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Geometric Properties for Analytic Functions Associated with Linear Operators

by

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Geometric Properties for Analytic Functions Associated with Linear Operators (선형 연산자와 관련된 해석 함수들의 기하학적 성질)

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요 약

20세기 초부터 집중적으로 발전해 온 기하함수이론은 해석함수들의 기하학적 성질을 연구하는 복소해석학의 분야 중 하나이며, 단엽 및 다엽함수들에 대한 이론은 현재 진행되고 있는 기하함수이론의 여러 주제들 중 특히 흥미롭고 중요한 연구이다.

본 논문에서는 여러 가지 선형 연산자를 사용하여 해석함수들의 다양한 부분족들을 소개하고 이 부분족들에 대해 유용한 여러 사상성질들을 조사하였으며 합성곱연산자 및 여러 주제들에 대한 부분족들의 기본 성질에 대해 연구하였다. 자세한 내용은 다음과 같다.

먼저 제 1장에서는 본 연구에서 기본적으로 필요한 정의와 보조 정리 및 여러 가지 부분족들을 소개하였다.

제 2장에서는 단위 개원판에서 정의된 β-볼록함수의 Choi-Saigo-Srivastva 연산자와 적분연산자에 관련된 다양한 종속 성질을 얻었다.

제 3장에서는 선형 연산자에 의해 정의된 다엽함수의 majorization 성질을 조사하였고, 제 4장에서는 Carathéodory 함수의 편각 성질을 조사하고 해석함수의 다양한 기하적인 성질을 찾았다.

제 5장에서 유리형 함수들에 관한 합성곱연산자를 소개하고 이 합성곱연산자와 관련된 유리형 함수의 여러가지 사상성질들과 편각 추정치를 얻었다.

제 6장에서는 선형 연산자의 역에 의해 정의된 승수 변환의 편각 성질과 보존성을 조사하였다.

마지막으로, 제 7장에서는 가우스 초기하 함수와 해석함수의 합성곱에 의해 정의된 선형 연산자를 사용하여 평등볼록성과 평등선형성에 관련된 함수들의 포함 및 사상성질을 얻었다.

Chapter 1

Introduction

Geometric Function Theory means the theory of conformal mappings which is induced by analytic functions. Historically, complex analysis and geometrical function theory have been concentrically developed from the beginning of the twentieth century. In the last years the theory of holomorphic mappings on complex spaces has been studied by many mathematicians with many applications to nonlinear analysis, functional analysis, differential equations, classical and quantum mechnics.

These mappings are mainly understood as univalent (or schlicht) mappings. The study of univalent functions dating from the early years of the twentieth century and is one of the popular areas of research in complex analysis. Initiated by the work of Bieberbach and his contemporaries, the famous conjecture of 1916 became one of the most celebrated problems in mathematics. The eventual solution of the Bieberbach conjecture by de Branges in 1984 employed nonelementary methods from several branches of analysis. Other interesting problems for univalent functions have also been raised, explored and solved. And lots of properties for these functions were obtained, but at the same time many unsolved problems remains still now. It is helpful to read books, such as Duren [16], Goluzin [21], Goodman [22], Graham and Kohr [26], Pommerenke [55] and Schober [60] for looking around the basics of univalent function theory.

In the present thesis, we introduce new subclasses of analytic functions defined by multiplier transformations and investigate properties of them. These classes are closely related to the class of univalent functions. And we will solve problems such as argument estimates, radius problems, majorization problems and subordination problems for functions in these classes. The following sections contain concepts related to this thesis.

1.1 Analytic functions and univalent functions

Let $\mathbb{C}$ be the planar complex plane, and let $\mathbb{U}_r = \{z \in \mathbb{C} : |z| < r\}$. In particular, we put $\mathbb{U} \equiv \mathbb{U}_1$. Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let $\mathcal{A}$ be the class of functions $f \in \mathcal{H}$ satisfying the normalized condition $f(0) = 0 = f'(0) - 1$. Therefore, if $f \in \mathcal{A}$ then f has its representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1.1}$$

A single-valued function f is said to be univalent in a domain $D \subset \mathbb{C}$ if it never takes the same value twice; that is, if $f(z_1) \neq f(z_2)$ for all points z_1

and z_2 in D with $z_1 \neq z_2$. Denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. The leading example of a function of class $\mathcal{S}$ is the Koebe function

$$k(z) = z(1-z)^{-2} = \sum_{n=1}^{\infty} n z^n.$$

Theorem 1.1.1. [16, p. 32] *For each $f \in \mathcal{S}$,*

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \leq \frac{4r}{1-r^2}, \quad |z| = r < 1.$$

Theorem 1.1.2. [16, p. 32] *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}, \quad |z| = r < 1.$$

For each $z \in \mathbb{U}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

Theorem 1.1.3. [16, p. 33] *For each $f \in \mathcal{S}$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}, \quad |z| = r < 1.$$

For each $z \in \mathbb{U}$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

1.2 Subclasses of $\mathcal{S}$

Definition 1.2.1. *A set D in the plane is called convex if for every pair of points w_1 and w_2 in the interior of D, the line segment joining w_1 and w_2 is also in the interior of D. If a function f maps $\mathbb{U}$ onto a convex domain, then f is called a convex function. We shall denote the class of convex functions in $\mathcal{A}$ by $\mathcal{K}$.*

Definition 1.2.2. *A set D in the plane is said to be starlike with respect to w_0 and interior point of D if each ray with initial point w_0 intersects the interior of D in a set that is either a line segment of a ray. If a function f maps $\mathbb{U}$ onto a domain that is starlike with respect to w_0, then we say that f is starlike with respect to w_0. In the special case that $w_0 = 0$, we say that f is a starlike function. We shall denote the class of starlike functions in $\mathcal{A}$ by $\mathcal{S}^*$.*

The analytic conditions for convexity and starlikeness were stated by Study [65] and Nevanlinna [45] as follows:

Theorem 1.2.1. *Let $f \in \mathcal{A}$. Then f belongs the class $\mathcal{K}$ if and only if*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{U}. \tag{1.2.1}$$

Theorem 1.2.2. *Let $f \in \mathcal{A}$. Then f belongs the class $\mathcal{S}^*$ if and only if*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

Definition 1.2.3. *A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a $g \in \mathcal{K}$ such that*

$$\Re\left\{\frac{f'(z)}{g'(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

We shall denote by $\mathcal{C}$ the class of functions $f \in \mathcal{A}$ satisfying the above condition.

It is known [16, p. 47] that f is close-to-convex if and only if the image of $|z| = r$ has no large hairpin turns, which means that there are no sections of the curve $f(C_r)$ in which the tangent vector turns backward through an angle greater than π. More precisely, we have

Theorem 1.2.3. [16, p. 48] *Let $f \in \mathcal{A}$ and $f'(z) \neq 0$ in $\mathbb{U}$. A necessary and sufficient condition that $f \in \mathcal{C}$ is that for every r in $(0, 1)$ and every pair θ_1, θ_2*

with $0 \leq \theta_2 - \theta_1 \leq 2\pi$*, we have*

$$\int_{\theta_1}^{\theta_2} \Re\left\{1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} d\theta > -\pi.$$

In view of (1.2.1), it is trivial that every convex function is close-to-convex. More generally, every starlike function is close-to-convex. Also, it is known [16, Theorem 2.17] that every close-to-convex function is univalent. These facts are summarized by the chain of inclusions

$$\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{C} \subset \mathcal{S}.$$

Now we give generalized concepts of the convex and starlike functions.

Definition 1.2.4. *Let* $0 \leq \alpha < 1$ *and* $f \in \mathcal{A}$*. Then we say that* $f \in \mathcal{K}(\alpha)$ *if* f *satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in \mathbb{U}. \tag{1.2.2}$$

And the function $f \in \mathcal{A}$ *satisfying* (1.2.2) *is called by a convex function of order* α*.*

Definition 1.2.5. *Let* $0 \leq \alpha < 1$ *and* $f \in \mathcal{A}$*. Then we say that* $f \in \mathcal{S}^*(\alpha)$ *if* f *satisfies*

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in \mathbb{U}. \tag{1.2.3}$$

And the function $f \in \mathcal{A}$ *satisfying* (1.2.3) *is called by a starlike function of order* α*.*

We remark that

(i) Taking $\alpha = 0$ in $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$ reduces the class $\mathcal{K}$ and $\mathcal{S}^*$, respectively;

(ii) $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ and $\mathcal{K}(\alpha) \subset \mathcal{K}$ hold for all $0 \leq \alpha < 1$;

(iii) $\mathcal{K} \subset \mathcal{S}^*(1/2)$ (see [41, p. 9]).

Let $\mathcal{T}$ denote the subclass of functions of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0;\ z \in \mathbb{U}). \tag{1.2.4}$$

The class $\mathcal{T}$ was introduced by Silverman [61]. We denote by $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the class of functions of the form (1.2.4) which are, respectively, starlike of order α and convex of order α with $0 \leq \alpha < 1$.

Theorem 1.2.4. [61, Corollary 1] *Let $f \in \mathcal{T}$ be of the form* (1.2.4)*. Then f is in $\mathcal{T}^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha.$$

Theorem 1.2.5. [61, Corollary 2] *Let $f \in \mathcal{T}$ be of the form* (1.2.4)*. Then f is in $\mathcal{C}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha.$$

1.3 Meromorphic functions

Let Σ denote the class of functions of the form [9]:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the annulus $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$ with a simple pole at origin with residue one there. The set Σ_0 is the subset of functions in Σ for which

$a_0 = 0$. Let Σ^s denote the class of univalent functions $f \in \Sigma$ in $\mathbb{U}^*$. We also set $\Sigma_0^s \equiv \Sigma^s \cap \Sigma_0$. Then the transformation

$$f(\zeta) = \frac{1}{g(\zeta)}.$$

which takes each g in $\mathcal{S}$ into a function in Σ_0^s.

1.4 Multivalent functions

A function f analytic in $\mathcal{D} \subset \mathbb{C}$ is called multivalent (p-valent) function, $p \in \mathbb{N}$ in $\mathcal{D}$ if for every complex number ω, the equation $f(z) = \omega$ does not have more than p roots in $\mathcal{D}$ and there exists a complex number ω_0 such that the equation $f(z) = \omega_0$, has exactly p roots in $\mathcal{D}$ (see [29]).

Now, for $p \in \mathbb{N}$, let $\mathcal{A}_p$ be the set of functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p},$$

which are analytic and p-valent in $\mathbb{U}$. For $p = 1$ we obtain the class $\mathcal{A}$ discussed earlier.

Definition 1.4.1. *A function $f(z) \in \mathcal{A}_p$, belongs to the class $\mathcal{CV}(p)$ of p-valent convex functions if and only if*

$$Re\left(\frac{(zf'(z))'}{pf'(z)}\right) > 0 \quad z \in \mathbb{U}.$$

A function $f(z) \in \mathcal{A}_p$, belongs to the class $\mathcal{ST}(p)$, of p-valent starlike functions, if and only if

$$Re\left(\frac{zf'(z)}{pf(z)}\right) > 0 \quad z \in \mathbb{U}.$$

Definition 1.4.2. *A function $f(z) \in \mathcal{A}_p$ is said to be p-valent convex of order β, $0 \leq \beta < p$ if and only if*

$$Re\left(\frac{(zf'(z))'}{pf'(z)}\right) > \beta \quad z \in \mathbb{U}.$$

Such class of functions shall be denoted by $\mathcal{CV}(p,\beta)$. Further a function $f(z) \in \mathcal{A}_p$, is said to be p-valent starlike of order β, $0 \leq \beta < p$, if and only if

$$Re\left(\frac{zf'(z)}{pf(z)}\right) > \beta \quad z \in \mathbb{U}.$$

Such a class of functions shall be denoted by $\mathcal{ST}(p,\beta)$.

Interesting results can be found in [19,35,54]. For $\beta = 0$, we obtain the classes $\mathcal{CV}(p)$ and $\mathcal{ST}(p)$ of p-valent starlike and convex functions with respect to the origin [25], and for $p = 1$ the class $\mathcal{CV}$ and $\mathcal{ST}$ are obtained.

1.5 Subordination and Majorization

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f [41, p. 4], if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$, such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{U}).$$

If the function F is univalent in $\mathbb{U}$, then we have (cf. [41])

$$f \prec F \quad \Longleftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Definition 1.5.1. [41, p. 16] *Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination*

$$\psi(p(z), zp'(z)) \prec h(z), \tag{1.5.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.5.1)*. A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of* (1.5.1) *is said to be the best dominant.*

Definition 1.5.2. [41, p. 16] *Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination*

$$\varphi(p(z), zp'(z)) \prec h(z), \tag{1.5.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.5.2)*. A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of* (1.5.2) *is said to be the best subordinant.*

Lemma 1.5.1. [40] *Let h be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with* $\mathrm{Re}\ \omega(z) \geq 0$. *If p is analytic in $\mathbb{U}$ and $p(0) = h(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathbb{U}).$$

A function $L(z,t)$ defined on $\mathbb{U}\times[0,\infty)$ is the subordination chain (or Löwner chain [12, p. 136]) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t\in[0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z\in\mathbb{U}$ and

$$L(z,s)\prec L(z,t)\quad (z\in\mathbb{U};\ 0\leq s<t).$$

Lemma 1.5.2. [55] *The function*

$$L(z,t)=a_1(t)z+\cdots$$

with

$$a_1(t)\neq 0\quad \text{and}\quad \lim_{t\to\infty}|a_1(t)|=\infty.$$

Suppose that $L(\cdot,t)$ ia analytic in $\mathbb{U}$ for all $t\geq 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z\in\mathbb{U}$. If $L(z,t)$ satisfies

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\}>0\quad (z\in\mathbb{U};\ 0\leq t<\infty)$$

and

$$|L(z,t)|\leq K_0|a_1(t)|\quad (|z|<r_0<1;\ 0\geq t<\infty))$$

for some positive constants K_0 and r_0, then $L(z,t)$ is a subordination chain.

Definition 1.5.3. [22, p. 178] *For f and g analytic in $\mathbb{U}$, we say that f is majorized by g in $\mathbb{U}_r$, and write $f(z)\ll g(z)$, $z\in\mathbb{U}_r$ (or, $f\ll g$ in $\mathbb{U}_r$), if there exists a function ϕ, analytic in $\mathbb{U}$, such that*

$$|\phi(z)|\leq 1\quad \textit{and}\quad f(z)=\phi(z)g(z),\ z\in\mathbb{U}_r.$$

We now look at some of the relations between $f(z)\prec F(z)$ and $f(z)\ll F(z)$. Lewandowski [31] has introduced a pair of symbols for these relations that are quite convenient. As usual we assume that all of the functions involved are regular in $\mathbb{U}$. Then Lewandowski writes (f, F, R_1) if $f(z) \prec F(z)$ for $|z| < R_1 \leq 1$ and he writes $|f, F, R_2|$ if $f(z) \ll F(z)$ for $|z| < R_2 \leq 1$.

It was Biernacki [5, 6] who first examined relations between (f, F, R_1) and $|f, F, R_2|$. He proved that there is a number R_2 such that whenever $(f, F, 1)$ and F is in $\mathcal{S}$, then $|f, F, R_2|$, and R_2 is in the interval $(1/4, 1)$.

Of course, the problem is to find the largest R_2 in this situation and to examine the relation for other sets of functions. Further, Lewandowski was the first to call attention to, and to make a contribution to, the inverse problem. For a given set $\mathcal{M}$ find the largest R such that $|f, F, 1|$ implies (f, F, R) for every F in $\mathcal{M}$. Here $f(z)$ may be subject to some additional restrictions such as being univalent in $\mathbb{U}$.

Let us introduce the following results of Biernacki [6]:

(i) If f and F are both univalent in $\mathbb{U}$, then $(f, F, 1)$ implies $|f, F, R|$, where $R \approx 0.390$ is the least positive root of

$$\ln(1 + x) - \ln(1 - x) + 2 \arctan x = \pi/2.$$

(ii) If f is in $\mathcal{S}^*$ and F is a starlike function, then $(f, F, 1)$ implies $|f, F, R|$, where $R = \sqrt{2} - 1$.

(iii) If f is in $\mathcal{K}$ and F is a convex function, then $(f, F, 1)$ implies $|f, F, R|$, where $R \approx 0.543$ is the least positive root of

$$\arcsin x + 2 \arctan x = \pi/2.$$

1.6 Carathéodory functions

Let $\mathcal{N}$ be the class of all functions which are analytic in the open unit disk $\mathbb{U}$ with $p(0) = 1$. We say that $p \in \mathcal{N}$ is a Carathéodory function [38, 48] if it satisfies the condition $\operatorname{Re} p(z) > 0$ in $\mathbb{U}$.

Lemma 1.6.1. [68] *Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exist two points $z_1, z_2 \in \mathbb{U}$ such that*

$$-\frac{\pi}{2}\delta_1 = \arg\{p(z_1)\} < \arg\{p(z)\} < \arg\{p(z_2)\} = \frac{\pi}{2}\delta_2 \tag{1.6.1}$$

for some δ_1 and δ_2 $(\delta_1, \delta_2 > 0)$ and for all z $(|z| < |z_1| = |z_2|)$, then

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\left(\frac{\delta_1+\delta_2}{2}m\right) \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i\left(\frac{\delta_1+\delta_2}{2}m\right), \tag{1.6.2}$$

where

$$m \geq \frac{1-|b|}{1+|b|} \quad \text{and} \quad b = i\tan\frac{\pi}{4}\left(\frac{\delta_2-\delta_1}{\delta_2+\delta_1}\right). \tag{1.6.3}$$

1.7 Multiplier transformations

For functions

$$f_j(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,j} z^{k+p} \quad (j = 1, 2; z \in \mathbb{U})$$

in the class $\mathcal{A}_p$, we define the convolution of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p} \quad (z \in \mathbb{U}).$$

As a similar way, for functions

$$f_j(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2; z \in \mathbb{D})$$

in the class Σ, we define the convolution of f_1 and f_2 [1] by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n \quad (z \in \mathbb{D}). \tag{1.7.1}$$

Making use of the convolution given by (1.7.1), we now define the following convolution operator D^α by

$$D^\alpha f(z) = \frac{1}{z(1-z)^{\alpha+1}} * f(z) \quad (\alpha > -1; f \in \Sigma; z \in \mathbb{D}). \tag{1.7.2}$$

Then it follows from (1.7.2) that

$$z(D^\alpha f(z))' = (\alpha+1) D^{\alpha+1} f(z) - (\alpha+2) D^\alpha f(z). \tag{1.7.3}$$

For $\alpha = n \in \mathbb{N}$, the operator D^α is introduced and studied by Ganigi and Uralegaddi [18] (see, also [69, 70]). Also, the operator D^α is closely related to Ruscheweyh derivative [58] for analytic functions defined in $\mathbb{U}$, which was extended by Goel and Sohi [19].

