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Fuzzy Information Aggregation in Decision Making

by

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Graduate School of Education

Pukyong National University

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Fuzzy Information Aggregation in Decision Making

(의사결정에서의 퍼지정보 집성)

Advisor : Prof. Jin Han Park

by

Seung Mi Yu

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요 약

본 논문에서는 퍼지평방평균(fuzzy quadratic mean)과 퍼지역조화평균 (fuzzy contra-aharmonic mean)과 같은 여러 가지의 퍼지수 집성연산자들을 소개하고, 이들 연산자들의 다양한 성질을 찾고, 또한 멱등성, 유계성, 단조성 및 교환성과 같은 성질을 만족함을 밝힌다. 그리고 퍼지가중평방평균 (FWQM) 연산자와 퍼지융합평방평균 (FHQM) 연산자 (또는 퍼지가중역조화평균 (FWCHM) 연산자와 퍼지가중역조화평균 (FHCHM) 연산자)를 이용하여 퍼지정보를 갖는 다속성집단의사결정문제의 해결방안을 제안하고, 에어컨 선택이라는 실제적인 예를 통하여 제안한 해결방안의 실제성과 효용성을 보인다.

1 Introduction

Information aggregation is an essential process of gathering relevant information from multiple sources by using a proper aggregation technique. Many techniques, such as the weighted average operator [1], the weighted geometric mean operator [2], harmonic mean operator [3], weighted harmonic mean (WHM) operator [3], ordered weighted average (OWA) operator [4], ordered weighted geometric operator [5, 6], weighted OWA operator [7], induced OWA operator [8], induced ordered weighted geometric operator [9], uncertain OWA operator [10], hybrid aggregation operator [11] and so on, have been developed to aggregate data information. However, yet most of existing aggregation operators do not take into account the information about the relationship between the values being fused. Yager [12] introduced a tool to provide more versatility in the information aggregation process, i.e., developed a power-average (PA) operator and a power OWA (POWA) operator. In some situations, however, these two operators are unsuitable to deal with the arguments taking the forms of multiplicative variables because of lack of knowledge, or data, and decision makers' limited expertise related to the problem domain. Based on this tool, Xu and Yager [29] developed additional new geometric aggregation operators, including the power-geometric (PG) operator, weighted PG operator and power-ordered weighted geometric (POWG) operator, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other.

In this thesis, we will develop some new fuzzy aggregation operators, and apply them to group decision making. In order to do this, the remainder of this thesis is arranged in following chapters. In Chapter 2, we first review some aggregation operators, including the WAA, WQM and WCHM operators. Then, in Chapter 3, we develop fuzzy aggregation operators including fuzzy quadratic mean (FQM), fuzzy weighted quadratic mean (FWQM), fuzzy ordered weighted quadratic mean (FOWQM), fuzzy contraharmonic mean (FCHM), fuzzy weighted contraharmonic mean (FWCHM), and fuzzy ordered weighted contraharmonic mean (FOWCHM) operators, and investigate some properties of the FOWQM and FOWCHM operators, such as commutativity, idempotency, monotonicity and boundedness. In Chapter 4, we utilize the FWQM and FHQM (or the FWCHM and FHCHM) operators to propose approaches to multiple attribute group decision making with triangular fuzzy information. Chapter 5 illustrates the presented approaches with a practical example.

2 Basic aggregation operators

In this chapter, we review some basic aggregation techniques and some of their fundamental characteristics.

Definition 2.1 [1] Let WAA : $R^n \to R$, if

$$\text{WAA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j a_j, \tag{1}$$

where R is the set of real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WAA is called the weighted arithmetic averaging (WAA) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\text{WAA}(a_1, a_2 \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WAA operator is reduced to the arithmetic averaging (AA) operator, i.e.,

$$\text{AA}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} a_j. \tag{2}$$

Definition 2.2 [3] Let WQM : $(R^+)^n \to R^+$, if

$$\text{WQM}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j a_j^2 \right)^{\frac{1}{2}}, \tag{3}$$

where R^+ is the set of all positive real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WQM is called the weighted quadratic mean (WQM) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WQM}(a_1, a_2 \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WQM operator is reduced to the quadratic mean (QM) operator, i.e.,

$$\mathrm{QM}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{j=1}^{n} a_j^2}{n} \right)^{\frac{1}{2}}. \tag{4}$$

Definition 2.3 [3] Let WCHM : $(R^+)^n \to R^+$, if

$$\mathrm{WCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} w_j a_j^2}{\sum_{j=1}^{n} w_j a_j}, \tag{5}$$

where R^+ is the set of all positive real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WCHM is called the weighted contraharmonic mean (WCHM) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WCHM}(a_1, a_2, \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WCHM operator is reduced to the contraharmonic mean (CHM) operator, i.e.,

$$\mathrm{CHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} a_j^2}{\sum_{j=1}^{n} a_j}. \tag{6}$$

The WAA, WQM and WCHM operators first weight all the given data, and then aggregate all these weighted data into a collective one. Yager [4] introduced and studied the OWA operator that weights the ordered positions of the data instead of weighting the data themselves.

