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Strong Convergence of Modified Hybrid Type Algorithms for Asymptotically Strict Quasi-pseudo-contractive Families

by

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Graduate School of Education
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(점근적 순-준-의 축약족에 대한 수정된 혼합형 알고리즘의 강수렴)

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점근적 순-준-의 축약족에 대한 수정된 혼합형 알고리즘의 강수렴

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요 약

본 논문에서는 먼저 Hilbert 공간 내에서 닫힌블록부분집합(closed convex subset)으로 정의된 자가수반사상(self-mapping)의 점근적 순-준-의 축약족(asymptotically strict quasi-pseudo-contractive family)에 대한 수정된 혼합형 알고리즘(modified hybrid type algorithms)을 소개한 후, 그러한 축약족에 대한 수정된 혼합형 알고리즘의 다음 강수렴(strong convergence) 정리를 밝혔다.

정리. 집합 C를 Hilbert공간 H 내의 공집합이 아닌 닫힌블록부분집합이고, $S=\{S_n : C \to C, n \geq 0\}$를 C상의 점근적 κ-순-준-의 축약족이라 하고, $[0,1]$ 내에 있는 수열 $\{\alpha_n\}$과 $\{\beta_n\}$이 다음 두 조건을 만족한다고 가정하자.

(i) $\lim_{n\to\infty} sup\, \alpha_n < 1$;

(ii) $\beta_n \in [\kappa, 1]$이고 $\lim_{n\to\infty} \beta_n = 1$이다.

더욱, $\omega_w(x_n) \subset F = \cap_{n=0}^{\infty} Fix(S_n)$이고 S가 연속조건, 즉 $\forall v \in C, \|S_n v - v\| \to 0 \Rightarrow v \in F$을 만족하고 $\theta_n = \gamma_n sup\ \{\|x_n - p\|^2 : p \in F\}$라 하자. 그 때, 임의로 주어진 한 점 $x_0 \in C$로부터 출발하여 다음과 같은 수정된 혼합형 알고리즘

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) S_n x_n, \\ z_n = \alpha_n y_n + (1-\alpha_n) S_n y_n, \\ C_n = \{p \in C : \|z_n - p\|^2 \leq \|x_n - p\|^2 + (1-\beta_n)\theta_n + (1-\alpha_n) \\ \qquad\qquad [\theta_n (1 + (1-\beta_n)\gamma_n) + (\kappa - \alpha_n)\|y_n - S_n y_n\|^2]\}, \\ Q_n = \{p \in C : \langle x_n - p, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases}$$

에 의하여 정의된 수열 $\{x_n\}$은 F의 점 $P_F x_0$에 강수렴한다.

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping. We use $Fix(T)$ to denote the set of fixed points of T; that is,

$$Fix(T) = \{x \in C : Tx = x\}.$$

Recall that $T : C \to C$ is said to be a *strict quasi-pseudo-contractive* [1] if $Fix(T) \neq \emptyset$ and there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \kappa\|x - Tx\|^2 \tag{1.1}$$

for all $x \in C$ and $p \in Fix(T)$. For such a case, T is said to be a κ-strict quasi-pseudo-contraction. A 0-strict quasi-pseudo-contraction T is quasi-nonexpansive; that is, T is quasi-nonexpansive if

$$\|Tx - p\| \leq \|x - p\|$$

for all $x \in C$ and $p \in Fix(T)$.

Recall also that a mapping $T : C \to C$ is said to be *asymptotically strict quasi-pseudo-contractive* [18] if $Fix(T) \neq \emptyset$ and there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ of nonnegative real numbers with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|T^n x - p\|^2 \leq (1 + \gamma_n)\|x - p\|^2 + \kappa\|x - T^n x\|^2 \tag{1.2}$$

for all $x \in C$, $p \in Fix(T)$ and $n \geq 1$; see also [7] or [16]. When (1.2) holds, T is afterward said to be an asymptotically κ-strict quasi-pseudo-contraction (with respect to the sequence $\{\gamma_n\}$ in case a distinction is needed). Note that if $\kappa = 0$, then T is asymptotically quasi-nonexpansive [4], that is,

$$\|T^n x - p\| \leq k_n\|x - p\|$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$, where $k_n := \sqrt{1+\gamma_n} \to 1$. It is also known [17] that the class of κ-strict quasi-pseudo-contractions and the class of asymptotically κ-strict quasi-pseudo-contractions are independent.