Now we define the $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \;\; (c \neq 0, -1, -2, \cdots),$$

where $(x)_k$ is the Pochhammer symbol(or the shifted factorial) defined by

$$(x)_n := \begin{cases} x(x+1)(x+2)\cdots(x+k-1), & \text{if } k \in \mathbb{N} = \{1, 2, \cdots\}, \\ 1 & \text{if } k = 0, \end{cases} \tag{1.7.4}$$

Let $f \in \mathcal{A}_p$. Denote by $L_p(a,c) : \mathcal{A}_p \to \mathcal{A}_p$ the operator defined by

$$L_p(a,c)f(z) = \phi_p(a,c;z) * f(z) \ \ (z \in \mathbb{U}),$$

where the symbol (*) stands for the Hadamard product (or convolution). We observe that

$$L_p(p+1,p)f(z) = zf'(z)/p \text{ and } L_p(n+p,1)f(z) = D^{n+p-1}f(z),$$

where n is any real number greater than $-p$, and the symbol D^n is the Ruscheweyh derivative [58] (also, see [19]) for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The operator $L_p(a,c)$ was introduced and studied by Saitoh [59]. This operator is an extension of the familiar Carlson-Shaffer operator $L(a,c)$ which has been used widely on the space of analytic and univalent functions in $\mathbb{U}$ (see, for details [8]; see also [63, 64]).

Corresponding to the function $\phi_p(a,c;z)$, let $\phi_p^{\dagger}(a,c;z)$ be defined such that

$$\phi_p(a,c;z) * \phi_p^{\dagger}(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}} \quad (\lambda > -p).$$

Analogous to $L_p(a,c)$, we now define a linear operator $\mathcal{I}_p^{\lambda}(a,c)$ on $\mathcal{A}$ as follows:

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = \phi_p^{\dagger}(a,c;z) * f(z) \quad (a, c \neq 0, -1, -2, \cdots; \ \lambda > -p; \ z \in \mathbb{U}). \quad (1.7.5)$$

We note that $\mathcal{I}_p^1(p+1,1)f(z) = f(z)$ and $\mathcal{I}_p^1(p,1)f(z) = \frac{zf'(z)}{p}$. It is easily verified from the definition of the operator $\mathcal{I}_p^{\lambda}(a,c)$ that

$$z(\mathcal{I}_p^{\lambda}(a+1,c)f(z))' = a\mathcal{I}_p^{\lambda}(a,c)f(z) - (a-p)\mathcal{I}_p^{\lambda}(a+1,c)f(z) \qquad (1.7.6)$$

and

$$z(\mathcal{I}_p^\lambda(a,c)f(z))' = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^\lambda(a,c)f(z). \qquad (1.7.7)$$

In particular, the operator $\mathcal{I}_1^\lambda(\mu+2,1)$ $(\lambda > -1,\ \mu > -2)$ were introduced by Choi, Saigo and Srivastava [10] and they investigated some inclusion properties of various classes defined by using the operator $\mathcal{I}_1^\lambda(\mu+2,1)$. For $a = n+1 (n \in \mathbb{N}_0)$ and $c = \lambda = 1$, we also note that the operator $\mathcal{I}_1^\lambda(a,c)f$ is the Noor integral operator of nth order of f studied by Liu [33] (also, see [34, 46, 47]). Also, let $\mathcal{I}_p^1 = \mathcal{I}_p$.

Let $F(a,b;c;z)$ be the Gaussian hypergeometric function defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (a,b \in \mathbb{C};\ c \neq 0,-1,-2,\cdots;\ z \in \mathbb{U}) \qquad (1.7.8)$$

where $(\nu)_n$ is the Pochhammer symbol (or the shifted factorial) defined by (1.7.4).

Then we see that the well-known formula

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \quad (\mathrm{Re}(c-a-b) > 0) \qquad (1.7.9)$$

We also recall (see [37]) that the function $F(a,b;c;z)$ is bounded if $\mathrm{Re}\{c-a-b\}) > 0$, and has a pole at $z = 1$ if $\mathrm{Re}\{c-a-b\} \leq 0$.

For $f \in \mathcal{A}$, we define the operator $I_{a,b;c}f$ by

$$I_{a,b;c}f(z) = zF(a,b;c;z) * f(z), \qquad (1.7.10)$$

where $*$ denotes the usual Hadamard product (or convolution) of power series.

1.8 Synopsis of the thesis

Now we give the outline of the thesis.

Chapter 2, is to obtain some interesting subordination properties by β-convex functions in the open unit disk associated with the Choi-Saigo-Srivastva operator. Moreover, applications for integral operators are also considered.

In chapter 3, we investigate majorization properties for p-valent functions defined by the linear operator.

Chapter 4, is to investigate argument properties of Carathéodory functions applying the recent result obtained by Nunokawa *et al.* [68]. We also obtain some geometric properties of analytic functions as special cases.

In chapter 5, we introduce a convolution operator for functions f belonging to the class Σ and we obtain some mapping properties and argument estimates for meromorphic functions associated with this convolution operator.

Chapter 6, is to derive some argument properties of multiplier transformations in the open unit disk defined by the inverse of a linear operator. We also investigate their integral preserving property in a sector.

In chapter 7, we obtain inclusion and mapping properties related to uniformly convex and uniformly starlike functions for a linear operator defined by means of Hadamard product (or convolution) with the Gaussian hypergeometric function.

Chapter 2

Subordination implications for certain analytic functions defined by convolution

2.1 Introduction

Let $\mathcal{Q}$ be the class of functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

In the present paper, making use of the principle of subordination, we investigate the subordination properties by certain univalent function for the linear operator $\mathcal{I}_p^\lambda(a,c)$ defined by (1.7.5). We also consider interesting applications to the integral operator.

The following lemmas will be required in our present investigation.

Lemma 2.1.1. [40] *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let*

$$q(z) = a + a_n z^n + \cdots$$

be analytic in $\mathbb{U}$ *with*

$$q(z) \not\equiv a \quad \text{and} \quad n \in \mathbb{N}.$$

If q *is not subordinate to* p, *then there exist points*

$$z_0 = r_0 \mathrm{e}^{i\theta} \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U} \backslash E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), \;\; q(z_0) = p(\zeta_0) \quad \textit{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

A functions $q \in \mathcal{H}$ with $q(0) = 0$ and $q'(0) \neq 0$ is said to be an β-convex function (not necessary normalized), if it satisfies the following condition:

$$\Re\left[(1-\beta)\frac{zq'(z)}{q(z)} + \beta\left(1 + \frac{zq''(z)}{q'(z)}\right)\right] > 0 \quad (\beta \in \mathbb{R};\; z \in \mathbb{U})$$

and we denote this class by $\mathcal{M}^*_\beta$. The class of β-convex functions was introduced by Mocanu [43]. We also note [42] that all β-convex functions univalent and starlike, and

$$\mathcal{M}^*_\beta \subset \mathcal{M}^*_\alpha \subset \mathcal{M}^*_0 \quad (0 \leq \frac{\alpha}{\beta} \leq 1)$$

Moreover, we note that $\mathcal{M}^*_1$ is the class of normalized convex functions in $\mathbb{U}$.

2.2 Main Results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation $\mathcal{I}_p^\lambda(a,c)$ defined by (1.7.5).

Theorem 2.2.1. *Let $f, g \in \mathcal{A}_p$ and suppose that*

$$\Re\left\{(1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{\mathcal{I}_p^\lambda(a+1,c)g(z)}+\beta\frac{z\left(\mathcal{I}_p^\lambda(a,c)g(z)\right)'-(p-1)\mathcal{I}_p^\lambda(a,c)g(z)}{z\left(\mathcal{I}_p^\lambda(a+1,c)g(z)\right)'-(p-1)\mathcal{I}_p^\lambda(a+1,c)g(z)}\right\}>\frac{(a-1)}{a}$$

$$(\beta \geq 0;\ a \geq 1;\ z \in \mathbb{U}). \tag{2.2.1}$$

Then the following subordination relation:

$$\left[\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}\right]^{\beta}\prec\left[\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{z^{p-1}}\right]^{\beta}\quad(z\in\mathbb{U}) \tag{2.2.2}$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}\prec\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}}\quad(z\in\mathbb{U}).$$

Proof. Let us define the functions F and G by

$$F(z):=\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}\quad\text{and}\quad G(z):=\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}}\quad(f,g\in\mathcal{A}_p;\ z\in\mathbb{U}). \tag{2.2.3}$$

By using the equation (1.7.6) to (2.2.3) and also, by a simple calculation, we have

$$\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{z^{p-1}}=\frac{(a-1)G(z)+zG'(z)}{a}. \tag{2.2.4}$$

Hence, combining (2.2.3) and (2.2.4), we obtain

$$\left[\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{z^{p-1}}\right]^{\beta}=G(z)\left[\frac{a-1+\frac{zG'(z)}{G(z)}}{a}\right]^{\beta} \tag{2.2.5}$$

Thus, from (2.2.5), we need to prove the following subordination implication:

$$\begin{aligned}F(z)\left[\frac{a-1+\frac{zF'(z)}{F(z)}}{a}\right]^{\beta}\prec G(z)\left[\frac{a-1+\frac{zG'(z)}{G(z)}}{a}\right]^{\beta}\quad (z\in\mathbb{U})\\ \Longrightarrow\quad F(z)\prec G(z)\ (z\in\mathbb{U}).\end{aligned} \tag{2.2.6}$$

Since $G\in\mathcal{M}_\beta^*$, without loss of generality, we can assume that G satisfies the conditions of Theorem 2.2.1 on the closed disk $\overline{\mathbb{U}}$ and

$$G'(\zeta)\neq 0\quad (\zeta\in\partial\mathbb{U}).$$

If not, then we replace F and G by

$$F_r(z)=F(rz)\text{ and, }G_r(z)=G(rz),$$

respectively, where $0<r<1$ and then G_r is univalent on $\overline{\mathbb{U}}$. Since

$$F_r(z)\left[\frac{a-1+\frac{zF_r'(z)}{F_r(z)}}{a}\right]^{\beta}\prec G_r(z)\left[\frac{a-1+\frac{zG_r'(z)}{G_r(z)}}{a}\right]^{\beta}\quad (z\in\mathbb{U}),$$

where

$$F_r(z)=F(rz)\quad (0<r<1;\ z\in\mathbb{U}),$$

we would then prove that

$$F_r(z)\prec G_r(z)\quad (0<r<1;\ z\in\mathbb{U}),$$

and by letting $r\to 1^-$, we obtain

$$F(z)\prec G(z)\quad (z\in\mathbb{U}).$$

If we suppose that the implication (2.2.6) is not true, that is,

$$F(z) \not\prec G(z) \quad (z \in \mathbb{U}),$$

then, from Lemma 2.1.1, there exist points

$$z_0 \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U}$$

such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = m\zeta_0 G'(\zeta_0) \ (m \geq 1). \tag{2.2.7}$$

To prove the implication (2.2.6), we define the function

$$L : \mathbb{U} \times [0, \infty) \longrightarrow \mathbb{C}$$

by

$$\begin{aligned} L(z,t) &= G(z) \left[\frac{a - 1 + (1+t)\frac{zG'(z)}{G(z)}}{a} \right]^{\beta} \\ &= a_1(t)z + \cdots , \end{aligned}$$

and we will show that $L(z,t)$ is a subordination chain. At first, we note that $L(z,t)$ is analytic in $|z| < r < 1$, for sufficient small $r > 0$ and for all $t \geq 0$. We also have that $L(z,t)$ is continuously differentiable on $[0, \infty)$ for each $|z| < r < 1$. A simple calculation shows that

$$a_1(t) = \frac{\partial L(0,t)}{\partial z} = G'(0) \left[\frac{a+t}{a} \right]^{\beta}.$$

Hence we obtain

$$a_1(t) \neq 0 \quad (t \geq 0)$$

and also we can see that

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

While, by a direct computation of $L(z,t)$, we have

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} = \frac{a-1}{\beta} + \frac{(1+t)}{\beta}\Re\left[(1-\beta)\frac{zG'(z)}{G(z)} + \beta\left(1+\frac{zG''(z)}{G'(z)}\right)\right]. \quad (2.2.8)$$

By using the assumption of Theorem 2.2.1 condition $\beta > 0$ to (2.2.8), we obtain

$$\Re\left\{\frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty),$$

which completes the proof of the first condition of Lemma 1.5.2. Moreover, we have

$$\begin{aligned}\left|\frac{L(z,t)}{a_1(t)}\right|^{1/\beta} &= \left|\frac{G(z)}{G'(0)}\right|^{1/\beta}\left|\frac{a-1+(1+t)\frac{zG'(z)}{G(z)}}{a+t}\right| \\ &\leq \left|\frac{G(z)}{G'(0)}\right|^{1/\beta}\left(\frac{a-1}{a+t} + \frac{1+t}{a+t}\left|\frac{zG'(z)}{G(z)}\right|\right) \\ &\leq \left(\frac{r}{(1-r)^2}\right)^{1/\beta}\left(\frac{a-1}{a} + \frac{1+r}{1-r}\right).\end{aligned} \quad (2.2.9)$$

Since G is univalent in $\mathbb{U}$, We have the following sharp growth and distortion results [55]:

$$\frac{r}{(1+r)^2} \leq |G(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1) \tag{2.2.10}$$

and

$$\frac{1-r}{(1+r)^3} \leq |G'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1) \tag{2.2.11}$$

Hence, by applying the equations (2.2.10) and (2.2.11) to (2.2.9), we can find easily an upper bound for the right-hand side of (2.2.9). Thus the function $L(z,t)$ satisfies the second condition of Lemma 1.5.2, which proves that $L(z,t)$ is a subordination chain. In particular, we note from the definition of subordination chain that

$$L(z,0) \prec L(z,t) \quad (z \in \mathbb{U};\ t \geq 0). \tag{2.2.12}$$

Now, by using the definition of $L(z,t)$ and the relation (2.2.7), we obtain

$$\begin{aligned} L(\zeta_0,t) &= G(\zeta_0)\left[\frac{a-1+(1+t)\frac{\zeta_0 G'(\zeta_0)}{G(\zeta_0)}}{a}\right]^{\beta} \\ &= F(z_0)\left[\frac{a-1+\frac{z_0 F'(z_0)}{F(z_0)}}{a}\right]^{\beta} \\ &= \left[\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z_0)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z_0)}{z^{p-1}}\right]^{\beta} \in L(\mathbb{U},0), \end{aligned}$$

by virtue of the subordination condition (2.2.2). This contradicts the above observation that

$$L(\zeta_0,t) \notin L(\mathbb{U},0).$$

Therefore, the subordination condition (2.2.2) must imply the subordination given by (2.2.6). This evidently completes the proof of Theorem 2.2.1.

□

Next, we give another subordination property by using the equation (1.7.6) in Theorem 2.2.2 below.

Theorem 2.2.2. *Let* $f, g \in \mathcal{A}_p$ *and suppose that*

$$\Re\left\{\frac{z\left(\mathcal{I}_p^\lambda(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^\lambda(a,c)g(z)}{z\left(\mathcal{I}_p^\lambda(a+1,c)g(z)\right)' - (p-1)\mathcal{I}_p^\lambda(a+1,c)g(z)}\right\} > \frac{a-1}{a} \quad (a \geq 1;\ z \in \mathbb{U}). \tag{2.2.13}$$

Then the following subordination relation:

$$\beta\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}} \ (0 \leq \beta \leq 1;\ z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Proof. Let us define the functions F and G as (2.2.3) and by using the equation (1.7.6) to (2.2.3), we have (2.2.4).

Hence, combining (2.2.3) and (2.2.4), we obtain

$$(1-\beta)\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^{p-1}} + \beta\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{z^{p-1}} = G(z)\left(\frac{\beta\left[a-1+\frac{zG'(z)}{G(z)}\right]}{a} + 1 - \beta\right) \tag{2.2.14}$$

Thus, from (2.2.14), we need to prove the following subordination implication:

$$F(z)\left(\frac{\beta\left[a-1+\frac{zF'(z)}{F(z)}\right]}{a}+1-\beta\right)\prec G(z)\left(\frac{\beta\left[a-1+\frac{zG'(z)}{G(z)}\right]}{a}+1-\beta\right)\ (z\in\mathbb{U})$$
$$\Longrightarrow\quad F(z)\prec G(z)\ (z\in\mathbb{U}). \tag{2.2.15}$$

Without loss of generality as in the proof of Theorem 2.2.1, we can assume that G satisfies the conditions of Theorem 2.2.1 on the closed disk $\overline{\mathbb{U}}$ and

$$G'(\zeta)\neq 0\quad(\zeta\in\partial\mathbb{U}).$$

To prove the implication (2.2.15), we consider the function

$$L:\mathbb{U}\times[0,\infty)\longrightarrow\mathbb{C}$$

by

$$\begin{aligned}L(z,t)&=G(z)\left(\frac{\beta\left[a-1+(1+t)\frac{zG'(z)}{G(z)}\right]}{a}+1-\beta\right)\\&=a_1(t)z+\cdots,\end{aligned}$$

and we want to prove that $L(z,t)$ is a subordination chain. But, the remaining part of the proof in Theorem 2.2.2 is similar to that of Theorem 2.2.1 and so we omit the detailed proof.

