))\end{aligned}$$

with $x(0) = x_0$. Hence, we have

$$\begin{aligned}x(t) =& S(t)x_0 + \int_0^t S(t-s)\{f(s, x(s), v(s)) + v(s))\}ds \\ =& S(t)x_0 + \int_0^t S(t-s)u(s))ds = y(t).\end{aligned}$$

Thus, we obtain that $R_T(0) \subset R_T(f)$. □

Example. We consider the semilinear heat equation dealt with by Zhou [85], and Naito [64]. Let

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi),$$

$$a(u, v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0, \pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$\begin{cases} \frac{d}{dt}y(x,t) & = Ay(x,t) + \int_0^t k(t-s)g(s,x(s),u(s))ds + Bu(t), \\ y(t,0) & = y(t,\pi) = 0, \quad t > 0, \\ y(0,x) & = \phi^0(x), \quad y(x,s) = \phi^1(x,s), \quad -h \le s < 0, \end{cases} \tag{5.3.7}$$

where k belongs to $L^2(0,T)$. The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Let

$$U = \{\sum_{n=2}^{\infty} u_n\phi_n : \sum_{n=2}^{\infty} u_n^2 < \infty\},$$

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n, \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n \in U.$$

It is easily seen that the operator B is one to one and R(B) is closed. It follows that the operator B satisfies hypothesis as in Theorem 5.3.3. We can see many examples which satisfy Assumption (DB) as seen in [85, 86].

For any $x = \sum_{n=1}^{\infty} x_n\phi_n \in L^2(0,\pi)$, consider the nonlinear term g given by

$$g(t,x,u) = \sum_{n=1}^{\infty}(\sin x_n)\phi_n(x) + \frac{\sqrt[n]{||u||}\phi_2(x)}{u}, \ \ n > 2.$$

It is easily seen that Assumption (DF) is satisfied. For $x \in L^2(0,T;V)$ and $k \in L^2(0,\pi)$, we set

$$f(t,x,u) = \int_0^t k(t-s)g(s,x(s),u(s))ds.$$

Then

$$||f(\cdot,x,u)||_{L^2(0,T;H)} = \left\{\int_0^T |\int_0^t k(t-s)\big(\sum_{n=1}^{\infty}(\sin x_n)\phi_n(x) + \sqrt{||u||}\phi_2(x)\big)ds|dt\right\}^{1/2}$$

$$\le \sqrt{T}||k||_{L^2(0,\pi)}\big(\sum_{n=1}^{\infty}|\sin x_n|^2 + \sqrt[n]{||u||^2}\big).$$

Hence, we have

$$\limsup_{||u||\to\infty} \frac{||f(\cdot,x,u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} = 0.$$

and $\mathrm{R}(g) \subset \mathrm{R}(B)$. From Theorem 5.3.3 it follows that the system of (5.3.7) is approximately controllable. Therefore, we obtain the approximate controllability of (5.3.7) without restrictions such as the uniform boundedness and inequality constraints for Lipschitz constant of f or compactness of $S(t)$.

Chapter 6
Controllability for abstract semilinear control systems with homogeneous properties

6.1 Introduction

In this paper, we deal with the approximate controllability for a semilinear control system in the form:

$$\begin{cases} \frac{d}{dt}x(t) & = Ax(t) + f(t, x(t)) + (Bu)(t), \quad 0 < t \leq T, \\ x(0) & = x_0. \end{cases} \tag{6.1.1}$$

Let V and H be complex Hilbert spaces forming a Gelgand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

by identifying the antidual of H with H, where V is a Hilbert space densely and continously embedded in H. Here, A is the operator associate with a sesquilinear form satisfying Gårding's inequality as detailed in Section 2. The motivation for the choice of Hilbert spaces setting for System (6.1.1) is the application to L^2-regularity using fact that the principal operator A generates an analytic semigroup $(S(t))_{t\geq 0}$ in both H and V^*(see Jeong,1999; Tanabe, 1979). The controller B is a bounded linear operator from another Hilbert space $L^2(0, T; U)(T > 0)$ to $L^2(0, T; U)$. k belongs to $L^2(0, T)$ and f is a nonlinear mapping satisfying Lipschitz continuity.