Iterative methods are often used to solve the fixed point equation $Tx = x$. The most well-known method is perhaps the Picard successive iteration method when T is a contraction. Picard's method generates a sequence $\{x_n\}$ successively as $x_n = Tx_{n-1}$ for $n \geq 2$ with $x_1 := x$ arbitrary, and this sequence converges in norm to the unique fixed point of T. However, if T is not a contraction (for instance, if T is nonexpansive), then Picard's successive iteration fails, in general, to converge. Instead, Mann's iteration method [11] prevails, which, an averaged process in nature, generates a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0, 1]$.

It is known that the Mann iteration method (1.3) is in general not strongly convergent [3] for either nonexpansive mappings or strict pseudo-contractions. In 2003, a method (called hybrid method) to modify the Mann iteration method (1.3) so that strong convergence is guaranteed has been proposed by Nakajo and Takahashi [15] for a single nonexpansive mapping T with $Fix(T) \neq \emptyset$ in a Hilbert space H:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases} \tag{1.4}$$

where P_K denotes the metric projection from H onto a nonempty closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{Fix(T)}x_0$. This result has been extended to the class of asymptotically nonexpansive mappings by Kim and Xu [6], and subsequently to the one of κ-strict pseudo-contractions by Marino and Xu [13] as follows.

Theorem MX (see Theorem 4.1 of [13]) *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a κ-strict pseudo-contraction for some $0 \le \kappa < 1$ and assume that the fixed point set $Fix(T)$ of T is nonempty. Let $\{x_n\}$ be the sequence generated by the following hybrid algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0. \end{cases} \tag{1.5}$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\alpha_n < 1$ for all $n \ge 0$. Then $\{x_n\}$ converges strongly to $P_{Fix(T)}x_0$.

Quite recently, Kim and Xu [7] gave an analogue of Theorem MX for the class of asymptotically κ-strict pseudo-contractions.

Theorem KX (see Theorem 4.1 of [7]) *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically κ-strict pseudo-contraction for some $0 \le \kappa < 1$. Assume that the fixed point set $Fix(T)$ of T is nonempty and bounded. Let $\{x_n\}$ be the sequence generated by the following* *hybrid algorithm:*

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(\kappa - \alpha_n) \\ \qquad\qquad\qquad\qquad \|x_n - T^n x_n\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \geq 0 \end{cases} \tag{1.6}$$

where

$$\theta_n = \triangle_n^2 (1-\alpha_n)\gamma_n \to 0 \text{ as } n \to \infty, \quad \triangle_n = \sup\{\|x_n - z\|^2 : z \in Fix(T)\} < \infty.$$

Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ *is chosen so that* $\limsup_{n\to\infty} \alpha_n < 1$. *Then* $\{x_n\}$ *converges strongly to* $P_{Fix(T)} x_0$.

From now on, motivated by definition of (1.2), we say that a family $\mathcal{S} = \{S_n : C \to C, n \geq 0\}$ of self-mappings of C is *asymptotically κ-strict quasi-pseudo-contractive* on C if $F := \cap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$ and there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}_{n=0}^{\infty}$ of nonnegative real numbers with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|S_n x - p\|^2 \leq (1+\gamma_n)\|x - p\|^2 + \kappa \|x - S_n x\|^2 \tag{1.7}$$

for all $x \in C$, $p \in F$ and all integers $n \geq 0$. When (1.7) holds, $\mathcal{S}$ is often said to be an asymptotically κ-strict quasi-pseudo-contractive family. Especially, when $\kappa = 0$ in (1.7), the family $\mathcal{S}$ is said to be *asymptotically quasi-nonexpansive*. Notice also that the asymptotically strict quasi-pseudo-contractive family $\mathcal{S} = \{S_n : C \to C, n \geq 0\}$ obviously includes the class of strict quasi-pseudo-contractions and the class of asymptotically strict quasi-pseudo-contractions, simply by taking $S_n := T$ (or T^n), $n \geq 0$, for a strict quasi-pseudo-contraction (or asymptotically strict quasi-pseudo-contraction) $T : C \to C$, respectively.

