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Subordination preserving properties for certain meromorphic functions

by

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Subordination preserving properties for certain meromorphic functions
(유리형 함수들에 대한 종속 보존 성질들)

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유리형 함수들에 대한 종속 보존 성질들

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요 약

Miller, Mocanu와 Reade 는 미분종속 원리를 이용하여, 개단위원 상에서 정의된 해석함수들의 비선형 적분연산자들에 대한 종속 보존 성질들을 조사하였다. 또한, Miller와 Mocanu는 미분 종속의 쌍대 개념으로서 미분 초종속 개념을 소개하였다.

본 논문에서는 Uralefaddi와 Path 에 의하여 소개된 선형연산자들의 종속 보존 성질 및 초종속 보존 성질들을 조사하였으며, 종속 및 초종속 보존 성질들을 결합한 Sandwich 형태의 새로운 결과를 연구하였다. 더욱, 유리형 함수들의 적분연산자에 대하여 종속 보존 성질들과 그 쌍대성 문제를 밝혔다.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [8]).

Definition 1.1 [7]. Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z), \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

Definition 1.2 [8]. Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3 [8]. We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Let $\mathcal{M}$ denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k$$

which are analytic in the annulus $\mathbb{D} = \{z : 0 < |z| < 1\}$ with an additional condition

$$\lim_{z\to 0} zf(z) \neq 0 \quad (z \in \mathbb{D}).$$

The Hadamard product or convolution of two functions f and g in $\mathcal{M}$ will be denoted by $f * g$.

Let

$$D^n f(z) = \frac{1}{z(1-z)^{n+1}} * f(z) \quad (z \in \mathbb{D}) \tag{1.3}$$

or, equivalently,

$$\begin{aligned} D^n f(z) =& \frac{1}{z}\left(\frac{z^{n+1}f(z)}{n!}\right)^{(n)} \\ =& \frac{1}{z} + (n+1)a_0 + \frac{(n+2)(n+1)}{2!}a_1 z + \cdots \\ & \cdots + \frac{(n+k+1)...(n+1)}{(k+1)!}a_k z^k + ... \quad (z \in \mathbb{D}). \end{aligned}$$

For various interesting developments involving the operators D^n for functions belonging to $\mathcal{M}$, the reader may be refereed to the recent works of Uralegaddi and Path[13], and others[14,15]. It is easily verified from (1.3) that

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+2)D^n f(z). \tag{1.4}$$

Making use of the principle of subordination between analytic functions, Miller et al. [9] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$. Also Owa and Srivastava [10] investigated the subordination properties of certain integral operators (see also [1]). Moreover, Miller and Mocanu [8] considered differential superordinations, as the dual problem of differential subordinations (see also [2]). In the present paper, we investigate the subordination and superordination preserving properties of the multiplier transformation D^n defined by (1.3)with the sandwich-type theorem.

The following lemmas will be required in our present investigation.

Lemma 1.1 [5]. *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition:*

$$\mathrm{Re}\{H(is, t)\} \leq 0,$$

for all real s and $t \leq -n(1+s^2)/2$, where n is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and

$$\mathrm{Re}\{H(p(z), zp'(z))\} > 0 \ \ (z \in \mathbb{U}),$$

then $\mathrm{Re}\{p(z)\} > 0$ in $\mathbb{U}$.

Lemma 1.2 [6]. *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \ \ (z \in \mathbb{U})$$

with $q(0) = c$ is analytic in $\mathbb{U}$ and satisfies $\mathrm{Re}\{\beta q(z) + \gamma\} > 0$ $(z \in \mathbb{U})$.

Lemma 1.3 [7]. *Let $p \in \mathcal{Q}$ with $p(0) = a$ and let $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \not\equiv a$ and $n \geq 1$. If q is not subordinate to p, then there exist points $z_0 = r_0 \mathrm{e}^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$, for which $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,*

$$q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \;\; (m \geq n).$$

A function $L(z,t)$ defined on $\mathbb{U} \times [0,\infty)$ is the subordination chain (or Löwner chain) if $L(\cdot,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for $0 \leq s < t$.