There are various approaches to obtain the sufficient conditions for approximate controllability of semilinear control equations; the range condition argument of controller as seen in Zhou (1983, 1984), the controllability of semilinear systems dominated by linear parts as in Dau er and Mahmudov(2002), Jeong and Kang (2018), Naito (1987) and Radhakrishnan and

Balachandran (2012). Another approach is to use a fixed point theorem combined with technique of operator transformations in Balachandran and Dauer (2002); Wang(2009). Recently, similar considerations of semilinear neutral equations have been studied by many authors(see Ren, Hu, and Sakthivel, 2011; Fu, Lu, and You, 2014; Jothimani, Mokkedem and Fu, 2017; Valliammal, and Ravichandran, 2018; Dhayal, Malik, and Abbas, 2019) as a continuous study. Moreover, Kang and Jeong (2019) dealt with the approximate controllability for System (6.1.1) assuming

$$\limsup_{||u||_{L^2(0,T;H)}\to\infty} \frac{||f(\cdot,u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} \neq 1$$

by using so called Fredholm theory: $(\lambda I - F)(u) = f$ for a given f is solvable in $L^2(0,T;H)$.

In this paper, authors want to use a new approach by using the surjectivity theorems similar to the Fredholm alternative for nonlinear operators motivated by the work Kang and Jeong (2019), which is about the solution of nonlinear operator equations $\lambda B(u) - F(u) = f$ provided that $\lambda B(u) - F(u) \neq 0$ for each u. In order to obtain the approximate controllability for System (6.1.1), it is necessary to suppose that B acts as an odd homeomorphism operator while F is odd completely continuous and homogeneous as defined in Section 3. By using this method, the approximate controllability of System (6.1.1) can be given as applicable conditions without restrictions such as the inequality constraints for Lipschitz constant of f or the compactness of $S(t)$.

Section 2 gives some properties of the strict solutions of System (6.1.1) and the continuity of the solution mapping on a control space with value in the terminal state space. In Section 3, in order to apply the surjective theory to the proof of the main theorem, we deal with the equation $\lambda B(u) - F(u) = f$ is solvable provided that $\lambda \neq 0$ is not an eigenvalue for the couple (T, F)(see Definition 6.3.3), which is equivalent that the nonlinear inverse considered as a multivalued mapping is bounded. Based on results in Section 3, we obtain the sufficient conditions for the approximate controllability of semilinear systems when the corresponding linear system is approximately controllable. Finally, a simple example to which our main result can be applied is given.

6.2 Semilinear functional equations

Throughout this paper, as seen in Section 2.2, V, H and V^* are complex Hilbert spaces forming a Gelgand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*.$$

Moreover, A in System (6.1.1) is also the operator in place of $-A$ in Section 2.2. It is known that A is a bounded linear operator from V to V^*, and A generates an analytic semigroup $S(t)(t \geq 0)$ in both of H and V^*(see [76]).

Let $f : \mathbb{R}^+ \times V \to H$ be a nonlinear mapping satisfying the following:

Assumption (EF).

(i) For any $x \in V$, the odd mapping $f(\cdot, x)$ is strongly measurable;

(ii) There exist positive constants $L_0,\ L$ such that for all $t \in \mathbb{R}^+$, $x,\ \hat{x} \in V$,

$$|f(t,x) - f(t,\hat{x})| \leq L||x - \hat{x}||,$$
$$|f(t,0)| \leq L_0.$$

Consider the following abstract semilinear system with initial values with the forcing term g;

$$\begin{cases} \frac{d}{dt}x(t) = & Ax(t) + f(t,x(t) + g(t), \quad 0 < t \leq T,, \\ x(0) = & x_0 \end{cases} \tag{6.2.1}$$

By virtue of Theorem 3.1 of Jeong, Kwun, and Park (1999), we have the following result on System (6.2.1).

Proposition 6.2.1. *Let Assumption (EF) be satisfied.*

1) Assume that for $(x_0, g) \in V \times L^2(0,T;H)$. *Then there exists a unique solution* x *of System (6.2.1) such that*

$$x \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H) \subset C([0,T];V),$$

and there exists a constant C_1 *such that*

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \leq C_1(||x_0|| + ||g||_{L^2(0,T;H)}). \tag{6.2.2}$$

2) Assume that for $(x_0, g) \in H \times L^2(0,T;V^*)$. *Then there exists a unique solution* x *of System (6.2.1) such that*

$$x \in L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H),$$

and there exists a constant C_1 *such that*

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|x_0| + ||g||_{L^2(0,T;V^*)}). \tag{6.2.3}$$

We refer to Theorem 3.3 of Di Blasio, Kunisch, and Sinestrari (1984) as for the similar result for the regularity of linear case. Let U be a Hilbert space and the controller operator B be a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$.

Now, we consider the semilinear control system (6.1.1) with Bu in place of g in System (6.2.1) as follows.

