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Surjective theory for retarded semilinear control systems

by

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Surjective theory for retarded semilinear control systems
(지연 준선형 제어계에 대한 전사 이론)

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by

Tae Wan Lee

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지연 준선형 제어계에 대한 전사 이론

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요 약

이 논문에서는 지연 준선형 방정식에 대한 제어 가능 문제를 전사이론에 기초하여 연구하였다. 이를 연구함의 목적은 자연과학, 경제학, 공학 등 많은 분야에서 현재의 상태 뿐만 아니라 과거의 상태에도 의존하는 수학적 모델로 나타나는 체계의 제어 가능성을 다루는 것으로, 초기치의 상태로부터 임의의 상태에서 제어 가능한가의 문제를 다루는 것이다.

Fredholm altnernative 이론과 유사한 전사 이론을 적용하기 위해 몇 가지 제한된 가정하에서 다루었다. 작용소 T는 odd (K, L, a)-homeomorphism 이고, 작용소 F는 odd strong continuous 하며, b-quais-homogeneous 일 때, $\lambda T(x) - F(x) = y$는 전사함수임을 알게 된다. 이를 전사 이론에 근거하여 제어 가능성의 결과를 얻을 수 있었다.

2장에서는 결과를 이끌어내는 Fredholm alternative 이론과 유사한 전사이론을 소개한다. 작용소 T의 (K, L, a)-homeomorphism 을 정의하고, 작용소 F의 b-quasi-homogeneous 를 정의한다.

3장에서는 몇 가지 예비 이론과 비선형항이 없는 방정식의 해의 초기치에 대한 정칙성과 연속성을 소개한다.

4장에서는 비선형항을 포함한 방정식의 정칙성을 다루고, 초기치와 컨트롤 항으로부터 방정식의 해로의 사상이 연속임을 알아내었다.

5장에서는 컨트롤 항의 충분한 조건 아래 앞의 이론을 바탕으로 제어 가능성을 증명하고, 예를 소개하며 마친다.

이 논문에서 다룬 방정식 외 다른 미분방정식의 제어 가능 문제도 생각해보았고, 이 부분에서의 더 많은 연구의 필요성을 느꼈다.

1 Introduction

In this paper, we deal with the approximate controllability for the following retarded functional differential control equation with the nonlinear integro-diffrential type in the form

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + f(t,x) + (Bu)(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0. \end{cases} \tag{1.1}$$

where

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds$$

for a k belonging to $L^2(0,T)$. Let V and H be complex Hilbert spaces forming a Gelfand triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

by identifying the antidual of H with H. Here, the principal operator A is associated with a sesquilinear form defined on $V \times V$ and satisfying Gårding's inequality, and B is a bounded linear operator from another Hilbert space $L^2(0,T;U)(T > 0)$ to $L^2(0,T;U)$. k belongs to $L^2(0,T)$ and g is a nonlinear mapping as detailed in Section 4.

The our purpose is to deal with the approximate controllability of semilinear retarded system (1.1) which appears in a great many practical mathematical models in the natural sciences, economics and engineering depends not only on the present but on the past state. The controllability problem is a question which is possible to steer a dynamic system from an initial state to an arbitrary final state using the set of admissible controls. Approximate controllability for the systems in infinite dimensional spaces is well developed, we refer to [8, 26] and references therein for the case of abstract linear control equations. In order to obtain similar results on semilinear systems dominated by linear parts (in case $f \equiv 0$), many researchers have used the following sufficient conditions as

(1) the corresponding linear system (1.1) when $f \equiv 0$ is approximately controllable [22, 33, 34],

(2) $S(t)$ is a compact operator or the embedding $D(A) \subset V$ is compact [6, 14, 13, 24, 32],

(3) use a fixed point theorem combined with technique of operator transformations by configuring the resolvent as seen in [4].

Recently, the approximate controllability of fractional order semilinear delay systems has been studied by authors [20, 27] as a continuous study. Similar considerations of semilinear stochastic systems have been dealt with in many references [2, 10, 18, 21, 19, 29].

In particular, with conditions the range condition of controller with a compact semigroup, [32] established the approximate controllability for the equation (1.1) assuming

$$\limsup_{||u||_{L^2(0,T;H)} \to \infty} \frac{||f(\cdot, u)||_{L^2(0,T;H)}}{||u||_{L^2(0,T;H)}} := \gamma \tag{1.2}$$

is sufficiently small. Sukavanam and Tomar [28] studied the approximate controllability for the general retarded initial value problem by assuming that the Lipschitz constant of the nonlinear term is less than 1. Moreover, [15] dealt with the approximate controllability for the system (1.1) even if $\gamma \neq 1$ of (1.2) by using so called Fredholm theory: $(\lambda I - F)(u) = f$ is solvable in $L^2(0, T; H)$.