Lemma 1.4 [8]. *Let $q \in \mathcal{H}[a,1]$, let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and set $\varphi(q(z), zq'(z)) \equiv h(z)$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, then*

$$h(z) \prec \varphi(p(z), zp'(z)) \;\; (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \;\; (z \in \mathbb{U}).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

Lemma 1.5 [11]. *The function $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. Suppose that $L(\cdot;t)$ ia analytic in $\mathbb{U}$ for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$. If $L(z;t)$ satisfies*

$$|L(z;t)| \leq K_0 |a_1(t)| \quad (|z| < r_0 < 1;\; 0 \leq t < \infty))$$

for some positive constants K_0 and r_0 and

$$\mathrm{Re}\left\{ \frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t} \right\} > 0 \quad (z \in \mathbb{U};\; 0 \leq t < \infty),$$

then $L(z;t)$ is a subordination chain.

2. Main Results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation D^n defined by (1.3).

Theorem 2.1. *Let $f, g \in \mathcal{M}$. Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \tag{2.1}$$

$$\big(z \in \mathbb{U};\ \phi(z) := (1-\alpha)zD^{n+1}g(z) + \alpha z D^n g(z);\ 0 \leq \alpha < 1\big),$$

where

$$\delta = \frac{(n+1)^2 + (1-\alpha)^2 - |(n+1)^2 - (1-\alpha)^2|}{4(1-\alpha)(n+1)} \tag{2.2}$$

If f and g satisfy the following subordination condition :

$$(1-\alpha)zD^{n+1}f(z) + \alpha z D^n f(z) \prec (1-\alpha)zD^{n+1}g(z) + \alpha z D^n g(z), \tag{2.3}$$

then

$$zD^n f(z) \prec zD^n g(z). \tag{2.4}$$

Moreover, the function $zD^n g(z)$ is the best dominant.

Proof. Let us define the functions F and G by

$$F(z) := zD^n f(z) \ \text{ and } \ G(z) := zD^n g(z), \tag{2.5}$$

respectively. Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. Otherwise, we replace F and G by $F_r(z) = F(rz)$ and $G_r(z) = G(rz)$ for $0 < r < 1$, respectively. Then these functions satisty the conditions of the theorem on $\overline{\mathbb{U}}$. We can prove that $F_r(z) \prec G_r(z)$, which enables us to obtain (2.4) on letting $r \to 1$.

We first show that, if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{2.6}$$

then

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Taking the logarithmic differentiation on both sides of the second equation in (2.5) and using (1.4) for $g \in \mathcal{M}$, we obtain

$$(n+1)\phi(z) = (n+1)G(z) + (1-\alpha)zG'(z) \tag{2.7}$$

Now, by differentiating both sides of (2.7), we obtain

$$(n+1)z\phi'(z) = (1-\alpha)zG'(z)\left(q(z) + \frac{(n+1)}{1-\alpha}\right),$$

which, in conjuction with (2.7), yields the relationship:

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + p(n+1)/(1-\alpha)} \\ &= q(z) + \frac{zq'(z)}{q(z) + (n+1)/(1-\alpha)} \equiv h(z). \end{aligned} \tag{2.8}$$

From (2.1), we have

$$\mathrm{Re}\left\{h(z) + \frac{(n+1)}{1-\alpha}\right\} > 0 \ \ (z \in \mathbb{U}),$$

and by using Lemma 1.2, we conclude that the differential equation (2.8) has a solution $q \in \mathcal{H}(\mathbb{U})$ with $q(0) = h(0) = 1$.

Let us put

$$H(u,v) = u + \frac{v}{u + (n+1)/(1-\alpha)} + \delta, \tag{2.9}$$

where δ is given by (2.2). From (2.1), (2.8) and (2.9), we obtain

$$\mathrm{Re}\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}).$$

Now we proceed to show that $\mathrm{Re}\{H(is,t)\} \leq 0$ for all real s and $t \leq -(1+s^2)/2$. From (2.9), we have

$$\begin{aligned}\mathrm{Re}\{H(is,t)\} &= \mathrm{Re}\left\{is + \frac{t}{is + (n+1)/(1-\alpha)} + \delta\right\} \\ &= \frac{t((n+1)/(1-\alpha))}{|(n+1)/(1-\alpha) + is|^2} + \delta \\ &\leq -\frac{E_\delta(s)}{2|(n+1)/(1-\alpha) + is|^2},\end{aligned} \tag{2.10}$$

where

$$E_\delta(s) := \left(\frac{(n+1)}{1-\alpha} - 2\delta\right) s^2 - \frac{(n+1)}{1-\alpha}\left(2\delta\frac{(n+1)}{1-\alpha} - 1\right). \tag{2.11}$$

For δ given by (2.2), we can prove easily that the expression $E_\delta(s)$ given by (2.11) is positive or equal to zero. Hence from (2.9), we see that $\mathrm{Re}\{H(is,t)\} \leq 0$ for all real s and $t \leq -(1+s^2)/2$. Thus, by using Lemma 1.1, we conclude that $\mathrm{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G is convex in $\mathbb{U}$.