If we take

$$a = p,\ c = \lambda = 1 \text{ and } \beta = 1$$

in Theorem 2.2.1 and Theorem 2.2.2, respectively, then we have the following result.

□

Corollary 2.2.1. *Let $f, g \in \mathcal{A}_p$ and suppose that*

$$\Re\left\{\frac{z}{p}\left(\frac{(zg'(z))' - (p-1)g'(z)}{zg'(z) - (p-1)g(z)}\right)\right\} > 0 \quad (a \geq p;\ z \in \mathbb{U}).$$

Then we have the following implication:

$$\frac{zf'(z)}{pz^{p-1}} \prec \frac{zg'(z)}{pz^{p-1}}\ (z \in \mathbb{U}) \quad \Longrightarrow \quad \frac{f(z)}{z^{p-1}} \prec \frac{g(z)}{z^{p-1}}\ (z \in \mathbb{U})$$

By using the same techniques as in the proof of Theorem 2.2.1 with the equation (1.7.7), we have the following result.

Theorem 2.2.3. *Let $f, g \in \mathcal{A}_p$ and suppose that*

$$\Re\left\{(1-\beta)\frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{\mathcal{I}_p^{\lambda}(a,c)g(z)} + \beta\frac{z\left(\mathcal{I}_p^{\lambda+1}(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{z\left(\mathcal{I}_p^{\lambda}(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^{\lambda}(a,c)g(z)}\right\} > \frac{\lambda+p-1}{\lambda+p}$$
$$(\beta \geq 0;\ \lambda \geq 0;\ z \in \mathbb{U}). \tag{2.2.16}$$

Then the following subordination relation:

$$\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{z^{p-1}}\right]^{\beta}\ (z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

The proof of Theorem 2.2.4 below is much akin to that of Theorem 2.2.3 and so the details may be omitted.

Theorem 2.2.4. *Let $f, g \in \mathcal{A}_p$ and suppose that*

$$\Re\left\{\frac{z\left(\mathcal{I}_p^{\lambda+1}(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{z\left(\mathcal{I}_p^{\lambda}(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^{\lambda}(a,c)g(z)}\right\} > \frac{\lambda+p-1}{\lambda+p} \quad (\lambda \geq 0;\; z \in \mathbb{U}). \tag{2.2.17}$$

Then the following subordination relation:

$$\beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}} \quad (0 \leq \beta \leq 1;\; z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Next, we consider the generalized Libera integral operator F_ν $(\nu > -p)$ defined by (cf. [4, 19, 35, 52])

$$F_\nu(f)(z) := \frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1}f(t)dt \quad (f \in \mathcal{A};\; \Re\{\nu\} > -1) \tag{2.2.18}$$

Now, we obtain the following subordination property involving the integral operator defined by (2.2.18).

Theorem 2.2.5. *Let* $f, g \in \mathcal{A}_p$ *and suppose that*

$$\Re\left\{(1-\beta)\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(g)(z)} + \beta\frac{z\left(\mathcal{I}_p^{\lambda}(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^{\lambda}(a,c)g(z)}{z\left(\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(g)(z)\right)' - (p-1)\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(g)(z)}\right\} > \frac{\nu+p-1}{\nu+p}$$

$$(\nu \geq 0;\ \beta \geq 0;\ z \in \mathbb{U}).$$

Then the following subordination relation:

$$\left[\frac{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(f)(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(g)(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}}\right]^{\beta}\ (z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(f)(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(g)(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Proof. Let us define the function F and G by

$$F(z) := \frac{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(f)(z)}{z^{p-1}} \quad \text{and} \quad G(z) := \frac{\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(g)(z)}{z^{p-1}} \quad (f, g \in \mathcal{A}_p;\ z \in \mathbb{U}). \tag{2.2.19}$$

From the definition of the integral operator F_{ν} defined by (2.2.18), we obtain

$$z(\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(f)(z))' = (\nu+p)\mathcal{I}_p^{\lambda}(a,c)f(z) - \nu\mathcal{I}_p^{\lambda}(a,c)F_{\nu}(f)(z) \tag{2.2.20}$$

Hence, by using (2.2.19), (2.2.20) and the same method as in the proof of Theorem 2.2.1, we can prove Theorem 2.2.5 and so we omit the details involved. □

Finally, we obtain the following Theorem 2.2.6 below by using a similar method as in the proof of Theorem 2.2.3 or Theorem 2.2.4.

Theorem 2.2.6. *Let* $f, g \in \mathcal{A}_p$ *and suppose that*

$$\Re\left\{\frac{z\left(\mathcal{I}_p^\lambda(a,c)g(z)\right)' - (p-1)\mathcal{I}_p^\lambda(a,c)g(z)}{z\left(\mathcal{I}_p^\lambda(a,c)F_\nu(g)(z)\right)' - (p-1)\mathcal{I}_p^\lambda(a,c)F_\nu(g)(z)}\right\} > \frac{\nu+p-1}{\nu+p} \quad (\nu \geq 0;\ z \in \mathbb{U}).$$

Then the following subordination relation:

$$\beta\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g)(z)}{z^{p-1}} \quad (0 \leq \beta \leq 1;\ z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g)(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

If we take

$$a = p+1,\ c = \lambda = 1 \text{ and } \beta = 1$$

Theorem 2.2.5 or Theorem 2.2.6, then we have the following result.

Corollary 2.2.2. *Let* $f, g \in \mathcal{A}_p$ *and suppose that*

$$\Re\left\{\frac{zg'(z) - (p-1)g(z)}{z\left(F_\nu(g)(z)\right)' - (p-1)F_\nu(g)(z)}\right\} > \frac{\nu+p-1}{\nu+p} \quad (\nu \geq 0;\ z \in \mathbb{U}).$$

Then we have the following implication:

$$\frac{f(z)}{z^{p-1}} \prec \frac{g(z)}{z^{p-1}}\ (z \in \mathbb{U}) \quad \Longrightarrow \quad \frac{F_\nu(f)(z)}{z^{p-1}} \prec \frac{F_\nu(g)(z)}{z^{p-1}}\ (z \in \mathbb{U})$$

Theorem 2.2.7. *Let* $f, g \in \mathcal{A}_p$*. suppose that the condition* (2.2.1) *is satisfied and the function*

$$\left[\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$

is univalent. Then the following subordination relation:

$$\left[\frac{\mathcal{I}_p^{\lambda}(a+1,c)g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{\beta} \quad (z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a+1,c)g(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.8. *Let* $f, g \in \mathcal{A}_p$*. suppose that the condition* (2.2.13) *is satisfied and the function*

$$\beta\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}$$

is univalent. Then the following subordination relation:

$$\beta\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_P^{\lambda}(a+1,c)g(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} \quad (z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a+1,c)g(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.9. *Let $f, g \in \mathcal{A}_p$. suppose that the condition* (2.2.16) *is satisfied and the function*

$$\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$

is univalent. Then the following subordination relation:

$$\left[\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}\right]^{\beta} \quad (z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.10. *Let $f, g \in \mathcal{A}_p$. suppose that the condition* (2.2.17) *is satisfied and the function*

$$\beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}$$

is univalent. Then the following subordination relation:

$$\beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} \quad (z \in \mathbb{U})$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.11. *Let* $f, g_k \in \mathcal{A}_p(k=1,2)$*. with* $\frac{\mathcal{I}_p^\lambda(a+1,c)g_k}{z^{p-1}} \in \mathcal{M}_\beta^*$*. suppose that*

$$\left[\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$

and

$$\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}$$

are univalent in $\mathbb{U}$*. Then the following subordination relation*:

$$\left[\frac{\mathcal{I}_p^\lambda(a+1,c)g_1(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)g_1(z)}{z^{p-1}}\right]^{\beta} \prec \left[\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$
$$\prec \left[\frac{\mathcal{I}_p^\lambda(a+1,c)g_2(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)g_2(z)}{z^{p-1}}\right]^{\beta} \quad (z \in \mathbb{U}).$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a+1,c)g_1(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a+1,c)g_2(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.12. *Let* $f, g_k \in \mathcal{A}_p(k=1,2)$*. with* $\frac{\mathcal{I}_p^\lambda(a+1,c)g_k}{z^{p-1}} \in \mathcal{M}_\beta^*$*. suppose that*

$$\beta\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}$$

and

$$\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^{p-1}}$$

are univalent in $\mathbb{U}$*. Then the following subordination relation:*

$$\beta\frac{\mathcal{I}_p^{\lambda}(a,c)g_1(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^{\lambda}(a+1,c)g_1(z)}{z^{p-1}}\prec\beta\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}$$
$$\prec\beta\frac{\mathcal{I}_p^{\lambda}(a,c)g_2(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^{\lambda}(a+1,c)g_2(z)}{z^{p-1}}$$
$$(z\in\mathbb{U}).$$

implies that

$$\frac{\mathcal{I}_p^{\lambda}(a+1,c)g_1(z)}{z^{p-1}}\prec\frac{\mathcal{I}_p^{\lambda}(a+1,c)f(z)}{z^{p-1}}\prec\frac{\mathcal{I}_p^{\lambda}(a+1,c)g_2(z)}{z^{P-1}}\quad(z\in\mathbb{U}).$$

Theorem 2.2.13. *Let* $f, g_k\in\mathcal{A}_p(k=1,2)$*. with* $\frac{\mathcal{I}_p^{\lambda}(a,c)g_k}{z^{p-1}}\in\mathcal{M}_{\beta}^{*}$*. suppose that*

$$\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$

and

$$\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}$$

are univalent in $\mathbb{U}$*. Then the following subordination relation:*

$$\left[\frac{\mathcal{I}_p^{\lambda}(a,c)g_1(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)g_1(z)}{z^{p-1}}\right]^{\beta}\prec\left[\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$
$$\prec\left[\frac{\mathcal{I}_p^{\lambda}(a,c)g_2(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^{\lambda+1}(a,c)g_2(z)}{z^{p-1}}\right]^{\beta}$$
$$(z\in\mathbb{U}).$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a,c)g_1(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)g_2(z)}{z^{P-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.14. *Let $f, g_k \in \mathcal{A}_p (k = 1,2)$. with $\frac{\mathcal{I}_p^\lambda(a,c)g_k}{z^{p-1}} \in \mathcal{M}_\beta^*$. suppose that*

$$\beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}$$

and

$$\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{P-1}}$$

are univalent in $\mathbb{U}$. Then the following subordination relation:

$$\beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)g_1(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)g_1(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)f(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}$$
$$\prec \beta\frac{\mathcal{I}_p^{\lambda+1}(a,c)g_2(z)}{z^{p-1}} + (1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)g_2(z)}{z^{p-1}}$$
$$(z \in \mathbb{U}).$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a,c)g_1(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)g_2(z)}{z^{p-1}} \quad (z \in \mathbb{U}).$$

Theorem 2.2.15. *Let $f, g_k \in \mathcal{A}_p (k = 1,2)$. with $\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_k)(z)}{z^{p-1}} \in \mathcal{M}_\beta^*$. suppose that*

$$\left[\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}\right]^\beta$$

and

$$\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}$$

are univalent in $\mathbb{U}$*. Then the following subordination relation:*

$$\left[\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_1)(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)g_1(z)}{z^{p-1}}\right]^{\beta}\prec\left[\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}\right]^{\beta}$$
$$\prec\left[\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_2)(z)}{z^{p-1}}\right]^{1-\beta}\left[\frac{\mathcal{I}_p^\lambda(a,c)g_2(z)}{z^{p-1}}\right]^{\beta}$$
$$(z\in\mathbb{U}).$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_1)(z)}{z^{p-1}}\prec\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}\prec\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_2)(z)}{z^{p-1}}\quad(z\in\mathbb{U}).$$

Theorem 2.2.16. *Let* $f, g_k\in\mathcal{A}_p(k=1,2)$*. with* $\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_k)(z)}{z^{p-1}}\in\mathcal{M}_\beta^*$*. suppose that*

$$\beta\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}$$

and

$$\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}$$

are univalent in $\mathbb{U}$*. Then the following subordination relation:*

$$\beta\frac{\mathcal{I}_p^\lambda(a,c)g_1(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_1)(z)}{z^{p-1}} \prec \beta\frac{\mathcal{I}_p^\lambda(a,c)f(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}}$$
$$\prec \beta\frac{\mathcal{I}_p^\lambda(a,c)g_2(z)}{z^{p-1}}+(1-\beta)\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_2)(z)}{z^{p-1}} \quad (z\in\mathbb{U}).$$

implies that

$$\frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_1)(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)F_\nu(f)(z)}{z^{p-1}} \prec \frac{\mathcal{I}_p^\lambda(a,c)F_\nu(g_2)(z)}{z^{p-1}} \quad (z\in\mathbb{U}).$$

Chapter 3

Majorization properties for certain analytic functions defined by convolution

3.1 Introduction

In this chapter, we investigate majorization properties for p-valent functions defined by the linear operator $\mathcal{I}_p^{\lambda}(a,c)$. More precisely, we will find radii R for which satisfy the following implications:

A. $|\delta\mathcal{I}_p^{\lambda}(a+1,c)f(z),\mathcal{I}_p^{\lambda}(a+1,c)g(z),1|$ implies $|\mathcal{I}_p^{\lambda}(a,c)f(z),\mathcal{I}_p^{\lambda}(a,c)g(z),R|$;

B. $|\delta\mathcal{I}_p^{\lambda}(a,c)f(z),\mathcal{I}_p^{\lambda}(a,c)g(z),1|$ implies $|\mathcal{I}_p^{\lambda+1}(a,c)f(z),\mathcal{I}_p^{\lambda+1}(a,c)g(z),R|$,

where $f,\ g\in\mathcal{A}_p$ and $\delta\in\overline{\mathbb{U}}:=\{z\in\mathbb{C}:|z|\le 1\}$ is a given constant.

We recall that for $f \in \mathcal{A}_p$, the following equality holds:

$$\mathcal{I}_p^{\lambda}(a,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(c)_k(\lambda+p)_k}{(a)_k(1)_k} z^{p+k}, \tag{3.1.1}$$

where $(a)_k$ is the Pochhammer symbol defined by (1.7.4). In particular, by using (3.1.1), we easily obtain the identities

$$\mathcal{I}_p^{\lambda}(\lambda+p,1)f(z) = f(z)$$

and

$$\mathcal{I}_p^{1}(p,1)f(z) = \frac{1}{p} z f'(z).$$

Also, we remark that the following recurrence equations hold for the operator $\mathcal{I}_p^{\lambda}(a,c)$:

$$z(\mathcal{I}_p^{\lambda}(a+1,c)f(z))' = a\mathcal{I}_p^{\lambda}(a,c)f(z) - (a-p)\mathcal{I}_p^{\lambda}(a+1,c)f(z)$$

and

$$z(\mathcal{I}_p^{\lambda}(a,c)f(z))' = (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^{\lambda}(a,c)f(z).$$

3.2 Main Results

At first, we give a radius for which satisfies the condition A.

Theorem 3.2.1. *Let $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, and let $a \in \mathbb{R}$ with $p \leq a < 2p$. Let $f \in \mathcal{A}_p$ and $\mathcal{I}_p^{\lambda}(a+1,c)g(z) \in \mathcal{S}_p^*$. Suppose that $\delta\mathcal{I}_p^{\lambda}(a+1,c)f(z) \ll \mathcal{I}_p^{\lambda}(a+1,c)g(z)$ in $\mathbb{U}$.*

(1) If $|\delta| = 1$, then $\mathcal{I}_p^{\lambda}(a,c)f(z) \ll \mathcal{I}_p^{\lambda}(a,c)g(z)$ in $\mathbb{U}$. In fact, we have $\mathcal{I}_p^{\lambda}(a,c)f(z) = \mathcal{I}_p^{\lambda}(a,c)g(z)$ in $\mathbb{U}$;

(2) If $|\delta| < 1$, then $\delta\mathcal{I}_p^\lambda(a,c)f(z) \ll \mathcal{I}_p^\lambda(a,c)g(z)$ in $\mathbb{U}_{r_0}$, where

$$r_0 = \frac{1+p-\sqrt{1+a^2+2p-2ap+p^2}}{a}. \tag{3.2.1}$$

Proof. **I.** Let $|\delta| = 1$. Since $\delta\mathcal{I}_p^\lambda(a+1,c)f(z) \ll \mathcal{I}_p^\lambda(a+1,c)g(z)$ in $\mathbb{U}$, there exists an analytic function ϕ with $|\phi(z)| \leq 1$, $z \in \mathbb{U}$, such that $\delta\mathcal{I}_p^\lambda(a+1,c)f(z) = \phi(z)\mathcal{I}_p^\lambda(a+1,c)g(z)$. Since both $\mathcal{I}_p^\lambda(a+1,c)f(z)$ and $\mathcal{I}_p^\lambda(a+1,c)g(z)$ are in $\mathcal{A}_p$, we have

$$\delta\frac{\mathcal{I}_p^\lambda(a+1,c)f(z)}{z^p} = \phi(z)\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{z^p}, \quad z \in \mathbb{U}, \tag{3.2.2}$$

and $1 = |\delta| = |\phi(0)|$. Since $|\phi(z)| \leq 1$ for all $z \in \mathbb{U}$, it follows from Maximum-Modulus Theorem that ϕ is constant in $\mathbb{U}$. Thus, from (3.2.2), we have $\phi(z) \equiv \phi(0) = \delta$, $z \in \mathbb{U}$. Thus we have $\mathcal{I}_p^\lambda(a+1,c)f(z) = \mathcal{I}_p^\lambda(a+1,c)g(z)$. So, we get

$$\begin{aligned}\mathcal{I}_p^\lambda(a,c)f(z) &= \frac{1}{a}[z(\mathcal{I}_p^\lambda(a+1,c)f(z))' + (a-p)\mathcal{I}_p^\lambda(a+1,c)f(z)]\\ &= \frac{1}{a}[z(\mathcal{I}_p^\lambda(a+1,c)g(z))' + (a-p)\mathcal{I}_p^\lambda(a+1,c)g(z)]\\ &= \mathcal{I}_p^\lambda(a,c)g(z).\end{aligned}$$

II. Now, let $|\delta| < 1$, and

$$\delta\mathcal{I}_p^\lambda(a+1,c)f(z) = \phi(z)\mathcal{I}_p^\lambda(a+1,c)g(z)$$

for some $\phi : \mathbb{U} \to \mathbb{C}$ analytic and satisfying $|\phi(z)| \leq 1$, for $z \in \mathbb{U}$.