Corollary 6.2.1. *Assume that the embedding* $D(A) \subset V$ *is completely continuous. Let Assumption (EF) be satisfied, and* x_u *be the solution of System (6.1.1) associated with* $u \in L^2(0,T;U)$. *Then the mapping* $u \mapsto x_u$ *is completely continuous from* $L^2(0,T;U)$ *to* $L^2(0,T;V)$.

Proof. If $u \in L^2(0,T;U)$, then in view of (6.2.3) in Proposition 6.2.1

$$||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|x_0| + ||B||\,||u||_{L^2(0,T;U)}).$$

Since $x_u \in L^2(0,T;V)$, we have $f(\cdot, x_u) \in L^2(0,T;H)$. Consequently, by 1) of Proposition 6.2.1, we have

$$x_u \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;H).$$

Hence, with aid of (6.2.2) and (6.2.3) of Proposition 6.2.1,

$$\begin{aligned}
&||x_u||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||x_0|| + ||f(\cdot, x_u) + Bu||_{L^2(0,T;H)}) \\
&\le C_1\big[||x_0|| + \big\{L||x_u||_{L^2(0,T;V)} + L_0\sqrt{T}\big\} + ||Bu||_{L^2(0,T;H)}\big] \\
&\le C_1\big[||x_0|| + \big\{C_1 L\big(|x_0| + ||Bu||_{L^2(0,T;V^*)}\big) + L_0\sqrt{T}\big\} + ||Bu||_{L^2(0,T;H)}\big].
\end{aligned}$$

Thus, if u is bounded in $L^2(0,T;H)$, then so is x_u in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$. Since $D(A)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A)) \cap W^{1,2}(0,T;V) \subset L^2(0,T;V)$$

is completely continuous in view of Theorem 2 of Aubin (1997), therefore, the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

6.3 Nonlinear operator equations

Let X and Y be Banach spaces with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively.

Lemma 6.3.1. *Let T be an odd (K,L,α)-homeomorphism of X onto Y (see Definition 5.3.2) and $F : X \to Y$ an odd completely continuous operator. Suppose that for $\lambda \neq 0$,*

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \tag{6.3.1}$$

Then $\lambda T - F$ maps X onto Y.

Proof. We follow the proof Theorem 1.1 in Chapter II of Fučik, Nečas, Souček, and Souček (1973). Suppose that there exists $y \in Y$ such that $\lambda T(x) = y$. Then from (6.3.1) it follows that $FT^{-1} : Y \to Y$ is an odd completely continuous operator and

$$\lim_{||y||_Y \to \infty} ||y - FT^{-1}(\frac{y}{\lambda})||_Y = \infty.$$

Let $y_0 \in Y$. There exists $r > 0$ such that

$$||y - FT^{-1}(\frac{y}{\lambda})||_Y > ||y_0||_Y \geq 0$$

for each $y \in Y$ satisfying $||y||_Y = r$. Let $Y_r = \{y \in Y : ||y||_Y < r\}$ be a open ball. Then by view of Lemma 6.3.1, we have $d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0]$ is an odd number. For each $y \in Y$ satisfying $||y||_Y = r$ and $t \in [0, 1]$, there is

$$||y - FT^{-1}(\frac{y}{\lambda}) - ty_0||_Y \geq ||y - FT^{-1}(\frac{y}{\lambda})||_Y - ||y_0||_Y > 0$$

and hence, by the homotopic property of degree, we have

$$d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, y_0] = d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0] \neq 0.$$

Hence, there exists a $y_1 \in Y_r$ such that

$$y_1 - FT^{-1}(\frac{y_1}{\lambda}) = y_0.$$

We can choose $x_0 \in X$ satisfying $\lambda T(x_0) = y_1$, and so, $\lambda T(x_0) - F(x_0) = y_0$. Thus, it implies that $\lambda T - F$ is a mapping of X onto Y. □

Definition 6.3.1. *Let F be mapping defined by on a Banach space X with value in a real Banach space Y and $b > 0$ a real number. F is said to be b-homogeneous if*

$$t^b F_0(u) = F_0(tu)$$

holds for each $t \geq 0$ and all $u \in X$.

Example 6.3.1. *Set $X = Y = \mathbb{R}$ and*

$$F(u) = \frac{\alpha\, u^3}{(1+\beta)|u|},$$

where α, β are positive numbers. Then F is said to be 2-homogeneous.

