Thesis for the Degree of
Master of Education

Remark on Soft Topological Spaces

by

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Graduate School of Education

Pukyong National University

August 2012

Remark on Soft Topological Spaces
(Soft 위상공간에 대한 고찰)

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by

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A thesis submitted in partial fulfillment of the requirement
for the degree of

Master of Education

Graduate School of Education
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August 24, 2012

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Soft 위상공간에 대한 고찰

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요 약

불확실성을 다루는 fuzzy 집합의 위상구조인 fuzzy 위상공간의 개념은 Chang에 의하여 소개되었고 그에 대한 몇가지 성질들이 여러 학자들에 의해 연구되었다. 최근에, Shabbir 와 Naz는 매개 변수의 고정된 soft 집합을 통해 정의되는 soft 위상공간 개념과 fuzzy 위상공간이 soft 위상공간의 특별한 경우라는 것을 보였다.

본 논문에서는 soft 집합을 이용한 몇가지 기본개념과 soft 위상공간에서의 폐포, 내부, 근방, 연속성과 피복성을 조사하고 soft 필터, soft 극대필터 및 soft 필터기저를 정의하고 몇가지 성질들을 밝혔다. 또한, soft 위상공간상에서 soft 필터의 폐포점과 수렴의 관계에 대하여 살펴보았다.

1 Introduction

Many disciplines, including engineering, medicine, economics, and sociology, are highly dependent on the task of modeling uncertain data. When the uncertainty is highly complicated and difficult to characterize, classical mathematical approaches are often insufficient to derive effective or useful models. Testifying to the importance of uncertainties that cannot be defined by classical means, researchers are introducing alternative theories every day. In addition to classical probability theory, some of the most important results on this topic are fuzzy sets [23], intuitionistic fuzzy sets [2,3], vague sets [8], interval mathematics [3,9], and rough sets [20]. However, all of these new theories have inherent difficulties which are pointed out in [19]. A possible reason is that these theories possess inadequate parameterization tools [17,19]. Molodtsov [19] introduced soft sets as a mathematical tool for dealing with uncertainties which is free from the above difficulties. Soft set theory has rich potential for practical applications in several domains, a few of which are indicated by Molodtsov in his pioneer work [19]. Maji et al. [18] described an application of soft set theory to a decision-making problem. Pei and Miao [21] investigated the relationships between soft sets and information systems. Research on the soft set theory has been accelerated [5,6,7,12,13,14].

The topological structures of set theories dealing with uncertainties were first studied by Chang [4]. Chang introduced the notion of fuzzy topology and also studied some of its basic properties. Lashin et al. [16] generalized rough set theory in the framework of topological spaces. Recently, Shabir and Naz [22] introduced the notion of soft topological spaces which are defined over an initial universe with a fixed set of parameters. They also studied some of basic concepts of soft topological spaces.

In the present study, we introduce some new concepts in soft topological spaces such as interior point, interior, neighborhood, continuity, and compactness. The notions of soft filters, ultra soft filters and bases of a soft filter are introduced and their basic properties are investigated. The adherent and convergence of soft filters in soft topological spaces with related results is also discussed.

2 Preliminaries

Molodtsov [19] defined soft sets in the following manner. Let U be an initial universe set and E be a set parameters. Let $P(U)$ denote the power set of U, and let $A \subseteq E$.

Definition 2.1 ([19]) A pair (F, A) is called a soft set over U, where F is a mapping given by

$$F : A \to P(U)$$

In other words, a soft set over U is a parametrized family of subsets of the universe U. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A).

For illustration, Molodtsov [19] considered several examples. The set of all soft sets over U is denoted by $S(U)$.

Definition 2.2 ([17]) The class of all value sets of a set (F, E) is called the value class of the soft set, and is denoted by $C_{(F,E)}$.

Definition 2.3 ([21]) For two soft sets (F, A) and (G, B) over a common universe U, (F, A) is a soft subset of (G, B), denoted by $(F, A)\tilde{\subseteq}(G, B)$, if $A \subseteq B$ and $\forall e \in A$, $F(e) \subseteq G(e)$.

Definition 2.4 ([17]) Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.5 ([10]) The complement of a soft set (F, A), denoted by $(F, A)^c$, is defined by $(F, A)^c = (F^c, A)$. $F^c : A \to P(U)$ is a mapping given by $F^c(\alpha) = U - F(\alpha)$, $\forall \alpha \in A$. F^c is called the soft complement function of F. Clearly, $(F^c)^c$ is the same as F and $((F, A)^c)^c = (F, A)$.

Definition 2.6 ([17]) A soft set (F, A) over U is said to be a null soft set, denoted by Φ_A, if $\forall e \in A$, $F(e) = \emptyset$.

Definition 2.7 ([17]) A soft set (F, A) over U is said to be an absolute soft set, denoted by U_A, if $\forall e \in A$, $F(e) = U$.

Clearly, ${U_A}^c = \Phi_A$ and ${\Phi_A}^c = U_A$.

Definition 2.8 ([17]) The union of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cup B$ and for all $e \in C$,

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B \\ G(e) & \text{if } e \in B \backslash A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

This relationship is written as $(F, A)\tilde{\cup}(G, B) = (H, C)$.

Definition 2.9 ([21]) The intersection of two soft sets (F, A) and (G, B) over the common universe U is the soft set (H, C), where $C = A \cap B$ and for all $e \in C$. $H(e) = F(e) \cap G(e)$. This relationship is written as $(F, A)\tilde{\cap}(G, B) = (H, C)$.

For other properties of these operations, we refer to references [17] and [21].

Zadeh's fuzzy set may be considered a special case of the soft set.

Example 2.10 ([19]) Let A be a fuzzy set, and μ_A be the membership function of the fuzzy set A. That is, μ_A is a mapping of U onto [0,1].

Consider the family of α-level sets for the function μ_A:

$$F(\alpha) = \{x \in U : \mu_A(x) \geq \alpha\}, \ \alpha \in [0, 1].$$

If we know the family F, we can find the functions $\mu_A(x)$ by means of the following formula:

$$\mu_A(x) = \sup_{\substack{\alpha \in [0,1] \\ x \in F(\alpha)}} \alpha$$

Thus, every Zadeh's fuzzy sets A may be considered the soft set $(F, [0, 1])$.

Example 2.11 ([19]) Let(X, τ) be a topological space. If $T(x)$ is the family of all open neighborhoods of a point x in X, i.e., $T(x) = \{V \in \tau : x \in V\}$, then the ordered pair (T, X) is a soft set over X.

3 The family $SS(U)_A$ of soft sets and basic properties

Inspired by Molodtsov [19], Maji et al. [17] proposed several operation on soft sets, and some basic properties of these operations are revealed. Recently, Ali et al. [10] point out that several assertions in this paper are not true in general. Also they proposed some new operations on soft sets. In order to efficiently discuss, we consider only soft sets (F, A) over a universe U in which all the parameter set A are same. We denote the family of these soft sets by $SS(U)_A$. In fact, for the family $SS(U)_A$, Ali et al. [11] investigated some properties for the algebraic structures on $SS(U)_A$ and Shabir and Naz [22] introduced the notion of soft topology on U. In this section, we investigate basic properties and operations induced by the family $SS(U)_A$.

Proposition 3.1 ([10]) *If (F, A) and (G, A) are two soft sets in $SS(U)_A$, then*

(a) $((F, A)\tilde{\cup}(G, A))^c = (F, A)^c\,\tilde{\cap}(G, A)^c$.

(b) $((F, A)\tilde{\cap}(G, A))^c = (F, A)^c\,\tilde{\cup}(G, A)^c$.

Definition 3.2 Let I be an arbitrary index set and $\{(F_i, A)\}_{i\in I}$ be a subfamily of $SS(U)_A$.

(a) The union of these soft sets is the soft set (H, A), where $H(e) = \cup_{i\in I}F_i(e)$ for each $e \in A$.

We write $\tilde{\cup}_{i\in I}(F_i, A) = (H, A)$.

(b)The intersection of these soft sets is the soft set (M, A), where $M(e) = \cap_{i\in I}F_i(e)$ for all $e \in A$.

We write $\tilde{\cap}_{i\in I}(F_i, A) = (M, A)$.

Proposition 3.3 *Let I be an arbitrary index set and $\{(F_i, A)\}_{i\in I}$ be a subfamily of $SS(U)_A$. Then*

(a) $[\tilde{\cup}_{i\in I}(F_i, A)]^c = \tilde{\cap}_{i\in I}(F_i, A)^c$, *and*

(b) $[\tilde{\cap}_{i\in I}(F_i, A)]^c = \tilde{\cup}_{i\in I}(F_i, A)^c$.