By a simple calculation, we have

$$z(\mathcal{I}_p^\lambda(a+1,c)f(z))' = \frac{1}{\delta}[z\phi'(z)\mathcal{I}_p^\lambda(a+1,c)g(z) + \phi(z)z(\mathcal{I}_p^\lambda(a+1,c)g(z))'].$$

Also, since

$$z(\mathcal{I}_p^\lambda(a+1,c)f(z))' = a\mathcal{I}_p^\lambda(a,c)f(z) - (a-p)\mathcal{I}_p^\lambda(a+1,c)f(z)$$
$$= a\mathcal{I}_p^\lambda(a,c)f(z) - \frac{a-p}{\delta}\phi(z)\mathcal{I}_p^\lambda(a+1,c)g(z),$$

we easily get the following identity:

$$\delta\mathcal{I}_p^\lambda(a,c)f(z) = \frac{1}{a}z\phi'(z)\mathcal{I}_p^\lambda(a+1,c)g(z) + \phi(z)\mathcal{I}_p^\lambda(a,c)g(z).$$

Since $\mathcal{I}_p^\lambda(a+1,c)g(z) \in \mathcal{S}_p^*$, it holds that

$$\left|\frac{z(\mathcal{I}_p^\lambda(a+1,c)g(z))'}{\mathcal{I}_p^\lambda(a+1,c)g(z)}\right| \geq \frac{p(1-|z|)}{1+|z|}.$$

Let $z \in \mathbb{U}$ with $|z| < (2p-a)/a$. Then, by the triangle inequality, we obtain

$$\left|\frac{\mathcal{I}_p^\lambda(a,c)g(z)}{\mathcal{I}_p^\lambda(a+1,c)g(z)}\right| \geq \frac{1}{a}\left|\frac{z(\mathcal{I}_p^\lambda(a+1,c)g(z))'}{\mathcal{I}_p^\lambda(a+1,c)g(z)}\right| - \frac{a-p}{a} \geq \frac{2p-a-ar}{a(1+r)} > 0. \tag{3.2.3}$$

Now we put $|z| = r$ and $|\phi(z)| = \rho$. Using (3.2.3) and the well-known inequality $|\phi'(z)| \leq (1-\rho^2)/(1-r^2)$ yields that

$$\begin{aligned} &|\delta||\mathcal{I}_p^\lambda(a,c)f(z)| \\ &\leq |\mathcal{I}_p^\lambda(a,c)g(z)||\phi(z)|\left(1+\frac{1}{a}\left|\frac{z\phi'(z)}{\phi(z)}\right|\left|\frac{\mathcal{I}_p^\lambda(a+1,c)g(z)}{\mathcal{I}_p^\lambda(a,c)g(z)}\right|\right) \\ &\leq |\mathcal{I}_p^\lambda(a,c)g(z)|D_1, \end{aligned} \tag{3.2.4}$$

where

$$D_1 = D_1(r,\rho) := \rho\left(1+\frac{r(1-\rho^2)}{\rho(1-r)(2p-a-ar)}\right).$$

Since $\rho \leq 1$, the inequality $D_1 \leq 1$ is equivalent to that

$$\frac{r(1+\rho)}{(1-r)(2p-a-ar)} \leq 1.$$

And, this inequality holds for r satisfying

$$\frac{2r}{(1-r)(2p-a-ar)} \leq 1. \tag{3.2.5}$$

Now, it is easy to see that (3.2.5) is true for $r \leq r_0$, where r_0 is given by (3.2.1).

Consequently, for $r \leq r_0$, we have $D_1 \leq 1$, which implies, by (3.2.4), that

$$|\delta\mathcal{I}_p^\lambda(a,c)f(z)| \leq |\mathcal{I}_p^\lambda(a,c)g(z)|, \quad |z| = r \leq r_0.$$

That is, $\delta\mathcal{I}_p^\lambda(a,c)f(z) \ll \mathcal{I}_p^\lambda(a,c)g(z)$ in $|z| < r_0$, as we asserted. $\square$

By taking $\lambda = 1$, $a = p$ and $c = 1$ in Theorem 3.2.1 we obtain the following result.

Corollary 3.2.1. *Let $\delta \in \mathbb{C}$ with $|\delta| \leq 1$. Let $f \in \mathcal{A}_p$ and $g \in \mathcal{S}_p^*$. Suppose that $\delta f(z) \ll g(z)$ in $\mathbb{U}$.*

(1) If $|\delta| = 1$, then $zf'(z) \ll zg'(z)$ in $\mathbb{U}$. In fact, we have $f'(z) = g'(z)$ in $\mathbb{U}$;

(2) If $|\delta| < 1$, then $\delta zf'(z) \ll zg'(z)$ in $\mathbb{U}_{r_0}$, where $r_0 = (1+p-\sqrt{1+2p})/p$.

Next, we give a radius for which satisfies the condition B.

Theorem 3.2.2. *Let $\delta \in \mathbb{C}$ with $|\delta| \leq 1$, and let $\lambda \in \mathbb{R}$ with $\lambda < p$. Let $f \in \mathcal{A}_p$ and $\mathcal{I}_p^\lambda(a,c)g(z) \in \mathcal{S}_p^*$. Suppose that $\delta\mathcal{I}_p^\lambda(a,c)f(z) \ll \mathcal{I}_p^\lambda(a,c)g(z)$ in $\mathbb{U}$.*

(1) If $|\delta| = 1$, then $\mathcal{I}_p^{\lambda+1}(a,c)f(z) \ll \mathcal{I}_p^{\lambda+1}(a,c)g(z)$ in $\mathbb{U}$. In fact, we have $\mathcal{I}_p^{\lambda+1}(a,c)f(z) = \mathcal{I}_p^{\lambda+1}(a,c)g(z)$ in $\mathbb{U}$;

(2) If $|\delta| < 1$, then $\delta\mathcal{I}_p^{\lambda+1}(a,c)f(z) \ll \mathcal{I}_p^{\lambda+1}(a,c)g(z)$ in $\mathbb{U}_{r_0}$, where

$$r_0 = \frac{1+p-\sqrt{1+\lambda^2+2p}}{\lambda+p}. \tag{3.2.6}$$

Proof. **I.** Let $|\delta| = 1$. Since $\delta\mathcal{I}_p^{\lambda}(a,c)f(z) \ll \mathcal{I}_p^{\lambda}(a,c)g(z)$ in $\mathbb{U}$, there exists an analytic function ϕ with $|\phi(z)| \leq 1$, $z \in \mathbb{U}$, such that $\delta\mathcal{I}_p^{\lambda}(a,c)f(z) = \phi(z)\mathcal{I}_p^{\lambda}(a,c)g(z)$. Since both $\mathcal{I}_p^{\lambda}(a,c)f(z)$ and $\mathcal{I}_p^{\lambda}(a,c)g(z)$ are in $\mathcal{A}_p$, we have

$$\delta\frac{\mathcal{I}_p^{\lambda}(a,c)f(z)}{z^p} = \phi(z)\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{z^p}, \quad z \in \mathbb{U}, \tag{3.2.7}$$

and $1 = |\delta| = |\phi(0)|$. Since $|\phi(z)| \leq 1$ for all $z \in \mathbb{U}$, it follows from Maximum-Modulus Theorem that ϕ is constant in $\mathbb{U}$. Thus, from (3.2.7), we have $\phi(z) \equiv \phi(0) = \delta$, $z \in \mathbb{U}$. Thus we have $\mathcal{I}_p^{\lambda}(a,c)f(z) = \mathcal{I}_p^{\lambda}(a,c)g(z)$. So, we get

$$\begin{aligned}
\mathcal{I}_p^{\lambda+1}(a,c)f(z) &= \frac{1}{\lambda+p}[z(\mathcal{I}_p^{\lambda}(a,c)f(z))' + \lambda\mathcal{I}_p^{\lambda}(a,c)f(z)] \\
&= \frac{1}{\lambda+p}[z(\mathcal{I}_p^{\lambda}(a,c)g(z))' + \lambda\mathcal{I}_p^{\lambda}(a,c)g(z)] \\
&= \mathcal{I}_p^{\lambda+1}(a,c)g(z).
\end{aligned}$$

II. Now, let $|\delta| < 1$, and

$$\delta\mathcal{I}_p^{\lambda}(a,c)f(z) = \phi(z)\mathcal{I}_p^{\lambda}(a,c)g(z)$$

for some $\phi : \mathbb{U} \to \mathbb{C}$ analytic and satisfying $|\phi(z)| \leq 1$, for $z \in \mathbb{U}$.

By a simple calculation, we have

$$z(\mathcal{I}_p^{\lambda}(a,c)f(z))' = \frac{1}{\delta}[z\phi'(z)\mathcal{I}_p^{\lambda}(a,c)g(z) + \phi(z)z(\mathcal{I}_p^{\lambda}(a,c)g(z))'].$$

Also, since

$$\begin{aligned}
z(\mathcal{I}_p^{\lambda}(a,c)f(z))' &= (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \lambda\mathcal{I}_p^{\lambda}(a,c)f(z) \\
&= (\lambda+p)\mathcal{I}_p^{\lambda+1}(a,c)f(z) - \frac{\lambda}{\delta}\phi(z)\mathcal{I}_p^{\lambda}(a,c)g(z),
\end{aligned}$$

we easily get the following identity:

$$\delta\mathcal{I}_p^{\lambda+1}(a,c)f(z) = \frac{1}{\lambda+p}z\phi'(z)\mathcal{I}_p^{\lambda}(a,c)g(z) + \phi(z)\mathcal{I}_p^{\lambda+1}(a,c)g(z).$$

Since $\mathcal{I}_p^{\lambda}(a,c)g(z) \in \mathcal{S}_p^*$, it holds that

$$\left| \frac{z(\mathcal{I}_p^{\lambda}(a,c)g(z))'}{\mathcal{I}_p^{\lambda}(a,c)g(z)} \right| \geq \frac{p(1-|z|)}{1+|z|}.$$

Let $z \in \mathbb{U}$ with $|z| < (p-\lambda)/(p+\lambda)$. Then, by the triangle inequality, we obtain

$$\left| \frac{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}{\mathcal{I}_p^{\lambda}(a,c)g(z)} \right| \geq \frac{1}{\lambda+p} \left| \frac{z(\mathcal{I}_p^{\lambda}(a,c)g(z))'}{\mathcal{I}_p^{\lambda}(a,c)g(z)} \right| - \frac{\lambda}{\lambda+p} \geq \frac{p(1-r)-\lambda(1+r)}{(\lambda+p)(1+r)} > 0. \tag{3.2.8}$$

Now we put $|z| = r$ and $|\phi(z)| = \rho$. Using (3.2.8) and the well-known inequality $|\phi'(z)| \leq (1-\rho^2)/(1-r^2)$ yields that

$$\begin{aligned} &|\delta||\mathcal{I}_p^{\lambda+1}(a,c)f(z)| \\ &\leq |\mathcal{I}_p^{\lambda+1}(a,c)g(z)||\phi(z)| \left(1 + \frac{1}{\lambda+p}\left|\frac{z\phi'(z)}{\phi(z)}\right|\left|\frac{\mathcal{I}_p^{\lambda}(a,c)g(z)}{\mathcal{I}_p^{\lambda+1}(a,c)g(z)}\right|\right) \\ &\leq |\mathcal{I}_p^{\lambda+1}(a,c)g(z)|D_2, \end{aligned} \tag{3.2.9}$$

where

$$D_2 = D_2(r,\rho) := \rho\left(1 + \frac{r(1-\rho^2)}{\rho(1-r)(p-\lambda-(p+\lambda)r)}\right).$$

Since $\rho \leq 1$, the inequality $D_2 \leq 1$ is equivalent to that

$$\frac{r(1+\rho)}{(1-r)(2p-a-ar)} \leq 1.$$

And, this inequality holds for r satisfying

$$\frac{2r}{(1-r)(p-\lambda-(p+\lambda)r)} \leq 1. \tag{3.2.10}$$

Now, it is easy to see that (3.2.10) is true for $r \leq r_0$, where r_0 is given by (3.2.6).

Consequently, for $r \leq r_0$, we have $D_2 \leq 1$, which implies, by (3.2.9), that

$$|\delta\mathcal{I}_p^{\lambda+1}(a,c)f(z)| \leq |\mathcal{I}_p^{\lambda+1}(a,c)g(z)|, \quad |z| = r \leq r_0.$$

That is, $\delta\mathcal{I}_p^{\lambda+1}(a,c)f(z) \ll \mathcal{I}_p^{\lambda+1}(a,c)g(z)$ in $|z| < r_0$, as we asserted. $\square$

Chapter 4

Argument estimates for Carathéodory functions

4.1 Introduction

Recently, Nunokawa *et al.* [68] investigated an argument property of $p \in \mathcal{N}$ at extremal points on the boundary of the circle $|z| = r < 1$, which is the more extened one of the result earlier studied by Nunokawa [49].

In the present paper, we give some applications of the result obtained by Nunokawa et al. [68], which contain argument properties of Carathéodory functions. We also improve the results by Darus and Thomas [14], Nunokawa [49] and Nunokawa and Thomas [51] with some special cases.

4.2 Main results

To prove the main theorems, we need the following lemma due to Nunokawa et al. [68].

Lemma 4.2.1. *Let* $p \in \mathcal{N}$ *and* $p(z) \neq 0$ *in* $\mathbb{U}$. *If there exist two points* $z_1, z_2 \in \mathbb{U}$ *such that*

$$-\frac{\pi}{2}\alpha = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\beta \tag{4.2.1}$$

for some $\alpha, \beta (\alpha, \beta > 0)$ *and for all* $z (|z| < |z_1| = |z_2|)$, *then we have*

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+s^2}{2s}\right)m \tag{4.2.2}$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+t^2}{2t}\right)m, \tag{4.2.3}$$

where

$$p(z_1)^{\frac{2}{\alpha+\beta}} = -is\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right) \quad (s>0) \tag{4.2.4}$$

and

$$p(z_2)^{\frac{2}{\alpha+\beta}} = it\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right) \quad (t>0) \tag{4.2.5}$$

when

$$m \geq \frac{1-|a|}{1+|a|}, \quad a = i\tan\frac{\pi}{4}\left(\frac{\beta-\alpha}{\alpha+\beta}\right). \tag{4.2.6}$$

At first, with the help of Lemma 4.2.1, we obtain the following result.

Theorem 4.2.1. *Let* $k \in \mathbb{N} = \{1, 2, \cdots\}$, $\eta \in [0,\ 1]$ *and* $\alpha, \beta > 0$ *with* $(\alpha+\beta)(k-1) < 2$. *If a function* $p \in \mathcal{N}$ *satisfies the condition*

$$-\frac{\pi}{2}c_1(a,k,\alpha,\beta,\eta) \ < \ \arg\left\{p(z)\left(1+\frac{zp'(z)}{p^k(z)}\right)^{\eta}\right\} \ < \ \frac{\pi}{2}c_2(a,k,\alpha,\beta,\eta)$$

where

$$c_1(a,k,\alpha,\beta,\eta) = \alpha + \frac{2\eta}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\alpha(k-1)}{2(1+|a|)d(k,\alpha,\beta)+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\alpha(k-1)}\right\} \tag{4.2.7}$$

and

$$c_2(a,k,\alpha,\beta,\eta) = \beta + \frac{2\eta}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\beta(k-1)}{2(1+|a|)d(k,\alpha,\beta)+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\beta(k-1)}\right\} \tag{4.2.8}$$

when a *is given by* (4.2.6) *and*

$$d(k,\alpha,\beta) = \left(1+\frac{(\alpha+\beta)(k-1)}{2}\right)^{\frac{2+(\alpha+\beta)(k-1)}{4}}\left(1-\frac{(\alpha+\beta)(k-1)}{2}\right)^{\frac{2-(\alpha+\beta)(k-1)}{4}}, \tag{4.2.9}$$

then

$$-\frac{\pi}{2}\alpha \;<\; \arg\ p(z) \;<\; \frac{\pi}{2}\beta.$$

Proof. We note that $p(z) \neq 0$ for $z \in \mathbb{U}$. Otherwise, $p(z) = (z-z_1)^l p_1(z)$ $(z \in \mathbb{U})$ for some $l \geq 1$ and $z_1 \in \mathbb{U}$, where p_1 is an analytic function in $\mathbb{U}$ such that $p_1(z_1) \neq 0$. Then

$$\frac{zp'(z)}{p^k(z)} = \frac{1}{(z-z_1)^{l(k-1)+1}}\left(\frac{lz}{p_1^{k-1}(z)} + \frac{(z-z_1)zp_1'(z)}{p_1^k(z)}\right) \quad (z \in \mathbb{U}),$$

and so the above expression has a pole at the point z_1. This contradicts the assumptions of the theorem.