Next, we prove that the subordination condition (2.3) implies that

$$F(z) \prec G(z) \tag{2.12}$$

for the functions F and G defined by (2.5). For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(1-\alpha)(1+t)}{n+1} zG'(z) \;\; (z \in \mathbb{U};\; 0 \leq t < \infty).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{n+1+(1-\alpha)(1+t)}{n+1}\right) \neq 0 \ \ (0 \leq t < \infty;\ \lambda > 0).$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0,\infty)$. By using the well-known growth and distortion theorems for convex functions, it is easy to check that the first part of Lemma 1.5 is satisfied. Furthermore, we have

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \mathrm{Re}\left\{\frac{(n+1)}{1-\alpha} + (1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0,$$

since G is convex and $(n+1)/(1-\alpha) > 0$. Therefore, by virtue of Lemma 1.5, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\phi(z) = G(z) + \frac{1-\alpha}{n+1} zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (0 \leq t < \infty).$$

This implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U})$$

for $\zeta \in \partial\mathbb{U}$ and $t \in [0,\infty)$.

Now suppose that F is not subordinate to G, then by Lemma 1.3, there exists points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{(1-\alpha)(1+t)}{n+1}\zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1-\alpha}{n+1} z_0 F'(z_0) \\ &= (1-\alpha) z_0 D^{n+1} f(z_0) + \alpha z_0 D^n f(z_0) \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (2.3). This contradicts the above observation that $L(\zeta_0, t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (2.3) must imply the subordination given by (2.12). Considering $F(z) = G(z)$, we see that the function G is best dominant. This evidently completes the proof of Theorem 2.1.

Remark 2.1. We note that δ given by (2.2) in Theorem 2.1 satisfies the inequality $0 < \delta \leq 1/2$.

We next prove a dual problem of Theorem 2.1, in the sense that the subordinations are replaced by superordinations.

Theorem 2.2. *Let $f, g \in \mathcal{M}$. Suppose that*

$$\text{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$
$$\left(z \in \mathbb{U};\ \phi(z) := (1-\alpha) z D^{n+1} g(z) + \alpha z D^n g(z);\ 0 \leq \alpha < 1\right),$$

where δ is given by (2.2). *If $(1-\alpha) z D^{n+1} f(z) + \alpha z D^n f(z)$ is univalent in $\mathbb{U}$ and $z D^n f(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$(1-\alpha) z D^{n+1} g(z) + \alpha z D^n g(z) \prec (1-\alpha) z D^{n+1} f(z) + \alpha z D^n f(z) \tag{2.13}$$

implies that

$$zD^n g(z) \prec zD^n f(z).$$

Moreover, the function $zD^n g(z)$ *is the best subordinant.*

Proof. The first part of the proof is similar to that of Theorem 2.1 and so we will use the same notation as in the proof of Theorem 2.1.

Now let us define the functions F and G, respectively, by (2.5). We first note that, if the function q is defined by (2.6), by using (2.7), then we obtain

$$\begin{aligned}\phi(z) &= G(z) + \frac{1-\alpha}{n+1} zG'(z) \\ &=: \varphi(G(z), zG'(z)).\end{aligned} \tag{2.14}$$

After a simple calculation, Eq. (2.13) yields the relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + (n+1)/(1-\alpha)}.$$

Then by using the same method as in the proof of Theorem 2.1, we can prove that $\mathrm{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G defined by (2.5) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (2.13) implies that

$$G(z) \prec F(z) \tag{2.15}$$

for the functions F and G defined by (2.5). Now consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{(1-\alpha)t}{n+1} zG'(z) \ \ (z \in \mathbb{U};\ 0 \leq t < \infty).$$

Since G is convex and $(1-\alpha)/(n+1) > 0$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 2.1. Therefore according to

Lemma 1.4, we conclude that the superordination condition (2.13) must imply the superordination given by (2.15). Furthermore, since the differential equation (2.14) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.2.

If we combine this Theorem 2.1 and Theorem 2.2, then we obtain the following sandwich-type theorem.