In this paper, authors want to look at approaches from other perspectives. Our used tool is the surjective theorems similar to the Fredholm alternative for nonlinear operators under restrictive assumption, which is on the solution of nonlinear operator equations $\lambda T(x) - F(x) = y$ in dependence on the real number λ, where T be odd (K, L, a)- homeomorphism, that is, there exist real numbers $K > 0$, $L > 0$, and $\alpha > 0$ such that

$$L||x||_X^\alpha \leq ||T(x)||_Y \leq K||x||_X^\alpha,$$

and $F : X \to Y$ an odd strong continuous and b-quasi-homogeneous operator. Here, F is said to be b-quasi-homogeneous if there exist nonlinear operators R and F_0 defined on X such that F_0 is b-homogeneous, strong continuous satisfying $F(u) = R(u)F_0(u)$ for every $u \in H$ satisfying

$$\lim_{||u||_X \to \infty} ||Ru||_Y = 1.$$

Based on this surjective results, we can obtain the approximate controllability for (1.1) without restrictions such as (1)-(3) mentioned above.

The structure of this paper is as follows. In Section 2, we deal with some surjectivity results similar to Fredholm alternative, which can be used to prove the main results. Section 3 is devoted to constructing principal operators $A_i (i = 0, 1, 2)$ associated with a sesquilinear form defined on $V \times V$ and satisfying Gårding's inequality, and gives the basic properties of the analytic semigroup generated by the principal operator A_0. Section 4 gives a variation of constant formula of L^2-regularity and properties of the strict solutions of (1.1)(see [7] in the linear case). The assumption required here is that the embedding $D(A) \subset V$ is completely continuous. Then by virtue of [1], we can show that the solution mapping of a control space to the terminal state space is completely continuous by means of regularities results, so is the nonlinear operator. Moreover, the sufficient conditions on the controller and nonlinear terms for approximate controllability for (1.1) can be obtained. Finally, we give an example of a partial functional differential control equation as an application of the preceding theory.

2 Surjectivity results

Let us introduce the theory of the degree for completely continuous perturbations of the identity operator, which is the infinite dimensional version of Borsuk's theorem. Let $0 \in D$ be a bounded open set in a Banach space X, $\overline{D}$ its closure and ∂D its boundary. The number $d[I - T; D, 0]$ is the degree of the mapping $I - T$ with respect to the set D and the point 0 (see [11] or [17]).

Theorem 2.1. *(Borsuk's theorem) Let D be a bounded open symmetric set in a Banach space X, $0 \in D$. Suppose that $T : \overline{D} \to X$ be odd completely continuous operator satisfying $T(x) \neq x$ for $x \in \partial D$. Then $d[I - T; D, 0]$ is odd integer. That is, there exists at least one point $x_0 \in D$ such that $(I - T)(x_0) = 0$.*

Definition 2.1. *Let T be a mapping defined by on a Banach space X with value in a real Banach space Y. The mapping T is said to be a (K, L, α)-homeomorphism of X onto Y if*

(i) T is a homeomorphism of X onto Y;

(ii) *there exist real numbers $K > 0$, $L > 0$, and $\alpha > 0$ such that*

$$L||x||_X^\alpha \leq ||T(x)||_Y \leq K||x||_X^\alpha, \quad \forall x \in X.$$

Lemma 2.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ a continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$, then

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty.$$

Proof. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\} \subset X$ such that

$$||\lambda T(x_n) - F(x_n)||_Y \leq M$$

as $x_n \to \infty$. From this it follows that

$$\frac{\lambda T(x_n)}{||x_n||_X^\alpha} - \frac{F(x_n)}{||x_n||_X^\alpha} \to 0.$$

Hence, we have

$$\limsup_{n \to \infty} \frac{|\lambda| ||T(x_n)||_Y}{||x_n||_X^\alpha} = N.$$

Since T is an odd (K, L, α)-homeomorphism of X onto Y, $|\lambda|K \geq N \geq |\lambda|L$. It is a contradiction with $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}]$. □

Proposition 2.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator. Suppose that for $\lambda \neq 0$,*

$$\lim_{||x||_X \to \infty} ||\lambda T(x) - F(x)||_Y = \infty. \tag{2.1}$$

Then $\lambda T - F$ maps X onto Y.

Proof. We follow the proof Theorem 1.1 in Chapter II of [11]. Suppose that there exists $y \in Y$ such that $\lambda T(x) = y$. Then from (2.1) it follows that $FT^{-1} : Y \to Y$ is an odd completely continuous operator and

$$\lim_{||y||_Y \to \infty} ||y - FT^{-1}(\frac{y}{\lambda})||_Y = \infty.$$

Let $y_0 \in Y$. There exists $r > 0$ such that

$$||y - FT^{-1}(\frac{y}{\lambda})||_Y > ||y_0||_Y \geq 0$$

for each $y \in Y$ satisfying $||y||_Y = r$. Let $Y_r = \{y \in Y : ||y||_Y < r\}$ be a open ball. Then by view of Theorem 2.1, we have $d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0]$ is an odd number. For each $y \in Y$ satisfying $||y||_Y = r$ and $t \in [0,1]$, there is

$$||y - FT^{-1}(\frac{y}{\lambda}) - ty_0||_Y \geq ||y - FT^{-1}(\frac{y}{\lambda})||_Y - ||y_0||_Y > 0$$

and hence, by the homotopic property of degree, we have

$$d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, y_0] = d[y - FT^{-1}(\frac{y}{\lambda}); Y_r, 0] \neq 0.$$

Hence, by the existence theory of the Leray-Schauder degree, there exists a $y_1 \in Y_r$ such that

$$y_1 - FT^{-1}(\frac{y_1}{\lambda}) = y_0.$$

We can choose $x_0 \in X$ satisfying $\lambda T(x_0) = y_1$, and so, $\lambda T(x_0) - F(x_0) = y_0$. Thus, it implies that $\lambda T - F$ is a mapping of X onto Y. $\square$

Combining Lemma 2.1. and Proposition 2.1, we have the following results.