If there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (4.2.1) is satisfied, then (by Lemma 4.2.1) we obtain (4.2.2) and (4.2.3) under the restrictions (4.2.4) and (4.2.5), respectively.

At first, we suppose that

$$p(z_2)^{\frac{2}{\alpha+\beta}} = it\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right) \quad (t > 0).$$

Then, using (4.2.3), we have

$$p(z_2)\left(1+\frac{z_2p'(z_2)}{p^k(z_2)}\right)^{\eta}=t^{\frac{\alpha+\beta}{2}}e^{i\frac{\pi}{2}\beta}\left(1+me^{i\frac{\pi}{2}(1-\beta(k-1))}\frac{(\alpha+\beta)(1+t^2)}{4t^{\frac{(\alpha+\beta)(k-1)}{2}+1}}\right)^{\eta}.$$

Let

$$r(t)=\frac{(\alpha+\beta)(1+t^2)}{4t^{\frac{(\alpha+\beta)(k-1)}{2}+1}}\ \ (t>0).$$

Noting that $(\alpha+\beta)(k-1)/2<1$, we can observe that the function r attains its minimum value $r(t_0)=(\alpha+\beta)/2d(k,\alpha,\beta)$, where $d(k,\alpha,\beta)$ is given by (4.2.9) and $t_0=\sqrt{(2+(\alpha+\beta)(k-1))/(2-(\alpha+\beta)(k-1))}$. Hence we obtain

$$\begin{aligned}\arg\left\{p(z_2)\left(1+\frac{z_2p'(z_2)}{p^k(z_2)}\right)^{\eta}\right\}&\geq\frac{\pi}{2}\beta+\eta\tan^{-1}\left\{\frac{mr(t_0)\cos\frac{\pi}{2}\beta(k-1)}{1+mr(t_0)\sin\frac{\pi}{2}\beta(k-1)}\right\}\\&\geq\frac{\pi}{2}\beta+\eta\tan^{-1}\left\{\frac{(1-|a|)r(t_0)\cos\frac{\pi}{2}\beta(k-1)}{1+|a|+(1-|a|r(t_0)\sin\frac{\pi}{2}\beta(k-1)}\right\}\\&=c_2(a,k,\alpha,\beta,\eta),\end{aligned}$$

where $c_2(a,k,\alpha,\beta,\eta)$ is given by (4.2.8). This contradicts the assumption of the theorem.

Next, we suppose that

$$p(z_1)^{\frac{2}{\alpha+\beta}}=-is\exp\left(i\frac{\pi}{2}\left(\frac{\beta-\alpha}{\alpha+\beta}\right)\right)\ \ (s>0),$$

applying the same method as the above and using (4.2.2) and (4.2.6), we have

$$\begin{aligned}\arg\left\{p(z_1)\left(1+\frac{z_1p'(z_1)}{p^k(z_1)}\right)^{\eta}\right\}&\leq-\frac{\pi}{2}\alpha-\eta\tan^{-1}\left\{\frac{(1-|a|)r(t_0)\cos\frac{\pi}{2}\alpha(k-1)}{1+|a|+(1-|a|r(t_0)\sin\frac{\pi}{2}\alpha(k-1)}\right\}\\&=-c_1(a,k,\alpha,\beta,\eta),\end{aligned}$$

where $c_1(a,k,\alpha,\beta,\eta)$ is given by (4.2.7), which contradiction to the assumption of the theorem. Therefore we complete the proof of Theorem 4.2.1. □

If we let $\alpha=\beta$ in Theorem 4.2.1, then we see easily the following corollary.

Corollary 4.2.1. *Let* $k \in \mathbb{N}$, $\eta \in [0,\ 1]$ *and* $\alpha > 0$ *with* $\alpha(k-1) < 1$. *If a function* $p \in \mathcal{N}$ *satisfies the condition*

$$\left|\arg\ \left\{p(z)\left(1+\frac{zp'(z)}{p^k(z)}\right)^{\eta}\right\}\right| \ < \ \frac{\pi}{2}c(k,\alpha,\eta)$$

where

$$c(k,\alpha,\eta) = \alpha + \frac{2\eta}{\pi}\tan^{-1}\left\{\frac{\alpha\cos\frac{\pi}{2}\alpha(k-1)}{(1+\alpha(k-1))^{\frac{1+\alpha(k-1)}{2}}(1-\alpha(k-1))^{\frac{1-\alpha(k-1)}{2}} + \alpha\sin\frac{\pi}{2}\alpha(k-1)}\right\},$$

then

$$|\arg\ p(z)| < \frac{\pi}{2}\alpha.$$

Remark 4.2.1. *For* $k = 2$ *and* $\eta = 1$, *Corollary 4.2.1 is the result obtained by Nunokawa and Thomas* [51].

Taking $p(z) = zf'(z)/f(z)$ in Theorem 4.2.1, we have

Corollary 4.2.2. *Let* $k \in \mathbb{N}$, $\eta \in [0,\ 1]$ *and* $\alpha, \beta > 0$ *with* $(\alpha+\beta)(k-1) < 2$. *If a function* $f \in \mathcal{A}$ *satisfies the conditon*

$$-\frac{\pi}{2}c_1(a,k,\alpha,\beta,\eta) < \arg\left\{\frac{zf'(z)}{f(z)}\left(1+\left(\frac{f(z)}{zf'(z)}\right)^{k-1}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right)^{\eta}\right\}$$
$$< \frac{\pi}{2}c_2(a,k,\alpha,\beta,\eta),$$

where $c_1(a,k,\alpha,\beta,\eta)$ *and* $c_2(a,k,\alpha,\beta,\eta)$ *are given by* (4.2.7) *and* (4.2.8), *respectively, then*

$$-\frac{\pi}{2}\alpha \ < \arg\ \frac{zf'(z)}{f(z)} \ < \ \frac{\pi}{2}\beta.$$

Remark 4.2.2. *(i) If we take* $k = 2$ *and* $\alpha = \beta$ *in Corollary 4.2.2, we have the corresponding result obtained by Darus and Thomas* [14]. *(ii) For the case of* $k = 2$, $\eta = 1$ *and* $\alpha = \beta$, *Corollary 4.2.2 is the result studied by Nunokawa* [49].

Letting $p(z) = f(z)/z$ in Theorem 4.2.1, we have the following result.

Corollary 4.2.3. *Let* $k \in \mathbb{N},\ \eta \in [0,\ 1]$ *and* $\alpha, \beta > 0$ *with* $(\alpha+\beta)(k-1) < 2$*. If a function* $f \in \mathcal{A}$ *satisfies the condition*

$$-\frac{\pi}{2}c_1(a,k,\alpha,\beta,\eta) < \arg\left\{\frac{f(z)}{z}\left(1+\left(\frac{z}{f(z)}\right)^{k-1}\left(\frac{zf'(z)}{f(z)}-1\right)\right)^{\eta}\right\}$$
$$< \frac{\pi}{2}c_2(a,k,\alpha,\beta,\eta),$$

where $c_1(a,k,\alpha,\beta,\eta)$ *and* $c_1(a,k,\alpha,\beta,\eta)$ *are given by* (4.2.7) *and* (4.2.8)*, respectively, then*

$$-\frac{\pi}{2}\alpha\ < \arg\ \frac{f(z)}{z}\ <\ \frac{\pi}{2}\beta.$$

Setting $p(z) = f'(z)$ in Theorem 4.2.1, we have the following corollary.

Corollary 4.2.4. *Let* $k \in \mathbb{N},\ \eta \in [0,\ 1]$ *and* $\alpha, \beta > 0$ *with* $(\alpha+\beta)(k-1) < 2$*. If a function* $f \in \mathcal{A}$ *satisfies the condition*

$$-\frac{\pi}{2}c_1(a,k,\alpha,\beta,\eta) < \arg\left\{f'(z)\left(1+\frac{zf''(z)}{(f'(z))^k}\right)^{\eta}\right\} < \frac{\pi}{2}c_2(a,k,\alpha,\beta,\eta),$$

where $c_1(a,k,\alpha,\beta,\eta)$ *and* $c_1(a,k,\alpha,\beta,\eta)$ *are given by* (4.2.7) *and* (4.2.8)*, respectively, then*

$$-\frac{\pi}{2}\alpha\ < \arg\ f'(z)\ <\ \frac{\pi}{2}\beta.$$

By using the similar method as in the proof of Theorem 4.2.1, we have the following three theorems below. The proof is much akin to that of Theorem 4.2.1 and so the details may be omitted.

Theorem 4.2.2. *Let* $\alpha, \beta \in (0, 1]$*. If a function* $f \in \mathcal{A}$ *satisfies the condition*

$$\left|\arg\ \frac{zf'(z)}{f(z)}\right|\ <\ \tan^{-1}\frac{(\alpha+\beta)(1-|a|)}{2(1+|a|)},$$

where a is given by (4.2.6)*, then*

$$-\frac{\pi}{2}\alpha \ < \arg\ \frac{f(z)}{z}\ <\ \frac{\pi}{2}\beta.$$

Remark 4.2.3. *For the case $\alpha=\beta$, Theorem 4.2.2 is the result obtained by Nunokawa and Thomas* [51]*.*

Theorem 4.2.3. *Let $\alpha,\beta\in(0,1/2]$. If a function $f\in\mathcal{A}$ satisfies the condition*

$$-\pi\alpha-\tan^{-1}\frac{(\alpha+\beta)(1-|a|)}{2(1+|a|)}\ < \arg\frac{f(z)f'(z)}{z}\ <\ \pi\beta+\tan^{-1}\frac{(\alpha+\beta)(1-|a|)}{2(1+|a|)},$$

where a is given by (4.2.6)*, then*

$$-\frac{\pi}{2}\alpha \ < \arg\ \frac{f(z)}{z}\ <\ \frac{\pi}{2}\beta.$$

Theorem 4.2.4. *Let $\alpha,\beta\in(0,1)$. If a function $f\in\mathcal{A}$ satisfies the condition*

$$-\frac{\pi}{2}d_1(a,\alpha,\beta)\ < \arg\ \left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\ <\ \frac{\pi}{2}d_2(a,\alpha,\beta),$$

where

$$d_1(a,\alpha,\beta)=\frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\alpha}{2(1+|a|)\left(1+\frac{\alpha+\beta}{2}\right)^{\frac{2+\alpha+\beta}{4}}\left(1-\frac{\alpha+\beta}{2}\right)^{\frac{2-\alpha-\beta}{4}}+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\alpha}\right\}$$

and

$$d_2(a,\alpha,\beta)=\frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha+\beta)(1-|a|)\cos\frac{\pi}{2}\beta}{2(1+|a|)\left(1+\frac{\alpha+\beta}{2}\right)^{\frac{2+\alpha+\beta}{4}}\left(1-\frac{\alpha+\beta}{2}\right)^{\frac{2-\alpha-\beta}{4}}+(\alpha+\beta)(1-|a|)\sin\frac{\pi}{2}\beta}\right\},$$

when where a is given by (4.2.6)*, then*

$$-\frac{\pi}{2}\alpha \ < \arg\ \frac{zf'(z)}{f(z)}\ <\ \frac{\pi}{2}\beta.$$

Taking $\alpha=\beta$ in Theorem 4.2.4, we have the following result.

Corollary 4.2.5. *Let* $\alpha \in (0,1)$. *If a function* $f \in \mathcal{A}$ *satisfies the condition*

$$\left|\arg\left\{\frac{f(z)}{zf'(z)}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}\right| < \tan^{-1}\left\{\frac{\alpha\cos\frac{\pi}{2}\alpha}{(1+\alpha)^{\frac{1+\alpha}{2}}(1-\alpha)^{\frac{1-\alpha}{2}}+\alpha\sin\frac{\pi}{2}\alpha}\right\},$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha.$$

Chapter 5

Properties of Meromorphic Functions Defined by a Convolution Operator

5.1 Introduction

In the present paper, we shall derive certain interesting properties of the convolution operator D^{α} defined by (1.7.2). We note that the contents of this chapter have been published by Nonlinear Functional Analysis and Applications [2].

5.2 Main Results

To prove our results, we need the following lemmas.

Lemma 5.2.1. [28] *Let* h *be analytic and convex in* $\mathbb{U}$ *with* $h(0) = a$, $\gamma \neq 0$, $\mathrm{Re}\{\gamma\} \geq 0$. *If* $p \in \mathcal{H}[a, n]$ *and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1}$$

and q *is the best dominant.*

Lemma 5.2.2. [40] *Let* Ω *be a set in the complex plane* $\mathbb{C}$ *and let* b *be a complex number with* $\mathrm{Re}\{b\} > 0$. *Suppose that the function*

$$\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$$

satisfies the condition:

$$\psi(ix, y; z) \notin \Omega,$$

for all real $x, y \geq -|b - ix|^2/(2\mathrm{Re}\{b\})$ *and all* $z \in \mathbb{U}$. *If the function* p *is analytic in* $\mathbb{U}$ *with* $p(0) = b$ *and if*

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then

$$\operatorname{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Theorem 5.2.1. *Let* $\alpha > -1,\ 0 \leq \lambda \leq 1$ *and* $\gamma > 1$. *If* $f \in \Sigma$, *then*

$$\operatorname{Re}\left\{(1-\lambda)\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} + \lambda\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}\right\} < \gamma \quad (z \in \mathbb{U}) \tag{5.2.1}$$

implies that

$$\operatorname{Re}\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\} < \beta \quad (z \in \mathbb{U}), \tag{5.2.2}$$

where $\beta \in (1, \infty)$ *is the positive root of the equation:*

$$(2(\alpha+1)(1-\lambda) + 2\lambda(\alpha+1))x^2 + (3\lambda - 2\gamma(\alpha+2))x - \lambda = 0. \tag{5.2.3}$$

Proof. Let

$$p(z) = \frac{1}{\beta - 1}\left(\beta - \frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right) \quad (z \in \mathbb{U}). \tag{5.2.4}$$

Then p is analytic in $\mathbb{U}$ and $p(0) = 1$. Differentiating (5.2.4) and using (1.7.3), we obtain

$$\begin{aligned}
&(1-\lambda)\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} + \lambda\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)} \\
&= (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} - \left((1-\lambda)(\beta-1) + \frac{\lambda(\alpha+1)(\beta-1)}{\alpha+2}\right)p(z) \\
&\qquad - \frac{\lambda(\beta-1)zp'(z)}{(\alpha+2)(\beta-(\beta-1)p(z))} \\
&= \psi(p(z), zp'(z)),
\end{aligned}$$

where

$$\psi(r,s) = (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} - \left((1-\lambda)(\beta-1) + \frac{\lambda(\alpha+1)(\beta-1)}{\alpha+2}\right) r - \frac{\lambda(\beta-1)s}{(\alpha+2)(\beta-(\beta-1)r)} \tag{5.2.5}$$

By virtue of (5.2.1) and (5.2.5), we have

$$\{\psi(p(z), zp'(z) : z \in \mathbb{U}\} \subset \Omega = \{w \in \mathbb{C} : \text{Re}\{w\} < \gamma\}.$$

Now for all real $x, y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\text{Re}\{\psi(ix,y)\} &= (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} - \frac{\lambda(\beta-1)\beta y}{(\alpha+2)(\beta^2-(\beta-1)^2x^2)} \\ &\geq (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} + \frac{\lambda(\beta-1)\beta(1+x^2)}{2(\alpha+2)(\beta^2-(\beta-1)^2x^2)} \\ &\geq (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} + \frac{\lambda(\beta-1)}{2(\alpha+2)\beta} = \gamma,\end{aligned}$$

where β is the positive root of the equation (5.2.3). Note that, if

$$g(x) = (2(1-\lambda) + 2\lambda(\alpha+1))x^2 + (2\lambda - 2\gamma(\alpha+2))x - \lambda,$$

then $g(0) = -\lambda < 0$ and $g(1) = -2((\alpha+1)(\gamma-\lambda) + (\gamma-1)) < 0$. This shows that $\beta \in (1, \infty)$. Hence for each $z \in \mathbb{U}$, $\psi(ix, y) \notin \Omega$. Therefore, by Lemma 5.2.2, $\text{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$, which proves (5.2.2). □

Theorem 5.2.2. *Let $\lambda \geq 0$, $\gamma > 1$ and $0 \leq \delta < 1$. Suppose also that*

$$\text{Re}\left\{\frac{D^{\alpha}g(z)}{D^{\alpha+1}g(z)}\right\} > \delta \quad (g \in \Sigma; z \in \mathbb{U}). \tag{5.2.6}$$

If $f \in \Sigma$ satisfies

$$\text{Re}\left\{(1-\lambda)\frac{D^{\alpha}f(z)}{D^{\alpha}g(z)} + \lambda\frac{D^{\alpha+1}f(z)}{D^{\alpha+1}g(z)}\right\} < \gamma \quad (z \in \mathbb{U}), \tag{5.2.7}$$

then

$$\text{Re}\left\{\frac{D^{\alpha}f(z)}{D^{\alpha}g(z)}\right\} < \frac{2\gamma(\alpha+1)+\lambda\delta}{2(\alpha+1)+\lambda\delta} \quad (z \in \mathbb{U}), \tag{5.2.8}$$

Proof. Let

$$\beta = \frac{2\gamma(\alpha+1)+\lambda\delta}{2(\alpha+1)+\lambda\delta} \quad (\beta > 1)$$

and

$$p(z) = \frac{1}{\beta-1}\left(\beta - \frac{D^{\alpha+1}f(z)}{D^{\alpha}g(z)}\right) \quad (z \in \mathbb{U}). \tag{5.2.9}$$