Proof (a) $[\tilde{\cup}_{i\in I}(F_i, A_i)]^c = (H, A)^c$. Since $(H, A)^c = (H^c, A)$, by definition. $H^c(e) = U - H(e) = U - \cup_{i\in I}F_i(e) = \cap_{i\in I}(U - F_i(e))$ for all $e \in A$. On the other hand, $\widetilde{\cap}_{i\in I}(F_i, A)^c = \widetilde{\cap}_{i\in I}(F_i{}^c, A) = (K, A)$. By definition, we have $K(e) = \cap_{i\in I} F_i{}^c(e) = \cap_{i\in I}(U - F_i(e))$ for all $e \in A$.

(b) Let $[\widetilde{\cap}_{i\in I}(F_i, A)]^c = (H, A)^c$. Since $(H, A)^c = (H^c, A)$, by definition. $H^c(e) = U - H(e) = U - \cap_{i\in I} F_i(e) = \cup_{i\in I}(U - F_i(e))$ for all $e \in A$. On the other hand, $\widetilde{\cup}_{i\in I}(F_i, A)^c = \widetilde{\cup}_{i\in I}(F_i{}^c, A) = (K, A)$. By definition, we have $K(e) = \cup F_i{}^c(e) = \cup_{i\in I}(U - F_i(e))$ for all $e \in A$. This completes the proof. □

Proposition 3.4 ([11]) *Let (F, A) and (G, A) be soft sets in $SS(U)_A$. Then the following are true.*

(a) $(F, A)\widetilde{\cap}\Phi_A = \Phi_A$.

(b) $(F, A)\widetilde{\cap}U_A = (F, A)$.

(c) $(F, A)\widetilde{\cup}\Phi_A = (F, A)$.

(d) $(F, A)\widetilde{\cup}U_A = U_A$.

Proposition 3.5 ([11]) *Let (F, A) and (G, A) be soft sets in $SS(U)_A$. Then the following are true.*

(a) $(F, A)\widetilde{\subseteq}(G, A)$ *iff* $(F, A)\widetilde{\cap}(G, A) = (F, A)$.

(b) $(F, A)\widetilde{\subseteq}(G, A)$ *iff* $(F, A)\widetilde{\cup}(G, A) = (G, A)$.

Proof (a) Suppose that $(F, A)\widetilde{\subseteq}(G, A)$. Then $F(e) \subseteq G(e)$ for all $e \in A$. Let $(F, A)\widetilde{\cap}(G, A) = (H, A)$. Since $H(e) = F(e) \cap G(e) = F(e)$ for all $e \in A$, by definition $(H, A) = (F, A)$. Suppose that $(F, A)\widetilde{\cap}(G, A) = (F, A)$. Let $(F, A)\widetilde{\cap}(G, A) = (H, A)$. Since $H(e) = F(e) \cap G(e) = F(e)$ for all $e \in A$, we know that $F(e) \subset G(e)$ for all $e \in A$. Hence $(F, A)\widetilde{\subseteq}(G, A)$.

(b) It is similar to the proof of (1). □

Proposition 3.6 *Let (F, A), (G, A), (H, A), $(S, A) \in SS(U)_A$. Then the following are true.*

(a) *If $(F, A)\widetilde{\cap}(G, A) = \Phi_A$, then $(F, A)\widetilde{\subseteq}(G, A)^c$.*

(b) $(F, A)\widetilde{\cup}(F, A)^c = U_A$ [11].

(c) *If $(F, A)\widetilde{\subseteq}(G, A)$ and $(G, A)\widetilde{\subseteq}(H, A)$, then $(F, A)\widetilde{\subseteq}(H, A)$.*

(d) *If $(F, A)\widetilde{\subseteq}(G, A)$ and $(H, A)\widetilde{\subseteq}(S, A)$, then $(F, A)\widetilde{\subseteq}(H, A)\widetilde{\subseteq}(G, A)\widetilde{\cap}(S, A)$.*

(e) $(F, A)\widetilde{\subseteq}(G, A)$ *iff* $(G, A)^c\widetilde{\subseteq}(F, A)^c$.

Proof We only prove (a) and (e). The other proofs follow similar lines.

(a) Suppose that $(F, A)\widetilde{\cap}(G, A) = \Phi_A$. Then $F(e) \cap G(e) = \emptyset$ and so $F(e) \subseteq U - G(e) = G^c(e)$ for all $e \in A$. Since $(G, A)^c = (G^c, A)$, we have $(F, A)\widetilde{\subseteq}(G, A)^c$.

(e) It follows from the following: $(F, A)\widetilde{\subseteq}(G, A)$ iff $F(e) \subseteq G(e)$ for all $e \in A$ iff $G(e)^c \subseteq F(e)^c$ for all $e \in A$ iff $G^c(e) \subseteq F^c(e)$ for all $e \in A$ iff $(G, A)^c\widetilde{\subseteq}(F, A)^c$. □

Definition 3.7 The soft set $(F, A) \in SS(U)_A$ is called a soft point in U_A, denoted by e_F, if for the element $e \in A$, $F(e) \neq \emptyset$ and $F(e') = \emptyset$ for all $e' \in A - \{e\}$.

Definition 3.8 The soft point e_F is said to be in the soft set (G, A), denoted by $e_F\widetilde{\in}(G, A)$, if for the element $e \in A$ and $F(e) \subseteq G(e)$

Proposition 3.9 *Let $e_F\widetilde{\in}U_A$ and $(G, A)\widetilde{\subseteq}U_A$. If $e_F\widetilde{\in}(G, A)$, then $e_F\widetilde{\notin}(G, A)^c$.*

Proof If $e_F\widetilde{\in}(G, A)$, then for $e \in A$ and $F(e) \subseteq G(e)$. This implies $F(e) \not\subseteq U - G(e) = G^c(e)$. Therefore, we have $e_F \notin (G^c, A) = (G, A)^c$. □

Remark 3.10 The converse of the above proposition is not true in general.

Example 3.11 Let $A = \{e_1, e_2, e_3\}$ be a parameter set and $U = \{h_1, h_2, h_3, h_4\}$ be a universe.
Let $e_{2_F} = (e_2, \{h_1, h_2, h_3\})$ and $(G, A) = \{(e_1\{h_1, h_4\}), (e_2, \{h_1, h_3\})\}\widetilde{\subseteq}U_A$. Then $e_{2_F}\widetilde{\notin}(G, A)$ and also $e_{2_F}\widetilde{\notin}(G, A)^c = \{(e_1, \{h_2, h_3\}), (e_2\{h_2, h_4\}), (e_3, U)\}$

Next, we will establish several properties of soft sets induced by mappings.

Definition 3.12 ([15]) Let $SS(U)_A$ and $SS(V)_B$ be families of soft sets. Let $u : U \to V$ and $p : A \to B$ be mappings. Then a mapping $f_{pu} : SS(U)_A \to SS(V)_B$ is defined as:

(a) Let (F, A) be a soft set in $SS(U)_A$. The image of (F, A) under f_{pu}, written as $f_{pu}(F, A) = (f_{pu}(F), p(A))$, is a soft set in $SS(V)_B$ such that

$$f_{pu}(F)(y) = \begin{cases} \cup_{x \in p^{-1}(y) \cap A} u(F(x)), & p^{-1}(y) \cap A \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $y \in B$.

(b) Let (G, B) be a soft set in $SS(V)_B$. Then the inverse image of (G, B) under f_{pu}, written as $f_{pu}^{-1}(G, B) = (f_{pu}^{-1}(G), p^{-1}(B))$, is a soft set in $SS(U)_A$ such that

$$f_{pu}^{-1}(G)(x) = \begin{cases} u^{-1}(G(p(x))), & p(x) \in B \\ \emptyset, & \text{otherwise} \end{cases}$$

for all $x \in A$.

Theorem 3.13 ([15]) *Let $SS(U)_A$ and $SS(V)_B$ be families of soft sets. For a function $f_{pu} : SS(U)_A \to SS(V)_B$, the following statements are true.*

(a) $f_{pu}(\Phi_A) = \Phi_B$.

(b) $f_{pu}(U_A) \tilde{\subseteq} U_B$.

(c) $f_{pu}((F, A) \tilde{\cup} (G, A)) = f_{pu}(F, A) \tilde{\cup} f_{pu}(G, A)$ *where* $(F, A), (G, B) \in SS(U)_A$.

In general $f_{pu}(\tilde{\cup}_i f_{pu}(F_i, A)) = \tilde{\cup}_i f_{pu}(F_i, A)$ *where* $(F_i, A) \in SS(U)_A$

(d) *If* $(F, A) \tilde{\subseteq} (G, A)$, *then* $f_{pu}((F, A)) \tilde{\subseteq} f_{pu}(G, A)$, *where* $(F, A), (G, A) \in SS(U)_A$.