Definition 6.3.2. *Let X and Y be two Banach spaces, $T : X \to Y$, $F : X \to Y$ operators and $\lambda \neq 0$ a real number. $\lambda \neq 0$ is said to be an eigenvalue for the couple (T, F) if there exists $u_0 \in X$ such that*

$$\lambda T(u_0) - F(u_0) = 0.$$

If the operator T is a-homogeneous and the operator F is b-homogeneous, we are going to prove the existence of a solution of the equation

$$\lambda T(x) - F(x) = y$$

for each $y \in Y$ provided $\lambda \neq 0$ with $a = b$.

Theorem 6.3.1. *Let X and Y be two Banach spaces. Let T be an odd (K, L, a) - homeomorphism of X onto Y which is a-homogeneous. Let $F : X \to Y$ be an odd completely continuous a-homogeneous operator. Suppose $\lambda \neq 0$ is not an eigenvalue for the couple (T, F). Then the operator $\lambda T - F$ maps X onto Y.*

Proof. In virtue of Lemma 6.3.1, it suffices to show that

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty.$$

Suppose that a constant $M > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $||x_n||_X \to \infty$ such that

$$||\lambda T(x_n) - F(x_n)||_Y \leq M$$

for each positive integer n. Here, we use symbols " $\to$ " to denote the strong convergence. Set

$$\frac{x_n}{||x_n||_X} = v_n.$$

Then we have

$$\frac{\lambda T(x_n) - F(x_n)}{||x_n||_X^a} = \frac{\lambda T(||x_n||v_n)}{||x_n||_X^a} - \frac{F(||x_n||v_n)}{||x_n||_X^a} = \lambda T(v_n) - F(v_n) \to 0.$$

The complete continuity of the operator F implies that there exists a subsequence $\{v_{n_k}\} \subset \{v_n\}$ and $v_0 \in X$ such that

$$F(v_{n_k}) \to \lambda T(v_0) \in Y.$$

Hence,

$$\lambda T(v_{n_k}) \to \lambda T(v_0) \in Y.$$

and since T is homeomorphic

$$v_{n_k} \to v_0.$$

We have $||v_0|| = 1$ and

$$\lambda T(v_0) - F(v_0) = 0.$$

Thus λ is the eigenvalue number for the couple (T, F), which is a contradiction. □

The following shows that if $\lambda \neq 0$ is not an eigenvalue for the couple (T, F) if and only the nonlinear inverse $(\lambda T - F)^{-1}$ considered as a multivalued mapping is bounded.

Corollary 6.3.1. *Suppose that assumptions of T and F in Theorem 6.3.1 are satisfied. Then if $\lambda \neq 0$ is not an eigenvalue for the couple (T, F) if and only $(\lambda T - F)(X) = Y$ and for each $L > 0$ there exists $r > 0$ such that $||x||_X \leq r$ for all $x \in X$ with $||\lambda T(x) - F(x)||_Y \leq L$.*

Proof. Let $\lambda \neq 0$ be not an eigenvalue for the couple (T, F). Suppose that there exists a sequence $\{x_n\} \subset X,\ ||x||_X = 1$ such that

$$\lambda T(x_n) - F(x_n) \to 0.$$

The complete continuity of F implies that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x_0 \in X$ such that

$$F(x_{n_k}) \to \lambda T(x_0) \in Y.$$

Hence,

$$\lambda T(x_{n_k}) \to \lambda T(x_0) \in Y.$$

and since T is homeomorphic,

$$x_{n_k} \to x_0.$$

We have $||x_0|| = 1$ and

$$\lambda T(x_0) - F(x_0) = 0.$$

This is a contraction proving

$$\inf_{||x||_X=1} ||\lambda T(x_n) - F(x_n)||_Y = c > 0$$

and hence,

$$||\lambda T(x_n) - F(x_n)||_Y \geq c||x||_X$$

for each $x \in X$. The assertion of $(\lambda T - F)(X) = Y$ is from Theorem 6.3.1. The proof of the converse is obvious. □

6.4 Surjectivity theory for controllability

As seem in Section 5.3, the reachable sets for System (6.1.1) are represented as

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$

$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

For y_g be the solution of System (6.2.1) with $B \equiv I$, we have

$$y_g(t) = \int_0^t S(t-s)\{f(s, y_g(s)) + g(s)\}ds.$$

By using the Krasnosel'skii theorem(see Aubin and Ekeland, 1984), we can define an operator $F : L^2(0,T;H) \to L^2(0,T;H)$ as

$$F(g) = -f(\cdot, y_g). \tag{6.4.1}$$

We shall make use of the following assumption:
Assumption (EA). The embedding $D(A) \subset V$ is completely continuous.
Assumption (EF1). F is 1-homogeneous, and satisfies Assumption (EF) stated in Section 2.