Then the function p is analytic in $\mathbb{U}$ and $p(0) = 1$. Setting

$$B(z) = \frac{D^{\alpha}g(z)}{D^{\alpha+1}g(z)} \quad (g \in \Sigma; z \in \mathbb{U}),$$

by assumption, we have

$$\text{Re}\{B(z)\} > \delta \quad (z \in \mathbb{U}).$$

Differentiating (5.2.9) and using (1.7.3), we have

$$(1-\lambda)\frac{D^{\alpha}f(z)}{D^{\alpha}g(z)}+\lambda\frac{D^{\alpha+1}f(z)}{D^{\alpha+1}g(z)}$$
$$=(1+\lambda\alpha)\beta-(\beta-1)p(z)-\frac{\lambda(\beta-1)B(z)zp'(z)}{\alpha+1}.$$

Letting

$$\psi(r,s)=(1+\lambda\alpha)\beta-(\beta-1)r-\frac{\lambda(\beta-1)sB(z)}{\alpha+1}\quad(z\in\mathbb{U}),$$

we deduce from (5.2.7) that

$$\{\psi(p(z),zp'(z);z\in\mathbb{U}\}\subset\Omega=\{w\in\mathbb{C}:\operatorname{Re}\{w\}<\gamma\}.$$

Now for all real $x, y\leq-(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\{\psi(ix,y)\}&=\beta-\frac{\lambda(\beta-1)y}{\alpha+1}\operatorname{Re}\{B(z)\}\\&\geq\beta+\frac{\lambda(\beta-1)\delta}{2(\alpha+1)}(1+x^2)\\&\geq\beta+\frac{\lambda(\beta-1)\delta}{2(\alpha+1)}=\gamma,\end{aligned}$$

Hence for each $z\in\mathbb{U}$, $\psi(ix,y)\notin\Omega$. Thus by Lemma 5.2.2, $\operatorname{Re}\{p(z)\}>0$ for $z\in\mathbb{U}$. Therefore we complete the proof of Theorem 5.2.2. □

Theorem 5.2.3. *Let $\alpha>-1$, $\beta\geq1$ and $\gamma>0$. If $f\in\Sigma$, then*

$$\operatorname{Re}\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\}<\frac{\alpha+1+\gamma}{\alpha+1}\quad(z\in\mathbb{U})\tag{5.2.10}$$

implies that

$$\mathrm{Re}\left\{(zD^{\alpha}f(z))^{-1/2\beta\gamma}\right\} > 2^{-1/\beta} \quad (z \in \mathbb{U}). \tag{5.2.11}$$

The bound $2^{-1/\beta}$ *is the best possible.*

Proof. From (1.7.3) and (5.2.10), we have

$$\mathrm{Re}\left\{\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right\} < -1+\gamma \quad (z \in \mathbb{U}).$$

That is,

$$\frac{1}{2\gamma}\left(\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}+1\right) \prec \frac{z}{1+z} \quad (z \in \mathbb{U}). \tag{5.2.12}$$

Let

$$p(z) = (zD^{\alpha}f(z))^{-1/2\gamma} \quad (z \in \mathbb{U}).$$

Then (5.2.12) may be written as

$$z\,(\log p(z))' \prec z\left(\log\frac{1}{1+z}\right)' \quad (z \in \mathbb{U}). \tag{5.2.13}$$

By using the well-known result [67] to (5.2.13), we obtain

$$p(z) \prec \frac{1}{1+z} \quad (z \in \mathbb{U}),$$

that is, that

$$(zD^{\alpha}f(z))^{-1/2\gamma\beta} = \left(\frac{1}{1+w(z)}\right)^{1/\beta} \quad (z \in \mathbb{U}), \tag{5.2.14}$$

where w is analytic function in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathbb{U}$. According to $\mathrm{Re}\{t^{1/\beta}\} \geq (\mathrm{Re}\{t\})^{1/\beta}$ for $\mathrm{Re}\{t\} > 0$ and $\beta \geq 1$, (5.2.14) yields

$$\begin{aligned}\mathrm{Re}\left\{(zD^{\alpha}\{f(z))^{-1/2\gamma\beta}\right\} &\geq \left(\mathrm{Re}\left\{\frac{1}{1+w(z)}\right\}\right)^{1/\beta} \\ &> 2^{-1/\beta} \quad (z \in \mathbb{U}).\end{aligned}$$

To see that the bound $2^{-1/\beta}$ cannot ne increased, we consider the function $g \in \Sigma$ such that

$$zD^{\alpha}g(z) = (1+z)^{2\gamma} \quad (z \in \mathbb{U}).$$

It is not so difficult to show that g satisfies (5.2.10) and

$$\mathrm{Re}\left\{(zD^{\alpha}\{g(z))^{-1/2\gamma\beta}\right\} \longrightarrow 2^{-1/\beta}$$

as $z = \mathrm{Re}\{z\} \to 1^{-}$. Therefore the proof of Theorem 5.2.3 is complete.

□

Theorem 5.2.4. *Let $\alpha > -1$, $\lambda \geq 0$ and $0 < \delta_1, \delta_2 \leq 1$. If $f \in \Sigma$ satisfies*

$$-\frac{\pi}{2}\delta_1 < \arg\{(1-\lambda)zD^{\alpha}f(z) + \lambda zD^{\alpha+1}f(z)\} < \frac{\pi}{2}\delta_2, \tag{5.2.15}$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\{zD^{\alpha}f(z)\} < \frac{\pi}{2}\eta_2, \tag{5.2.16}$$

where η_1 and η_2 are the solutions of the equations:

$$\delta_1 = \eta_1 + \frac{2}{\pi} \arctan \left\{ \frac{\lambda(\eta_1 + \eta_2)}{2(\alpha + 1)} \left(\frac{1 - |a|}{1 + |a|} \right) \right\} \tag{5.2.17}$$

and

$$\delta_2 = \eta_2 + \frac{2}{\pi} \arctan \left\{ \frac{\lambda(\eta_1 + \eta_2)}{2(\alpha + 1)} \left(\frac{1 - |a|}{1 + |a|} \right) \right\}, \tag{5.2.18}$$

when

$$a = i \tan \left\{ \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \right\}.$$

Proof. Let

$$p(z) = zD^{\alpha} f(z) \quad (z \in \mathbb{U}).$$

Then by using (1.7.3), we have

$$(1 - \lambda)zD^{\alpha} f(z) + \lambda zD^{\alpha+1} f(z) = p(z) + \frac{\lambda}{\alpha + 1} zp'(z). \tag{5.2.19}$$

Let h be the function which maps $\mathbb{U}$ onto the angular domain $\{w \in \mathbb{C} : -\pi\delta_1/2 < \arg\{w\} < \pi\delta_2/2\}$ with $h(0) = 1$. Then from (5.2.15) and (5.2.19), we get

$$p(z) + \frac{\lambda}{\alpha + 1} zp'(z) \prec h(z).$$

Therefore an application of Lemma 5.2.1 yields $\text{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$ and hence $p(z) \neq 0$ for $z \in \mathbb{U}$.

Suppose that there exists two points $z_1, z_2 \in \mathbb{U}$ such that the condition (1.6.1) is satisfied. Then by Lemma 1.6.1, we obtain (1.6.2) under the restriction (1.6.3). Therefore we have

$$\begin{aligned}\arg\left\{p(z_1)+\frac{\lambda}{\alpha+1}z_1p'(z_1)\right\} &= \arg\left\{p(z_1)\right\}+\arg\left\{\alpha+1+\lambda\frac{z_1p'(z_1)}{p(z_1)}\right\}\\ &= -\frac{\pi}{2}\eta_1+\arg\left\{\alpha+1-i\frac{\lambda(\eta_1+\eta_2)}{2}m\right\}\\ &\leq -\frac{\pi}{2}\eta_1-\arctan\left\{\frac{\lambda(\eta_1+\eta_2)}{2(\alpha+1)}\left(\frac{1-|a|}{1+|a|}\right)\right\}\\ &= -\frac{\pi}{2}\delta_1\end{aligned}$$

and

$$\begin{aligned}\arg\left\{p(z_2)+\frac{\lambda}{\alpha+1}z_2p'(z_2)\right\} &\geq \frac{\pi}{2}\eta_1+\arctan\left\{\frac{\lambda(\eta_1+\eta_2)}{2(\alpha+1)}\left(\frac{1-|a|}{1+|a|}\right)\right\}\\ &= \frac{\pi}{2}\delta_2,\end{aligned}$$

which contradict the assumption (5.2.15). Therefore we have the assertion (5.2.16). □

For $\delta_1=\delta_2=\delta$ in Theorem 5.2.4, we have the following result.

Corollary 5.2.1. *Let $\alpha>-1$, $\lambda\geq 0$ and $0<\delta\leq 1$. If $f\in\Sigma$ satisfies*

$$|\arg\{(1-\lambda)zD^{\alpha}f(z)+\lambda zD^{\alpha+1}f(z)\}|<\frac{\pi}{2}\delta,$$

then

$$|\arg\{zD^{\alpha}f(z)\}|<\frac{\pi}{2}\eta,$$

where η is the solutions of the equation:

$$\delta = \eta + \frac{2}{\pi}\arctan\left\{\frac{\lambda}{\alpha+1}\right\}.$$

Now we consider the following integral operator F_c (see [3,20,41,44]) defined by

$$F_c(f)(z) = \frac{c}{z^{c+1}}\int_0^z f(t)t^c dt \quad (\mathrm{Re}\{c\} \geq 0). \tag{5.2.20}$$

Theorem 5.2.5. *Let $\alpha > -1$, $c \geq 0$ and $0 < \delta_1, \delta_2 \leq 1$. If $f \in \Sigma$ satisfies*

$$-\frac{\pi}{2}\delta_1 < \arg\{zD^\alpha f(z)\} < \frac{\pi}{2}\delta_2,$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\{zD^\alpha F_c(z)\} < \frac{\pi}{2}\eta_2,$$

where F_c is the integral operator defined by (5.2.20), *and η_1 and η_2 are the solutions of the equations* (5.2.17) *and* (5.2.18) *with $\alpha = c-1$ and $\lambda = 1$.*

Proof. Let

$$p(z) = zD^\alpha F_c(z) \quad (z \in \mathbb{U}).$$

From the definition of F_c, it can be verified that

$$z(D^\alpha F_c(z))' = cD^\alpha f(z) - (c+1)D^\alpha F_c(z). \tag{5.2.21}$$

Therefore, using (5.2.21) and (1.7.3) for F_c, we have

$$zD^{\alpha}f(z) = p(z) + \frac{1}{c}zp'(z).$$

The remaining part of the proof is similar to that of Theorem5.2.4 and so we omit for details. $\square$

Chapter 6

Argument estimates of multiplier transformations defined by a linear operator

6.1 Introduction

In the present paper, we give some argument properties of certain class of analytic functions in $\mathcal{A}_p$ involving the linear operator $\mathcal{I}_p(a, c)$ defined by (1.7.5). An application of a certain integral operator is also considered. The results obtained here besides extending the works of Cho et. al. [11] and Fukui, Kim and Srivastava [17] yields a number of new results.

6.2 Main Results

Now we derive

Theorem 6.2.1. *If $f \in \mathcal{A}_p$ satisfies*

$$-\frac{\pi}{2}\delta_1 \ < \ \arg \ \left\{(1-l)\frac{\mathcal{I}_p(a+1,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)} + l\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a,c)g(z)} - \beta\right\} \ < \ \frac{\pi}{2}\delta_2$$

$$(a > 0;\ l \geq 0;\ 0 \leq \beta < 1;\ 0 < \delta_1, \delta_2 \leq 1;\ z \in \mathbb{U})$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\frac{\mathcal{I}_p(a,c)g(z)}{\mathcal{I}_p(a+1,c)g(z)} \ \prec \ \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1;\ z \in \mathbb{U}), \tag{6.2.1}$$

then

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg \ \left\{\frac{\mathcal{I}_p(a+1,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)} - \beta\right\} \ < \ \frac{\pi}{2}\alpha_2 \quad (z \in \mathbb{U}),$$

where α_1 and $\alpha_2 (0 < \alpha_1, \alpha_2 \leq 1)$ are the solutions of the equations :

$$\delta_1 = \begin{cases} \alpha_1 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)l(1-|b|)\cos\frac{\pi}{2}t_1}{\frac{2a(1+A)(1+|b|)}{1+B} + (\alpha_1+\alpha_2)l(1-|b|)\sin\frac{\pi}{2}t_1}\right\} & \text{for } B \neq -1, \\ \alpha_1 & \text{for } B = -1, \end{cases} \tag{6.2.2}$$

and

$$\delta_2 = \begin{cases} \alpha_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)l(1-|b|)\cos\frac{\pi}{2}t_1}{\frac{2a(1+A)(1+|b|)}{1+B} + (\alpha_1+\alpha_2)l(1-|b|)\sin\frac{\pi}{2}t_1}\right\} & \text{for } B \neq -1, \\ \alpha_2 & \text{for } B = -1, \end{cases} \tag{6.2.3}$$

when b is given by (1.6.3) *and*

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB}\right). \tag{6.2.4}$$

Proof. Let

$$\frac{\mathcal{I}_p(a+1,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)} = \beta + (1-\beta)q(z). \tag{6.2.5}$$

Then q is analytic in $\mathbb{U}$ with $q(0) = 1$. On differentiating both sides of (6.2.5) and using the identity (1.7.6) in the resulting equation, we deduce that

$$(1-l)\frac{\mathcal{I}_p(a+1,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)} + l\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a,c)g(z)} - \beta = (1-\beta)\left\{q(z) + \frac{lzq'(z)}{ar(z)}\right\},$$

where

$$r(z) = \frac{\mathcal{I}_p(a,c)g(z)}{\mathcal{I}_p(a+1,c)g(z)}.$$

While, by using the result of Silverman and Silvia [62], we have

$$\left|r(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (z \in \mathbb{U}\ ;\ B \neq -1) \tag{6.2.6}$$

and

$$\operatorname{Re}\{r(z)\} > \frac{1-A}{2} \qquad (z \in \mathbb{U}\ ;\ B = -1). \tag{6.2.7}$$

Then, from (6.2.6) and (6.2.7), we obtain

$$r(z) = \rho e^{\pi\phi i/2},$$

where

$$\begin{cases} \frac{1-A}{1-B} \ < \ \rho \ < \ \frac{1+A}{1+B} \\ -t_1 \ < \ \phi \ < \ t_1 \ \text{for } B \neq -1, \end{cases}$$

when t_1 is given by (6.2.4), and

$$\begin{cases} \frac{1-A}{2} \;<\; \rho \;<\; \infty \\ -1 \;<\; \phi \;<\; 1 \text{ for } B=-1. \end{cases}$$

Let h be the function which maps onto the angular domain $\{w : -(\pi/2)\delta_1 < \arg\{w\} < (\pi/2)\delta_2\}$ with $h(0) = 1$. Applying Lemma 1.5.1 for this h with $\omega(z) = l/(ar(z))$, we see that Re $\{q(z)\} > 0$ in $\mathbb{U}$ and hence $q(z) \neq 0$ in $\mathbb{U}$.

If there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (1.6.1) is satisfied, then (by Lemma 1.6.1 we obtain (1.6.2) under the restriction (1.6.3). At first, for the case $B \neq -1$, we obtain

$$\begin{aligned}
&\arg\left\{(1-l)\frac{\mathcal{I}_p(a+1,c)f(z_1)}{\mathcal{I}_p(a+1,c)g(z_1)} + l\frac{\mathcal{I}_p(a,c)f(z_1)}{\mathcal{I}_p(a,c)g(z_1)} - \beta\right\} \\
&= \arg\left\{q(z_1) + \frac{l}{ar(z_1)} z_1 q'(z_1)\right\} \\
&= -\frac{\pi}{2}\alpha_1 + \arg\left\{1 - i\frac{\alpha_1+\alpha_2}{2}\frac{lm}{a}(\rho e^{i\frac{\pi\phi}{2}})^{-1}\right\} \\
&\leq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)lm\sin\frac{\pi}{2}(1-\phi)}{2a\rho + (\alpha_1+\alpha_2)lm\cos\frac{\pi}{2}(1-\phi)}\right\} \\
&\leq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)l(1-|b|)\cos\frac{\pi}{2}t_1}{\frac{2a(1+A)(1+|b|)}{1+B} + (\alpha_1+\alpha_2)l(1-|b|)\sin\frac{\pi}{2}t_1}\right\} \\
&= -\frac{\pi}{2}\delta_1,
\end{aligned}$$

and

$$\begin{aligned}
&\arg\left\{(1-l)\frac{\mathcal{I}_p(a+1,c)f(z_2)}{\mathcal{I}_p(a+1,c)g(z_2)} + l\frac{\mathcal{I}_p(a,c)f(z_2)}{\mathcal{I}_p(a,c)g(z_2)} - \beta\right\} \\
&\geq \frac{\pi}{2}\alpha_2 + \tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)l(1-|b|)\cos\frac{\pi}{2}t_1}{\frac{2a(1+A)(1+|b|)}{1+B} + (\alpha_1+\alpha_2)l(1-|b|)\sin\frac{\pi}{2}t_1}\right\} \\
&= \frac{\pi}{2}\delta_2,
\end{aligned}$$

where we have used the inequality (1.6.3), and $\delta_1,\ \delta_2$ and t_1 are given by (6.2.2), (6.2.3) and (6.2.4), respectively. Similarly, for the case $B=-1$, we have

$$\arg\ \left\{(1-l)\frac{\mathcal{I}_p(a+1,c)f(z_1)}{\mathcal{I}_p(a+1,c)g(z_1)}+l\frac{\mathcal{I}_p(a,c)f(z_1)}{\mathcal{I}_p(a,c)g(z_1)}-\beta\right\}\leq -\frac{\pi}{2}\alpha_1$$

and

$$\arg\ \left\{(1-l)\frac{\mathcal{I}_p(a+1,c)f(z_2)}{\mathcal{I}_p(a+1,c)g(z_2)}+l\frac{\mathcal{I}_p(a,c)f(z_2)}{\mathcal{I}_p(a,c)g(z_2)}-\beta\right\}\geq \frac{\pi}{2}\alpha_2$$

These are contradictions to the assumption of Theorem 6.2.1. Therefore we complete the proof of Theorem 6.2.1. □

If we take $a=p,\ c=1$ and $\delta_1=\delta_2$ in Theorem 6.2.1, we have

Corollary 6.2.1. *If* $f\in\mathcal{A}_p$ *satisfies*

$$\left|\arg\ \left\{(1-l)\frac{f(z)}{g(z)}+l\frac{f'(z)}{g'(z)}-\beta\right\}\right| < \frac{\pi}{2}\delta$$

$$(l\geq 0;\ 0\leq\beta<1;\ 0<\delta\leq 1;\ z\in\ U)$$

for some $g\in\mathcal{A}_p$ *satisfying the condition*

$$\frac{zg'(z)}{pg(z)}\ \prec\ \frac{1+Az}{1+Bz}\quad(-1\leq B<A\leq 1;\ z\in\mathbb{U}),$$

then

$$\left|\arg\ \left\{\frac{f(z)}{g(z)}-\beta\right\}\right|\ <\ \frac{\pi}{2}\alpha\quad(z\in\mathbb{U}),$$

where $\alpha(0<\alpha\leq 1)$ *is the solution of the equation :*

$$\delta = \begin{cases} \alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha l \cos\frac{\pi}{2}t_1}{\frac{p(1+A)}{1+B}+\alpha l \sin\frac{\pi}{2}t_1}\right\} & \textit{for } B \neq -1, \\ \alpha & \textit{for } B = -1, \end{cases}$$

when t_1 *is given by* (6.2.4)

Letting $B \to A$ $(A < 1)$ and $g(z) = z^p$ in Corollary 6.2.1, we get

Corollary 6.2.2. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\arg\left\{(1-l)\frac{f(z)}{z^p} + l\frac{f'(z)}{pz^{p-1}} - \beta\right\}\right| < \frac{\pi}{2}\delta$$
$$(l \geq 0;\ 0 \leq \beta < 1;\ 0 < \delta \leq 1;\ z \in \mathbb{U}),$$

then

$$\left|\arg\left\{\frac{f(z)}{g(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{l\alpha}{p}\right\}.$$

Remark 6.2.1. *Taking* $l = 1$ *and* $\beta = 0$ *in Corollary 6.2.2, we get the corresponding result obtained by Cho et al.* [11].