Definition 2.4 [4] An OWA operator of dimension n is a mapping OWA : $R^n \to R$ that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore,

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} \omega_j b_j, \tag{7}$$

where b_j is the jth largest of a_i $(i = 1, 2, \ldots, n)$. Especially, if $\omega_i = 1$, $\omega_j = 0$, $j \neq i$, then $b_n \leq \mathrm{OWA}(a_1, a_2, \ldots, a_n) = b_i \leq b_1$; if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} b_j = \frac{1}{n} \sum_{j=1}^{n} a_j = \mathrm{AA}(a_1, a_2, \ldots, a_n). \tag{8}$$

3 New fuzzy aggregation operators

The above aggregation techniques can only deal with the situation that the arguments are represented by the exact numerical values, but are invalid if the aggregation information is given in other forms, such as triangular fuzzy number [15], which is a widely used tool to deal with uncertainty and fuzzyness, described as follows:

Definition 3.1 [15] A triangular fuzzy number $\hat{a}$ can be defined by a triplet $[a^L, a^M, a^U]$. The membership function $\mu_{\hat{a}}(x)$ is defined as:

$$\mu_{\hat{a}}(x) = \begin{cases} 0, & x < a^L; \\ \frac{x-a^L}{a^M-a^L}, & a^L \leq x \leq a^M; \\ \frac{x-a^U}{a^M-a^U}, & a^M \leq x \leq a^U; \\ 0, & x > a^U \end{cases}$$

where $a^U \geq a^M \geq a^L \geq 0$, a^L and a^U stand for the lower and upper values of $\hat{a}$, respectively, and a^M stands for the modal value [15]. Especially, if and two of a^L, a^M and a^U are equal,then $\hat{a}$ is reduced to an interval number; if all a^L, a^M and a^U are equal, then $\hat{a}$ is reduced to a real number. For convenience, we let Ω be the set of all triangular fuzzy numbers.

Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then some operational laws defined as follows [15]:

1) $\hat{a} + \hat{b} = [a^L, a^M, a^U] + [b^L, b^M, b^U] = [a^L + b^L, a^M + b^M, a^U + b^U]$;

2) $\lambda\hat{a} = \lambda[a^L, a^M, a^U] = [\lambda a^L, \lambda a^M, \lambda a^U]$;

3) $\hat{a} \times \hat{b} = [a^L, a^M, a^U] \times [b^L, b^M, b^U] = [a^L b^L, a^M b^M, a^U b^U]$

4) $\frac{1}{\hat{a}} = \frac{1}{[a^L, a^M, a^U]} = [\frac{1}{a^U}, \frac{1}{a^M}, \frac{1}{a^L}]$.

In order to compare two triangular fuzzy numbers, Xu [13] provided the following definition:

Definition 3.2 [13] Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then the degree of possibility of $\hat{a} \geq \hat{b}$ is defined as follows:

$$p(\hat{a} \geq \hat{b}) = \delta \max\left\{1 - \max\left(\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0\right), 0\right\} + (1-\delta)\max\left\{1 - \max\left(\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0\right), 0\right\}, \delta \in [0,1] \quad (9)$$

which satisfies the following properties:

$$0 \leq p(\hat{a} \geq \hat{b}) \leq 1, \quad p(\hat{a} \geq \hat{a}) = 0.5, \quad p(\hat{a} \geq \hat{b}) + p(\hat{b} \geq \hat{a}) = 1. \quad (10)$$

Here, δ reflects the decision maker's risk-bearing attitude. If $\delta > 0.5$, then the decision maker is risk lover; If $\delta = 0.5$, then the decision maker is neutral to risk; If $\delta < 0.5$, then the decision maker is risk avertor.

In the following, we shall give a simple procedure for ranking of the triangular fuzzy numbers $\hat{a}_i$ $(i = 1, 2, \ldots, n)$. First, by using Equation (9), we compare each $\hat{a}_i$ with all the $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, for simplicity, let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$, then we develop a possibility matrix [16, 10] as

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}, \quad (11)$$

where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = \frac{1}{2}$, $i, j = 1, 2, \ldots, n$.

Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$. Then, in accordance with the values of p_i $(i = 1, 2, \ldots, n)$, we rank the $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order.

3.1 Fuzzy quadratic mean operators

Based on operational laws, we extend the WQM operator (3) to fuzzy environment:

Definition 3.3 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, and let FWQM : $\Omega^n \to \Omega$, if

$$\text{FWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\sum_{j=1}^{n} w_j \hat{a}_j^2\right)^{\frac{1}{2}} \tag{12}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then FWQM is called a fuzzy weighted quadratic mean (FWQM) operator.

Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\text{FWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FWQM operator is reduced to the fuzzy quadratic mean (FQM) operator:

$$\text{FQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\frac{\sum_{j=1}^{n} \hat{a}_j^2}{n}\right)^{\frac{1}{2}}. \tag{13}$$

By the operational laws and Equation (12), we have

$$\begin{aligned} \text{FWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \left(\sum_{j=1}^{n} w_j \hat{a}_j^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} w_j [a_j^L, a_j^M, a_j^U]^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} w_j (a_j^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^U)^2\right)^{\frac{1}{2}}\right] \end{aligned} \tag{14}$$

and then by Equation (13), we have

$$\text{FQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left[\left(\frac{\sum_{j=1}^{n} (a_j^L)^2}{n}\right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^{n} (a_j^M)^2}{n}\right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^{n} (a_j^U)^2}{n}\right)^{\frac{1}{2}}\right] \tag{15}$$

Especially, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FWQM operator is reduced to the uncertain weighted quadratic mean (UWQM) operator:

$$\begin{aligned}\text{UWQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \left(\sum_{j=1}^{n} w_j \tilde{a}_j^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} w_j (a_j^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^U)^2\right)^{\frac{1}{2}}\right]. \qquad (16)\end{aligned}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the UWQM operator is reduced to the uncertain quadratic mean (UQM) operator:

$$\text{UQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\frac{\sum_{j=1}^{n}(\tilde{a}_j^2)}{n}\right)^{\frac{1}{2}} = \left[\left(\frac{\sum_{j=1}^{n}(a_j^L)^2}{n}\right)^{\frac{1}{2}}, \left(\frac{\sum_{j=1}^{n}(a_j^U)^2}{n}\right)^{\frac{1}{2}}\right]. \quad (17)$$

If $a_j^L = a_j^U = a_j$, for all $j = 1, 2, \ldots, n$, then Equations (16)and (17) are, respectively, reduced to the WQM operator (3) and the QM operator (4).

