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Argument estimates of meromorphic functions associated with certain differential operators

by

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Argument estimates of meromorphic functions associated with certain differential operators
(미분 연산자와 관련된 유리형 함수들의 편각추정)

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미분 연산자와 관련된 유리형 함수들의 편각추정

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요　　약

지금까지 많은 학자들에 의하여 유리형 함수들 및 이와 관련된 함수의 족들에 대한 기하학적 성질들이 연구되어 왔다. 특히 Uralegaddi와 Somanatha [8]는 유리형 함수들의 족들을 소개하고 그 족들 사이의 포함관계를 연구하였다. 또한 Libera와 Robertson [3], Singh [7]은 유리형 close-to-convex 함수들의 족을 소개하였으며, 그 족에 속하는 함수들에 대하여 다양한 기하학적 성질들을 조사하였다.

본 논문에서는 Uralegaddi와 Somanatha [8,9]에 의해 소개된 미분 연산자를 이용하여 유리형 함수들의 부분 족들을 소개하였으며, Miller와 Mocanu [1,4], NunoKawa [5]에 의하여 연구된 결과들을 응용하여 유리형 함수들의 편각 추정과 족들 사이의 포함관계를 조사하였다. 더욱이 유리형 close-to-convex 함수의 적분보존성질들을 연구하였으며, Goel와 Sohi [2]에 의해 얻은 결과를 확장하였다.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{a_{-1}}{z} + \sum_{m=0}^{\infty} a_m z^m \quad (a_{-1} \neq 0),$$

which are analytic in the punctured open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function w such that $w(0) = 0, |w(z)| < 1$ for $z \in \mathbb{U} = \mathbb{D} \cup \{0\}$, and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

Following Uralegaddi and Somanatha [8,9], we define

$$D^0 f(z) = f(z),$$
$$D^1 f(z) = \frac{a_{-1}}{z} + 2a_0 + 3a_1 z + 4a_2 z^2 + \cdots,$$
$$D^2 f(z) = D^1(D^1 f(z)),$$

and

$$\begin{aligned} D^n f(z) &= D^1(D^{n-1} f(z)) \\ &= \frac{a_{-1}}{z} + \sum_{m=2}^{\infty} m^n a_{m-2} z^{m-2} \ \ (n \in \mathbb{N} = \{1, 2, \cdots\},\ z \in \mathbb{D}). \end{aligned} \tag{1.1}$$

Let

$$\Sigma[n; A, B] = \left\{ f \in \Sigma : -\frac{z(D^n f(z))'}{D^n f(z)} \prec \frac{1 + Az}{1 + Bz},\ z \in \mathbb{U} \right\}, \tag{1.2}$$

where $-1 \leq B < A \leq 1$. In particular, we note that $\Sigma[0; 1 - 2\eta, -1] (0 \leq \eta < 1)$ is the well-known class of meromorphic starlike functions of order η. Further, Uraligaddi and Somanatha [8] introduced the classes $\Sigma[n; 1-2\eta, -1]$ and obtained the inclusion relationship among the classes $\Sigma[n; 1-2\eta, -1]$. From (1.2), we observe [6] that a function f is in $\Sigma[n; A, B]$ if and only if

$$\left| \frac{z(D^n f(z))'}{D^n f(z)} + \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \;\; (-1 < B < A \leq 1;\; z \in \mathbb{U}). \tag{1.3}$$

For any nonnegative integer n, let $\Sigma_c[n, \gamma; A, B]$ be the class of functions $f \in \Sigma$ satisfying the condition

$$-\mathrm{Re}\left\{ \frac{z(D^n f(z))'}{D^n g(z)} \right\} > \gamma \qquad (0 \leq \gamma < 1\;;\; z \in \mathbb{U}).$$

for some $g \in \Sigma[n; A, B]$. In particular, $\Sigma_c[0, 0; 1, -1]$ is the class of meromorphic close-to-convex functions introduced by Libera and Robertson [3] and the class $\Sigma_c[0, \gamma; 1, -1]$ have extensively studied by Singh [7].