Theorem 6.2.2. *If* $f \in \mathcal{A}_p$ *satisfies*

$$-\frac{\pi}{2}\delta_1 < \arg\left\{\frac{\mathcal{I}_p(a,c)f(z)}{z^p} - \beta\right\} < \frac{\pi}{2}\delta_2$$
$$(0 \leq \beta < 1;\ 0 < \delta_1, \delta_2 \leq 1;\ z \in \mathbb{U}),$$

then

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg \left\{\frac{\mathcal{I}_p(a,c)F_\mu(f)(z)}{z^p} - \beta\right\} \ < \ \frac{\pi}{2}\alpha_2,$$

where F_μ *is the integral operator defined by*

$$F_\mu(f) := F_\mu(f)(z) = \frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1}f(t)dt \quad (\mu \geq -p; z \in \mathbb{U}) \tag{6.2.8}$$

and, α_1 *and* α_2 $(0 < \alpha_1, a_2 \leq 1)$ *are the solutions of the equations :*

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|b|)}{2(1+|b|)(\mu+p)} \ \ and \ \ \delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\frac{(\alpha_1+\alpha_2)(1-|b|)}{2(1+|b|)(\mu+p)}$$

when b *is given by* (1.6.3)

Proof. Consider the function q defined in $\mathbb{U}$ by

$$\frac{\mathcal{I}_p(a,c)F_\mu(f)(z)}{z^p} = \beta + (1-\beta)q(z). \tag{6.2.9}$$

Then q is analytic in $\mathbb{U}$ with $q(0) = 1$. Differentiating both sides of (6.2.9) and simplifying, we get

$$\frac{\mathcal{I}_p(a,c)f(z)}{z^p} - \beta = (1-\beta)\left\{q(z) + \frac{zq'(z)}{\mu+p}\right\}.$$

Now, by using Lemma 1.5.1 and a similar method as in the proof of Theorem 6.2.1, we get Theorem 6.2.2. □

Taking $a = p, c = 1, \beta = \rho/p$ and $\delta_1 = \delta_2 = 1$ in Theorem 6.2.2, we have

Corollary 6.2.3. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\mathrm{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \rho \ \ (0 \leq \rho < p; \ z \in \mathbb{U})$$

then

$$\left|\arg\left\{\frac{(F_\mu(f)(z))'}{z^{p-1}} - \rho\right\}\right| < \frac{\pi}{2}\alpha,$$

where F_μ *is given by* (6.2.8) *and* α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha}{\mu + p}\right\} = 1.$$

By using the same mathod as in the proof of Theorem 6.2.2, we have

Theorem 6.2.3. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\arg\left\{\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)f(z)} - \beta\right\}\right| < \frac{\pi}{2}\delta$$

$$(a > 0; \ 0 \leq \beta < 1; \ 0 < \delta \leq 1; \ z \in \mathbb{U})$$

then

$$\left|\arg\left\{\frac{\mathcal{I}_p(a+1,c)f(z)}{z^p}\right\}\right| < \frac{\pi}{2}\alpha \ \ (z \in \mathbb{U}),$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\delta = \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha}{a(1-\beta)}\right\}.$$

Letting $a = p, c = 1, \beta = \rho/p$ and $\delta = 1$ in Theorem 6.2.3, we have

Corollary 6.2.4. *If $f \in \mathcal{A}_p$ satisfies*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \ (0 \le \rho < p;\ z \in \mathbb{U})$$

then

$$\left|\arg\left\{\frac{f(z)}{z^p}\right\}\right| < \frac{\pi}{2}\alpha \ (z \in \mathbb{U}),$$

where α $(0 < \alpha \le 1)$ is the solution of the equation :

$$\frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha}{p-\rho}\right\} = 1.$$

Theorem 6.2.4. *Let $f \in \mathcal{A}_p$ and suppose that*

$$A \le B + \frac{p(1-B)}{a} \ (a > 0;\ -1 \le B < A \le 1).$$

If

$$-\frac{\pi}{2}\delta_1 < \arg\left\{(1-l)\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)} + l\frac{(\mathcal{I}_p(a,c)f(z))'}{(\mathcal{I}_p(a+1,c)g(z))'} - \beta\right\} < \frac{\pi}{2}\delta_2$$
$$(l \ge 0;\ 0 \le \beta < 1;\ 0 < \delta_1, \delta_2 \le 1;\ z \in \mathbb{U}),$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\frac{\mathcal{I}_p(a,c)g(z)}{\mathcal{I}_p(a+1,c)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

then

$$-\frac{\pi}{2}\alpha_1 \;<\; \arg\left\{\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)}-\beta\right\} \;<\; \frac{\pi}{2}\alpha_2 \quad (z\in\mathbb{U}),$$

where α_1 *and* α_2 $(0<\alpha_1, a_2\leq 1)$ *is the solutions of the equations :*

$$\delta_1=\begin{cases}\alpha_1+\frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)l(1-|b|)\cos\frac{\pi}{2}t_2}{2\frac{p(1+B)+a(A-B)}{1+B}(1+|b|)+(\alpha_1+\alpha_2)l(1-|b|)\sin\frac{\pi}{2}t_2}\right\} & \textit{for } B\neq -1,\\ \alpha_1 & \textit{for } B=-1,\end{cases}$$

and

$$\delta_2=\begin{cases}\alpha_2+\frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)l(1-|b|)\cos\frac{\pi}{2}t_2}{2\frac{p(1+B)+a(A-B)}{1+B}(1+|b|)+(\alpha_1+\alpha_2)l(1-|b|)\sin\frac{\pi}{2}t_2}\right\} & \textit{for } B\neq -1,\\ \alpha_2 & \textit{for } B=-1,\end{cases}$$

when b *is given by* (1.6.3) *and*

$$t_2=\frac{2}{\pi}\sin^{-1}\left(\frac{a(A-B)}{(p-a)(1-B^2)-a(1-AB)}\right).$$

Proof. Letting

$$\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)}=\beta+(1-\beta)q(z) \text{ and } r(z)=\frac{\mathcal{I}_p(a,c)g(z)}{\mathcal{I}_p(a+1,c)g(z)},$$

we have

$$(1-l)\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)}+l\frac{(\mathcal{I}_p(a,c)f(z))'}{(\mathcal{I}_p(a+1,c)g(z))'}-\beta=(1-\beta)\left\{q(z)+\frac{lzq'(z)}{ar(z)+p-a}\right\}.$$

The remaining part of the proof of Theorem 6.2.4 is similar to that of Theorem 6.2.1. So we omit the details. □

Putting $a = p,\ c = 1,\ l = 1,\ A = \eta/p,\ B = 0$ and $\delta_1 = \delta_2$ in Theorem 6.2.4, we have

Corollary 6.2.5. *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\arg\left\{\frac{(zf'(z))'}{g'(z)} - \beta\right\}\right| < \frac{\pi}{2}\delta \ \ (0 \leq \beta < p;\ 0 < \delta \leq 1;\ z \in \mathbb{U})$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\left|\frac{zg'(z)}{g(z)} - p\right| < \eta \ \ (0 < \eta \leq p;\ z \in \mathbb{U}),$$

then

$$\left|\arg\left\{\frac{zf'(z)}{g(z)} - \beta\right\}\right| < \frac{\pi}{2}\alpha \ \ (z \in \mathbb{U}),$$

where $\alpha (0 < \alpha \leq 1)$ is the solution of the equation :

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha \sin\left(\frac{\pi}{2} - \sin^{-1}\eta/p\right)}{p + \eta + \alpha\cos\left(\frac{\pi}{2} - \sin^{-1}\eta/p\right)}\right\}.$$

Lemma 6.2.1. *Let*

$$\eta = \xi + \frac{\xi}{\mu + p + a\xi} \ \ (0 \leq (a-1)/a < \xi < \eta < 1). \tag{6.2.10}$$

If $g \in \mathcal{A}_p$ satisfies

$$\left|\frac{\mathcal{I}_p(a, c)g(z)}{\mathcal{I}_p(a+1, c)g(z)} - 1\right| < \eta \ \ (z \in \mathbb{U}), \tag{6.2.11}$$

then

$$\left|\frac{\mathcal{I}_p(a,c)F_\mu(g)(z)}{\mathcal{I}_p(a+1,c)F_\mu(g)(z)}-1\right| \;<\; \xi \;\; (z\in\mathbb{U}).$$

where $F_\mu(g)$ *be defined by* (6.2.8) *for* $\mu > (a\xi^2+(p+1-a)\xi-p)/(1-\xi)$.

Proof. Defining the function w by

$$\frac{\mathcal{I}_p(a,c)F_\mu(g)(z)}{\mathcal{I}_p(a+1,c)F_\mu(g)(z)} = 1+\xi w(z), \tag{6.2.12}$$

we see that w is analytic in $\mathbb{U}$ with $w(0)=0$. Now, using the identities

$$z(\mathcal{I}_p(a+1,c)F_\mu(g))'(z) = a\mathcal{I}_p(a,c)F_\mu(g)(z)-(a-p)\mathcal{I}_p(a+1,c)F_\mu(g)(z) \tag{6.2.13}$$

and

$$z(\mathcal{I}_p(a+1,c)F_\mu(g))'(z) = (\mu+p)\mathcal{I}_p(a+1,c)g(z)-\mu\mathcal{I}_p(a+1,c)F_\mu(g)(z) \tag{6.2.14}$$

in (6.2.12), we get

$$\frac{\mathcal{I}_p(a+1,c)F_\mu(g)(z)}{\mathcal{I}_p(a+1,c)g(z)} = \frac{\mu+p}{\mu+p+a\xi w(z)}. \tag{6.2.15}$$

Making use of the logarithmic differentiation of both sides of (6.2.15) and using the identity (6.2.13) for both g and f in the resulting equation, we deduce that

$$\left|\frac{\mathcal{I}_p(a,c)g(z)}{\mathcal{I}_p(a+1,c)g(z)}-1\right| \;=\; \xi\left|w(z)+\frac{zw'(z)}{\mu+p+a\xi w(z)}\right|.$$

Let us assume that there exists a point $z_0\in\mathbb{U}$ such that $\max_{|z|<|z_0|}|w(z)| = |w(z_0)| = 1$. Then by Jack's Lemma [30], we have $z_0w'(z_0) = kw(z_0)(k\geq 1)$. Letting $w(z_0)=e^{i\theta}$, and applying this result to $w(z)$ at $z_0\in\mathbb{U}$, we get

$$\left|\frac{\mathcal{I}_p(a,c)g(z_0)}{\mathcal{I}_p(a+1,c)g(z_0)}-1\right| = \xi\left|1+\frac{k}{\mu+p+a\xi e^{i\theta}}\right|$$

$$= \xi\left[\frac{(\mu+p+k)^2+2a\xi(\mu+p+k)\cos\theta+(a\xi)^2}{(\mu+p)^2+2a\xi(\mu+p)\cos\theta+(a\xi)^2}\right]^{\frac{1}{2}}. \tag{6.2.16}$$

Since the right side of (6.2.16) is decreasing for $0 \le \theta < 2\pi$ and $\mu > \{a\xi^2+(p+1-a)\xi-p\}/(1-\xi)$, we obtain

$$\left|\frac{\mathcal{I}_p(a,c)g(z_0)}{\mathcal{I}_p(a+1,c)g(z_0)}-1\right| \ge \frac{\xi(\mu+p+1+a\xi)}{\mu+p+a\xi},$$

which contradicts our hypothesis and hence we get

$$|w(z)| = \frac{1}{\xi}\left|\frac{\mathcal{I}_p(a,c)F_\mu(g)(z)}{\mathcal{I}_p(a+1,c)F_\mu(g)(z)}-1\right| < 1 \ \ (z \in \mathbb{U}).$$

This completes the proof of Lemma 6.2.1. □

Remark 6.2.2. *We note that for $a=c=p=1$, Lemma 6.2.1 yields the corresponding result obtained by Fukui, Kim and Srivastava* [17].

Theorem 6.2.5. *Let η be as given in* (6.2.10) *and* $\mu^* > \max\left\{\frac{a\xi^2+(p+1-a)\xi-p}{1-\xi},\ a\xi-p\right\}$. *If $f \in \mathcal{A}_p$ satisfies*

$$-\frac{\pi}{2}\delta_1 < \arg\left\{\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)}-\beta\right\} < \frac{\pi}{2}\delta_2$$

$$(0 \le \beta < 1;\ 0 < \delta_1, \delta_2 \le 1;\ z \in \mathbb{U}),$$

for some $f \in \mathcal{A}_p$ satisfying the condition (6.2.11)*, then*

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg \left\{\frac{\mathcal{I}_p(a,c)F_{\mu^*}(f)(z)}{\mathcal{I}_p(a+1,c)F_{\mu^*}(g)(z)} - \beta\right\} \ < \ \frac{\pi}{2}\alpha_2 \ \ (z \in \mathbb{U}),$$

where the operator F_{μ^*} is defined by (6.2.8) *for μ^*, and α_1 and α_2* $(0 < \alpha_1, a_2 \leq 1)$ *are the solutions of the equations* :

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\frac{\pi}{2}t_3}{2(\mu^*+p+a\xi)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin\frac{\pi}{2}t_3}\right\}$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left\{\frac{(\alpha_1+\alpha_2)(1-|b|)\cos\frac{\pi}{2}t_3}{2(\mu^*+p+a\xi)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin\frac{\pi}{2}t_3}\right\}$$

when b is given by (1.6.3) *and*

$$t_3 = \frac{2}{\pi}\sin^{-1}\left(\frac{a\xi}{\mu^*+p}\right).$$

Proof. Consider the function q defined in $\mathbb{U}$ by

$$\frac{\mathcal{I}_p(a,c)F_{\mu^*}(f)(z)}{\mathcal{I}_p(a+1,c)F_{\mu^*}(g)(z)} = \beta + (1-\beta)q(z). \tag{6.2.17}$$

Then q is analytic in $\mathbb{U}$ with $q(0) = 1$. Taking logarithmic differentiation on both sides of (6.2.17) and using the identity (6.2.13) in the resulting equation, we get

$$\frac{z(\mathcal{I}_p(a,c)F_{\mu^*}(f))'(z)}{\mathcal{I}_p(a,c)F_{\mu^*}(f)(z)} = p - a + a\frac{\mathcal{I}_p(a,c)F_{\mu^*}(g)(z)}{\mathcal{I}_p(a+1,c)F_{\mu^*}(g)(z)} + (1-\beta)\frac{zq'(z)}{\beta+(1-\beta)q(z)} \tag{6.2.18}$$

From the definition of $F_{\mu^*}(f)$, we have

$$(\mu^* + p)\mathcal{I}_p(a,c)f(z) = z(\mathcal{I}_p(a,c)F_{\mu^*}(f)(z))' + \mu^*\mathcal{I}_p(a,c)F_{\mu^*}(f)(z). \qquad (6.2.19)$$

Again, from (6.2.13) and (6.2.14), it follows that

$$(\mu^* + p)\mathcal{I}_p(a+1,c)g(z) = a\mathcal{I}_p(a,c)F_{\mu^*}(g)(z) + (p + \mu^* - a)\mathcal{I}_p(a+1,c)F_{\mu^*}(g)(z). \qquad (6.2.20)$$