Theorem 6.4.1. *Let Assumptions (EA), and (EF1) be satisfied, and let $F(g) \neq g$ for every $g \neq 0$. Then if the linear system (6.1.1) with $f \equiv 0$ is approximately controllable, then so is the semilinear system (6.1.1).*

Proof. Let $\eta \in L^2(0,T;D(A))$. Then there exists $p \in C^1(0,T;X)$ such that

$$\eta = \int_0^T S(T-s)p(s)ds,$$

for instance, $p(s) = (\eta + sA\eta)/T$. Since the linear system (6.1.1) with $f \equiv 0$ is approximately controllable, that is,

$$\overline{R_T(0)} = H,$$

for any $\epsilon > 0$, there exists $v \in L^2(0,T;U)$ such that

$$|\eta - \int_0^T S(T-s)(Bv)(s)ds| \leq \epsilon. \tag{6.4.2}$$

Let

$$N = \{q \in L^2(0,T;H) : \int_0^T S(T-s)q(s)ds = 0\}$$

and denote by $N^\perp$ be the orthogonal complement of N in $L^2(0,T;H)$. We denote the range of the operator B by H_B. In view of (6.4.2) we have $L^2(0,T;H) = \overline{H}_B + N$, where $\overline{H}_B$ is the closure of H_B in $L^2(0,T;H)$.

For $u \in N^\perp$, let Pu be the unique minimum norm element of $\{u+N\} \cap \overline{H}_B$. Then the proof of Lemma 1 of Naito (1987) can be applied to show that P is a linear and continuous operator from $N^\perp$ to $\overline{H}_B$. Let $\tilde{Y} = L^2(0,T;H)/N$ be the quotient space and the norm of a coset $\tilde{u} = u + N \in \tilde{Y}$ is defined of $||\tilde{u}|| = \inf\{|u+f| : f \in N\}$.

We define by Q the isometric isomorphism from $\tilde{Y}$ onto $N^\perp$, that is, $Q\tilde{u}$ is the minimum norm element in $\tilde{u} = \{u + f : f \in N\}$. Let

$$\mathcal{F}\tilde{u} = F(PQ\tilde{u}) + N$$

for $\tilde{u} \in \tilde{Y}$. Then $\mathcal{F}$ is a compact mapping from $\tilde{Y}$ to itself.

We are going to show that $\eta \in \overline{R_T(f)}^V$, where $\overline{R_T(f)}^V$ is the closure of $R_T(f)$ in V. In the sense of Corollary 6.2.1, from Assumption (EF1), we get that $\mathcal{F}$ defined by (6.4.1) is also a completely continuous mapping from $L^2(0,T;H)$ to itself. Since the identity operator I on $\tilde{Y}$ is an odd $(1,1,1)$-homeomorphism and 1-homogenous, and $F(g) \neq g$ for every $g \neq 0$, we know that $\lambda = 1$ is not an eigenvalue for the couple $(I, \mathcal{F})$. Hence, from Theorem 6.3.1, it follows that that $I - \mathcal{F}$ maps $\tilde{Y}$ onto itself. Let $z = Bv$, where v is the control in (6.4.2). Then $\tilde{z} = z + N \in \tilde{Y}$, and there exists $\tilde{u} \in \tilde{Y}$ such that

$$\tilde{z} = \tilde{u} - \mathcal{F}\tilde{u}.$$

Put $u = Q\tilde{u}$ and $u_B = PQ\tilde{u}$. Then we have that $u_B = Pu$ and $u - u_B = u - Pu \in N$. Hence

$$\tilde{z} = u - F(u_B) + N = u_B - F(u_B) + N.$$

Therefore,

$$\begin{aligned}\eta &= \int_0^T W(T-s)\{-F(u_B)(s) + u_B(s)\}ds \\ &= \int_0^T W(T-s)\{f(s, y_{u_B}) + u_B(s)\})ds.\end{aligned}$$

Since $u_B \in \overline{H}_B$, there exists a sequence $\{v_n\} \in L^2(0,T;U)$ such that $Bv_n \mapsto u_B$ in $L^2(0,T;H)$. Then by the last part of Corollary 6.2.1, we have that $x(\cdot;g,v_n) \mapsto y_{u_B}$ in $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset C([0,T];V)$, and hence $x(T;g,v_n) \mapsto y_{u_B}(T) = \eta$ in V. Thus we conclude $\eta \in \overline{R_T(g)}^V$. □