In this thesis, we first propose the following hybrid iteration method

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ y_n = \beta_n x_n + (1-\beta_n) S_n x_n, \\ z_n = \alpha_n y_n + (1-\alpha_n) S_n y_n, \\ C_n = \{p \in C : \|z_n - p\|^2 \le \|x_n - p\|^2 + (1-\beta_n)\theta_n + (1-\alpha_n) \\ \qquad\qquad [\theta_n(1+(1-\beta_n)\gamma_n) + (\kappa - \alpha_n)\|y_n - S_n y_n\|^2]\}, \\ Q_n = \{p \in C : \langle x_n - p, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0, \end{cases} \tag{1.8}$$

where

$$\theta_n = \gamma_n \cdot \sup\{\|x_n - p\|^2 : p \in F := \cap_{n=0}^{\infty} Fix(S_n)\},$$

and the sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ lie in the interval $[0,1]$. We next study strong convergence of the hybrid algorithm (1.8) for such an asymptotically strict pseudo-contractive family $\mathcal{S} = \{S_n : C \to C,\, n \ge 0\}$.

2 Preliminaries

Let H be a real Hilbert space with the duality product $\langle \cdot, \cdot \rangle$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We also denote the weak ω-limit set of $\{x_n\}$ by

$$\omega_w(x_n) = \{x : \exists\, x_{n_j} \rightharpoonup x\}.$$

We now need some facts and tools in a real Hilbert space H which are listed as lemmas below (see [14] for necessary proofs of Lemmas 2.2 and 2.4).

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities (which will be used in the various places in the proofs of the results of this thesis).*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \quad x, y \in H.$

(ii) *For all $\lambda_i \in [0,1]$ with $\sum_{i=0}^{N-1} \lambda_i = 1$, and $x, y \in H$, the following equality holds:*

$$\|\sum_{i=0}^{N-1} \lambda_i x_i\|^2 = \sum_{i=0}^{N-1} \lambda_i \|x_i\|^2 - \sum_{i<j}^{N-1} \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{2.1}$$

In particular, for $N = 2$ we have

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \quad t \in [0,1]. \tag{2.2}$$

Lemma 2.2. ([14]) *Let H be a real Hilbert space. Given a nonempty closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is convex (and closed).

Recall that given a nonempty closed convex subset K of a real Hilbert space H, the nearest point projection P_K from H onto K assigns to each $x \in H$ its nearest point denoted $P_K x$ in K from x to K; that is, $P_K x$ is the unique point in K with the property

$$\|x - P_K x\| \leq \|x-y\|, \quad y \in K.$$

Lemma 2.3. *Let K be a nonempty closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation:*

$$\langle x-z, y-z\rangle \leq 0, \quad y \in K.$$

Lemma 2.4. ([14]) *Let K be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $x_0 \in H$. Let $q = P_K x_0$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset K$ and satisfies the condition*

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad n \geq 1. \tag{2.3}$$

Then $x_n \to q$.

We also need the following lemmas.

Lemma 2.5. ([23]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 + \gamma_n) a_n, \quad n \geq n_0$$

for some positive integer n_0, where $\{\gamma_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

3 Strong convergence theorems

Note that the common fixed point set $F := \cap_{n=0}^{\infty} Fix(S_n)$ is closed, but we don't know whether it is convex or not. However, we firstly prove that F is convex provided the family $\mathcal{S} = \{S_n : C \to C, n \geq 0\}$ satisfies the following *continuity* condition:

$$\forall\, v \in C, \;\; \|S_n v - v\| \to 0 \;\; \Rightarrow \;\; v \in F. \tag{3.1}$$

Lemma 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\mathcal{S} = \{S_n : C \to C, n \geq 0\}$ be asymptotically κ-strict quasi-pseudo-contractive on C. Assume that the family $\mathcal{S}$ satisfies the following continuity condition (3.1). Then the common fixed point set F is convex.*