Theorem 2.3. *Let* $f, g_k \in \mathcal{M}(k = 1, 2)$*. Suppose also that*

$$\mathrm{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$

$$\big(z \in \mathbb{U};\ \phi_k(z) := (1-\alpha)zD^{n+1}g_k(z) + \alpha zD^n g_k(z);\ k = 1, 2;\ 0 \le \alpha < p\big), \tag{2.16}$$

where δ *is given by* (2.2)*. If* $(1-\alpha)zD^{n+1}f(z) + \alpha zD^n f(z)$ *is univalent in* $\mathbb{U}$ *and* $zD^n f(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$*, then*

$$\begin{aligned}(1-\alpha)zD^{n+1}g_1(z) + \alpha zD^n g_1(z) &\prec (1-\alpha)zD^{n+1}f(z) + \alpha zD^n f(z)\\ &\prec (1-\alpha)zD^{n+1}g_2(z) + \alpha zD^n g_2(z)\end{aligned}$$

implies that

$$zD^n g_1(z) \prec zD^n f(z) \prec zD^n g_2(z).$$

Moreover, the functions $zD^n g_1(z)$ *and* $zD^n g_2(z)$ *are the best subordinant and the best dominant, respectively.*

Since the assumption of Theorem 2.3, that the functions $(1-\alpha)zD^{n+1}f(z)+\alpha zD^n f(z)$ and $zD^n f(z)$ need to be univalent in $\mathbb{U}$, is not so easy to check, we will replace these conditions by another conditions in the following result.

Corollary 2.1. *Let* $f, g_k \in \mathcal{M}(k = 1, 2)$*. Suppose that the condition* (2.16) *is satisfied and*

$$\text{Re}\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta$$
$$\left(z\in\mathbb{U};\ \psi(z) := (1-\alpha)zD^{n+1}f(z)+\alpha zD^n f(z)\right), \tag{2.17}$$

where δ *is given by* (2.2). *Then*

$$\begin{aligned}(1-\alpha)zD^{n+1}g_1(z)+\alpha zD^n g_1(z) &\prec (1-\alpha)zD^{n+1}f(z)+\alpha zD^n f(z)\\ &\prec (1-\alpha)zD^{n+1}g_2(z)+\alpha zD^n g_2(z)\end{aligned}$$

implies that

$$zD^n g_1(z) \prec zD^n f(z) \prec zD^n g_2(z).$$

Moreover, the functions $zD^n g_1(z)$ *and* $zD^n g_2(z)$ *are the best subordinant and the best dominant, respectively.*

Proof. In order to prove Corollary 2.1, we have to show that the condition (2.17) implies the univalence of $\psi(z)$ and $F(z) := zD^n f(z)$. Since $0<\delta\leq 1/2$ from Remark 2.1, the condition (2.17) means that ψ is a close-to-convex function in $\mathbb{U}$ (see [4]) and hence ψ is univalent in $\mathbb{U}$. Furthermore, by using the same tecnniques as in the proof of Theorem 2.1, we can prove the convexity(univalence) of F and so the details may be omitted. Therefore, from Theorem 2.3, we obtain Corollary 2.1.

Setting $n=0,\ \alpha=0$ in Theorem 2.3, we have the following result.

Corollary 2.2. *Let* $f, g_k\in\mathcal{M}(k=1,2)$. *Suppose that*

$$\text{Re}\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{1}{2}$$
$$\left(z\in\mathbb{U};\ \phi_k(z) := z^2 g_k'(z)+2zg_k(z);\ k=1,2\right).$$

If $z^2f'(z)+2zf(z)$ *is univalent in* $\mathbb{U}$ *and* $zf(z)\in\mathcal{H}[1,1]\cap\mathcal{Q}$, *then*

$$z^2 g_1'(z) + 2zg_1(z) \prec z^2 f'(z) + 2zf(z) \prec z^2 g_2'(z) + 2zg_2(z)$$

implies that

$$zg_1(z) \prec zf(z) \prec zg_2(z).$$

Moreover, the functions $zg_1(z)$ *and* $zg_2(z)$ *are the best subordinant and the best dominant, respectively.*

Next, we consider the integral operator F_c defined by (cf. [3,14,15])

$$F_c(f)(z) := \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \quad (c > 0). \tag{2.18}$$

Now, we obtain the following result involving the integral operator defined by (2.18).