Assumption (EB). There exist positive constants β, γ such that

$$\beta\|u\| \leq |Bu| \leq \gamma\|u\|, \quad \forall u \in L^2(0,T;U).$$

Corollary 6.4.1. *Let Assumptions (EA), (EF1), and (EB) be satisfied. Suppose that $\lambda = 1$ is not an eigenvalue for the couple (B,F). Then the semilinear control system (6.1.1) is approximately controllable.*

Proof. Since B is an odd $(\gamma,\beta,1)$- homeomorphism of $L^2(0,T;U)$ onto $L^2(0,T;H)$, From Theorem 6.4.1, it follows that then $B-F$ maps $L^2(0,T;U)$ onto $L^2(0,T;H)$ for any $\lambda \neq 0$. □

Example 6.4.1. *We consider the semilinear heat equation dealt with by Naito (1987); Zhou (1983, 1984). Let*

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$

$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Moreover, by the result known as Sobolev's imbedding theorem, the embedding $D(A) \subset V$ is completely continuous. Let

$$U = \{\sum_{n=2}^{\infty} u_n\phi_n : \sum_{n=2}^{\infty} u_n^2 < \infty\},$$

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n, \quad \text{for} \quad u = \sum_{n=2}^{\infty} u_n \in U.$$

Now we can define bounded linear operator $\hat{B}$ from $L^2(0,T;U)$ to $L^2(0,T;H)$ by $(\hat{B}u) = Bu(t)$, $u \in L^2(0,T;U)$. It is easily known that the operator $\hat{B}$ is one to one and satisfies Assumption (EB). We can see many examples which satisfy Assumption (EB) as seen in Zhou (1983, 1984). Moreover, the linear system of (6.1.1) with $f \equiv 0$ is approximately controllable. The nonlinear term is given by

$$f(t,x) = \frac{x\phi_1}{1+\alpha(t)} + \sigma x\phi_2, \quad \sigma > 0, \;\; \alpha(t) \in C[0,T]$$

It is easily seen that Assumption (EF1) is satisfied. Therefore, from Theorem 6.4.1 or Corollary 6.4.1, it follows that the system of (6.1.1) is approximately controllable.

Example 6.4.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let $\mathcal{A}(x, D_x)$ be an elliptic differential operator of second order in $L^2(\Omega)$(see Yamamoto and Park, 1990):*

$$\mathcal{A}(x, D_x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{i,j}(x)\frac{\partial}{\partial x_i})) + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x).$$

We consider a diffusion and reaction process differential equation defined as $Au = -\mathcal{A}(x, D_x)u$:

$$\begin{cases} x^{'}(t,\xi) = Ax(t,\xi) + f(t, x(t,\xi) + (Bu)(t) \\ x|_{\partial\Omega} = 0, \\ x(\xi,t) = x_0, \end{cases} \tag{6.4.3}$$

where the controller B is defined by Example 6.4.1. We define the following spaces:

$$H^1(\Omega) = \Big\{x : x, \;\; \frac{\partial x}{\partial x_i} \in L^2(\Omega), \;\; i = 1, 2, \cdots, n\Big\},$$

$$H^2(\Omega) = \Big\{x : x, \;\; \frac{\partial x}{\partial x_i}, \;\; \frac{\partial^2 x}{\partial x_i \partial x_j} \in L^2(\Omega), \;\; i, j = 1, 2, \cdots, n\Big\},$$

$$H_0^1(\Omega) = \{x : x \in H^1(\Omega), \;\; x|_{\partial\Omega} = 0\} = \text{the closure of } C_0^\infty(\Omega) \text{ in } H^1(\Omega),$$

where $\partial/\partial\xi_i x$ and $\partial^2/\partial\xi_i\partial\xi_j x$ are the derivative of x in the distribution sense. The norm of $H_0^1(\Omega)$ is definued by

$$||x|| = \Big\{\int_\Omega \sum_{i=1}^n (\frac{\partial x(x)}{\partial x_i})^2 dx\Big\}^{1/2}.$$

Hence $H_0^1(\Omega)$ is a Hilbert space. Let $H^{-1}(\Omega)$=$H_0^1(\Omega)^*$ be a dual space of $H_0^1(\Omega)$. For any $l \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$, the notation (l, v) denotes the value l at v. In what follows, we consider the regularity for given equations in the spaces

$$V = H_0^1(\Omega) = \{x \in H^1(\Omega); x = 0 \ \ on \ \partial\Omega\}, \quad H = L^2(\Omega), \quad \text{and } V^* = H^{-1}(\Omega)$$

as introduced in Section 6.2. We deal with the Dirichlet condition's case as follows.