Example 3.4 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 3, 4]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 6]$, $\hat{a}_4 = [1, 3, 5]$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of $\hat{a}_i$ $(i = 1, 2, 3, 4)$, then by Equation (14), we have

$$\begin{aligned}\text{FWQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) &= \left[\left(\sum_{j=1}^{n} w_j (a_j^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} w_j (a_j^U)^2\right)^{\frac{1}{2}}\right] \\ &= [1.5811, 3.1464, 4.8580].\end{aligned}$$

Based on the OWA and FQM operators and Definition 3.2, we define a fuzzy ordered weighted quadratic mean (FOWQM) operator as below:

Definition 3.5 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A FOWQM operator of dimension n is a mapping FOWQM : $\Omega^n \rightarrow \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^{n} \omega_j = 1$. Furthermore,

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \left(\sum_{j=1}^{n} \omega_j \hat{a}_{\sigma(j)}^2\right)^{\frac{1}{2}}$$

$$= \left[\left(\sum_{j=1}^{n} \omega_j (a^L_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (a^M_{\sigma(j)})^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (a^U_{\sigma(j)})^2 \right)^{\frac{1}{2}} \right], \quad (18)$$

where $a_{\sigma(j)} = \left[a^L_{\sigma(j)}, a^M_{\sigma(j)}, a^U_{\sigma(j)}\right]$ $(j = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{a}_{\sigma(j-1)} \geq \hat{a}_{\sigma(j)}$ for all j.

However, if there is a tie between $\hat{a}_i$ and $\hat{a}_j$ by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average. The weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ can be determined by using some weight determining methods like the normal distribution based method, see Refs [17, 18, 19] for more details.

Similarly to the OWA operator, the FOWQM operator has the following properties:

Theorem 3.6 Let $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency**: If all $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ are equal, i.e., $\hat{a}_j = \hat{a}$, for all i, then

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness**:

$$\hat{a}^- \leq \text{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \hat{a}^+,$$

where $\hat{a}^- = [\min_j(a^L_j), \min_j(a^M_j), \min_j(a^U_j)]$, $\hat{a}^+ = [\max_j(a^L_j), \max_j(a^M_j), \max_j(a^U_j)]$.

(3) **Monotonicity**: Let $\hat{a}^*_j = [a^{L*}_j, a^{M*}_j, a^{U*}_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then if $a^L_j \leq a^{L*}_j$, $a^M_j \leq a^{M*}_j$ and $a^U_j \leq a^{U*}_j$ for all j, then

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \text{FOWQM}(\hat{a}^*_1, \hat{a}^*_2, \ldots, \hat{a}^*_n).$$

(4) **Commutativity**: Let $\hat{a}'_j = [a^{L'}_j, a^{M'}_j, a^{U'}_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then

$$\text{FOWQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \text{FOWQM}(\hat{a}'_1, \hat{a}'_2, \ldots, \hat{a}'_n),$$

where $(\hat{a}'_1, \hat{a}'_2, \ldots, \hat{a}'_n)$ is any permutation of $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FOWQM operator is reduced to the FQM operator; if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FOWQM operator is reduced to the uncertain ordered weighted quadratic mean (UOWQM) operator:

$$\begin{aligned} \text{UOWQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \left(\sum_{j=1}^{n} \omega_j \tilde{a}_{\sigma(j)}^2 \right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} \omega_j (a_{\sigma(j)}^L)^2 \right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (a_{\sigma(j)}^U)^2 \right)^{\frac{1}{2}} \right], \quad (19) \end{aligned}$$

where $\tilde{a}_\sigma(j) = [a_{\sigma(j)}^L, a_{\sigma(j)}^U]$, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all j. If there is a tie between $\tilde{a}_i$ and $\tilde{a}_j$, then we replace each of $\tilde{a}_i$ and $\tilde{a}_j$ by their average $(\tilde{a}_i + \tilde{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \ldots, n$, then the UOWQM operator is reduced to the ordered weighted quadratic mean (OWQM)operator:

$$\text{OWQM}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \omega_j b_j^2 \right)^{\frac{1}{2}} \quad (20)$$

where b_j is the jth largest of $a_j (j = 1, 2, \ldots, n)$. The OWQM operator (20) has some special cases:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\text{OWQM}(a_1, a_2, \ldots, a_n) = \max\{a_i\} = b_1. \quad (21)$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\text{OWQM}(a_1, a_2, \ldots, a_n) = \min\{a_i\} = b_n. \quad (22)$$

(3) If $\omega_j = 1,\ w_i = 0,\ i \neq j$, then

$$b_n \leq \mathrm{OWQM}(a_1, a_2, \ldots, a_n) = b_j \leq b_1. \tag{23}$$

(4) If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWQM}(a_1, a_2, \ldots, a_n) = \left(\frac{\sum_{j=1}^{n} b_j^2}{n}\right)^{\frac{1}{2}} = \left(\frac{\sum_{j=1}^{n} a_j^2}{n}\right)^{\frac{1}{2}} = \mathrm{QM}(a_1, a_2, \ldots, a_n). \tag{24}$$