Corollary 2.1. *Let T be an odd (K, L, α)-homeomorphism of X onto Y and $F : X \to Y$ an odd completely continuous operator satisfying*

$$\limsup_{||x||_X \to \infty} \frac{||F(x)||_Y}{||x||_X^\alpha} = N \in \mathbb{R}^+.$$

Then if $|\lambda| \notin [\frac{N}{K}, \frac{N}{L}] \cup \{0\}$ then $\lambda T - F$ maps X onto Y. Therefore, if $N = 0$, then for all $\lambda \neq 0$ the operator $\lambda T - F$ maps X onto Y.

Let X be a Banach space with the norm $\| \cdot \|_X$. Denote by X^* the adjoint space of all bounded linear functionals on X. The pairing between $x^* \in X^*$ and $x \in X$ is denoted by (x^*, x). Unless otherwise stated, we use symbols " $\to$ " and " $\rightharpoonup$ " to denote the strong and weak convergence, respectively, i.e., the sequence $\{x_n\}$, $x_n \in X$ converges strongly (weakly) to the point $x_0 \in X$, denote by $x_n \to x_0$ ($x_n \rightharpoonup x_0$), if

$$\lim_{n\to\infty} \|x_n - x_0\|_X = 0 \ (\lim_{n\to\infty} (x^*, x_n) = (x^*, x_0) \quad \text{for each} \quad x* \in X^*).$$

Let F be mapping (nonlinear, in general) with the domain $M \subset X$ and the range in the Banach space Y. F is said to strongly (weakly) continuous on M if $x_n \to x_0$ ($x_n \rightharpoonup x_0$) in X implies $F(x_n) \to F(x_0)$ in Y for $x_n,\ x_0 \in M$, and F is said to be completely continuous on M if F is continuous on M and for each bounded subset $D \subset M$, $F(D)$ is compact subset in Y.

Definition 2.2. *Let F be a mapping defined by on a Banach space X with value in a real Banach space Y and $b > 0$ a real number.*

(a) *F is said to be b-homogeneous if*

$$t^b F(u) = F(tu)$$

holds for each $t \geq 0$ and all $u \in X$.

(b) F is said to be b-quasi-homogeneous if there exist nonlinear operators R and F_0 defined on X with value in Y such that F_0 is b-homogeneous and $F(u) = R(u)F_0(u)$ for every $u \in X$ satisfying

$$\lim_{||u||_X \to \infty} ||Ru||_Y \in \mathbb{R}^+.$$

Example 2.1. *Set $X = Y = \mathbb{R}$ and*

$$F(u) = \frac{|u|}{1+|u|}u^3.$$

Then F is said to be 3-quasi-homogeneous considering as $R(u) = \frac{|u|}{1+|u|}$.

Remark 2.1. *In [11], the relationship between F and F_0 is defined in other words as F is said to be b-strongly quasi-homogeneous with respect to F_0, if*

$$t_n > 0 \to 0, u_n \rightharpoonup u_0 \Rightarrow t_n^b F(u_n/t_n) \to F_0(u_0) \in Y.$$

If F is the strong continuous and b-quasi-homogeneous, then F is a b-strongly quasi-homogeneous with respect to F_0. So our basic results follow theorems of [11].

Theorem 2.2. *Let X be a reflexive space, and let T be odd (K, L, α)- homeomorphism of X onto Y, $F : X \to Y$ an odd strong continuous and b-quasi-homogeneous operator. If $\alpha > b$, then $\lambda T - F$ maps X onto Y for any $\lambda \neq 0$.*

Proof. Since X is a reflexive space, we know that every strong continuous operator $F : X \to Y$ is also completely continuous. Hence according to Corollary 2.1 it is sufficient to prove that

$$\lim_{x\to\infty} \frac{\|F(x)\|_Y}{\|x\|_X^\alpha} = 0.$$

Since F is b-quasi-homogeneous, there exist R and F_0 be a mappings defined by on a Banach space X with value in Y and a real Banach space Y, respectively, such that $F = RF_0$ satisfying

$$\lim_{\|u\|_X \to \infty} \|R(u)\|_Y = c_0$$

for some a constant $c_0 > 0$ holds and F_0 is b-homogeneous. Suppose that there exist $\epsilon > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ such that

$$\frac{x_n}{\|x_n\|_X} = v_n \rightharpoonup v_0$$

and

$$\frac{\|F(x_n)\|_Y}{\|x\|_X^{\alpha}} \geq \epsilon$$

for any positive integer n. Then

$$\frac{F(x_n)}{\|x_n\|_X^b} = \frac{F(\|x_n\|_X v_n)}{\|x_n\|_X^b} = R(\|x_n\|_X v_n)F_0(v_n) \to c_0 F_0(v_0).$$