Proof. Let $p, q \in F$ and $v := \lambda p + (1-\lambda)q \in C$ with $\lambda \in (0,1)$. To show the convexity of F, we must show that $\|S_n v - v\| \to 0$. Now use (ii) of Lemma 2.1 and (1.7) to get

$$\begin{aligned}
\|S_n v - v\|^2 &= \|\lambda(S_n v - p) + (1-\lambda)(S_n v - q)\|^2 \\
&= \lambda\|S_n v - p\|^2 + (1-\lambda)\|S_n v - q\|^2 - \lambda(1-\lambda)\|p-q\|^2 \\
&\le \lambda[(1+\gamma_n)\|v-p\|^2 + \kappa\|v - S_n v\|^2] + \\
&\quad (1-\lambda)[(1+\gamma_n)\|v-q\|^2 + \kappa\|v - S_n v\|^2] - \lambda(1-\lambda)\|p-q\|^2.
\end{aligned}$$

Thus we have

$$\begin{aligned}
(1-\kappa)\|S_n v - v\|^2 &\le \lambda(1-\lambda)(1+\gamma_n)\|p-q\|^2 - \lambda(1-\lambda)\|p-q\|^2 \\
&= \lambda(1-\lambda)\gamma_n\|p-q\|^2 \to 0
\end{aligned}$$

because $\gamma_n \to 0$ as $n \to \infty$. So, we obtain that $\|S_n v - v\| \to 0$. □

Lemma 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\mathcal{S} = \{S_n : C \to C,\ n \ge 0\}$ be asymptotically κ-strict quasi-pseudo-contractive on C. Assume that F is a nonempty bounded subset of C, and also that two control sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are chosen in $[0,1]$ so that*

(i) $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\beta_n \in [\kappa, 1]$ *and* $\lim_{n\to\infty} \beta_n = 1$.

Let $\{x_n\}$ be the sequence generated by the hybrid algorithm (1.8), starting from an arbitrarily given $x_0 \in C$. Then there hold the following properties.

(a) $\|x_n - x_0\| \le \|q - x_0\|$ *for all* $n \ge 1$, *where* $q := P_{\overline{co}(F)} x_0$.

(b) $\|x_n - x_{n+1}\| \to 0$ *and, furthermore,* $\|y_n - S_n y_n\| \to 0$ *as* $n \to \infty$.

Proof. First observe that C_n is closed convex by Lemma 2.2 and also that Q_n is closed convex for all $n \geq 0$. Next we show that $F \subset C_n$ for $n \geq 0$. Indeed, let $p \in F$. By virtue of (1.7), we see

$$\|S_n x_n - p\|^2 \leq (1+\gamma_n)\|x_n - p\|^2 + \kappa\|x_n - S_n x_n\|^2.$$

Then this jointed with the identity (2.2) and the hypothesis (ii) yields

$$\begin{aligned}
& \|y_n - p\|^2 = \|\beta_n(x_n - p) + (1-\beta_n)(S_n x_n - p)\|^2 \\
= \;& \beta_n\|x_n - p\|^2 + (1-\beta_n)\|S_n x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - S_n x_n\|^2 \\
\leq \;& [1+\gamma_n(1-\beta_n)]\|x_n - p\|^2 - (1-\beta_n)(\beta_n - \kappa)\|x_n - S_n x_n\|^2 \\
\leq \;& [1+\gamma_n(1-\beta_n)]\|x_n - p\|^2 \qquad (3.2) \\
\leq \;& \|x_n - p\|^2 + (1-\beta_n)\theta_n. \qquad (3.3)
\end{aligned}$$

Using the identity (2.2) again and the hypothesis (iii) we similarly compute

$$\begin{aligned}
& \|z_n - p\|^2 = \|\alpha_n(y_n - p) + (1-\alpha_n)(S_n y_n - p)\|^2 \\
= \;& \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|S_n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|y_n - S_n yn\|^2 \\
\leq \;& [1+\gamma_n(1-\alpha_n)]\|y_n - p\|^2 - (1-\alpha_n)(\alpha_n - \kappa)\|y_n - S_n y_n\|^2 \\
\leq \;& \|y_n - p\|^2 + (1-\alpha_n)[\theta_n(1+(1-\beta_n)\gamma_n) + (\kappa - \alpha_n)\|y_n - S_n y_n\|^2] \\
\leq \;& \|x_n - p\|^2 + (1-\beta_n)\theta_n + (1-\alpha_n) \\
& [\theta_n(1+(1-\beta_n)\gamma_n) + (\kappa - \alpha_n)\|y_n - S_n y_n\|^2] \qquad (3.4)
\end{aligned}$$

and thus $p \in C_n$ for all $n \geq 0$. This shows $F \subset C_n$ for each $n \geq 0$.