Example 3.7 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [3, 4, 6]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 5]$, $\hat{a}_4 = [1, 3, 5]$, and $\hat{a}_5 = [2, 5, 7]$. To rank these triangular fuzzy numbers, we first compare each triangular fuzzy number $\hat{a}_i$ with all triangular fuzzy number $\hat{a}_j (j = 1, 2, 3, 4, 5)$ by using Equation (9) (without loss of generality, set $\sigma = 0.5$), let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$ $(j = 1, 2, 3, 4, 5)$, then we utilize these possibility degrees to construct the following matrix $P = (p_{ij})_{5\times 5}$:

$$P = \begin{pmatrix} 0.5000 & 1 & 0.6667 & 0.8750 & 0.3750 \\ 0 & 0.5000 & 0 & 0.2917 & 0 \\ 0.3333 & 1 & 0.5000 & 0.7083 & 0.2000 \\ 0.1250 & 0.7083 & 0.2917 & 0.5000 & 0.1000 \\ 0.6250 & 1 & 0.8000 & 0.9000 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 3.4167, \quad p_2 = 0.7917, \quad p_3 = 2.7417, \quad p_4 = 1.7250, \quad p_5 = 3.8250$$

and then we rank the triangular fuzzy number $\hat{a}_i$ $(i = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4, 5)$:

$$\hat{a}_{\sigma(1)} = \hat{a}_5,\ \hat{a}_{\sigma(2)} = \hat{a}_1,\ \hat{a}_{\sigma(3)} = \hat{a}_3,\ \hat{a}_{\sigma(4)} = \hat{a}_4,\ \hat{a}_{\sigma(5)} = \hat{a}_2.$$

Suppose that the weighting vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)^T$ of the FOWQM operator is $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ (derived by the normal distribution based method [17]), then by Equation (18), we get

$$\begin{aligned} &\mathrm{FOWQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) \\ &= \left[\left(\sum_{j=1}^{n} \omega_j (a_{\sigma(j)}^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (a_{\sigma(j)}^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (a_{\sigma(j)}^U)^2\right)^{\frac{1}{2}} \right] \\ &= [2.0562, 3.8496, 5.6149]. \end{aligned}$$

Clearly, the fundamental characteristic of the FWQM operator is that it considers the importance of each given triangular fuzzy number, whereas the fundamental characteristic of the FOWQM operator is the reordering step, and it weights all the ordered positions of the triangular fuzzy numbers instead of weighing the given triangular fuzzy numbers themselves. By combining the advantages of the FWQM and FOWQM operators, in the following, we develop a fuzzy hybrid quadratic mean (FHQM) operator that weights both the given triangular fuzzy numbers and their ordered positions.

Definition 3.8 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A FHQM operator of dimension n is a mapping FHQM : $\Omega^n \to \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$, such that

$$\begin{aligned}\mathrm{FHQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \left(\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^U)^2\right)^{\frac{1}{2}}\right], \end{aligned} \qquad (25)$$

where $\dot{\hat{a}}_{\sigma(j)} = \left[\dot{a}_{\sigma(j)}^L, \dot{a}_{\sigma(j)}^M, \dot{a}_{\sigma(j)}^U\right]$ is the jth largest of the weighted triangular fuzzy numbers $\dot{\hat{a}}_j$ $(\dot{\hat{a}}_j = n w_j \hat{a}_j,\ j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, and n is the balancing coefficient.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\hat{a}}_j = \hat{a}_j$, $j = 1, 2, \ldots, n$, in this case, the FHQM operator is reduced to the FOWQM operator; if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\begin{aligned}\mathrm{FHQM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \left(\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^n n w_j^2 (a_{\sigma(j)}^U)^2\right)^{\frac{1}{2}}\right] \end{aligned} \qquad (26)$$

which we call the generalized fuzzy weighted quadratic mean (GFWQM) operator.

Moreover, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FHQM operator is reduced to the uncertain hybrid quadratic mean (UHQM) operator:

$$\text{UHQM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \left(\sum_{j=1}^{n} \omega_j \dot{\tilde{a}}_{\sigma(j)}^2\right)^{\frac{1}{2}} = \left[\left(\sum_{j=1}^{n} n w_j^2 (a_{\sigma(j)}^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} n w_j^2 (a_{\sigma(j)}^U)^2\right)^{\frac{1}{2}}\right], \quad (27)$$

where $\dot{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{\tilde{a}}_j$ $(\dot{\tilde{a}}_j = n w_j \tilde{a}_j, j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\tilde{a}}_j = \tilde{a}_j$, $j = 1, 2, \ldots, n$, in this case, the UHQM operator is reduced to the UOWQM operator.

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \ldots, n$, then the UHQM operator is reduced to the hybrid quadratic mean (HQM) operator:

$$\text{HQM}(a_1, a_2, \ldots, a_n) = \left(\sum_{j=1}^{n} \omega_j \dot{a}_{\sigma(j)}^2\right)^{\frac{1}{2}}, \quad (28)$$

where $\dot{a}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{a}_j$ $(\dot{a}_j = n w_j \tilde{a}_j, j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{a}_j = a_j$, $j = 1, 2, \ldots, n$, in this case, the HQM operator is reduced to the OWQM operator.