The purpose of the present paper is to give some argument estimates of meromorphic functions belonging to Σ which imply the basic inclusion relationship among the classes $\Sigma_c[n, \gamma; A, B]$ and the integral preserving properties for meromorphic close-to-convex functions in a sector in connection with the differential operators D^n defined by (1.1). Further, we extend the previous result of Goel and Sohi [2].

2. Main results

In proving our results below, we need the following lemmas.

Lemma 2.1 [1]. *Let h be convex univalent in $\mathbb{U}$ with $h(0) = 1$ and* $\mathrm{Re}\{\lambda h(z) + \mu\} > 0 (\lambda, \mu \in \mathbb{C})$*. If p is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\lambda p(z) + \mu} \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.2 [4]. *Let* h *be convex univalent in* $\mathbb{U}$ *and* η *be analytic in* $\mathbb{U}$ *with* $\mathrm{Re}\{\eta(z)\} \geq 0$. *If* p *is analytic in* $\mathbb{U}$ *and* $p(0) = h(0)$, *then*

$$p(z) + \eta(z)zp'(z) \prec h(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathbb{U}).$$

Lemma 2.3 [5]. *Let* p *be analytic in* $\mathbb{U}$ *with* $p(0) = 1$ *and* $p(z) \neq 0$ *in* $\mathbb{U}$. *If there exist two points* $z_1, z_2 \in \mathbb{U}$ *such that*

$$-\frac{\pi}{2}\alpha_1 = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi}{2}\alpha_2 \tag{2.1}$$

for some $\alpha_1, \alpha_2 (\alpha_1, \alpha_2 > 0)$ *and for all* $z(|z| < |z_1| = |z_2|)$, *then we have*

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha_1 + \alpha_2}{2}m \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha_1 + \alpha_2}{2}m, \tag{2.2}$$

where

$$m \geq \frac{1 - |a|}{1 + |a|} \quad \text{and} \quad a = i \tan \frac{\pi}{4}\left(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}\right). \tag{2.3}$$

At first, with the help of Lemma 2.1, we obtain the following result.

Proposition 2.1. *Let* h *be convex univalent in* $\mathbb{U}$ *with* $h(0) = 1$ *and* $\mathrm{Re}\{h(z)\}$ *be bounded in* $\mathbb{U}$. *If* $f \in \Sigma$ *satisfies the condition*

$$-\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} \prec h(z) \quad (z \in \mathbb{U}),$$

then

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathbb{U})$$

for $\max_{z\in\mathbb{U}} \mathrm{Re}\{h(z)\} < 2$ *(provided* $D^n f(z) \neq 0$ *in* $\mathbb{D}$*).*

Proof. Let

$$p(z) = -\frac{z(D^n f(z))'}{D^n f(z)},$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. By using the equation

$$z(D^n f(z))' = D^{n+1} f(z) - 2\ D^n f(z), \tag{2.4}$$

we get

$$p(z) - 2 = -\frac{D^{n+1} f(z)}{D^n f(z)}. \tag{2.5}$$

Taking logarithmic derivatives in both sides of (2.5) and multiplying by z, we have

$$\frac{zp'(z)}{-p(z)+2} + p(z) = -\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

From Lemma 2.1, it follows that $p(z) \prec h(z)$ for $\mathrm{Re}\{-h(z)+2\} > 0$ $(z \in \mathbb{U})$, which means

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathbb{U})$$

for $\max_{z \in \mathbb{U}} \mathrm{Re}\{h(z)\} < 2$ (provided $D^n f(z) \neq 0$ in $\mathbb{D}$).

Taking $h(z) = (1 + Az)/(1 + Bz)(-1 \leq B < A \leq 1)$ in Proposition 2.1, we have the following result.