Next we show that

$$F \subset Q_n, \quad n \geq 0. \qquad (3.5)$$

We prove this by induction. For $n = 0$, we have $F \subset C = Q_0$. Assume that

$F \subset Q_k$. Since x_{k+1} is the projection of x_0 onto $C_k \cap Q_k$, by Lemma 2.3 we have

$$\langle x_{k+1} - z, x_0 - x_{k+1} \rangle \geq 0, \quad z \in C_k \cap Q_k.$$

As $F \subset C_k \cap Q_k$ by the induction assumption, the last inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{k+1} implies that $F \subset Q_{k+1}$. Hence (3.5) holds for all $n \geq 0$, and x_n is well defined for all n. Furthermore, since $C_n \cap Q_n$ is closed and convex, it follows from $F \subset C_n \cap Q_n$ that

$$\overline{co}(F) \subset C_n \cap Q_n, \quad n \geq 0.$$

Notice that the definition of Q_n actually implies $x_n = P_{Q_n} x_0$. This together with the fact $\overline{co}(F) \subset Q_n$ further implies

$$\|x_n - x_0\| \leq \|p - x_0\|, \quad p \in \overline{co}(F).$$

In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|q - x_0\|, \quad \text{where } q := P_{\overline{co}(F)} x_0. \tag{3.6}$$

Hence (a) is fulfilled.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. This together with Lemma 2.1 (i) implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.7}$$

This implies that the sequence $\{\|x_n - x_0\|\}$ is increasing. Since it is also bounded, we see that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Note that since $\{x_n\}$ is bounded, so is

$\{S_n x_n\}$. Then it turns out from (3.7) that

$$\|x_{n+1} - x_n\| \to 0. \tag{3.8}$$

To prove the second part of (b), i.e., $\|y_n - S_n y_n\| \to 0$, since $y_n = \beta_n x_n + (1-\beta_n) S_n x_n$, it follows from (3.8) and $\beta_n \to 1$ that

$$\begin{aligned} \|y_n - x_{n+1}\| &\le \|y_n - x_n\| + \|x_n - x_{n+1}\| & (3.9) \\ &= (1-\beta_n)\|S_n x_n - x_n\| + \|x_n - x_{n+1}\| \to 0. & (3.10) \end{aligned}$$

Now use the fact $x_{n+1} \in C_n$ to get

$$\begin{aligned} \|z_n - x_{n+1}\|^2 \le\ & \|x_n - x_{n+1}\|^2 + (1-\beta_n)\theta_n + (1-\alpha_n) \\ & [\theta_n(1 + (1-\beta_n)\gamma_n) + (\kappa - \alpha_n)\|y_n - S_n y_n\|^2]. \end{aligned} \tag{3.11}$$

On the other hand, by virtue of $z_n = \alpha_n y_n + (1-\alpha_n) S_n y_n$ and (2.2) in Lemma 2.1, we have

$$\begin{aligned} \|z_n - x_{n+1}\|^2 &= \|\alpha_n(y_n - x_{n+1}) + (1-\alpha_n)(S_n y_n - x_{n+1})\|^2 \\ &= \alpha_n\|y_n - x_{n+1}\|^2 + (1-\alpha_n)\|S_n y_n - x_{n+1}\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|y_n - S_n y_n\|^2. \end{aligned}$$

After substituting this equality into (3.11), by simplifying and dividing both sides by $(1-\alpha_n)$ (note that $\alpha_n < 1$ for all n), we arrive at

$$\begin{aligned} \|x_{n+1} - S_n y_n\|^2 \le\ & \frac{1}{1-\alpha_n}\{\|x_{n+1} - x_n\|^2 + (1-\beta_n)\theta_n - \alpha_n\|y_n - x_{n+1}\|^2\} \\ & \theta_n[1 + (1-\beta_n)\gamma_n] + \kappa\|y_n - S_n y_n\|^2. \end{aligned} \tag{3.12}$$