(e) *If* $(G, B) \tilde{\subseteq} (H, B)$, *then* $f_{pu}^{-1}(G, B) \tilde{\subseteq} f_{pu}^{-1}(H, B)$, *where* $(G, B), (H, B) \in SS(V)_B$.

The soft function f_{pu} is called surjective if p and u are surjective. The soft function f_{pu} is called injective if p and u are injective.

Theorem 3.14 *Let $SS(U)_A$ and $SS(V)_B$ be families of soft sets. For a function $f_{pu} : SS(U)_A \to SS(V)_B$, the following statements are true.*

(a) $f_{pu}^{-1}((G, B)^c) = (f_{pu}^{-1}(G, B))^c$ *for any soft set* (G, B) *in* $SS(V)_B$.

(b) $f_{pu}(f_{pu}^{-1}((G, B))) \tilde{\subseteq} (G, B)$ *for any soft set* (G, B) *in* $SS(V)_B$.

If f_{pu} is surjective, the equality holds.

(c) $(F, A) \tilde{\subseteq} f_{pu}^{-1}(f_{pu}(F, A))$ *for any soft set* (F, A) *in* $SS(U)_A$.

If f_{pu} is injective, the equality holds.

Proof We only prove (a). The other proofs follow similar lines.

(a)Firstly, we will prove $f_{pu}^{-1}(G^c) = f_{pu}^{-1}(G)^c_{p^{-1}(B)}$. For every x$\in A$, we have

$$f_{pu}^{-1}(G)^c_{p^{-1}(B)}(x) = \begin{cases} U - f_{pu}^{-1}(G)(x), & p(x) \in B \\ U, & p(x) \notin B \end{cases}$$
$$= \begin{cases} U - u^{-1}(G(p(x))), & p(x) \in B \\ U, & p(x) \notin B \end{cases}$$

On the other hand, for every $x \in A$, we have

$$(f_{pu}^{-1}(G^c))(x) = \begin{cases} u^{-1}(V - G(p(x))), & p(x) \in B \\ U, & p(x) \notin B \end{cases}$$
$$= \begin{cases} U - u^{-1}(G(p(x))), & p(x) \in B \\ U, & p(x) \notin B \end{cases}$$

Consequently, $f_{pu}^{-1}(G^c) = f_{pu}^{-1}(G)^c_{P^{-1}(B)}$. Hence,

$$\begin{aligned} f_{pu}^{-1}((G,B)^c) &= f^{-1}(G^c, B) = (f_{pu}^{-1}(G^c), p^{-1}(B)) \\ &= (f_{pu}^{-1}(G)^c_{p^{-1}(B)}, p^{-1}(B)) = (f_{pu}^{-1}(G)^c_{p^{-1}(B)}, A) \\ &= (f_{pu}^{-1}(G), p^{-1}(B))^c = (f_{pu}^{-1}(G,B))^c. \end{aligned}$$

This completes the proof. □

4 Soft topology on U

In this section, we investigate some properties of soft topology which are construct by elements of $SS(U)_A$,

Definition 4.1 ([22]) Let τ be a collection of soft sets over a universe U with a fixed set A of parameters, then $\tau \subseteq SS(U)_A$ is called a soft topology on U with a fixed set A if

($T1$) Φ_A, U_A belong to τ;

($T2$) the union of any number of soft sets in τ belongs to τ;

($T3$) the intersection of any two soft sets in τ belongs to τ.

The triplet (U, τ, A) is called soft topological space over U. The members of τ

are called soft open sets in U and complements of their are called soft closed sets in U.

The proof of the following theorem is an obvious application of De Morgans laws in conjunction with the definition of a soft topology on X, and can be omitted.

Theorem 4.2 *If* $\mathbf{F}$ *is a collection of soft closed sets in a soft topological space* (U,τ,A)*, then*

$(F1)$ $\Phi_A, U_A \in \mathbf{F}$.

$(F2)$ *Any finite union of members of* $\mathbf{F}$ *belongs to* $\mathbf{F}$.

$(F3)$ *Any intersection of members of* $\mathbf{F}$ *belongs to* $\mathbf{F}$.

Example 4.3 Let X = {very costly, costly, cheap, beautiful, surrounded by green space, wooden, modern, in good repair, in bad repair}. Consider the soft set (F, A) which describes the "cost of the houses" and the soft set (G, B) which describes the "attractiveness of the houses". Suppose $U = \{h_1, h_2, , h_10\}$, A = {very costly, costly, cheap} and B = {beautiful, surrounded by green space, cheap}. Let F(very costly) $= \{h_2, h_4, h_7, h_8\}$, F(costly) $= \{h_1, h_3, h_5\}$, F(cheap) $= \{h_6, h_9\}$, G(surrounded by green space) $= \{h_5, h_6, h_8\}$, G(beautiful) $= \{h_2, h_3, h_7\}$, and G(cheap) $= \{h_6, h_9, h_10\}$. Then C = {cheap}, H(cheap) $= \{h_6, h_9\}$, and D = {very costly, costly, cheap, beautiful, surrounded by green space}. Also, T(cheap) $= \{h_6, h_9, h_10\}$, T(very costly) $= \{h_2, h_4, h_7, h_8\}$, T(costly) $= \{h_1, h_3, h_5\}$, T(surrounded by green space) $= \{h_5, h_6, h_8\}$, and T(beautiful) $= \{h_2, h_3, h_7\}$.

The family $\tau = \{\Phi_X, U_X, (F, A), (G, B), (H, C), (T, D)\}$ is a soft topology, because $(F, A)\tilde{\cap}(G, B) = (H, C)$ and $(F, A)\tilde{\cup}(G, B) = (T, D)$.

Example 4.4 ([22]) Let A be a set of parameters and let U be an initial universe. Then the indiscrete soft topology on U is the family $\tau = \{\Phi_A, U_A\}$ and the discrete soft topology on U is the family $\tau = SS(U)_A$.

In the following examples, we show that an ordinary topological space can be considered a soft topological space. However, every soft topological space is not an ordinary topological space.

Example 4.5 Let (X, τ) be a topological space. For every $A \subseteq X$, we define the characteristic function $\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$. Then

$$\chi_\tau = \{\chi_A | A \in \tau, \chi_A : X \to \{0,1\}\}$$

is a fuzzy topology on X. Thus, an ordinary topological space can be considered a fuzzy topological space.

Example 4.6 ([1]) Suppose that there are six alternatives in the universe of houses $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, and that we consider the single parameter "quality of houses" a linguistic variable. For this variable we might define the set of linguistic terms $T(\text{quality}) = \{best, good, fair, poor\}$. Each linguistic term is associated with its own fuzzy set. Two of them might be defined as follows:

$F_{best} = \{(h_1, 0.2), (h_2, 0.7), (h_5, 0.9), (h_6, 1.0)\}$,

$F_{poor} = \{(h_1, 0.9), (h_2, 0.3), (h_3, 1.0), (h_4, 1.0), (h_5, 0.2)\}$.

Consider the fuzzy sets F_{poor} and F_{best}. Their α-level sets are

$F_{poor}(0.2) = \{h_1, h_2, h_3, h_4, h_5\}$, $F_{poor}(0.3) = \{h_1, h_2, h_3, h_4\}$, $F_{poor}(0.9) = \{h_1, h_3, h_4\}$, $F_{poor}(1.0) = \{h_3, h_4\}$ and

$F_{best}(0.2) = \{h_1, h_2, h_5, h_6\}$, $F_{best}(0.7) = \{h_2, h_5, h_6\}$, $F_{best}(0.9) = \{h_5, h_6\}$, $F_{best}(1.0) = \{h_6\}$.

The values $A = \{0.2, 0.3, 0.9, 1.0\} \in [0,1]$ can be treated as a set of parameters, such that the mapping $F_{poor} : X \to P(U)$ gives approximate value sets $F_{poor}(\alpha)$ for $\alpha \in A$. We can thus write the equivalent soft set as $(F_{poor}, [0,1]) = \{(0.2, \{h_1, h_2, h_3, h_4, h_5\}), (0.3, \{h_1, h_2, h_3, h_4\}), (0.9, \{h_1, h_3, h_4\}), (1.0, \{h_3, h_4\})\}$.

Similarly, for $F_{best} : X \to P(U)$, we have $(F_{best}, [0,1]) = \{(0.2, \{h_1, h_2, h_5, h_6\}), (0.7, \{h_2, h_5, h_6\}), (0.9, \{h_5, h_6\}), (1.0, \{h_6\})\}$.