Corollary 2.1. *The inclusion relation,* $\Sigma[n+1; A, B] \subset \Sigma[n; A, B]$, *holds for any* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Remark 2.1. If we take $A = 1 - \eta (0 \leq \eta < 1)$ and $B = 0$ in (1.2), then we see from Proposition 2.1 that every function belonging to the class $\Sigma[n; 1 - \eta, 0]$ for any nonnegative integer n is meromorphic starlike of order η, which is the corresponding result obtained by Uralegaddi and Somanatha [8].

Proposition 2.2. *Let h be convex univalent in $\mathbb{U}$ with $h(0) = 1$ and $\mathrm{Re}\{h(z)\}$ be bounded in $\mathbb{U}$. Let $F_c(f)$ be the integral operator defined by*

$$F_c(f) := F_c(f)(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \quad (c > 0). \tag{2.6}$$

If $f \in \Sigma$ satisfies the condition

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathbb{U}),$$

then

$$-\frac{z(D^n F_c(f))'}{D^n F_c(f)} \prec h(z) \quad (z \in \mathbb{U})$$

for $\max_{z\in\mathbb{U}} \mathrm{Re}\{h(z)\} < c+1$ *(provided $D^n F_c(f) \neq 0$ in $\mathcal{D}$).*

Proof. From (2.6), we have

$$z(D^n F_c(f))'(z) = cD^n f(z) - (c+1)D^n F_c(f)(z). \tag{2.7}$$

Let

$$p(z) = -\frac{z(D^n F_c(f))'}{D^n F_c(f)},$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Then, by using (2.7), we get

$$p(z) - (c+1) = -c\frac{D^n f(z)}{D^n F_c(f)}. \tag{2.8}$$

Taking logarithmic derivatives in both sides of (2.8) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{-p(z) + (c+1)} = -\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathbb{U}).$$

Therefore, by Lemma 2.1, we have

$$-\frac{z(D^n F_c(f))'}{D^n F_c(f)} \prec h(z) \quad (z \in \mathbb{U})$$

for $\max_{z\in\mathbb{U}} \mathrm{Re}\{h(z)\} < c+1$ (provided $D^n F_c(f) \neq 0$ in $\mathbb{D}$).

Letting $h(z) = (1 + Az)/(1 + Bz)(-1 \le B < A \le 1)$ in Proposition 2.2, we have immediately

Corollary 2.2. *If $f \in \Sigma[n; A, B]$, then $F_c(f) \in \Sigma[n; A, B]$, where F_c is the integral operator defined by* (2.6)*.*

Applying Proposition 2.1, we now derive

Theorem 2.1. *Let $0 < \delta_1, \delta_2 \le 1$, $0 \le \gamma < 1$ and $1+A < 2(1+B)$ $(-1 < B < A \le 1)$. If a function $f \in \Sigma$ satisfies the condition :*

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \Sigma[n+1; A, B]$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right) < \frac{\pi}{2}\alpha_2.$$

where α_1 and $\alpha_2 (0 < \alpha_1, \alpha_2 \le 1)$ are the solutions of the equations :

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{2(1-B)+A-1}{1-B}\right)(1+|a|) + (\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right) \tag{2.9}$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{2(1-B)+A-1}{1-B}\right)(1+|a|) + (\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right) \tag{2.10}$$

when a is given by (2.3) *and*

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{2(1-B^2)-(1-AB)}\right). \tag{2.11}$$

Proof. Let

$$p(z) = -\frac{1}{1-\gamma}\left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right).$$

By (2.4), we have

$$\begin{aligned}(1-\gamma)zp'(z)D^n g(z) + (1-\gamma)p(z)z(D^n g(z))' - 2z(D^n f(z))' \\ = -z(D^{n+1} f(z))' - \gamma z(D^n g(z))'.\end{aligned} \tag{2.12}$$