Also, since

$$\begin{aligned} & \|x_{n+1} - S_n y_n\|^2 = \|(x_{n+1} - y_n) + (y_n - S_n y_n)\|^2 \\ = \ & \|x_{n+1} - y_n\|^2 + \|y_n - S_n y_n\|^2 - 2\langle y_n - x_{n+1}, y_n - S_n y_n\rangle \end{aligned}$$

by the parallelogram law, substituting this equality into (3.12) and simplifying, we have

$$\begin{aligned}
& (1-\kappa)\|x_n - S_n x_n\|^2 \\
\leq\ & \frac{1}{1-\alpha_n}\{\|x_{n+1}-x_n\|^2 + (1-\beta_n)\theta_n - \|y_n - x_{n+1}\|^2\} \\
& +\theta_n[1+(1-\beta_n)\gamma_n] + 2\langle y_n - x_{n+1}, y_n - S_n y_n\rangle \\
\leq\ & \frac{1}{1-\alpha_n}\{\|x_{n+1}-x_n\|^2 + (1-\beta_n)\theta_n - \|y_n - x_{n+1}\|^2\} \\
& +\theta_n[1+(1-\beta_n)\gamma_n] + 2\|y_n - x_{n+1}\|\|y_n - S_n y_n\|. \qquad (3.13)
\end{aligned}$$

From (3.8) and (3.9) together with $\theta_n \to 0$, it follows that the right hand side of (3.13) converges to zero as $n \to \infty$. Hence (b) is proven. □

Now we present the strong convergence of the hybrid algorithm (1.8) for an asymptotically strict pseudo-contractive family $\mathcal{S} = \{S_n : C \to C,\, n \geq 0\}$.

Theorem 3.3. *Under the same hypotheses with Lemma 3.2, assume, in addition, that $\omega_w(x_n) \subset F$ and $\mathcal{S}$ satisfies the continuity condition (3.1). Then $x_n \to P_F x_0$.*

Proof. Obviously, F is closed and convex. Combined the assumption $\omega_w(x_n) \subset F$ with (a) of Lemma 3.2, an application of Lemma 2.4 (with $K := F$) ensures that $x_n \to q$, where $q = P_F x$. □

As a special case, taking $\beta_n \equiv 1$ in Lemma 3.2 (in this case, notice that the control condition $\limsup_{n\to\infty} \alpha_n < 1$ can be weaken with $0 < \alpha_n < 1$), we obtain the strong convergence of the following modified hybrid type iteration

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily,} \\ z_n = \alpha_n x_n + (1-\alpha_n) S_n x_n, \\ C_n = \{p \in C : \|z_n - p\|^2 \le \|x_n - p\|^2 + (1-\alpha_n) \\ \qquad\qquad\qquad\qquad [\theta_n + (\kappa - \alpha_n)\|x_n - S_n x_n\|^2]\}, \\ Q_n = \{p \in C : \langle x_n - p, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \ge 0, \end{cases} \tag{3.14}$$

where

$$\theta_n = \gamma_n \cdot \sup\{\|x_n - p\|^2 : p \in F := \cap_{n=0}^{\infty} Fix(S_n)\},$$

and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ lies in the interval $[0,1]$.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let a family $\mathcal{S} = \{S_n : C \to C, n \ge 0\}$ be asymptotically κ-strict quasi-pseudo-contractive on C. Assume that F is a nonempty bounded subset of C, and also that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen in $[0,1]$ so that $0 < \alpha_n < 1$. Let $\{x_n\}$ be the sequence generated by the hybrid type algorithm (3.14), starting from an arbitrarily given $x_0 \in C$. Assume, in addition, that $\omega_w(x_n) \subset F$ and $\mathcal{S}$ satisfies the continuity condition (3.1). Then $x_n \to P_F x_0$.*

We here give an example of an asymptotically strict quasi-pseudo-contractive family of self-mappings which is not asymptotically nonexpansive.