Example 4.7 For the above example, we can define the fuzzy topology $\tau = \{0, 1, F_{poor}, F_{best}, F_{best} \wedge F_{poor}, F_{best} \vee F_{poor}\}$. Moreover, we can define the equivalent soft topology

$$\tau = \{\Phi_I, U_I, (F_{poor}, I), (F_{best}, I), F_{poor}, I)\tilde{\cap}(F_{best}, I), (F_{best}, I)\tilde{\cup}(F_{poor}, I)\}.$$

Here $\Phi_I = 0, U_I = 1,$ $(F_{poor}, I)\tilde{\cap}(F_{best}, I) = (F_{poor} \wedge F_{best}, I)$ and
$(F_{poor}, I)\tilde{\cup}(F_{best}, I) = (F_{poor} \vee F_{best}, I)$.

Definition 4.8 A soft set (G, A) in a soft topological space (U, τ, A) is called a soft neighborhood (briefly: nbd) of the soft point $e_F \tilde{\in} U_A$ if there exists a soft open set (H, A) such that $e_F \tilde{\in} (H, A) \tilde{\subseteq} (G, A)$.

The neighborhood system of a soft point e_F , denoted by $N_\tau(e_F)$, is the family of all its neighborhoods.

Definition 4.9 A soft set (G, A) in a soft topological space (U, τ, A) is called a soft neighborhood (briefly: nbd) of the soft set (F, A) if there exists a soft open set (H, A) such that $(F, A) \tilde{\subseteq} (H, A) \tilde{\subseteq} (G, A)$.

Theorem 4.10 *The neighborhood system $N_\tau(e_F)$ at e_F in a soft topological space (U, τ, A) has the following properties:*

(a) *If $(G, A) \in N_\tau(e_F)$, then $e_F \tilde{\in} (G, A)$.*

(b) *If $(G, A) \in N_\tau(e_F)$ and $(G, A) \tilde{\subseteq} (H, A)$, then $(H, A) \in N_\tau(e_F)$.*

(c) *If $(G, A), (H, A) \in N_\tau(e_F)$, then $(G, A) \tilde{\cap} (H, A) \in N_\tau(e_F)$.*

(d) *If $(G, A) \in N_\tau(e_F)$, then there is a $(M, A) \in N_\tau(e_F)$ such that $(G, A) \in N_\tau(e'_H)$ for each $e'_H \tilde{\in} (M, A)$.*

Proof (a) If $(G, A) \in N_\tau(e_F)$, then there is a $(H, A) \in \tau$ such that $e_F \tilde{\in} (H, A) \tilde{\subseteq} (G, A)$. Therefore, we have $e_F \tilde{\in} (G, A)$.

(b) Let $(G, A) \in N_\tau(e_F)$ and $(G, A) \tilde{\subseteq} (H, A)$. Since $(G, A) \in N_\tau(e_F)$, then there is a $(M, A) \in \tau$ such that $e_F \tilde{\in} (M, A) \tilde{\subset} (G, A)$. Therefore, we have $e_F \tilde{\in} (M, A) \tilde{\subseteq} (G, A) \tilde{\subseteq} (H, A)$ and so $(H, A) \in N_\tau(e_F)$.

(c) If $(G, A), (H, A) \in N_\tau(e_F)$, then there exist $(M, A), (S, A) \in \tau$ such that $e_F \tilde{\in} (M, A) \tilde{\subseteq} (G, A)$ and $e_F \tilde{\in} (S, A) \tilde{\subseteq} (H, A)$. Hence $e_F \tilde{\in} (M, A) \tilde{\cap} (S, A) \tilde{\subseteq} (G, A) \tilde{\cap} (H, A)$. Since $(M, A) \tilde{\cap} (S, A) \in \tau$, we have $(G, A) \tilde{\cap} (H, A) \in N_\tau(e_F)$.

(d) If $(G, A) \in N_\tau(e_F)$, then there is a $(S, A) \in \tau$ such that $e_F \tilde{\in} (S, A) \tilde{\subseteq} (G, A)$. Put $(M, A) = (S, A)$. Then for every $e'_H \tilde{\in} (M, A)$, $e'_H \tilde{\in} (M, A) \tilde{\subseteq} (S, A) \tilde{\subseteq} (G, A)$. This implies $(G, A) \in N_\tau(e'_H)$. □

Definition 4.11 Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U.

(a) The soft closure of (G, A) is the soft set

$$\overline{(G,A)} = \tilde{\cap}\{(S,A) : (S,A) \text{ is soft closed and } (G,A)\tilde{\subseteq}(S,A)\} \text{ (see [22]);}$$

(b) The soft interior of (G, A) is the soft set

$$(G,A)^\circ = \tilde{\cup}\{(S,A) : (S,A) \text{ is soft open and } (S,A)\tilde{\subseteq}(G,A)\}.$$

By property $(T3)$ for soft open sets, $(G,A)^\circ$ is soft open. It is the largest soft open set contained in (G, A).

Corollary 4.12 *Let (U, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over U. Then*

(a) *(F, A) is soft closed iff $(F,A) = \overline{(F,A)}$* (see [22]).

(b) *(G, A) is soft open iff $(G,A) = (G,A)^\circ$.*

Theorem 4.13 *A soft set (G, A) is soft open if and only if for each soft set (F, A) contained in (G, A), (G, A) is a soft neighborhood of (F, A).*

Proof ($\Rightarrow$)Obvious.

($\Leftarrow$) Since $(G,A)\tilde{\subseteq}(G,A)$, there exists a soft open set (H, A) such that $(G,A)\tilde{\subseteq}(H,A)\tilde{\subseteq}(G,A)$. Hence $(H, A) = (G, A)$ and (G, A) is soft open. □

Proposition 4.14 *Let (U, τ, A) be a soft topological space and let (F, A) and (G, A) be a soft sets over U. Then*

(a) *If $(F,A)\tilde{\subseteq}(G,A)$, then $\overline{(F,A)}\tilde{\subseteq}\overline{(G,A)}$ (see [22]).*

(b) *If $(F,A)\tilde{\subseteq}(G,A)$, then $(F,A)^\circ\tilde{\subseteq}(G,A)^\circ$.*

Proof It is clear. □

Theorem 4.15 *Let (U, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over U. Then*

(a) $(\overline{(G,A)})^c = ((G,A)^c)^\circ$.

(b) $((G,A)^\circ)^c = \overline{((G,A)^c)}$.

Proof (a) By Proposition 3.3,

$$\begin{aligned}\overline{((G,A))}^c &= (\tilde{\cap}\{(S,A) : (S,A) \text{ is soft closed and } (G,A)\tilde{\subseteq}(S,A)\})^c \\ &= \tilde{\cup}\{(S,A)^c : (S,A) \text{ is soft closed and } (G,A)\tilde{\subseteq}(S,A)\} \\ &= \tilde{\cup}\{(S,A)^c : (S,A)^c \text{ is soft open and } (S,A)^c\tilde{\subseteq}(G,A)^c\} \\ &= ((G,A)^c)^\circ\end{aligned}$$

The other can be proved similarly. □

Definition 4.16 Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U. The soft point $e_F\tilde{\in}U_A$ is called a soft interior point of a soft set (G, A) if there exists a soft open set (H, A) such that $e_F\tilde{\in}(H, A)\tilde{\subseteq}(G, A)$.

Proposition 4.17 *Let $e_F\tilde{\in}U_A$ for all $e \in A$ and (G, A) be a soft open set in a topological space (U, τ, A). Then the following statements hold:*

(a) *Every soft point $e_F\tilde{\in}(G, A)$ is a soft interior point.*

(b) *For each $e \in A$, let us consider a mapping $[e]_G : A \to P(U)$ defined as follows*

$$[e]_G(e') = \begin{cases} G(e), & if\ e' = e \\ \emptyset, & if\ e' \neq e. \end{cases}$$

(c) $\tilde{\cup}_{e\in A}[e]_G = (G, A)$.

Proof (a) Obvious.

(b) Since (G, A) is a soft open set, the soft interior point $[e]_G$ is the largest soft interior point of (G, A) determined by $e \in A$ and so $[e]_G = \tilde{\cup}e_F$ for every soft interior point e_F of (G, A).