Dividing (2.12) by $D^n g(z)$ and simplifying, we get

$$p(z) + \frac{zp'(z)}{-q(z)+2} = -\frac{1}{1-\gamma}\left(\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} + \gamma\right),$$

where

$$q(z) = -\frac{z(D^n g(z))'}{D^n g(z)}.$$

Since $g \in \Sigma[n+1; A, B]$, from Corollary 2.1, we have

$$q(z) \prec \frac{1+Az}{1+Bz}.$$

From (1.3), we have

$$-q(z) + 2 = \rho e^{i\frac{\pi}{2}\phi},$$

where

$$\begin{cases} \frac{2(1+B)-(1+A)}{1+B} < \rho < \frac{2(1-B)+A-1}{1-B} \\ -t_1 < \phi < t_1 \end{cases}$$

when t_1 is given by (2.11). Let h be a function which maps $\mathbb{U}$ onto the angular domain $\{w : -\frac{\pi}{2}\delta_1 < \arg w < \frac{\pi}{2}\delta_2\}$ with $h(0) = 1$. Applying Lemma 2.2 for this h with $\lambda(z) = 1/(-q(z)+2)$, we see that $\mathrm{Re}\{p(z)\} > 0$ in $\mathbb{U}$ and hence $p(z) \neq 0$ in $\mathbb{U}$.

If there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (2.1) is satisfied, then (by Lemma 2.3) we obtain (2.2) under the restriction (2.3). Then we obtain

$$
\begin{aligned}
&\arg\left[-\frac{1}{1-\gamma}\left(\frac{z_1(D^{n+1}f(z_1))'}{D^{n+1}g(z_1)}+\gamma\right)\right] \\
&= -\frac{\pi}{2}\alpha + \arg\left(1 - i\frac{\alpha+\beta}{2}m(\rho e^{i\frac{\pi}{2}\phi})^{-1}\right) \\
&\leq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\sin\frac{\pi}{2}(1-\phi)}{2\rho+(\alpha_1+\alpha_2)m\cos\frac{\pi}{2}(1-\phi)}\right) \\
&\leq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{2(1-B)+A-1}{1-B}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right) \\
&= -\frac{\pi}{2}\delta_1,
\end{aligned}
$$

and

$$
\begin{aligned}
&\arg\left[-\frac{1}{1-\gamma}\left(\frac{z_2(D^{n+1}f(z_2))'}{D^{n+1}g(z_2)}+\gamma\right)\right] \\
&\geq \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_1}{2\left(\frac{2(1-B)+A-1}{1-B}\right)(1+|a|)+(\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_1}\right) \\
&= \frac{\pi}{2}\delta_2,
\end{aligned}
$$

where δ_1, δ_2 and t_1 are given by (2.9), (2.10) and (2.11), respectively. This is a contradiction to the assumption of our theorem. Therefore we complete the proof of our theorem.

Letting $\delta_1 = \delta_2$ in Theorem 2.1, we have the following result.

Corollary 2.3. *Let* $0 \leq \gamma < 1,\ 0 < \delta \leq 1$ *and* $1+A < 2(1+B)$ $(-1 < B < A \leq 1)$. *If a function* $f \in \Sigma$ *is satisfies the condition :*

$$
\left|\arg\left(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta
$$

for some $g \in \Sigma[n+1; A, B]$, *then*

$$\left|\arg\ \left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_1}{\frac{2(1-B)+A-1}{1-B} + \alpha\sin\frac{\pi}{2}t_1}\right)$$

and t_1 *is given by* (2.11).

Remark 2.2. From Corollary 2.1, we see that the inclusion relation $\Sigma_c[n+1, \gamma; A, B] \subset \Sigma_c[n, \gamma; A, B]$ holds for any nonnegative integer n, and so every function belonging to the class $\Sigma_c[n, \gamma; A, B]$ for any nonnegative integer n is meromorphic close-to-convex function.

Taking $n = 0,\ \gamma = 0,\ B \to A$ and $g(z) = 1/z$ in Corollary 2.3, we obtain the following result.

Corollary 2.4. *If a function* $f \in \Sigma$ *is satisfies the condition* :

$$\left|\arg\ \left(-z^2(zf''(z) + 3f'(z))\right)\right| < \frac{\pi}{2}\delta,$$

then

$$\left|\arg\ \left(-z^2 f'(z)\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ *is the solution of the equation* :

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\alpha.$$

By the same techniques as in the proof of Theorem 2.1, we have the following result.