Example 3.5. *Let $C = H = \ell^2$ and $t > 1$, $p \ge 1$. Then we can define*

$$S_n x = -\Big(t + \frac{1}{n^p}\Big)x, \qquad x \in C$$

for each $n \ge 1$ and let $S_0 = I$, the identity mapping on C. Then there hold the following properties:

(a) $F := \cap_{n=0}^{\infty} Fix(S_n) = \{0\}$;

(b) *the family* $\mathcal{S} = \{S_n : C \to C,\, n \geq 0\}$ *is not asymptotically nonexpansive;*

(c) $\mathcal{S}$ *is asymptotically* κ*-strict pseudo-contractive on* C *for any* $\kappa \in [\frac{t-1}{t+1}, 1)$;

(d) $\mathcal{S}$ *satisfies the continuity condition (3.1).*

Proof. (b) Let $x, y \in C$ and $\frac{t-1}{t+1} \leq \kappa < 1$. Since

$$\|S_n x - S_n y\|^2 = \Big(t + \frac{1}{n^p}\Big)^2 \|x - y\|^2$$

for $x, y \in C$, $\mathcal{S} = \{S_n : C \to C,\, n \geq 0\}$ is not asymptotically nonexpansive.

(c) Since

$$\|S_n x\|^2 = \Big(t + \frac{1}{n^p}\Big)^2 \|x\|^2,$$
$$\|(I - S_n)x\|^2 = \Big(1 + t + \frac{1}{n^p}\Big)^2 \|x\|^2,$$

and

$$\begin{aligned}\Big(t + \frac{1}{n^p}\Big)^2 - \kappa\Big(1 + t + \frac{1}{n^p}\Big)^2 \;&\leq\; \Big(t + \frac{1}{n^p}\Big)^2 - \frac{t-1}{t+1}\Big(1 + t + \frac{1}{n^p}\Big)^2 \\ &=\; 1 + \frac{2}{n^p} + \frac{2}{t+1}\Big(\frac{1}{n^p}\Big)^2 \\ &<\; 1 + \frac{3}{n^p},\end{aligned}$$

we have

$$\begin{aligned}\|S_n x - 0\|^2 \;&=\; \Big[\Big(t + \frac{1}{n}\Big)^2 - \kappa\Big(1 + t + \frac{1}{n}\Big)^2\Big]\|x\|^2 \\ &\quad + \kappa\Big(1 + t + \frac{1}{n}\Big)^2 \|x\|^2 \\ &\leq\; \Big(1 + \frac{3}{n^p}\Big)\|x\|^2 + \kappa\Big(1 + t + \frac{1}{n}\Big)^2 \|x\|^2 \\ &=\; (1 + \gamma_n)\|x - 0\|^2 + \kappa\|(I - S_n)x\|^2\end{aligned}$$

for $p = 0 \in F$, where $\gamma_n := \frac{3}{n^p}$. Therefore, $\mathcal{S} = \{S_n : C \to C, n \geq 0\}$ is asymptotically κ-strict quasi-pseudo-contractive on C for any κ satisfying $\frac{t-1}{t+1} \leq \kappa < 1$.

(d) Let $v - S_n v \to 0$ for $v \in C$. Then

$$\|v - S_n v\| = \left|1 + \left(t + \frac{1}{n^p}\right)\right| \|v\| \to (1+t)\|v\| = 0.$$

Hence $v = 0 \in F$. □

Remark 3.6. Note that if the family $\mathcal{S} = \{S_n : C \to C, n \geq 0\}$ is given as in Example 3.5 with $p > 1$, and if $\{x_n\}$ is generated by the hybrid algorithm (1.8) and $\limsup_{n\to\infty} \alpha_n < 1$, then $\omega_w(x_n) = \{0\}$. Indeed, assume that $x_{n_k} \rightharpoonup z \in C$. Since $y_n - x_n = (1-\beta_n)(S_n x_n - x_n) \to 0$, we also have $y_{n_k} \rightharpoonup z \in C$. From $\|y_n - S_n y_n\| = \left(1 + t + \frac{1}{n^p}\right)\|y_n\| \to 0$, it follows that $y_n \to 0$. By uniqueness of weak limit, we have $z = 0$ and so $\omega_w(x_n) \subset \{0\}$. For the converse inclusion, since $\{x_n\}$ is bounded; hence $\omega_w(x_n) \neq \emptyset$, say $z \in \omega_w(x_n)$. Then there is a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \rightharpoonup z$. The same argumentation as above gives

$$x_{n_k} \rightharpoonup z = 0 \in \omega_w(x_n),$$

which concludes that $\omega_w(x_n) = \{0\}$. Consequently, the sequence $\{x_n\}$ generated by (1.8) converges strongly to $P_F x_0 = 0$ under the condition $\limsup_{n\to\infty} \alpha_n < 1$ by applying Theorem 3.3.

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