Example 3.9 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 4, 5]$, $\hat{a}_2 = [1, 3, 4]$, $\hat{a}_3 = [2, 3, 5]$, $\hat{a}_4 = [3, 4, 5]$, and $\hat{a}_5 = [2, 5, 8]$, and $w = (0.20, 0.25, 0.15, 0.25, 0.15)^T$ be the weight vector of $\hat{a}_j$ $(j = 1, 2, 3, 4, 5)$. Then we get the weighted triangular fuzzy numbers:

$$\dot{\hat{a}}_1 = [2, 4, 5],\ \dot{\hat{a}}_2 = [1.25, 3.75, 5],\ \dot{\hat{a}}_3 = [1.5, 2.25, 3.75],$$
$$\dot{\hat{a}}_4 = [3.75, 5, 6.25],\ \dot{\hat{a}}_5 = [1.5, 3.75, 6].$$

By using Equation (11) (without loss of generality, set $\delta = 0.5$), we construct the following matrix:

$$P = \begin{pmatrix} 0.5000 & 0.5833 & 0.9545 & 0.0385 & 0.4864 \\ 0.4167 & 0.5000 & 0.8462 & 0 & 0.4154 \\ 0.0455 & 0.1538 & 0.5000 & 0 & 0.1250 \\ 0.9615 & 1 & 1 & 0.5000 & 0.8571 \\ 0.5136 & 0.5846 & 0.8750 & 0.1429 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5628,\quad p_2 = 2.1782,\quad p_3 = 0.8243,\quad p_4 = 4.3187,\quad p_5 = 2.6160$$

and then we rank the triangular fuzzy number $\hat{a}_i$ $(i = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4, 5)$:

$$\dot{\hat{a}}_{\sigma(1)} = \dot{\hat{a}}_4,\ \dot{\hat{a}}_{\sigma(2)} = \dot{\hat{a}}_5,\ \dot{\hat{a}}_{\sigma(3)} = \dot{\hat{a}}_1,\ \dot{\hat{a}}_{\sigma(4)} = \dot{\hat{a}}_2,\ \dot{\hat{a}}_{\sigma(5)} = \dot{\hat{a}}_3.$$

Suppose that $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ is the weighting vector of the FHQM operator (derived by the normal distribution based method [17]), then by Equation (25), we get

$$\begin{aligned} \mathrm{FHQM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) &= \left(\sum_{j=1}^{n} \omega_j \dot{\hat{a}}_{\sigma(j)}^2\right)^{\frac{1}{2}} \\ &= \left[\left(\sum_{j=1}^{n} \omega_j (\dot{a}_{\sigma(j)}^L)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (\dot{a}_{\sigma(j)}^M)^2\right)^{\frac{1}{2}}, \left(\sum_{j=1}^{n} \omega_j (\dot{a}_{\sigma(j)}^U)^2\right)^{\frac{1}{2}}\right] \\ &= [2.0196, 4.0166, 5.4955]. \end{aligned}$$

3.2 Fuzzy contraharmonic mean operators

Similar to the FWQM operator, based on operational laws, we extend the WCHM operator (3) to fuzzy environment:

Definition 3.10 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U](j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, and let FWCHM : $\Omega^n \to \Omega$, if

$$\begin{aligned} \text{FWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n w_j \hat{a}_j^2}{\sum_{j=1}^n w_j \hat{a}_j} \\ &= \left[\frac{\sum_{j=1}^n w_j ({a_j}^L)^2}{\sum_{j=1}^n w_j {a_j}^U}, \frac{\sum_{j=1}^n w_j ({a_j}^M)^2}{\sum_{j=1}^n w_j {a_j}^M}, \frac{\sum_{j=1}^n w_j ({a_j}^U)^2}{\sum_{j=1}^n w_j {a_j}^L}\right], \end{aligned} \tag{29}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{a}_j(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then FWCHM is called a fuzzy weighted contraharmonic mean (FWCHM) operator.

Especially, if $w_i = 1, w_j = 0, j \neq i$, then $\text{FWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FWCHM operator is reduced to the FCHM operator:

$$\begin{aligned} \text{FCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n \hat{a}_j^2}{\sum_{j=1}^n \hat{a}_j} \\ &= \left[\frac{\sum_{j=1}^n ({a_j}^L)^2}{\sum_{j=1}^n {a_j}^U}, \frac{\sum_{j=1}^n ({a_j}^M)^2}{\sum_{j=1}^n {a_j}^M}, \frac{\sum_{j=1}^n ({a_j}^U)^2}{\sum_{j=1}^n {a_j}^L}\right]. \end{aligned} \tag{30}$$

Especially,if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FWCHM operator (29) is reduced to the uncertain ordered weighted contraharmonic mean (UCHM) operator:

$$\text{UWCHM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{j=1}^n w_j \tilde{a}_j^2}{\sum_{j=1}^n w_j \tilde{a}_j} = \left[\frac{\sum_{j=1}^n w_j ({a_j}^L)^2}{\sum_{j=1}^n w_j {a_j}^U}, \frac{\sum_{j=1}^n w_j ({a_j}^U)^2}{\sum_{j=1}^n w_j {a_j}^L}\right]. \tag{31}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the UWCHM operator is reduced to the uncertain contraharmonic mean (UCHM) operator:

$$\text{UCHM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{j=1}^n \tilde{a}_j^2}{\sum_{j=1}^n \tilde{a}_j} = \left[\frac{\sum_{j=1}^n ({a_j}^L)^2}{\sum_{j=1}^n {a_j}^U}, \frac{\sum_{j=1}^n ({a_j}^U)^2}{\sum_{j=1}^n {a_j}^L}\right]. \tag{32}$$

If $a_j^L = a_j^U = a_j$, for all $j = 1, 2, \ldots, n$, then Equations (31) and (32) are respectively reduced to the WCHM operator (5) and the CHM operator(6).