(c) Obvious. □

Proposition 4.18 *Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U. Then*

$$(G, A)^\circ = \tilde{\cup}_{e\in A}\{e_F : e_F \text{ is any soft interior point of } (G, A) \text{ for } e \in A\}.$$

Proof For the proof, let $(G, A)^\circ = (H, A)$, where $H(e) = \cup S(e)$ for each soft open set (S, A) such that $(S, A)\tilde{\subseteq}(G, A)$. Since $(G, A)^\circ$ is a soft open set, by the above Proposition 4.17(c), $(G, A)^\circ = \widetilde{\cup}_{e\in A}[e]_H$ and for each $[e]_H$, $[e]_H$ is a soft interior point of (G, A) because of $[e]_H \widetilde{\in} (G, A)^\circ \ \widetilde{\subseteq} (G, A)$. Therefore,

$$(G, A)^\circ \widetilde{\subseteq} \widetilde{\cup}_{e\in A}\{e_F : e_F \text{ is any soft interior point of } (G, A) \text{ for } e \in A\}.$$

For the other hand, let e_{F_i} be any soft interior point of (G, A) for each $e \in A$. Then there exists a soft open set $(K^e_{F_i}, A) \in SS(U)_A$ for each $e \in A$ such that $e_{F_i} \widetilde{\in} (K^e_{F_i}, A) \widetilde{\subseteq} (G, A)$. So for each $e \in A$, we have $\widetilde{\cup}_i e_{F_i} \widetilde{\subseteq} (K^e_{F_i}, A) \widetilde{\subseteq} (G, A)$ and it implies

$$\begin{aligned} &\widetilde{\cup}_{e\in A}\{e_F : e_F \text{ is any soft interior point of} (G, A) \text{ for } e \in A\} \\ &\quad = \widetilde{\cup}_e \widetilde{\cup}_i e_{F_i} \widetilde{\subseteq} \widetilde{\cup}_e \widetilde{\cup}_i (K^e_{F_i}, A) \widetilde{\subseteq} (G, A). \end{aligned}$$

Since $\widetilde{\cup}_e \widetilde{\cup}_f (K^e_{F_i}, A)$ is soft open and $(G, A)^\circ$ is the largest soft open subset of (G, A),
we have $\widetilde{\cup}_{e\in A}\{e_F : e_F$ is any soft interior point of (G, A) for $e \in A\} \widetilde{\subseteq} (G, A)^\circ$. □

Proposition 4.19 *Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U. Then for every soft interior point e_F of (G, A), $[e]_G = \widetilde{\cup} e_F$ iff (G, A) is soft open.*

Proof It follows from Propositions 4.17 and 4.18. □

5 Soft filter

Definition 5.1 A soft filter on U_A is a non-empty subfamily $\mathcal{F}$ of $SS(U)_A$ having the following properties:

(a) Every soft subset of $SS(U)_A$ which includes a soft set in $\mathcal{F}$ belongs to $\mathcal{F}$;

(b) The intersection of each finite family of soft sets in $\mathcal{F}$ belongs to $\mathcal{F}$;

(c) All the soft sets in $\mathcal{F}$ are not null soft set.

Let $\mathcal{F}$ be a soft filter on U_A. A collection $\mathcal{B}$ of soft subsets of $SS(U)_A$ is called a base for the soft filter $\mathcal{F}$ if (1)$\mathcal{B} \subseteq \mathcal{F}$ and (2) for every soft set (F, A) in $\mathcal{F}$, there is soft set (G, A) in $\mathcal{B}$ such that $(G, A) \widetilde{\subseteq} (F, A)$ we say also that $\mathcal{B}$ generates $\mathcal{F}$.

Theorem 5.2 *Let $\mathcal{F}$ and $\mathcal{G}$ be soft filters on U_A. Then a soft set (H, A) in $SS(U)_A$ belongs to both $\mathcal{F}$ and $\mathcal{G}$ if and only if there are soft sets $(F, A) \in \mathcal{F}$ and $(G, A) \in \mathcal{G}$ such that $(H, A) = (F, A)\tilde{\cup}(G, A)$.*

Proof Suppose $(H, A) \in \mathcal{F} \cap \mathcal{G}$. Then $(H, A) = (H, A)\tilde{\cup}(H, A)$, $(H, A) \in \mathcal{F}$ and $(H, A) \in \mathcal{G}$.

Conversely, suppose $(H, A) = (F, A)\tilde{\cup}(G, A)$ where $(F, A) \in \mathcal{F}$ and $(G, A) \in \mathcal{G}$. Then $(F, A)\tilde{\subseteq}(H, A)$, so $(H, A) \in \mathcal{F}$, and $(G, A)\tilde{\subseteq}(H, A)$, so $(H, A) \in \mathcal{G}$. □

Theorem 5.3 *Let $\mathcal{B}$ be a collection of soft sets in $SS(U)_A$. Then $\mathcal{B}$ is a base for a soft filter on U_A if and only if (1) the finite intersection of members of $\mathcal{B}$ includes a member of $\mathcal{B}$ and (2) $\mathcal{B}$ is non-empty and Φ_A does not belong to $\mathcal{B}$.*

Proof Suppose that $\mathcal{B}$ is a base for a soft filter $\mathcal{F}$ on U_A. Let $\{(B_i, A) : i = 1, \cdots, n\}$ be a finite family of soft sets in $\mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{F}$, it follows that $\tilde{\cap}_{i=1}^{n}(B_i, A) \in \mathcal{F}$ and so $\tilde{\cap}_{i=1}^{n}(B_i, A)$ includes a soft set in $\mathcal{B}$. Since $\mathcal{F}$ is non-empty and every soft set in $\mathcal{F}$ includes a soft set in $\mathcal{B}$, it follows that $\mathcal{B}$ is non-empty. Since $\Phi_A \notin \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{F}$, we have $\Phi_A \notin \mathcal{B}$.

Conversely, suppose the conditions are satisfied. Let $\mathcal{F} = \{(G, A) \in SS(U)_A : (G, A)$ includes a soft set in $\mathcal{B}\}$. Then $\mathcal{F}$ is a soft filter on U_A with base $\mathcal{B}$. □

Let $\mathbf{A}$ be a collection of soft subsets of U_A; let $\mathbf{A}'$ be the collection of intersections of all finite families of soft sets in $\mathbf{A}$. If $\mathbf{A}'$ does not contain the null soft set Φ, then it satisfies the conditions of Theorem 5.3 and hence is a base for a soft filter $\mathcal{F}$ on U_A. We call $\mathcal{F}$ the soft filter generated by $\mathbf{A}$.

Theorem 5.4 *Let $\mathcal{F}$ and $\mathcal{G}$ be soft filters on U_A. Suppose that for every pair of soft subsets (F, A), (G, A) of U_A in $\mathcal{F} \cup \mathcal{G}$, we have $(F, A)\tilde{\cap}(G, A) \neq \Phi_A$. Then the soft filter generated by $\mathcal{F} \cup \mathcal{G}$ consists of all soft sets of the form $(H, A)\tilde{\cap}(S, A)$ where $(H, A) \in \mathcal{F}$ and $(S, A) \in \mathcal{G}$.*

Proof Let $\mathcal{H}$ be the soft filter generated by $\mathcal{F} \cup \mathcal{G}$. Let $\mathcal{S}$ be the set of intersections of all finite families of soft sets from $\mathcal{F} \cup \mathcal{G}$. Let $(F, A) \in \mathcal{H}$. Then (F, A)

includes a soft set in $\mathcal{S}$. Every soft set in $\mathcal{S}$ has the form $(H,A)\tilde{\cap}(S,A)$ where $(H,A) \in \mathcal{F}$ and $(S,A) \in \mathcal{G}$. If $(H,A)\tilde{\cap}(S,A) \subseteq (F,A)$ where $(H,A) \in \mathcal{F}$ and $(S,A) \in \mathcal{G}$, then it follows that we have

$$\begin{aligned}(F,A) &= (F,A)\tilde{\cup}((H,A)\tilde{\cap}(H,A)) \\ &= ((F,A)\tilde{\cup}(H,A))\tilde{\cap}((F,A)\tilde{\cup}(S,A)).\end{aligned}$$

since $(F,A)\tilde{\cup}(H,A)\tilde{\supseteq}(H,A)$, $(F,A)\tilde{\cup}(S,A)\tilde{\supseteq}(S,A)$, we have $(F,A)\tilde{\cup}(H,A) \in \mathcal{F}$ and $(F,A)\tilde{\cup}(S,A) \in \mathcal{G}$. So $(F,A) \in \mathcal{S}$. Thus $\mathcal{H} = \mathcal{S}$, as required. □

Theorem 5.5 *The set of all soft filters on a non-null soft set U_A is inductively ordered by inclusion.*

Proof Let $\mathbf{F} = \{\mathcal{F} : \mathcal{F}$ is a soft filter on $U_A\}$ be totally ordered by inclusion $\subseteq$. Let $\mathbf{A}$ be the union of $\mathbf{F}$. Let $\{(F_i,A) : i \in I\}$ be a finite family of soft sets in $\mathbf{A}$. For each $i \in I$, there is a soft filter $\mathcal{F}_i$ in $\mathbf{F}$ such that $(F_i,A) \in \mathcal{F}_i$. Since $\mathbf{F}$ is $\subseteq$-totally ordered, there is an index j in I such that $(F_i,A) \in F_j$ for all $i \in I$. Hence $\tilde{\cap}_{i\in I}(F_i,A) \neq \Phi_A$. By Theorem 5.3, $\mathbf{A}$ generates a soft filter $\mathcal{F}$ on U_A which is clearly the $\subseteq$-supremum of $\mathbf{F}$. □

It follows from Theorem 5.5 by the application of Zorn's Lemma that the collection of soft filters on a non-null soft set U_A has $\subseteq$-maximal elements: these maximal soft filters are called ultra soft filters. It is also easy to show that for every soft filter $\mathcal{F}$ on a soft set U_A there is an ultra soft filter on U_A which includes $\mathcal{F}$.