Theorem 2.2. *Let* $0 < \delta_1, \delta_2 \le 1, \gamma > 1$ *and* $1+A < 2(1+B)$ $(-1 < B < A \le 1)$. *If a function* $f \in \Sigma$ *satisfies the condition :*

$$-\frac{\pi}{2}\delta_1 < \arg\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} + \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \Sigma[n+1; A, B]$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right) < \frac{\pi}{2}\alpha_2.$$

where α_1 *and* α_2 $(0 < \alpha_1, \alpha_2 \le 1)$ *are the solutions of the equations* (2.9) *and* (2.10).

Next, we prove

Theorem 2.3. *Let* $0 < \delta_1, \delta_2 \le 1$, $0 \le \gamma < 1$ *and* $1+A < (c+1)(1+B)$ $(-1 < B < A \le 1)$. *If a function* $f \in \Sigma$ *satisfies the condition :*

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g \in \Sigma[n; A, B]$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(-\frac{z(D^n F_c(f))'}{D^n F_c(g)} - \gamma\right) < \frac{\pi}{2}\alpha_2,$$

where F_c *is defined by* (2.6), *and* α_1 *and* $\alpha_2 (0 < \alpha_1, \alpha_2 \le 1)$ *are the solutions of the equations :*

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(c+1)(1-B)+A-1}{1-B}\right)(1+|a|) + (\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_2}\right) \quad (2.13)$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|a|)\cos\frac{\pi}{2}t_2}{2\left(\frac{(c+1)(1-B)+A-1}{1-B}\right)(1+|a|) + (\alpha_1+\alpha_2)(1-|a|)\sin\frac{\pi}{2}t_2}\right) \quad (2.14)$$

when a is given by (2.3) *and*

$$t_2 = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{(c+1)(1-B^2)-(1-AB)}\right) \tag{2.15}$$

Proof. Let

$$p(z) = -\frac{1}{1-\gamma}\left(\frac{z(D^nF_c(f))'}{D^nF_c(g)}+\gamma\right).$$

Since $g \in \Sigma[n;A,B]$, from Corollary 2.2, $F_c(g) \in \Sigma[n;A,B]$. Using (2.7), we have

$$(1-\gamma)p(z)D^nF_c(g) - (c+1)D^nF_c(f) = -cD^nf(z) - \gamma D^nF_c(g).$$

Then, by a simple calculation, we get

$$(1-\gamma)(zp'(z)+p(z)(-q(z)+c+1)) = -\frac{cz(D^nf(z))'}{D^nF_c(g)} - \gamma(-q(z)+c+1),$$

where

$$q(z) = -\frac{z(D^nF_c(g))'}{D^nF_c(g)}.$$

Hence we have

$$p(z) + \frac{zp'(z)}{-q(z)+c+1} = -\frac{1}{1-\gamma}\left(\frac{z(D^nf(z))'}{D^ng(z)}+\gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we omit it.

Taking $\delta_1 = \delta_2$ in Theorem 2.3, we obtain the following result.

Corollary 2.5. *Let* $0 \le \gamma < 1,\ 0 < \delta \le 1$ *and* $1+A < (c+1)(1+B)$ $(-1 < B < A \le 1)$. *If a function* $f \in \Sigma$ *satisfies the condition* :

$$\left|\arg\left(-\frac{z(D^n f(z))'}{D^n g(z)}-\gamma\right)\right|<\frac{\pi}{2}\delta$$

for some $g\in\Sigma[n;A,B]$, *then*

$$\left|\arg\left(-\frac{z(D^n F_c(f))'}{D^n F_c(g)}-\gamma\right)\right|<\frac{\pi}{2}\alpha,$$

where F_c *is the integral operator given by* (2.6) *and* $\alpha(0<\alpha\le 1)$ *is the solution of the equation* :

$$\delta=\alpha+\frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos\frac{\pi}{2}t_2}{\frac{(c+1)(1-B)+A-1}{1-B}+\alpha\sin\frac{\pi}{2}t_2}\right)$$

where t_2 *is given by* (2.15).