Example 3.11 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 3, 4]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 6]$, $\hat{a}_4 = [1, 3, 5]$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of $\hat{a}_i (i = 1, 2, 3, 4)$, then by Equation (29), we have

$$\begin{aligned} \text{FWCHM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) &= \left[\frac{\sum_{j=1}^n w_j ({a_j}^L)^2}{\sum_{j=1}^n w_j {a_j}^U}, \frac{\sum_{j=1}^n w_j ({a_j}^M)^2}{\sum_{j=1}^n w_j {a_j}^M}, \frac{\sum_{j=1}^n w_j ({a_j}^U)^2}{\sum_{j=1}^n w_j {a_j}^L} \right] \\ &= [0.5208, 3.1935, 15.7333]. \end{aligned}$$

Based on the OWA and FCHM operators, we define a fuzzy ordered weighted contraharmonic mean (FOWCHM) operator as below:

Definition 3.12 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A FOWCHM operator of dimension n is a mapping FOWCHM : $\Omega^n \to \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$. Furthermore,

$$\begin{aligned} \text{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n \omega_j \hat{a}_{\sigma(j)}^2}{\sum_{j=1}^n w_j \hat{a}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^L)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^U}, \frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^M)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^M}, \frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^U)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^L} \right], \end{aligned} \tag{33}$$

where $a_{\sigma(j)} = \left[a_{\sigma(j)}^L, a_{\sigma(j)}^M, a_{\sigma(j)}^U \right]$ $(j = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{a}_{\sigma(j-1)} \geq \hat{a}_{\sigma(j)}$ for all j.

Similarly to the Theorem 3.6, the FOWCHM operator has the following properties:

Theorem 3.13 Let $\hat{a}_j = [a_j^L, a_j^M, a_j^U] (j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency**: If all $\hat{a}_j (j = 1, 2, \ldots, n)$ are equal, i.e., $\hat{a}_j = \hat{a}$, for all j, then

$$\text{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness**:

$$\hat{a}^- \leq \text{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \hat{a}^+,$$

where $\hat{a}^- = [\min_j(a_j^L), \min_j(a_j^M), \min_j(a_j^U)]$, $\hat{a}^+ = [\max_j(a_j^L), \max_j(a_j^M), \max_j(a_j^U)]$.

(3) **Monotonicity**: Let $\hat{a}_j^* = [a_j^{L*}, a_j^{M*}, a_j^{U*}](j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then if $a_j^L \leq a_j^{L*}, a_j^M \leq a_j^{M*}$ and $a_j^U \leq a_j^{U*}$ for all j, then

$$\text{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \text{FOWCHM}(\hat{a}_1^*, \hat{a}_2^*, \ldots, \hat{a}_n^*).$$

(4) **Commutativity**: Let $\hat{a}_j' = [a_j^{L'}, a_j^{M'}, a_j^{U'}](j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then

$$\text{FOWCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \text{FOWCHM}(\hat{a}_1', \hat{a}_2', \ldots, \hat{a}_n'),$$

where $(\hat{a}_1', \hat{a}_2', \ldots, \hat{a}_n')$ is any permutation of $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$.

Especially, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FOWCHM operator is reduced to the FCHM operator; if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FOWCHM operator is reduced to the uncertain ordered weighted contraharmonic mean (UOWCHM) operator:

$$\begin{aligned} \text{UOWCHM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{\sum_{j=1}^n \omega_j \tilde{a}_{\sigma(j)}^2}{\sum_{j=1}^n \omega_j \tilde{a}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^L)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^U}, \frac{\sum_{j=1}^n \omega_j (a_{\sigma(j)}^U)^2}{\sum_{j=1}^n \omega_j a_{\sigma(j)}^L} \right], \end{aligned} \tag{34}$$

where $\tilde{a}_\sigma(j) = [a_{\sigma(j)}^L, a_{\sigma(j)}^U]$, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\sigma(j-1)} \geq \tilde{a}_{\sigma(j)}$ for all j. If there is a tie between $\tilde{a}_i$ and $\tilde{a}_j$, then we replace each of $\tilde{a}_i$ and $\tilde{a}_j$ by their average $(\tilde{a}_i + \tilde{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $a_j^L = a_j^U = a_j$, for all $j = 1, 2, \ldots, n$, then the UOWCHM operator is reduced to the ordered weighted contraharmonic mean (OWCHM) operator:

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} \omega_j {b_j}^2}{\sum_{j=1}^{n} \omega_j b_j}, \tag{35}$$

where b_j is the jth largest of a_j $(j = 1, 2, \ldots, n)$. Then the OWCHM operator (35) has some special cases:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \max\{a_i\} = b_1. \tag{36}$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \min\{a_i\} = b_n. \tag{37}$$

(3) If $\omega_j = 1$, $\omega_i = 0$, $i \neq j$, then

$$b_n \leq \mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = b_j \leq b_1. \tag{38}$$

(4) If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} {b_j}^2}{\sum_{j=1}^{n} b_j} = \frac{\sum_{j=1}^{n} {a_j}^2}{\sum_{j=1}^{n} a_j} = \mathrm{CHM}(a_1, a_2, \ldots, a_n). \tag{39}$$