Theorem 5.6 *Let $\mathcal{F}$ be an ultra soft filter on a soft set U_A. If (F,A) and (G,A) are soft sets in $SS(U)_A$ such that $(F,A)\tilde{\cup}(G,A) \in \mathcal{F}$, then either $(F,A) \in \mathcal{F}$ or $(G,A) \in \mathcal{F}$.*

Proof Suppose $(F,A) \notin \mathcal{F}$ and $(G,A) \notin \mathcal{F}$. Let $\mathcal{F}' = \{(H,A) \in SS(U)_A : (F,A)\tilde{\cup}(H,A) \in \mathcal{F}\}$. Then

(a) Let $(H,A) \in \mathcal{F}'$ and $(S,A) \in SS(U)_A$ with $(H,A)\tilde{\subseteq}(S,A)$. Since $(F,A)\tilde{\cup}$ $(H,A) \in \mathcal{F}$ and $((F,A)\tilde{\cup}(H,A))\tilde{\subseteq}((F,A)\tilde{\cup}(S,A))$, we have $(F,A)\tilde{\cup}(S,A)$ $\in \mathcal{F}$. So $(S,A) \in \mathcal{F}'$.

(b) Let $\{(H_i, A) : i \in I\}$ be a finite family of soft sets in $\mathcal{F}'$. Since $\mathcal{F}$ is a soft filter, $(F, A)\tilde{\cup}(\tilde{\cap}_{i\in I}(H_i, A)) = \tilde{\cap}_{i\in I}((F, A)\tilde{\cup}(H_i, A)) \in \mathcal{F}$. So $\tilde{\cap}_{i\in I}(H_i, A) \in \mathcal{F}'$

(c) Since $(F, A) \notin \mathcal{F}$, we have $\Phi_A \notin \mathcal{F}'$. Thus $\mathcal{F}'$ is a soft filter on U_A. Clearly, $F \subseteq \mathcal{F}'$ and $(G, A) \in \mathcal{F}'$ although $(G, A) \notin \mathcal{F}$. So $\mathcal{F} \subseteq \mathcal{F}'$, which contradicts the fact that $\mathcal{F}$ is an ultra soft filter. □

Theorem 5.7 *Let U_A be a non-null soft set and $\boldsymbol{A}$ be a collection of soft sets in $SS(U)_A$ which generates a soft filter $\mathcal{F}$ on U_A. If for every soft set $(F, A) \in SS(U)_A$ we have either $(F, A) \in \boldsymbol{A}$ or $(F, A)^c \in \boldsymbol{A}$, then $\boldsymbol{A}$ is an ultra soft filter on U_A.*

Proof Let $\mathcal{F}$ be the soft filter generated by $\mathbf{A}$ and $\mathcal{F}'$ be any ultra soft filter which includes $\mathcal{F}$. Then clearly $\mathbf{A} \subseteq \mathcal{F}'$. Let (F, A) be any soft set in $\mathcal{F}'$. Then $(F, A)^c \notin \mathbf{A}$, for if $(F, A)^c \in \mathbf{A}$ then $(F, A)^c \in \mathcal{F}'$ and $(F, A)\tilde{\cap}(F, A)^c = \Phi_A \in \mathcal{F}'$. This is a contradiction since $\mathcal{F}'$ is a soft filter. Hence $(F, A) \in \mathbf{A}$ and so $\mathcal{F}' \subseteq \mathbf{A}$. So $\mathbf{A} = \mathcal{F}'$, an ultra soft filter. □

Theorem 5.8 *Every soft filter $\mathcal{F}$ on non-null soft set U_A is the intersection of the family of ultra soft filters which include $\mathcal{F}$.*

Proof Let $(F, A) \in SS(U)_A$ be a soft set which does not belong to $\mathcal{F}$. Then for each soft set (G, A) in $\mathcal{F}$ we cannot have $(G, A)\tilde{\subset}(F, A)$ and hence we must have $(G, A)\tilde{\cap}(F, A)^c \neq \Phi_A$. So $\mathcal{F} \cup \{(F, A)^c\}$ generates a soft filter on U_A, which is included in some ultra soft filter $\mathcal{F}_{(F,A)}$. Since $(F, A)^c \in \mathcal{F}_{(F,A)}$ we must have $(F, A) \notin \mathcal{F}_{(F,A)}$. Thus (F, A) does not belong to the intersection of the set of all ultra soft filters which include $\mathcal{F}$. Hence this intersection is just the soft filter $\mathcal{F}$ itself. □

Now, let (U, τ, A) be a soft topological space and $\mathcal{F}$ be a soft filter on U_A. A soft set (F, A) in $SS(U)_A$ is said to be a limit or a limit soft set of the soft filter $\mathcal{F}$ and $\mathcal{F}$ is said to converge to (F, A) or to be convergent to U_A if the τ-nbd soft filter $N_{(F,A)}$ of (F, A) is included in the soft filter $\mathcal{F}$. If $\mathcal{B}$ is a base for a soft filter on U_A then (F, A) is a limit of $\mathcal{B}$ and $\mathcal{B}$ converges to (F, A) if the soft filter generated by $\mathcal{B}$ converges to (F, A).

Theorem 5.9 *Let τ and τ' be soft topologies on a soft set U_A. Then τ is finer than τ' if and only if every soft filter $\mathcal{F}$ on U_A which converges to (F, A) for the soft topology τ also converges to (F, A) for the soft topology τ'.*

Proof Suppose τ is finer than τ'. Let $\mathcal{F}$ be a soft filter which is τ-convergent to (F, A). Then $\mathcal{F} \supseteq N^{\tau}_{(F,A)}$, the τ-nbd soft filter of (F, A). Since τ is finer than τ', every τ'-nbd of (F, A) is a τ-nbd. So $\mathcal{F} \supseteq N^{\tau'}_{(F,A)}$, the τ'-nbd soft filter of (F, A), and hence $\mathcal{F}$ is τ'-convergent to (F, A).

Conversely, suppose that every soft filter on U_A which is τ-convergent to (F, A) is also τ'-convergent to (F, A). Let (G, A) be any τ'-open soft set and (G', A) be any soft subset of (G, A). Then $(G, A) \in N^{\tau}_{(G',A)}$. Since $N^{\tau}_{(G',A)}$ is τ-convergent to (G', A), it follows from our hypothesis that it is τ'-convergent to (G', A). Thus $N^{\tau}_{(G',A)} \supseteq N^{\tau'}_{(G',A)}$ and in particular $(G, A) \in N^{\tau}_{(G',A)}$. Thus (G, A) is a τ-nbd of each of its soft subsets and hence by Theorem 4.13, (G, A) is τ-open. so $\tau' \subseteq \tau$, i.e., τ is finer than τ'. □

Again let (U, τ, A) be a soft topological space and $\mathcal{F}$ be a soft filter on U_A. A soft set (F, A) in $SS(U)_A$ is said to be an adherent soft set of $\mathcal{F}$ if (F, A) is an adherent soft set of every soft set in $\mathcal{F}$. The adherence of $\mathcal{F}$, Adh(F), is the set of all adherent soft sets of $\mathcal{F}$; so Adh$(F) = \widetilde{\cap}_{(H,A)\in F}\overline{(H, A)}$. If $\mathcal{B}$ is a base for a soft filter on U_A, then (F, A) is an adherent soft set of $\mathcal{B}$ if it is an adherent soft set of the soft filter generated by $\mathcal{B}$. The adherence of $\mathcal{B}$, Adh$(\mathcal{B})$, is the set of its adherent soft set.