Since $a > b$,

$$\frac{\|x_n\|_X^b}{\|x_n\|_X^{\alpha}} \to 0.$$

Thus

$$0 < \epsilon \leq \frac{\|F(x_n)\|_Y}{\|x_n\|^{\alpha}} = \frac{\|x_n\|^b}{\|x_n\|^{\alpha}} \cdot \frac{\|F(x_n)\|_Y}{\|x_n\|^b} \to 0,$$

which is a contradiction. $\square$

Theorem 2.3. *Let X be a Hilbert space, and let T be odd (K, L, α)- homeomorphism of X onto Y, $F : X \to Y$ an odd strongly continuous and b-quasi-homogeneous operator. If $F_0(v) = 0$ imply $v = 0$, and $\alpha < b$, then $\lambda T - F$ maps X onto Y for any $\lambda \neq 0$.*

Proof. According to Proposition 2.1, we shall prove

$$\lim_{x\to\infty} \|\lambda T(x) - F(x)\|_Y = \infty.$$

Since F is b-quasi-homogeneous, there exist R and F_0 be mappings defined by on a Banach space X with value in Y and a real Banach space Y, respectively, such that $F = RF_0$ satisfying

$$\lim_{\|u\|_X\to\infty} \|R(u)\|_Y = c_0$$

for some a constant $c_0 > 0$ holds and F_0 is b-homogeneous. Suppose that there exist a constant $M > 0$ and a sequence $\{x_n\}$, $x_n \in X$, $\|x_n\|_X \to \infty$ such that

$$\frac{x_n}{\|x_n\|_X} = v_n \rightharpoonup v_0$$

and

$$\|\lambda T(x_n) - F(x_n)\|_Y \le M$$

for any positive integer n. Here, we note that $v_n \to v_0$ since X is a Hilbert space and $\|v_0\|_X = 1$. Then

$$\frac{\lambda T(\|x_n\|_X\, v_n)}{\|x_n\|^b} - \frac{F(\|x_n\|_X\, v_n)}{\|x_n\|^b} \to 0,$$

and so

$$\frac{\lambda T(\|x_n\|_X\, v_n)}{\|x_n\|^b} \to c_0 F_0(v_0).$$

But since T is (K, L, α)- homeomorphism, we have

$$K|\lambda|\frac{\|x_n\|^\alpha}{\|x_n\|^b} \ge \frac{\|\lambda T(x_n)\|_Y}{\|x_n\|^b} \ge L|\lambda|\frac{\|x_n\|^\alpha}{\|x_n\|^b}.$$

Thus, noting that $\alpha < b$, it holds

$$\frac{\|\lambda T(x_n)\|_Y}{\|x_n\|^b} \to 0,$$

and $F_0(v_0) = 0$. From our assumption $v_0 = 0$ and this is a contradiction with $\|v_0\|_X = 1$. $\square$

3 Preliminaries

The notations $|\cdot|$, $||\cdot||$ and $||\cdot||_*$ denote the norms of $H,\ V$ and V^*, respectively as usual. Therefore, for the brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V.$$

Let $a(u, v)$ be a bounded sesquilinear form defined in $V \times V$ satisfying Gårding's inequality

$$\mathrm{Re}\ \ a(u, u) \geq c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \geq 0.$$

Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u,\ v \in V.$$

Then A is a bounded linear operator from V to V^*. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. Moreover, for each $T > 0$, by using interpolation theory, we have

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

From the following inequalities

$$c_0||u||^2 \leq \mathrm{Re}\, a(u, u) + c_1|u|^2 \leq |Au|\, |u| + c_1|u|^2 \leq (|Au| + c_1|u|)|u|,$$

it follows that there exists a constant $C_0 > 0$ such that

$$||u|| \le C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}.$$

Therefore, in terms of the intermediate theory, we can see that

$$(D(A), H)_{1/2,2} = V, \ \text{ and } \ (V, V^*)_{1/2,2} = H,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [3], [31]). For the sake of simplicity, we assume that $c_1 = 0$ and hence the closed half plane $\{\lambda : \operatorname{Re}\lambda \ge 0\}$ is contained in the resolvent set of A. It is known that A generates an analytic semigroup $S(t)$ in both H and V^*. As seen in Lemma 3.6.2 of [30], there exists a constant $M > 0$ such that

$$|S(t)x| \le M|x| \ \text{ and } \ ||S(t)x||_* \le M||x||_*, \tag{3.1}$$

moreover, for all $t > 0$ and every $x \in H$ or V^*:

$$|S(t)x| \le Mt^{-1/2}||x||_*, ||S(t)x|| \le Mt^{-1/2}|x|.$$

We consider the following initial value problem

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + h(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0. \end{cases} \tag{3.2}$$

The operators A_1 and A_2 are bounded linear operators from V to V^* such that their restrictions to $D(A_0)$ are bounded linear operators from $D(A_0)$ equipped with the graph norm of A_0 to H. The function $a(\cdot)$ is assumed to be real valued and Hölder continuous in the interval $[-h, 0]$.