Assume that $a_{ij} = a_{ji}$ are continuous and bounded on $\overline{\Omega}$ and $\{a_{ij}(x)\}$ is positive definite uniformly in Ω, i.e., there exists a positive number δ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2, \quad \forall \xi \in \bar{\Omega}. \tag{6.4.4}$$

Let

$$b_i \in L^\infty(\Omega), \quad c \in L^\infty(\Omega) \quad \text{and} \quad \beta_i = \sum_{j=1}^n \partial a_{ij}/\partial x_j + b_i.$$

For each $x, y \in H_0^1(\Omega)$, let us consider the following sesquilinear form:

$$a(x,y) = \int_\Omega \{ \sum_{i,j=1}^n a_{ij} \frac{\partial x}{\partial \xi_i} \overline{\frac{\partial y}{\partial \xi_j}} + \sum_{j=1}^n \beta_i \frac{\partial x}{\partial \xi_i} \overline{y} + cx\overline{y} \} dx.$$

Since $\{a_{ij}\}$ is real symmetric, by (6.4.4) the inequality

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\bar{\xi_j} \geq \delta|\xi|^2$$

holds for all complex vectors $\xi = (\xi_1, \cdots, \xi_n)$*. By hypothesis, there exists a constant* K *such that* $|\beta_i(x)| \le K$ *and* $c(x) \le K$ *hold a.e., hence*

$$\begin{aligned}\operatorname{Re} a(x,x) &\ge \int_\Omega \delta \sum_{i=1}^n |\frac{\partial x}{\partial \xi_i}|^2 dx - K\int_\Omega \sum_{i=1}^n |\frac{\partial x}{\partial \xi_i}||x|dx - K\int_\Omega |x|^2 dx \\ &\ge \delta \int_\Omega \sum_{i=1}^n |\frac{\partial x}{\partial \xi_i}|^2 dx - K\int_\Omega \sum_{i=1}^n \big(\frac{\epsilon}{2}|\frac{\partial x}{\partial \xi_i}|^2 + \frac{1}{2\epsilon}|x|^2\big)dx - K\int_\Omega |x|^2 dx \\ &= (\delta - \frac{\epsilon}{2}K)\sum_{i=1}^n \int_\Omega |\frac{\partial x}{\partial \xi_i}|^2 dx - \big(\frac{nK}{2\epsilon} + K\big)\int_\Omega |x|^2 dx.\end{aligned}$$

By choosing $\epsilon = \delta K^{-1}$*, we have*

$$\begin{aligned}\operatorname{Re} a(x,x) &\ge \frac{\delta}{2}\sum_{i=1}^n \int_\Omega |\frac{\partial x}{\partial \xi_i}|^2 dx - \big(\frac{nK^2}{2\delta} + K\big)\int_\Omega |x|^2 dx \\ &= \frac{\delta}{2}||x||_1^2 - \big(\frac{nK^2}{2\delta} + K + \frac{\delta}{2}\big)||x||^2.\end{aligned}$$

By virtue of Lax-Milgram theorem, we know that for any $y \in V$ *there exists* $f \in V^*$ *such that*

$$a(x,y) = (f,y).$$

Therefore, we know that the associated operator $A : V \to V^*$ *defined by*

$$(Ax, y) = -a(x,y), \quad x, y \in V$$

is a bounded linear operator from V *to* V^**, and* A *generates an analytic semigroup* $S(t)(t \ge 0)$ *in both of* H *and* V^**(see [76]), which satisfies conditions mentioned as Section 2.*

We introduce a nonlinear mapping $f : [0,T] \times V \to H$ *defined by*

$$f(t,x) = \int_0^t k(t-s)\int_0^s \sum_{i=1}^n \frac{\partial}{\partial \xi_i}\sigma_i(\tau, \nabla x(\tau,\xi))d\tau ds,$$

where k belongs to $L^2(0,T)$, and

$$g(t, x(t,\xi)) = \int_0^t \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \sigma_i(s, \nabla x(s,\xi)) ds.$$

We assume the following:

Assumption (EF2).