Theorem 5.10 *Let (U, τ, A) be a soft topological space and $\mathcal{B}$ be a base for a soft filter on U_A. Then $Adh(\mathcal{B}) = \widetilde{\cap}_{(F,A)\in\mathcal{B}}\overline{(F, A)}$.*

Proof Let $\mathcal{F}$ be the soft filter which $\mathcal{B}$ is a base. Then, according to the definition of the adherence of a soft filter base,

$$Adh(\mathcal{B}) = Adh(\mathcal{F}) = \widetilde{\cap}_{(F,A)\in\mathcal{F}}\overline{(F, A)}\widetilde{\subseteq}\widetilde{\cap}_{(F,A)\in\mathcal{B}}\overline{(F, A)}.$$

Let (G, A) be any soft set in $\mathcal{F}$. Then there is a soft set (H, A) in $\mathcal{B}$ such that $(H, A)\widetilde{\subset}(G, A)$ and so $\overline{(G, A)}\widetilde{\supset}\overline{(H, A)}$. Thus $\widetilde{\cap}_{(F,A)\in\mathcal{F}}\overline{(F, A)}\widetilde{\supseteq}\widetilde{\cap}_{(F,A)\in\mathcal{B}}\overline{(F, A)}$. Hence $\widetilde{\cap}_{(F,A)\in\mathcal{F}}\overline{(F, A)} = \widetilde{\cap}_{(F,A)\in\mathcal{B}}\overline{(F, A)}$. □

Theorem 5.11 *Let (U, τ, A) be a soft topological space and (G, A) be a soft set in $SS(U)_A$. Then a soft set (F, A) in $SS(U)_A$ is adherent to (G, A) if and only if there is a soft filter $\mathcal{F}$ on U_A such that $(G, A) \in \mathcal{F}$ and $\mathcal{F}$ converges to (F, A).*

Proof Suppose (F, A) is adherent to (G, A). Then every τ-neighborhood (H, A) of (F, A) meets (G, A), i.e., $(H, A)\tilde{\cap}(G, A) \neq \Phi_A$. Thus $N_{(F,A)} \cup \{(G, A)\}$, where $N_{(F,A)}$ is the τ-neighborhood soft filter of (F, A), generates a soft filter which contains (G, A) and is τ-convergent to (F, A).

Conversely, suppose there is a soft filter $\mathcal{F}$ such that $(G, A) \in \mathcal{F}$ and $\mathcal{F}$ is τ-convergent to (F, A). Let (H, A) be any τ-neighborhood of (F, A). Then $(H, A) \in \mathcal{F}$, and since $(G, A) \in \mathcal{F}$ it follows that $(G, A)\tilde{\cap}(H, A) \neq \Phi_A$. so (F, A) is adherent to (G, A). □

6 Sequences of soft sets in $SS(U)_A$

Definition 6.1 A sequence of soft sets, say $\{(F_n, A) : n \in \mathbf{N}\}$, is eventually contained in a soft set (F, A) if and only if there is an integer m such that, if $n \geq m$, then $(F_n, A)\tilde{\subseteq}(F, A)$. The sequence is frequently contained in (F, A) if and only if for each integer m, there is an integer n such that $n \geq m$ and $(F_n, A)\tilde{\subseteq}(F, A)$. If the sequence is in a soft topological space (U, τ, A), then we say that the sequence converges to a soft set (F, A) if it is eventually contained in each nbd of (F, A).

Definition 6.2 Let f be a mapping over the set of non-negative integers. Then the sequence $\{(G_i, A) : i = 1, 2, \ldots\}$ is a subsequence of a sequence $\{(F_n, A) : n = 1, 2, \ldots\}$ iff there is a map f such that $(G_i, A) = (F_{f(i)}, A)$ and for each integer m, there is an integer n_0 such that $f(i) \geq m$ whenever $i \geq n_0$.

Definition 6.3 A soft set (F, A) in a soft topological space (U, τ, A) is a cluster soft set of a sequence of soft sets if the sequence is frequently contained in every nbd of (F, A).

Theorem 6.4 *If the nbd system of each soft set in a soft topological space (U, τ, A) is countable, then*

(a) *A soft set (F, A) is open if and only if each sequence $\{(F_n, A) : n = 1, 2, \ldots\}$ of soft sets which converges to a soft set (G, A) contained in (F, A) is eventually contained in (F, A).*

(b) *If (F, A) is a cluster soft set of a sequence $\{(F_n, A) : n = 1, 2, \ldots\}$ of soft sets, then there is a subsequence of the sequence converging to (F, A).*

Proof (a) ($\Rightarrow$) Since (F, A) is open, (F, A) is a nbd of (G, A). Hence, $\{(F_n, A) : n = 1, 2, \ldots\}$ is eventually contained in (F, A).

($\Leftarrow$) For each $(G, A)\tilde{\subseteq}(F, A)$, let $(G_1, A), (G_2, A), \ldots, (G_n, A), \ldots$ be the nbd system (G, A). Let $(H_n, A) = \tilde{\cap}_{i=1}^{n}\{(G_i, A)\}$. Then $(H_1, A), (H_2, A), \ldots, (H_n, A), \ldots$ is a sequence which is eventually contained in each nbd of (G, A), i.e., $(H_1, A), (H_2, A), \ldots, (H_n, A), \ldots$ converges to (G, A). Hence, there is an m such that for $n \geq m$, $(H_n, A)\tilde{\subseteq}(F, A)$. The (H_n, A) are nbds of (G, A). Therefore, by Theorem 4.13, (F, A) is soft open.

(b) Let $(K_1, A), (K_2, A), \ldots, (K_n, A), \ldots$ be the nbd system of (F, A) and let $(L_n, A) = \tilde{\cap}_{i=1}^{n}\{(K_i, A)\}$. Then $(L_1, A), (L_2, A), \ldots, (L_n, A), \ldots$ is a sequence such that $(L_{n+1}, A) \subseteq (L_n, A)$ for each n. For every non-negative integer i, choose $f(i)$ such that $f(i) \geq i$ and $(F_{f(i)}, A) \subseteq (L_i, A)$. Then surely $\{(F_{f(i)}, A) : i = 1, 2, \ldots\}$ is a subsequence of the sequence $\{(F_n, A) : n = 1, 2, \ldots\}$. Clearly this subsequence converges to (F, A). □

7 Soft pu-continuous functions between $SS(U)_A$ and $SS(V)_B$

In this section, we introduce the notion of soft pu-continuity of functions induced by two mappings $u : U \to V$ and $p : A \to B$ on soft topological spaces (U, τ, A) and (V, τ^*, B).

Definition 7.1 Let (U, τ, A) and (V, τ^*, B) be soft topological spaces. Let $u : U \to V$ and $p : A \to B$ be mappings. Let $f_{pu} : SS(U)_A \to SS(V)_B$ be a function and $e_F \tilde{\in} U_A$.

(a) f_{pu} is soft pu-continuous at $e_F \tilde{\in} U_A$ if for each $(G, B) \in N_{\tau^*}(f_{pu}(e_F))$, there exits a $(H, A) \in N_\tau(eF)$ such that $f_{pu}(H, A) \tilde{\subseteq} (G, B)$.

(b) f_{pu} is soft pu-continuous on U_A if f_{pu} is soft continuous at each soft point in U_A.

Theorem 7.2 *Let (U, τ, A) and (V, τ^*, B) be soft topological spaces. Let $f_{pu} : SS(U)_A \to SS(V)_B$ be a function and $e_F \tilde{\in} U_A$. Then the following statements are equivalent.*

(a) *f_{pu} is soft pu-continuous at e_F.*

(b) *For each $(G, B) \in N_{\tau^*}(f_{pu}(eF))$, there exits a $(H, A) \in N_\tau(e_F)$ such that $(H, A) \tilde{\subseteq} f_{pu}^{-1}(G, B)$.*

(c) *For each $(G, B) \in N_{\tau^*}(f_{pu}(e_F))$, $f_{pu}^{-1}(G, B) \in N_\tau(e_F)$.*

Proof This is trivial. □

Theorem 7.3 *Let (U, τ, A) and (V, τ^*, B) be soft topological spaces. Let $f_{pu} : SS(U)_A \to SS(V)_B$ be a function. Then the following statements are equivalent.*

(a) *f_{pu} is soft pu-continuous.*

(b) *For each $(H, B) \in \tau^*$, $f^{-1}((H, B)) \in \tau$.*

(c) *For each soft closed set (F, B) over V, $f_{pu}^{-1}(F, B)$ is soft closed over U.*

Proof (a)⇒(b). Let $(H, B) \in \tau^*$ and $e_F \tilde{\in} f_{pu}^{-1}(H, B)$. We will show that $f_{pu}^{-1}(H, B) \in N_\tau(e_F)$. Since $f_{pu}(e_F) \tilde{\in} (H, B)$ and $(H, B) \in \tau^*$, we have $(H, B) \in N_{\tau^*}(f_{pu}(eF))$. Since f_{pu} is soft pu-continuous at e_F, there exits $(M, A) \in N_\tau(e_F)$ such that $f_{pu}(M, A) \tilde{\subseteq} (H, B)$. Therefore, we have $e_F \tilde{\in} (M, A) \tilde{\subseteq} f_{pu}^{-1}(H, B)$ and so $f_{pu}^{-1}(H, B) \in N_\tau(e_F)$.

(b)⇒(c). Let (F, B) be soft closed over V. Then $(F, B)^c \in \tau^*$ and by (b), $f_{pu}^{-1}((F, B)^c) \in \tau$. Since $f_{pu}^{-1}((F, B)^c) = (f_{pu}^{-1}(F, B))^c$, we have that $f_{pu}^{-1}(F, B)$ is soft closed over U.