By virtue of Theorem 3.3 of [7](or Theorem 3.1 of [14]), we have the following results on the corresponding linear equation (3.2).

Proposition 3.1. *Suppose that the assumptions for the principal operator A_0 stated above are satisfied. Then the following properties hold:*

1) Let $V = (D(A_0), H)_{1/2,2}$. For $(\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A_0))$ and $h \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (3.2) belonging to

$$L^2(0, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$

and satisfying

$$||x||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \leq C_1(||\phi^0|| + ||\phi^1||_{L^2(-h,0;D(A_0))} + ||h||_{L^2(0,T;H)}), \quad (3.3)$$

where C_1 is a constant depending on T.

2) For $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $h \in L^2(0, T; V^*)$, $T > 0$, there exists a unique solution x of (3.2) belonging to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \leq C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||h||_{L^2(0,T;V^*)}), \quad (3.4)$$

where C_1 is a constant depending on T.

Lemma 3.1. *Suppose that $k \in L^2(0, T; H)$ and $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \leq t \leq T$. Then there exists a constant C_2 such that*

$$||x||_{L^2(0,T;H)} \leq C_2 T ||k||_{L^2(0,T;H)}, \quad (3.5)$$

and

$$||x||_{L^2(0,T;V)} \leq C_2 \sqrt{T} ||k||_{L^2(0,T;H)}. \quad (3.6)$$

Proof. By a consequence of (3.3), it is immediate that

$$||x||_{L^2(0,T;D(A_0))} \leq C_1||k||_{L^2(0,T;H)}. \tag{3.7}$$

Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2dt \leq M\int_0^T(\int_0^t |k(s)|ds)^2dt \\ &\leq M\int_0^T t\int_0^t |k(s)|^2dsdt \leq M\frac{T^2}{2}\int_0^T |k(s)|^2ds, \end{aligned}$$

where M is the constant of (3.1), it follows that

$$||x||_{L^2(0,T;H)} \leq T\sqrt{M/2}||k||_{L^2(0,T;H)}. \tag{3.8}$$

From (3.7), and (3.8) it holds that

$$||x||_{L^2(0,T;V)} \leq C_0\sqrt{C_1T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M/2}, C_0\sqrt{C_1}(M/2)^{1/4}\}.$$

Thus (3.5) and (3.6) are satisfied. □

4 Semilinear equation systems

We consider the following retarded semilinear equation systems

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^0 a(s)A_2x(t+s)ds + f(t,x) + (Bu)(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \leq s < 0. \end{cases} \tag{4.1}$$

Let U be a Hilbert space and the controller operator B be a bounded linear operator from $L^2(0,T;U)$ to $L^2(0,T;H)$. Let $g : \mathbb{R}^+ \times V \to H$ be a nonlinear mapping satisfying the following:

Assumption (F)

(i) For any $x \in V$, the mapping $g(\cdot, x)$ is strongly measurable;

(ii) There exist positive constants L_0, L_1 such that

(a) $x \mapsto g(t,x)$ is odd mapping $(g(\cdot,-x) = -g(\cdot,x))$;

(b) for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$,

$$|g(t,x) - g(t,\hat{x})| \le L_1 ||x - \hat{x}||,$$
$$|g(t,0)| \le L_0.$$

For $x \in L^2(0,T;V)$, we set

$$f(t,x) = \int_0^t k(t-s) g(s, x(s)) ds$$

where k belongs to $L^2(0,T)$.

Lemma 4.1. *Let Assumption (F) be satisfied. Assume that $x \in L^2(0,T;V)$ for any $T > 0$. Then $f(\cdot, x) \in L^2(0,T;H)$ and*

$$||f(\cdot,x)||_{L^2(0,T;H)} \le L_0 ||k||_{L^2(0,T)} \sqrt{T} + L_1 ||k||_{L^2(0,T)} \sqrt{T} ||x||_{L^2(0,T;V)}. \qquad (4.2)$$

Moreover if $x,\ \hat{x} \in L^2(0,T;V)$, then

$$||f(\cdot,x) - f(\cdot,\hat{x})||_{L^2(0,T;H)} \le L_1 ||k||_{L^2(0,T)} \sqrt{T} ||x - \hat{x}||_{L^2(0,T;V)}. \qquad (4.3)$$

Proof. From Assumption (F) and using the Hölder inequality, it is easily seen that

$$
\begin{aligned}
&||f(\cdot,x)||_{L^2(0,T;H)} \le ||f(\cdot,0)||_{L^2(0,T;H)} + ||f(\cdot,x)-f(\cdot,0)||_{L^2(0,T;H)}\\
&\le \left(\int_0^T |\int_0^t k(t-s)g(s,0)ds|^2dt\right)^{1/2}\\
&\quad + \left(\int_0^T |\int_0^t k(t-s)\{g(s,x(s))-g(s,0)\}ds|^2dt\right)^{1/2}\\
&\le L_0\left(\int_0^T 1^2ds\right)^{1/2}\left(\int_0^T (k(t-s))^2ds\right)^{1/2}\\
&\quad + L_1||x||_{L^2(0,T;V)}\left(\int_0^T 1^2ds\right)^{1/2}\left(\int_0^T (k(t-s))^2ds\right)^{1/2}\\
&\le L_0||k||_{L^2(0,T)}\sqrt{T} + ||k||_{L^2(0,T)}\left(\int_0^T |L_1||x(s)-0)||\ ds|^2dt\right)^{1/2}\\
&\le L_0||k||_{L^2(0,T)}\sqrt{T} + L_1||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)}.
\end{aligned}
$$

The proof of (4.3) is similar. □

By virtue of Theorem 2.1 of [12], we have the following result on (4.1).