Example 3.14 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [3, 4, 6]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 5]$, $\hat{a}_4 = [1, 3, 5]$, and $\hat{a}_5 = [2, 5, 7]$. To rank these triangular fuzzy numbers, we first compare each triangular fuzzy number $\hat{a}_i$ with all triangular fuzzy number $\hat{a}_j$ $(j = 1, 2, 3, 4, 5)$ by using Equation (11) (without loss of generality, set $\sigma = 0.5$), let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$ $(j = 1, 2, 3, 4, 5)$, then we utilize these possibility degrees to construct the following matrix $P = (p_{ij})_{5\times 5}$:

$$P = \begin{pmatrix} 0.5000 & 1 & 0.6667 & 0.8750 & 0.3750 \\ 0 & 0.5000 & 0 & 0.2917 & 0 \\ 0.3333 & 1 & 0.5000 & 0.7083 & 0.2000 \\ 0.1250 & 0.7083 & 0.2917 & 0.5000 & 0.1000 \\ 0.6250 & 1 & 0.8000 & 0.9000 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 3.4167,\ p_2 = 0.7917,\ p_3 = 2.7417,\ p_4 = 1.7250,\ p_5 = 3.8250$$

and then we rank the triangular fuzzy number $\hat{a}_i$ $(i = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4, 5)$:

$$\hat{a}_{\sigma(1)} = \hat{a}_5,\ \hat{a}_{\sigma(2)} = \hat{a}_1,\ \hat{a}_{\sigma(3)} = \hat{a}_3,\ \hat{a}_{\sigma(4)} = \hat{a}_4,\ \hat{a}_{\sigma(5)} = \hat{a}_2.$$

Suppose that the weighting vector $w = (w_1, w_2, w_3, w_4)^T$ of the FOWCHM operator is $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ (derived by the normal distribution based method [17]), then by Equation (33), we get

$$\begin{aligned}
&\text{FOWCHM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5)\\
&= \left[\frac{\sum_{j=1}^n \omega_j (a^L_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j a^U_{\sigma(j)}}, \frac{\sum_{j=1}^n \omega_j (a^M_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j a^M_{\sigma(j)}}, \frac{\sum_{j=1}^n \omega_j (a^U_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j a^L_{\sigma(j)}}\right]\\
&= [0.7292, 3.7787, 15.9364].
\end{aligned}$$

Similar to the FHQM operator, by combining the advantages of the FWCHM and FOWCHM operators, in the following, we develop a fuzzy hybrid contraharmonic mean (FHCHM) operator that weights both the given triangular fuzzy numbers and their ordered positions.

Definition 3.15 Let $\hat{a}_j = [a^L_j, a^M_j, a^U_j]$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A FHCHM operator of dimension n is a mapping FHCHM : $\Omega^n \to \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_j \geq 0$ and $\sum_{j=1}^n \omega_j = 1$, such that

$$\begin{aligned}
\text{FHCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^n \omega_j \dot{\hat{a}}^2_{\sigma(j)}}{\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}}\\
&= \left[\frac{\sum_{j=1}^n \omega_j (\dot{a}^L_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j (\dot{a}^U_{\sigma(j)})^2}, \frac{\sum_{j=1}^n \omega_j (\dot{a}^M_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j (\dot{a}^M_{\sigma(j)})^2}, \frac{\sum_{j=1}^n \omega_j (\dot{a}^U_{\sigma(j)})^2}{\sum_{j=1}^n \omega_j (\dot{a}^L_{\sigma(j)})^2}\right],
\end{aligned} \tag{40}$$

where $\dot{\hat{a}}_{\sigma(j)} = \left[\dot{a}^L_{\sigma(j)}, \dot{a}^M_{\sigma(j)}, \dot{a}^U_{\sigma(j)}\right]$ is the jth largest of the weighted triangular fuzzy numbers $\dot{\hat{a}}_j$ $(\dot{\hat{a}}_j = nw_j\hat{a}_j,\ j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\hat{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\hat{a}}_j = \hat{a}_j, j = 1, 2, \ldots, n$, in this case, the FHCHM operator is reduced to the FOWCHM operator; if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\begin{aligned} \text{FHCHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{\sum_{j=1}^{n} \omega_j \dot{\hat{a}}_{\sigma(j)}^2}{\sum_{j=1}^{n} \omega_j \dot{\hat{a}}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^{n} n w_j^2 (a_{\sigma(j)}^L)^2}{\sum_{j=1}^{n} w_j (a_{\sigma(j)}^U)}, \frac{\sum_{j=1}^{n} n w_j^2 (a_{\sigma(j)}^M)^2}{\sum_{j=1}^{n} w_j (a_{\sigma(j)}^M)^2}, \frac{\sum_{j=1}^{n} n w_j^2 (a_{\sigma(j)}^U)^2}{\sum_{j=1}^{n} w_j (a_{\sigma(j)}^L)} \right] \end{aligned} \quad (41)$$

which we call the generalized fuzzy weighted contraharmonic mean (GFWCHM) operator. Moreover, if the triangular fuzzy numbers $\hat{a}_j = [a_j^L, a_j^M, a_j^U]$ $(j = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$, then the FHCHM operator is reduced to the uncertain hybrid contraharmonic mean (UHCHM) operator:

$$\text{UHCHM}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{\sum_{j=1}^{n} \omega_j \dot{\tilde{a}}_{\sigma(j)}^2}{\sum_{j=1}^{n} \omega_j \dot{\tilde{a}}_{\sigma(j)}}, \quad (42)$$

where $\dot{\tilde{a}}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{\tilde{a}}_j$ $(\dot{\tilde{a}}_j = n w_j \tilde{a}_j$, $j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\tilde{a}}_j = \tilde{a}_j, j = 1, 2, \ldots, n$, in this case, the UHCHM operator is reduced to the UOWCHM operator.