(c)⇒(b). It is similar to that of (b)⇒(c).

(b)⇒(a). Let $e_F \tilde{\in} U_A$ and $(G, B) \in N_{\tau^*}(f_{pu}(eF))$. Then there is a soft open set $(H, B) \in \tau^*$ such that $f_{pu}(eF) \tilde{\in} (H, B) \tilde{\subseteq} (G, B)$. By (b), $f_{pu}^{-1}(H, B) \in \tau$ and

$e_F\tilde{\in}f_{pu}^{-1}(H,B)\tilde{\subseteq}f_{pu}^{-1}(G,B)$. This shows that $f_{pu}^{-1}(G,B)\in N_\tau(eF)$. Therefore, we have f_pu is soft pu-continuous at every point $e_F\tilde{\in}U_A$. □

Theorem 7.4 *Let (U,τ,A) and (V,τ^*,B) be soft topological spaces. For a function $f_{pu}:SS(U)_A\to SS(V)_B$ consider the following statements.*

(a) *f_{pu} is soft pu-continuous.*

(b) *for each soft set (F,A) over U, the inverse image of every nbd of $f_{pu}(F,A)$ is a nbd of (F,A).*

(c) *for each soft set (F,A) over U and each nbd (H,B) of $f_{pu}(F,A)$, there is a nbd (G,A) of (F,A) such that $f_{pu}(G,A)\tilde{\subseteq}(H,B)$.*

(d) *For each each sequence $\{(F_n,A):n=1,2,\ldots\}$ of soft sets over U which converges to a soft set (F,A) over U, the sequence $\{f((F_n,A)):n=1,2,\ldots\}$ converges to $f_{pu}(F,A)$.*

Then we have (a)⇔(b)⇔(c)⇔(d). Moreover, if the nbd system of each soft set over U is countable, then (d) implies (a) and hence all of the above statements are equivalent.

Proof (a)⇔(b). Let f_{pu} be soft pu-continuous. If (H,B) is a nbd of $f_{pu}(F,A)$, then (H,B) contains a soft open nbd (G,B) of $f_{pu}(F,A)$. Since $f_{pu}(F,A)\tilde{\subseteq}(G,B)$ $\tilde{\subseteq}(H,B)$, $f_{pu}^{-1}(f_{pu}(F,A))\tilde{\subseteq}f_{pu}^{-1}(G,B)\tilde{\subseteq}f_{pu}^{-1}(H,B)$. But $(F,A)\tilde{\subseteq}f_{pu}^{-1}(f_{pu}(F,A))$ and $f_{pu}^{-1}(G,B)$ is soft open. Consequently, $f_{pu}^{-1}(H,B)$ is a nbd of (F,A).

(b)⇒(a). We will use previous theorem. Let (G,B) be soft open over V. Then $f_{pu}^{-1}(G,B)$ is a soft subset of U_A. Let (F,A) be any soft subset of $f_{pu}^{-1}(G,B)$. Then (G,B) is a soft open nbd of $f_{pu}(F,A)$, and by (b), $f_{pu}^{-1}(G,B)$ is a soft nbd of (F,A). This shows that $f_{pu}^{-1}(G,B)$ is a soft open set by Theorem 4.13.

(b)⇒(c). Let (F,A) be any soft set over U and let (H,B) be any nbd of $f_{pu}(F,A)$. By (b), $f_{pu}^{-1}(H,B)$ is a nbd of (F,A). Then there exists a soft open set (G,A) in U_A such that $(F,A)\tilde{\subseteq}(G,A)\tilde{\subseteq}f_{pu}^{-1}(H,B)$. Thus, we have a soft open nbd (G,A) of (F,A) such that $f_{pu}(F,A)\tilde{\subseteq}f_{pu}(G,A)\tilde{\subseteq}(H,B)$.

(c)⇒(b). Let (H,B) be a nbd of $f_{pu}(F,A)$. Then there is a nbd (G,A) of (F,A) such that $f_{pu}(G,A)\tilde{\subseteq}(H,B)$. Hence $f_{pu}^{-1}(f_{pu}(G,A))\tilde{\subseteq}f_{pu}^{-1}(H,B)$. Furthermore, since $(G,A)\tilde{\subseteq}f_{pu}^{-1}(f_{pu}(G,A))$, $f_{pu}^{-1}(H,B)$ is a nbd of (F,A).

(c)⇒(d). If (H, B) is a nbd of $f_{pu}(F, A)$, there is a nbd (G, A) of (F, A) such that $f_{pu}(G, A)\tilde{\subseteq}(H, B)$. Since $\{(F_n, A) : n = 1, 2, \ldots\}$ is eventually in (G, A), we have $f_{pu}(F_n, A)\tilde{\subseteq}f_{pu}(G, A)\tilde{\subseteq}(H, B)$ for $n \geq m$; i.e., there is an m such that for $n \geq m$, $(F_n, A)\tilde{\subseteq}(G, A)$. Therefore, $\{f_{pu}(F_n, A) : n = 1, 2, \ldots\}$ converges to $f_{pu}(F, A)$.

(d)⇒(a). Suppose that the nbd system of each soft set over U is countable. Let (G, B) be any soft open set over V. Then $f_{pu}^{-1}(G, B)$ is a soft subset of U_A. Let (F, A) be any soft subset of $f_{pu}^{-1}(G, B)$, and let $(F_1, A), (F_2, A), \ldots, (F_n, A), \ldots$ be the nbd system (F, A). Let $(H_n, A) = \tilde{\cap}_{i=1}^{n}(F_i, A)$. Then $(H_1, A), (H_2, A), \ldots,$ $(H_n, A), \ldots$ is a sequence which is eventually contained in each nbd of (F, A), i.e., $(H_1, A), (H_2, A), \ldots, (H_n, A), \ldots$ converges to (F, A). Hence, there is an m such that for $n \geq m$, $(H_n, A)\tilde{\subseteq}f_{pu}^{-1}(G, B)$. Since for each n, (H_n, A) is a nbd of (F, A), $f_{pu}^{-1}(G, B)$ is a nbd of (F, A). This shows that $f_{pu}^{-1}(G, B)$ is soft open. □

8 Compact soft spaces

We now consider a soft compact space constructed around a soft topology.

Definition 8.1 A family Ψ of soft sets is a cover of a soft set (F, A) if

$$(F, A)\tilde{\subseteq}\tilde{\cup}\{(F_i, A) : (F_i, A) \in \Psi, i \in I\}.$$

It is a soft open cover if each member of Ψ is a soft open set. A subcover of Ψ is a subfamily of Ψ which is also a cover.

Definition 8.2 A family Ψ of soft sets has the finite intersection property if the intersection of the members of each finite subfamily of Ψ is not null soft set.

Definition 8.3 A soft topological space (U, τ, A) is compact if each soft open cover of U_A has a finite subcover.

Theorem 8.4 *A soft topological space is compact if and only if each family of soft closed sets with the finite intersection property has a nonnull intersection.*

Proof If Ψ is a family of soft sets in a soft topological space (U, τ, A), then Ψ is a cover of U_A if and only if one of the following conditions holds.

(a) $\tilde{\cup}_{i\in I}\{(F_i, A) : (F_i, A) \in \Psi\} = U_A$.

(b) $\{\tilde{\cup}_{i\in I}\{(F_i, A) : (F_i, A) \in \Psi\}\}^c = (U_A)^c = \Phi_A$.

(c) $\tilde{\cap}_{i\in I}\{(F_i, A)^c : (F_i, A) \in \Psi\} = \Phi_A$.

Hence the soft topological space (U, τ, A) is compact if and only if each family of soft open sets over U such that no finite subfamily covers U_A, fails to be a cover, and this is true if and only if each family of soft closed sets which has the finite intersection property has a nonnull intersection. □

Theorem 8.5 *Let f_{pu} be a soft pu-continuous function carrying the compact soft topological space (U, τ, A) onto the soft topological space (V, τ^*, B). Then (V, τ^*, B) is compact.*

Proof Let $\Psi = \{(G_i, B) : i \in I\}$ be a cover of V_B by soft open sets. Then since f_{pu} is soft *pu*-continuous, the family of all soft sets of the form $f_{pu}^{-1}(G_i, B)$, for $(G_i, B) \in \Psi$, is a soft open cover of U_A which has a finite subcover. However, since f_{pu} is surjective, then it is easily seen that $f_{pu}(f_{pu}^{-1}(G, B)) = (G, B)$ for any soft set (G, B) over V. Thus, the family of images of members of the subcover is a finite subfamily of Ψ which covers V_B. Consequently, (V, τ^*, B) is compact. □

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