Proposition 4.1. *Suppose that the Assumption (F) is satisfied.*

1) For any $(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$ and $u\in L^2(0,T;U)$, $T>0$, the solution x of (4.1) exists and is unique in $L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$, and there exists a constant C_3 depending on T such that

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3(1+|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+||u||_{L^2(0,T;U)}). \quad (4.4)$$

2) For any $(\phi^0,\phi^1)\in V\times L^2(-h,0;D(A_0))$ and $u\in L^2(0,T;U)$, $T>0$, the solution x of (4.1) is unique in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$.

3) The mapping $V \times L^2(-h,0;D(A_0)) \times L^2(0,T;U) \ni (\phi^0,\phi^1,u) \quad \mapsto \quad x \in L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)$ is continuous.

Corollary 4.1. *Assume that the embedding $D(A_0) \subset V$ is completely continuous. Let Assumption (F) be satisfied and x_u be the solution of equation (4.1) associated with $u \in L^2(0,T;U)$. Then the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$.*

Proof. If $u \in L^2(0,T;U)$, then in view of (4.4) in Proposition 4.1

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_3(1 + ||B||\,||u||_{L^2(0,T;U)}). \tag{4.5}$$

Since $x_u \in L^2(0,T;V)$, from Lemma 4.1, we have $f(\cdot,x_u) \in L^2(0,T;H)$. Consequently

$$x_u \in L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H).$$

Hence, with aid of (3.3) of Proposition 3.1, (4.2) and (4.5),

$$\begin{aligned}
&||x_u||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \le C_1(||f(\cdot,x_u) + Bu||_{L^2(0,T;H)}) \\
&\le C_1(L_0||k||_{L^2(0,T)}\sqrt{T} + ||k||_{L^2(0,T)}\sqrt{T}L_1||x||_{L^2(0,T;V)} + ||Bu||_{L^2(0,T;H)}) \\
&\le C_1(L_0||k||_{L^2(0,T)}\sqrt{T} + ||k||_{L^2(0,T)}\sqrt{T}L_1C_3(1 + ||B||\,||u||_{L^2(0,T;U)}) \\
&\quad + ||Bu||_{L^2(0,T;H)}).
\end{aligned}$$

Thus, if u is bounded in $L^2(0,T;U)$, then so is x_u in $L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)$. Since $D(A_0)$ is compactly embedded in V by assumption, the embedding

$$L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H) \subset L^2(0,T;V)$$

is completely continuous in view of Theorem 2 of [1]. Therefore the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. □

5 Approximate controllability

Let $x(T; f, u)$ be a state value of the system (4.1) at time T corresponding to the nonlinear term f and the control u. We define the reachable sets for the system (2.12) as follows:

$$R_T(f) = \{x(T; f, u) : u \in L^2(0, T; U)\},$$
$$R_T(0) = \{x(T; 0, u) : u \in L^2(0, T; U)\}.$$

Definition 5.1. *The system (4.1) is said to be approximately controllable in the time interval $[0, T]$ if for every desired final state $x_1 \in H$ and $\epsilon > 0$ there exists a control function $u \in L^2(0, T; U)$ such that the solution $x(T; f, u)$ of (4.1) satisfies $|x(T; f, u) - x_1| < \epsilon$, that is, if $\overline{R_T(f)} = H$ where $\overline{R_T(f)}$ is the closure of $R_T(f)$ in H, then the system (4.1) is called approximately controllable at time T.*

Now, we consider the approximate controllability for the following semilinear control system with initial data $(\phi^0, \phi^1) = (0, 0)$:

$$\begin{cases} \frac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) + \int_{-h}^{0} a(s)A_2x(t+s)ds + f(t, x) + (Bu)(t), \\ x(0) = 0, \quad x(s) = 0 \quad -h \le s < 0. \end{cases} \tag{5.1}$$

Let U be a Hilbert space and the controller operator B be a bounded linear operator from $L^2(0, T; U)$ to $L^2(0, T; H)$. Let $W(\cdot)$ be the fundamental solution of the linear equation associated with (5.1) which is the operator-valued function satisfying

$$\begin{cases} W(t) = S(t) + \int_0^t S(t-s)\{A_1 W(s-h) + \int_{-h}^0 a(\tau) A_2 W(s+\tau) d\tau\} ds, \quad t > 0 \\ W(0) = I, \qquad W(t) = 0 \qquad -h \le t < 0, \end{cases}$$

where $S(\cdot)$ is the semigroup generated by A_0. Then

$$x(t; f, u) = \int_0^t W(t-s)\{f(s, x(\cdot; g, u)) + Bu(s)\} ds,$$

and in view of Proposition 4.1

$$||x(\cdot; f, u)||_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} \le C_3(1 + ||B|| \, ||u||_{L^2(0,T;U)}).$$

In order to obtain approximate controllability for the system (5.1), we need to impose the following assumptions :

Assumption (A) The embedding $D(A_0) \subset V$ is completely continuous.