Theorem 2.4 *Let* $f, g \in \mathcal{M}$. *Suppose that*

$$\text{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$
$$\left(z \in \mathbb{U};\ \phi(z) := zD^n g(z)\right), \tag{2.19}$$

where

$$\delta = \frac{1 + c^2 - |1 - c^2|}{4c} \ (c > 0). \tag{2.20}$$

If f *and* g *satisfy the following subordination condition* :

$$zD^n f(z) \prec zD^n g(z),$$

then

$$zD^n F_c(f)(z) \prec zD^n F_c(g)(z).$$

Moreover, the function $zD^nF_c(g)(z)$ *is the best dominant.*

Proof. Let us define the functions F and G by

$$F(z) := zD^nF_c(f)(z) \text{ and } G(z) := zD^nF_c(g)(z),$$

respectively. Without loss of generality, as in the proof of Theorem 2.1, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$.

From the definition of the integral operator F_c defined by (2.18), we obtain

$$z(D^nF_c(f)(z))' = cD^nf(z) - (c+1)D^nF_c(f)(z) \tag{2.21}$$

Then from (2.19) and (2.21), we have

$$c\phi(z) = cG(z) + zG'(z). \tag{2.22}$$

Setting

$$q(z) = 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}),$$

and differentiating both sides of (2.22), we obtain

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z)+c}.$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we may omit for the proof involved.

We state a dual problem of Theorem 2.4, which can be obtained by using the similar techniques as in the proof of Theorem 2.2.

Theorem 2.5. *Let* $f, g \in \mathcal{M}$. *Suppose also that*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$
$$(z \in \mathbb{U};\ \phi(z) := zD^ng(z)),$$

where δ *is given by* (2.20)*. If* $zD^n f(z)$ *is univalent in* $\mathbb{U}$ *and* $zD^n F_c(f)(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$*, then*

$$zD^n g(z) \prec zD^n f(z)$$

implies that

$$zD^n F_c(g)(z) \prec zD^n F_c(f)(z).$$

Moreover, the function $zD^n F_c(g)(z)$ *is the best subordinant.*

If we combine this Theorem 2.5 and Theorem 2.6, then we obtain the following result.

Theorem 2.6. *Let* $f, g_k \in \mathcal{M}(k = 1, 2)$*. Suppose also that*

$$\operatorname{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$
$$(z \in \mathbb{U};\ \phi_k(z) := zD^n g_k(z);\ k = 1, 2), \tag{2.23}$$

where δ *is given by* (2.20)*. If* $zD^n f(z)$ *is univalent in* $\mathbb{U}$ *and* $zD^n F_c(f)(z) \in \mathcal{H}[1,1] \cap \mathcal{Q}$*, then*

$$zD^n g_1(z) \prec zD^n f(z) \prec zD^n g_2(z)$$

implies that

$$zD^n F_c(g_1)(z) \prec zD^n F_c(f)(z) \prec zD^n F_c(g_2)(z).$$

Moreover, the functions $zD^n F_c(g_1)(z)$ *and* $zD^n F_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

By using the same methods as in the proof of Corollary 2,1, we have the following result.

Corollary 2.3. *Let $f, g_k \in \mathcal{M}(k = 1, 2)$. Suppose that the condition* (2.23) *is satisfied and*

$$\operatorname{Re}\left\{1 + \frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta$$
$$(z \in \mathbb{U};\ \psi_k(z) := zD^n f(z)),$$

where δ is given by (2.20)*. Then*

$$zD^n g_1(z) \prec zD^n f(z) \prec zD^n g_2(z)$$

implies that

$$zD^n F_c(g_1)(z) \prec zD^n F_c(f)(z) \prec zD^n F_c(g_2)(z).$$

Moreover, the functions $zD^n F_c(g_1)(z)$ and $zD^n F_c(g_2)(z)$ are the best subordinant and the best dominant, respectively.

Taking $n = 0$ in Theorem 2.6, we have the following result.

Corollary 2.4. *Let $f, g_k \in \Sigma(k = 1, 2)$. Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta$$
$$(z \in \mathbb{U};\ \phi_k(z) := zg_k(z);\ k = 1, 2),$$

where δ is given by (2.20)*. If $zf(z)$ is univalent in $\mathbb{U}$ and $zF_c(f)(z) \in \mathcal{H}[1, 1] \cap \mathcal{Q}$, then*

$$zg_1(z) \prec zf(z) \prec zg_2(z)$$

implies that

$$zF_c(g_1)(z) \prec zF_c(f)(z) \prec zF_c(g_2)(z).$$

Moreover, the functions $zF_c(g_1)(z)$ *and* $zF_c(g_2)(z)$ *are the best subordinant and the best dominant, respectively.*

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