Thus, by using (6.2.19) and (6.2.20) followed by (6.2.18), we obtain

$$\frac{\mathcal{I}_p(a,c)f(z)}{\mathcal{I}_p(a+1,c)g(z)} - \beta = (1-\beta)\left\{q(z) + \frac{zq'(z)}{ar(z) + \mu^* + p - a}\right\},$$

where

$$r(z) = \frac{\mathcal{I}_p(a,c)F_{\mu^*}(g)(z)}{\mathcal{I}_p(a+1,c)F_{\mu^*}(g)(z)}.$$

By using Lemma 6.2.1, we have

$$r(z) \prec 1 + \xi z \ \ (z \in \mathbb{U}),$$

where ξ is given by (6.2.10). Letting

$$ar(z) + \mu^* + p - a = \rho e^{i\pi\theta/2}$$

and using the techniques of Theorem 6.2.1, the remaining part of the proof of Theorem 6.2.5 follows. □

Remark 6.2.3. *We easily find the following* :

$$\mu^* \ > \begin{cases} a\xi - p, & \text{if } \frac{a-1}{a} < \xi < \frac{2a-1}{2a}, \\ \frac{2(a-p)-1}{2}, & \text{if } \xi = \frac{2a-1}{2a}, \\ \frac{a\xi^2+(p+1-a)\xi-p}{1-\xi}, & \text{if } \frac{2a-1}{2a} < \xi < 1. \end{cases}$$

Taking $a = p,\ c = 1$ and $\delta_1 = \delta_2$ in Theorem 6.2.5, we get

Corollary 6.2.6. *Let*

$$\eta = \xi + \frac{\xi}{\mu^* + p(1+\xi)} \quad ((p-1)/p < \xi < \eta < 1),$$

where $\mu^* > \max\{(p\xi^2 + \xi - p)/(1-\xi), p(\xi - 1)\}$. *If* $f \in \mathbb{A}_p$ *satisfies*

$$\left|\arg\ \left\{\frac{zf'(z)}{g(z)} - \beta\right\}\right| < \frac{\pi}{2}\delta\ \ (0 \le \beta < p;\ 0 < \delta \le 1;\ z \in \mathbb{U})$$

for some $g \in \mathbb{A}_p$ *satisfying the condition*

$$\left|\frac{zg'(z)}{g(z)} - p\right| \ < \ p\eta\ \ (z \in \mathbb{U}),$$

then

$$\left|\arg\ \left\{\frac{z(F_{\mu^*}(f))'(z)}{F_{\mu^*}(g)(z)} - \beta\right\}\right| \ < \ \frac{\pi}{2}\alpha\ \ (z \in \mathbb{U}),$$

where $\alpha (0 < \alpha \le 1)$ *is the solution of the equation* :

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left\{\frac{\alpha \sin\left(\frac{\pi}{2} - \sin^{-1}\frac{p\xi}{\mu^*+p}\right)}{\mu^* + p(1+\xi) + \alpha\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{p\xi}{\mu^*+p}\right)}\right\}.$$

Chapter 7

Properties of hypergeometric functions related to uniformly convex and uniformly starlike functions

7.1 Introduction

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^t(A,B)$ if

$$\left|\frac{f'(z)-1}{t(A-B)-B(f'(z)-1)}\right| < 1 \quad (-1 \leq B < A \leq 1; t \in \mathbb{C}\backslash\{0\}; z \in \mathbb{U}), \quad (7.1.1)$$

The class $\mathcal{R}^t(A,B)$ was introduced by Dixit and Pal [15]. By giving specific values to t, A and B in (7.1.1), we obtain the following subclasses studied by various researchers in earlier works [7, 13, 27, 53, 56].

A functions f of the form (1.2.4) is said to be in $\mathcal{UCT}(\alpha)$ if it satisfies the condition

$$\text{Re}\Big\{1+\frac{zf''(z)}{f'(z)}\Big\} \geq \alpha\Big|\frac{zf''(z)}{f'(z)}\Big| \ (\alpha \geq 0;\ z \in U).$$

A functions f of the form (1.2.4) is said to be in $\mathcal{USTN}(\alpha)$ if it satisfies the condition

$$\text{Re}\Big\{\frac{f(z)-f(\xi)}{(z-\xi)f'(z)}\Big\} \geq \alpha \ (0 \leq \alpha \leq 1;\ z, \xi \in \mathbb{U}).$$

The classes $\mathcal{UCT}(\alpha)$ and $\mathcal{USTN}(\alpha)$ were introduced in [66]. We note that $\mathcal{UCT}(1)$ (resp., $\mathcal{USTN}(0)$ is the subclasses of uniformly convex (resp., uniformly starlike) functions in $\mathbb{U}$ defined by Goodman [23, 24]. On later, the classes of uniformly convex and uniformly starlike functions have been extensively studied by Ma and Minda [36] and Rϕnning [57].

The object of the present paper is to give some applications of hypergeometric functions related to the classes $\mathcal{UCT}(\alpha)$ and $\mathcal{USTN}(\alpha)$. We also investigate a distortion theorem for the operator $I_{a,b;c}(f)$ when a function f belongs to the class $\mathcal{UCT}(\alpha)$. Furthermore, we consider the relationships among the classes $\mathcal{UCT}(\alpha)$, $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$.

Now we introduce several lemmas which are needed for the proof of our main results.

Lemma 7.1.1. [15] *Let a function f of the form* (1.1.1) *be in $\mathcal{R}^t(A,B)$. Then*

$$|a_n| \leq \frac{(A-B)|t|}{n}.$$

The result is sharp for the function

$$f(z) = \int_0^z \left(1 + \frac{(A-B)tz^{n-1}}{1+Bz^{n-1}}\right) dz \quad (n \geq 2;\ z \in \mathbb{U}).$$

Lemma 7.1.2. [56]

(i) For $a, b \in \mathbb{C}\backslash\{0,1\}$ and $c \in \mathbb{C}\backslash\{1\}$ with $c > \max\{0, a+b-1\}$

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} = \frac{1}{(a-1)(b-1)} \left(\frac{\Gamma(c+1-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - (c-1)\right).$$

(ii) For $a, b \in \mathbb{C}\backslash\{0\}$ with $a > 0$ and $b > 0$ and $c > a+b+1$,

$$\sum_{n=0}^{\infty} \frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n} = \left(\frac{ab}{c-a-b-1} + 1\right) \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}.$$

Lemma 7.1.3. [61] *A function f of the form (1.2.4) is in $\mathcal{T}^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha)a_n \leq 1-\alpha \quad (0 \leq \alpha < 1)$$

and is in $\mathcal{C}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha \quad (0 \leq \alpha < 1).$$

Lemma 7.1.4. [66] *A function f of the form (1.2.4) is in $\mathcal{UCT}(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)a_n \leq 1 \quad (\alpha \geq 1)$$

and in $\mathcal{USTN}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} ((3-\alpha)n-2)a_n \leq 1-\alpha \quad (0 \leq \alpha \leq 1).$$

7.2 Main Results

Theorem 7.2.1. *Let* $a, b > 0$*,* $c > a + b + 1$ *and let function* f *of the form* (1.2.4) *be in* $\mathcal{R}^t(A, B)$*. If*

$$(A - B)|t| \left[\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left(\frac{(\alpha+1)|ab|}{c-a-b-1} + 1 \right) - 1 \right] \leq 1 \quad (\alpha \geq 0), \quad (7.2.1)$$

then $I_{a,b;c} f \in \mathcal{UCT}(\alpha)$*.*

Proof. We note that

$$I_{a,b;c}(f) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n$$

in $\mathcal{A}$. Then by Lemma 7.1.4, we need only to show that

$$S_1 =; \sum_{n=2}^{\infty} n(n(\alpha+1) - \alpha) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n \leq 1. \quad (7.2.2)$$

Since $f \in \mathcal{R}^t(A, B)$, from Lemma 7.1.1, we have

$$a_n \leq \frac{(A - B)|t|}{n}.$$

By using the formula (1.7.9), (7.2.1) and (i) of Lemma 7.1.2, we have

$$
\begin{aligned}
S_1 &\leq (A-B)|t|\sum_{n=2}^{\infty}(n(\alpha+1)-\alpha)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= (A-B)|t|\Big[(\alpha+1)\sum_{n=1}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n} - \alpha\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\Big] \\
&= (A-B)|t|\Big[(\alpha+1)\Big(\sum_{n=0}^{\infty}\frac{(n+1)(a)_n(b)_n}{(c)_n(1)_n} - 1\Big) - \alpha\Big(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\Big)\Big] \\
&= (A-B)|t|\ \Big[(\alpha+1)\Big(\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\Big(\frac{ab}{c-a-b-1}+1\Big)-1\Big) \\
&\quad -\alpha\Big(\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}-1\Big)\Big] \\
&= (A-B)|t|\Big[\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\Big(\frac{(\alpha+1)ab}{c-a-b-1}+1\Big)-1\Big] \\
&\leq 1.
\end{aligned}
$$

Therefore (7.2.2) holds. Therefore we conclude that the function $I_{a,b;c}f \in \mathcal{UCT}(\alpha)$.

$\square$

Theorem 7.2.2. *Let* $a, b > 0$, $c > \max\{0, a+b-1\}$ *with* $a \neq 1$, $b \neq 1$ *and* $c \neq 1$ *and a function* f *of the form* (1.2.4) *be in* $\mathcal{R}^t(A,B)$. *If*

$$
(A-B)|t|\Big[\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\Big(3-\alpha-\frac{2(c-a-b)}{(a-1)(b-1)}\Big)+\frac{2(c-1)}{(a-1)(b-1)}-1+\alpha\Big]
$$
$$
\leq 1-\alpha \quad (0 \leq \alpha \leq 1), \tag{7.2.3}
$$

then $I_{a,b;c}f \in \mathcal{USTN}(\alpha)$.

Proof. By Lemma 7.1.4, it is sufficient to show that

$$
S_2 =; \sum_{n=2}^{\infty}((3-\alpha)n-2)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n \leq 1-\alpha. \tag{7.2.4}
$$

By using Lemma 7.1.1, (1.7.9), (7.2.3) and (ii) of Lemma 7.1.2, we have

$$
\begin{aligned}
S_2 &\leq (A-B)|t|\Big[(3-\alpha)\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} - 2\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n}}\Big] \\
&= (A-B)|t|\Big[(3-\alpha)\Big(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\Big) - 2\Big(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} - 1\Big)\Big] \\
&= (A-B)|t|\ \Big[(3-\alpha)\Big(\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - 1\Big) \\
&\quad -\frac{2}{(a-1)(b-1)}\Big(\frac{\Gamma(c-a-b+1)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - (c-1) - 1\Big)\Big] \\
&= (A-B)|t|\Big[\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\Big(3-\alpha-\frac{2(c-a-b)}{(a-1)(b-1)}\Big) + \frac{2(c-1)}{(a-1)(b-1)} + \alpha - 1\Big] \\
&\leq 1-\alpha.
\end{aligned}
$$

Therefore (7.2.4) holds. This completes the proof of Theorem 7.2.2. □

Next, we prove the following properties for the operator $I_{a,b;c}f$, when a function f belongs to the class $\mathcal{UCT}(\alpha)$.

Theorem 7.2.3. *Let* $a, b > 0$, $c \geq \max\{0, a+b-1, (1/2)(ab+a+b-1)\}$ *and let a function* f *of the form* (1.2.4) *be in* $\mathcal{UCT}(\alpha)$. *Then*

$$|z| - \frac{ab}{2c(\alpha+2)}|z|^2 \leq |I_{a,b;c}f(z)| \leq |z| + \frac{ab}{2c(\alpha+2)}|z|^2 \tag{7.2.5}$$

and

$$1 - \frac{ab}{c(\alpha+2)}|z| \leq |(I_{a,b;c}f(z))'| \leq 1 + \frac{ab}{c(\alpha+2)}|z|. \tag{7.2.6}$$

The results are sharp.

Proof. We note that

$$I_{a,b;c}f(z) = \Big(zF(a,b;c;z) * f\Big)(z) = z - \sum_{n=2}^{\infty}\Phi(n)a_n z^n,$$

where

$$\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0;\ n \geq 2)$$

and $0 < \Phi(n+1) \leq \Phi(n)$ $(n \geq 2)$ under the assumption for c. Since $f \in \mathcal{UCT}(\alpha)$, by Lemma 7.1.4, we have

$$2(\alpha+2)\sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)a_n \leq 1. \tag{7.2.7}$$

Therefore, by using (7.2.7), we obtain

$$\begin{aligned} |I_{a,b;c}(f)| &\leq |z| + \sum_{n=2}^{\infty} \Phi(n) a_n |z|^n \\ &\leq |z| + \Phi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\leq |z| + \frac{ab}{2c(\alpha+2)}|z|^2 \end{aligned}$$

and

$$\begin{aligned} |I_{a,b;c}(f)| &\geq |z| - \sum_{n=2}^{\infty} \Phi(n) a_n |z|^n \\ &\geq |z| - \Phi(2)|z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{ab}{2c(\alpha+2)}|z|^2. \end{aligned}$$

From (7.2.7), we note that

$$\sum_{n=2}^{\infty} n a_n \leq \frac{1}{\alpha+2}. \tag{7.2.8}$$

By using (7.2.8), we obtain (7.2.6). The results are sharp for the function

$$f(z) = z - \frac{1}{2(\alpha+2)} z^2.$$

$\square$

Finally, Now we find the order $\beta\ (0 \leq \beta < 1)$ for which the operator $I_{a,b;c}f$ belongs to the classes $\mathcal{T}^*(\beta)$ and $\mathcal{C}(\beta)$ when a function f belongs to the class $\mathcal{UCT}(\alpha)$.

Theorem 7.2.4. *Let* $a, b > 0,\ \max\{a+b-1, (1/2)(ab+a+b-1)\} \leq c \leq ab$ *and let a function* f *of the form* (1.2.4) *be in* $\mathcal{UCT}(\alpha)$. *Then* $I_{a,b;c}f \in \mathcal{C}(\beta)$, *where*

$$\beta = \frac{c(\alpha+2) - 2ab}{c(\alpha+2) - ab} \quad (\alpha \geq 0) \tag{7.2.9}$$

Proof. Let $\mathcal{UCT}(\alpha)$. Consider the operator

$$I_{a,b;c}f(z) = z + \sum_{n=2}^{\infty} \Phi(n) a_n z^n,$$

where

$$\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0;\ n \geq 2).$$

Since $\Phi(n)$ is decreasing function for n, by Lemma 7.1.3, we need to find $\beta\ (0 \leq \beta < 1)$ that

$$\Phi(2) \sum_{n=2}^{\infty} \frac{n(n-\beta)}{1-\beta} a_n \leq 1.$$

Since $f \in \mathcal{UCT}(\alpha)$, by Lemma 7.1.4, we have

$$\sum_{n=2}^{\infty} n(n(\alpha+1)-\alpha)a_n \leq 1.$$

To complete the proof, it is sufficient to find β such that

$$\frac{n-\beta}{1-\beta}\Phi(2) \leq n(\alpha+1)-\alpha. \tag{7.2.10}$$

From (7.2.10), we obtain

$$\beta \leq g(n),$$

where

$$g(n) = \frac{n(\alpha+1-\Phi(2))-\alpha}{n(\alpha+1)-(\alpha+\Phi(2))} \tag{7.2.11}$$

By the assumption of the theorem, it is easy to see that $g(n)$ is an increasing function for n $(n \geq 2)$. Setting $n = 2$ in (7.2.11), we have (7.2.9). Therefore we complete the proof of Theorem 7.2.4. □

Taking $a = b = c = 1$ and using Lemma 7.1.1 and Lemma 7.1.3 in Theorem 7.2.4, we have the following result [66]

Corollary 7.2.1.

$$\mathcal{UCT}(\alpha) \subset \mathcal{C}(\alpha/(\alpha+1)) \ (\alpha \geq 0).$$

Theorem 7.2.5. *Let* $a, b > 0$, $c \geq \max\{ab/2, a+b-1, (1/2)(ab+a+b-1)\}$ *and let a function* f *of the form* (1.2.4) *be in* $\mathcal{UCT}(\alpha)$. *Then* $I_{a,b;c}(f) \in \mathcal{T}^*(\beta)$, *where*

$$\beta = \frac{2c(\alpha+2) - 2ab}{2c(\alpha+2) - ab} \quad (\alpha \geq 0) \tag{7.2.12}$$

Proof. Let $f \in \mathcal{UCT}(\alpha)$ and let

$$\Phi(n) = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \quad (a, b > 0;\ n \geq 2).$$

Then by Lemma 7.1.4, we want to find β $(0 \leq \beta < 1)$

$$\Phi(2) \sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} a_n \leq 1,$$

because $\Phi(n)$ is decreasing function for n $(n \geq 2)$. Since $f \in \mathcal{UCT}(\alpha)$, by Lemma 7.1.4, we have

$$\sum_{n=2}^{\infty} n(n(\alpha+1) - \alpha) a_n \leq 1.$$

Therefore it suffices to find β such that

$$\frac{n-\beta}{1-\beta} \Phi(2) \leq n(n(\alpha+1) - \alpha). \tag{7.2.13}$$

Solving (7.2.13), we have

$$\beta \leq h(n),$$

where

$$h(n) = \frac{n^2(\alpha+1) - n(\alpha + \Phi(2))}{n^2(\alpha+1) - \alpha n - \Phi(2)}. \tag{7.2.14}$$

By a simple calculation, we can see that $h(n)$ is an increasing function for n $(n \geq 2)$. Setting $n = 2$ in (7.2.14), we have (7.2.12) and hence the result follows. $\square$

Taking $a = b = c = 1$ in Theorem 7.2.5, we have the following result.

Corollary 7.2.2.

$$\mathcal{UCT}(\alpha) \subset \mathcal{T}^*((2\alpha+2)/(2\alpha+3)) \ (\alpha \geq 0)$$

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