By using the Krasnosel'skii theorem(see [2]), we can define an operator $F : L^2(0, T; U) \to L^2(0, T; H)$ as

$$F(v) = -f(\cdot, x_v). \tag{5.2}$$

Assumption (F1) F is b-quasi-homogeneous.

Theorem 5.1. *Under Assumptions (A), (F) and (F1), if $1 > b$, then we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Therefore, if the linear system (4.1) with $f \equiv 0$ is approximately controllable, then so is the nonlinear system (4.1).

Proof. Thanks to Corollary 4.1, F defined by (5.2) is a completely continuous mapping from $L^2(0,T;U)$ to $L^2(0,T;H)$. We shall show that F is strongly continuous. Given a sequence $\{u_n\}$, $u_n \in L^2(0,T;H)$, $u_n \rightharpoonup u$, we claim that $F(u_n) \to F(u)$. By (4.3) and (4.4), we have

$$||F(u_n) - F(u)||_{L^2(0,T;H)} \le L_1 ||k||_{L^2(0,T)} \sqrt{T} ||u_n - u||_{L^2(0,T;V)}.$$

and by Corollary 4.1, the mapping $u \mapsto x_u$ is completely continuous from $L^2(0,T;U)$ to $L^2(0,T;V)$. Thus, $F(u_n) \rightharpoonup F(u)$. By virtue of the the compactness of F, $\{F(u_n)\}$ is sequencially compact, and so we can choose a subsequence of $\{F(u_n)\}$, denoted again by $\{F(u_n)\}$, such that $F(u_n) \to w \in Y$. Since every subsequence of $\{F(u_n)\}$ has the same limit point, we have $F(u_n) \to F(u)$. Since $1 > b$ and the identity operator I on $L^2(0,T;H)$ is an odd $(1,1,1)$-homeomorphism, from Theorem 2.2, it follows that $\lambda I - F$ maps $L^2(0,T;H)$ onto itself for any $\lambda \ne 0$. Let

$$\eta = \int_0^T W(T-s)(Bv)(s)ds \in R_T(0).$$

We are going to show that there exists w such that

$$\eta = x(T;f,w) \in \overline{R_T(f)}^V,$$

where $\overline{R_T(f)}^V$ is the closure of $R_T(f)$ in V, Here, we note that $\overline{R_T(f)}^V \subset \overline{R_T(f)}$. We denote the range of the operator B by H_B, its closure $\overline{H}_B$ in $L^2(0,T;H)$. Let $\overline{H}_B^{\perp}$ be the orthogonal complement of $\overline{H}_B$ in $L^2(0,T;H)$. Let $X = L^2(0,T;H)/\overline{H}_B^{\perp}$ be the quotient space and the norm of a coset $\tilde{y} = y_B + \overline{H}_B^{\perp} \in X$ is defined of $||\tilde{y}|| = ||y_B + \overline{H}_B^{\perp}|| = \inf\{|y_B + g| : y_B \in \overline{H}_B,\ g \in \overline{H}_B^{\perp}\}$.

We define by Q the isometric isomorphism from X onto $\overline{H}_B$, that is, $Q\tilde{y} = Q(y_B + g : y_B \in \overline{H}_B, \quad g \in \overline{H}_B^{\perp}) = y_B$. Let

$$\mathcal{F}\tilde{y} = F(Q\tilde{y}) + \overline{H}_B^{\perp}$$

for $\tilde{y} \in X$. Then $\mathcal{F}$ is also a completely continuous mapping from X to itself. Set $z = Bv$. Then $z \in \overline{H}_B$ and $\tilde{z} = z + \overline{H}_B^{\perp} \in X$. Hence, by Theorem 2.2 with $\lambda = 1$, there exists $\tilde{w} \in X$ such that

$$\tilde{z} = \tilde{w} - \mathcal{F}\tilde{w}. \tag{5.3}$$

Put $w_B = Q\tilde{w}$. Then we have that $w - w_B \in \overline{H}_B^{\perp}$. Hence,

$$\tilde{z} = w - F(Q\tilde{w}) + \overline{H}_B^{\perp} = w_B - F(w_B) + \overline{H}_B^{\perp}. \tag{5.4}$$

Thus, from (5.3) and (5.4) it follows that

$$\begin{aligned} \eta &= \int_0^T W(T-s)(-F(w_B)(s) + w_B(s))ds \\ &= \int_0^T W(T-s)(f(s, \hat{x}_{w_B}) + w_B(s))ds. \end{aligned}$$

Since $w_B \in \overline{H}_B$, there exists a sequence $\{v_n\}$, $v_n \in L^2(0,T;U)$ such that $Bv_n \mapsto w_B$ in $L^2(0,T;H)$. Then by the second part of Proposition 4.1, we have that $x(\cdot; f, v_n) \mapsto y_{w_B}$ in $L^2(0,T;D(A_0)) \cap W^{1,2}(0,T;H)$, and hence $x(T; f, v_n) \mapsto y_{w_B}(T) = \eta$ in V. Thus, we conclude that $\eta \in \overline{R_T(f)}$.