(i) *The partial derivatives $\sigma_i(s,x)$, $\partial/\partial t\, \sigma_i(t,x)$ and $\partial/\partial \xi_j\, \sigma_i(s,x)$ exist and continuous for $i = 1, 2$, $j = 1, 2, \cdots, n$;*

(ii) *$t \mapsto \sigma_i(s,x)$ is odd mapping ($\sigma_i(s,-x) = -\sigma_i(s,x)$, and 1-homogeneous ($\sigma_i(\cdot, tx) = t\sigma_i(\cdot, tx)$);*

(iii)

$$|\sigma_i(s,x) - \sigma_i(s,\widehat{x})| \le L|x - \widehat{x}|$$

where $|\cdot|$ denotes the norm of $L^2(\Omega)$.

For instance, we can give

$$\sigma_i(s,x) = \frac{\alpha |x|^2}{(1+\beta)x},$$

where, α, β are positive numbers. Then σ_i satisfies Assumption (EF1).

Lemma 6.4.1. *If Assumption (EF1) is satisfied, then the mapping $t \mapsto g(t,\cdot)$ is continuously differentiable on $[0,T]$ and $x \mapsto g(\cdot, x)$ is Lipschitz continuous on V.*

Proof. Put

$$g_1(s,x) = \sum_{i=1}^n \frac{\partial}{\partial \xi_i} \sigma_i(s, \nabla x).$$

Then we have $g_1(s,x) \in H^{-1}(\Omega)$. For each $z \in H_0^1(\Omega)$, we satisfy the following that

$$(g_1(s,x), z) = -\sum_{i=1}^{n}(\sigma_i(s,\nabla x), \frac{\partial}{\partial x_i}z).$$

The nonlinear term is given by

$$g(t,x) = \int_0^t g_1(s,x)ds.$$

For any $z \in H_0^1(\Omega)$, if x and $\widehat{x}$ belong to $H_0^1(\Omega)$, by Assumption (EF1) we obtain

$$|(g(t,x) - g(t,\widehat{x})), w| \leq LT||x - \widehat{x}||\,||z||.$$

□

Now in virtue of Lemma 6.4.1, we can apply the results of Theorem 6.4.1 as follows.

Theorem 6.4.2. *Let Assumptions (EA), and (EF2) be satisfied, and let $F(g) \neq 0$ for every $g \neq 0$, where F is defines as (6.4.1). Then if the linear system (6.4.3) with $f \equiv 0$ is approximately controllable, then so is the semilinear system (6.4.3).*

6.5 Conclusions

We have dealt with the approximate controllability of abstract semilinear functional control equations by solving nonlinear operator equations. The nonlinear equation is given as $\lambda B(x) - F(x) = y$ in dependence on the real number λ, where B is a given controller operator and F is a nonlinear operator. Similar results in linear functional analysis are well known and they are sometimes called Fredholm theorems. To thes end, we have proved that $\lambda B - F$ maps for any $\lambda \neq 0$ provided that B is an odd (K, L, a)-

homeomorphism and a-homogeneous, F an odd completely continuous b-homogeneous operator. Suppose that $a = b$, $\lambda B(u) \neq F(u)$ for every u, that is, $\lambda \neq 0$ is not an eigenvalue for the couple (T, F). Then the operator $\lambda B - F$ maps X onto Y. Based on this consideration, we have established the approximate controllability for a class of abstract semilinear control systems. But, in the case where $a \neq b$, it seems to be unsolved up on our terms to this time in infinite dimensional space. We shall prove the similar assertion under the assumption $a \neq b$ in a forthcoming work.

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감사의 글

먼저 이 학위 논문을 완성하기까지 석사 및 박사과정기간 동안 끝없는 지도와 아낌없는 지지해주시고 때로는 흔들리고 방황할 때마다 올바른 방향으로 길을 제시해주신 지도교수 정진문 교수님께 깊은 감사의 글을 전하고 싶습니다.

아울러 논문 심사 때 바쁘신 와중에도 소중한 조언과 격려를 아끼지 않으시고 세미나를 통해 많은 지식을 알려주셨던 부경대학교 조낙은 교수님과 이완석 교수님, 바쁜 시간 내주시어 멀리서 오셔서 논문 심사를 해주신 박종열 교수님, 김은채 교수님께도 감사의 말을 전하고 싶습니다. 그리고 졸업할 수 있게 많은 도움을 주셨던 부경대학교 응용수학과 교수님들께도 감사의 말을 전하고 싶습니다.

또한 저를 낳아주시고 길러 주시면서 오랜 대학원 생활동안 공부를 계속 할 수 있도록 저를 지지해주고, 격려해주신 부모님, 힘들어할 때면 연락해서 응원해주던 언니와 동생들 덕분에 무사히 논문을 완성할 수 있었다는 것에 감사를 표하고 싶습니다.

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