Remark 2.3. From Theorem 2.3 or Corollary 2.5, we see easily that every function in the class $\Sigma_c[n,\gamma;A,B]$ preserves the angles under the integral operator defined by (2.6).

Letting $A=1$, $B=0$, $n=0$ and $\delta=1$ in Corollary 2.5, we have the following result.

Corollary 2.6. *Let* $0\le\gamma<1$. *If a function* $f\in\Sigma$ *satisfies the condition* :

$$-\text{Re}\left\{\frac{zf'(z)}{g(z)}\right\}>\gamma$$

for some $g\in\Sigma$ *satisfying the condition* :

$$\left|\frac{zg'(z)}{g(z)}+1\right|<1,$$

then

$$-\text{Re}\left\{\frac{zF_c'(f)}{F_c(g)}\right\}>\gamma,$$

where F_c *is given by* (2.6).

Remark 2.4. If we put $\gamma = 0$ in Corollary 2.6, then we have the result obtained by Goel and Sohi [2].

Taking $n = \gamma = 0$, $B \to A$ and $g(z) = 1/z$ in Theorem 2.3, we have the following result.

Corollary 2.7. *Let* $0 < \delta_1, \delta_2 \leq 1$. *If a function* $f \in \Sigma$ *satisfies the condition :*

$$-\frac{\pi}{2}\delta_1 \ < \ \arg\ (-z^2 f'(z)) \ < \ \frac{\pi}{2}\delta_2,$$

then

$$-\frac{\pi}{2}\alpha_1 \ < \ \arg\ \left(-z^2 F_c'(f)\right) \ < \ \frac{\pi}{2}\alpha_2,$$

where F_c *is defined by* (2.6), *and* α_1 *and* $\alpha_2 (0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the equations :*

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1 + \alpha_2)(1 - |a|)}{2c(1 + |a|)}\right)$$

and

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1 + \alpha_2)(1 - |a|)}{2c(1 + |a|)}\right)$$

when a *is given by* (2.3).

By using the same methods as in proving Theorem 2.3, we have the following result.

Theorem 2.4. *Let* $0 < \delta_1, \delta_2 \leq 1$, $\gamma > 1$ *and* $1 + A < (c+1)(1+B)$ $(-1 < B < A \leq 1)$. *If a function* $f \in \Sigma$ *satisfies the condition :*

$$-\frac{\pi}{2}\delta_1 < \ \arg\ \left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right) \ < \ \frac{\pi}{2}\delta_2$$

for some $g \in \Sigma[n; A, B]$, *then*

$$-\frac{\pi}{2}\alpha_1 \;<\; \arg\left(\frac{z(D^nF_c(f))'}{D^nF_c(g)}+\gamma\right) \;<\; \frac{\pi}{2}\alpha_2,$$

where F_c *is defined by* (2.6), *and* α_1 *and* $\alpha_2 (0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the equations* (2.13) *and* (2.14).

Finally, we derive

Theorem 2.5. *Let* $0 \leq \gamma < 1,\; 0 < \delta \leq 1$ *and* $1+A < 2(1+B)\ (-1 < B < A \leq 1)$. *If a function* $f \in \Sigma$ *satisfies the condition* :

$$\left|\arg\left(-\frac{z(D^nf(z))'}{D^ng(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in \Sigma[n+1; A, B]$, *then*

$$\left|\arg\left(-\frac{z(D^{n+1}F_c(f))'}{D^{n+1}F_c(g)}-\gamma\right)\right| < \frac{\pi}{2}\delta,$$

where F_c *is defined by* (2.6) with $c = 1$.

Proof. From (2.4) and (2.7) with $c = 1$, we have $D^nf(z) = D^{n+1}F_c(f)$
Therefore

$$\frac{z(D^nf(z))'}{D^ng(z)} = \frac{z(D^{n+1}F_c(f))'}{D^{n+1}F_c(g)}$$

and the result follows.

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