$\square$

Theorem 5.2. *Let Assumptions (A), (F) and (F1) hold. If* $1 > b$*(or* $1 < b$ *and* $F(v) = 0$ *imply* $v = 0$*), then we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

Proof. If $1 > b$, it holds from Theorem 5.1. The case if $1 < b$ is obvious from Theorem 2.3. $\square$

We need to impose following assumption:

Assumption (B). There exist positive constants β, γ such that

$$\beta\|u\| \le |Bu| \le \gamma\|u\|, \quad \forall u \in L^2(0,T;U).$$

Theorem 5.3. *Under Assumptions (A), (F), (F1), and (B), if $1 > b$ then the semilinear control system (4.1) is approximately controllable.*

Proof. Since B is odd $(\gamma, \beta, 1)$- homeomorphism of $L^2(0,T;U)$ onto $L^2(0,T;H)$, $F : L^2(0,T;U) \to L^2(0,T;H)$ an odd strong continuous b-homogeneous operator. From Theorem 2.2, it follows that if $1 > b$ then $\lambda B - F$ maps $L^2(0,T;U)$ onto $L^2(0,T;H)$ for any $\lambda \ne 0$. Let $\xi \in D(A_0)$. Then there exists a function $p \in C^1(0,T;H)$ such that

$$\xi = \int_0^T W(T-s)p(s)ds,$$

for instance, put $p(s) = (\xi + sA_0\xi)/T$. Hence, there exists a function $u \in L^2(0,T;U)$ such that

$$p = (\lambda B - F)u,$$

that is,

$$\xi = \int_0^T W(T-s)\{f(s,x(s)) + (Bu)(s))ds.$$

Therefore, if $1 > b$, then $D(A_0) \subset R_T(f)$, which complete the proof. $\square$

Theorem 5.4. *Let Assumptions (A), (B), (F) and (F1) hold. If $F(v) = 0$ imply $v = 0$ and $1 \ne b$, then the semilinear control system (4.1) is approximately controllable.*

Proof. This theorem is obvious from Theorems 5.1 and 5.2. $\square$

Example 5.1. *We consider the semilinear heat equation dealt with by [22] and [33]. Let*

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi),$$
$$a(u,v) = \int_0^\pi \frac{du(x)}{dx}\overline{\frac{dv(x)}{dx}}dx$$

and

$$A = d^2/dx^2 \quad \text{with} \quad D(A) = \{y \in H^2(0,\pi) : y(0) = y(\pi) = 0\}.$$

We consider the following retarded functional differential equation

$$\frac{d}{dt}x(t) = Ax(t) + f(x(t)) + Bw(t), \tag{*}$$

where

$$f(x) = \frac{\sigma x}{1+|x|}x^3, \quad \sigma > 0.$$

For $x,\ y \in H$, set $\max\{|x(\xi)|, |x(\xi)|\}$ *for almost all $\xi \in (0,\pi)$. Then we have*

$$|f(x(\xi)) - f(y(\xi))| \le 3\sigma m^3(1+m)^{-1}|x(\xi) - y(\xi)|$$

for almost all $\xi \in (0,\pi)$. It is easily seen that Assumption (F) is satisfied and f is 3*-quasi-homogeneous. The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $\phi_n(x) = \sin nx$, respectively. Let*

$$U = \{\sum_{n=2}^\infty u_n\phi_n : \sum_{n=2}^\infty u_n^2 < \infty\},$$
$$Bu = 2u_2\phi_1 + \sum_{n=2}^\infty u_n\phi_n, \quad \text{for} \quad u = \sum_{n=2}^\infty u_n \in U.$$

Now we can define bounded linear operator $\hat{B}$ from $L^2(0,T;U)$ to $L^2(0,T;H)$ by $(\hat{B}u) = Bu(t)$, $u \in L^2(0,T;U)$. It is easily known that the operator $\hat{B}$ is one to one and the range of $\hat{B}$ is closed. It follows that the operator satisfies Assumption (B). We can see many examples which satisfy Assumption (B) as seen in [33, 34]. The solution of the following equation

$$\frac{d}{dt}x(t) = Ax(t) + Bw(t)$$

with initial datum 0 *is*

$$x(t) = \int_0^t e^{(t-s)A_0} Bw(s)ds.$$

Let $\xi \in D(A_0)$ and

$$u(s) = B^{-1}(\xi + sA\xi)/T.$$

Then it follows that $x(T) = \xi$, which says that the reachable set $R_T(0)$ for linear system is a dense subspace. Moreover, from Theorem 5.4 with $\lambda = 1$, it follows that the system of (*) is approximately controllable.

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