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \ldots, n$, then the UHCHM operator is reduced to the hybrid contraharmonic mean (HCHM) operator:

$$\text{HCHM}(a_1, a_2, \ldots, a_n) = \frac{\sum_{j=1}^{n} \omega_j \dot{a}_{\sigma(j)}^2}{\sum_{j=1}^{n} \omega_j \dot{a}_{\sigma(j)}}, \quad (43)$$

where $\dot{a}_{\sigma(j)}$ is the jth largest of the weighted interval numbers $\dot{a}_j$ $(\dot{a}_j = n w_j \tilde{a}_j$, $j = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$ with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, and n is the balancing coefficient. Especially, if

$w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{a}_j = a_j, j = 1, 2, \ldots, n$, in this case, the HCHM operator is reduced to the OWCHM operator.

Example 3.16 Given a collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 4, 5]$, $\hat{a}_2 = [1, 3, 4]$, $\hat{a}_3 = [2, 3, 5]$, $\hat{a}_4 = [3, 4, 5]$, and $\hat{a}_5 = [2, 5, 8]$, and let $w = (0.20, 0.25, 0.15, 0.25, 0.15)^T$ be the weight vector of $\hat{a}_j$ ($j = 1, 2, 3, 4, 5$). Then we get the weighted triangular fuzzy numbers:

$$\dot{\hat{a}}_1 = [2, 4, 5],\ \dot{\hat{a}}_2 = [1.25, 3.75, 5],\ \dot{\hat{a}}_3 = [1.5, 2.25, 3.75],$$
$$\dot{\hat{a}}_4 = [3.75, 5, 6.25],\ \dot{\hat{a}}_5 = [1.5, 3.75, 6].$$

By using Equation (11) (without loss of generality, set $\delta = 0.5$), we construct the following matrix:

$$P = \begin{pmatrix} 0.5000 & 0.5833 & 0.9545 & 0.0385 & 0.4864 \\ 0.4167 & 0.5000 & 0.8462 & 0 & 0.4154 \\ 0.0455 & 0.1538 & 0.5000 & 0 & 0.1250 \\ 0.9615 & 1 & 1 & 0.5000 & 0.8571 \\ 0.5136 & 0.5846 & 0.8750 & 0.1429 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5628,\ p_2 = 2.1782,\ p_3 = 0.8243,\ p_4 = 4.3187,\ p_5 = 2.6160$$

and then we rank the triangular fuzzy number $\hat{a}_i$ ($i = 1, 2, 3, 4, 5$) in descending order in accordance with the values of p_i ($i = 1, 2, 3, 4, 5$):

$$\dot{\hat{a}}_{\sigma(1)} = \dot{\hat{a}}_4,\ \dot{\hat{a}}_{\sigma(2)} = \dot{\hat{a}}_5,\ \dot{\hat{a}}_{\sigma(3)} = \dot{\hat{a}}_1,\ \dot{\hat{a}}_{\sigma(4)} = \dot{\hat{a}}_2,\ \dot{\hat{a}}_{\sigma(5)} = \dot{\hat{a}}_3.$$

Suppose that $\omega = (0.1117, 0.2365, 0.3036, 0.3265, 0.1117)^T$ is the weighting vector of the FHCHM operator, then by Equation (40), we get

$$\begin{aligned} \text{FHCHM}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) &= \frac{\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}^2}{\sum_{j=1}^n \omega_j \dot{\hat{a}}_{\sigma(j)}} \\ &= \left[\frac{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^L)^2}{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^U)}, \frac{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^M)^2}{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^M)}, \frac{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^U)^2}{\sum_{j=1}^n \omega_j (\dot{a}_{\sigma(j)}^L)} \right] \\ &= [0.7173, 3.9011, 15.4360]. \end{aligned}$$

4 Approaches to multiple attribute group decision making with triangular fuzzy information

For a group decision making with triangular fuzzy information, let X=$\{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives, and $G = \{G_1, G_2, \ldots, G_m\}$ be the set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$ with $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_s\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_s)^T$, where $v_k \geq 0$ and $\sum_{k=1}^{s} v_k = 1$. Suppose that $A^{(k)} = (\hat{a}_{ij}^{(k)})_{m\times n}$ is the decision matrix, where $\hat{a}_{ij}^{(k)} = \left[a_{ij}^{L(k)}, a_{ij}^{M(k)}, a_{ij}^{U(k)}\right]$ is an attribute value, which takes the form of triangular fuzzy number, of the alternative $x_j \in X$ with respect to the attribute $G_i \in G$.

Then, we utilize the FWQM and FHQM (or the FWCHM and FHCHM) operators to propose an approach to multiple attribute group decision making with triangular fuzzy information, which involves the following steps:

Step 1. Normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrix $A^{(k)}$ into a corresponding element in the matrix $R^{(k)} = (\hat{r}_{ij}^{(k)})_{m\times n}$ ($\hat{r}_{ij}^{(k)} = \left[{r_{ij}}^{L(k)}, {r_{ij}}^{M(k)}, {r_{ij}}^{U(k)}\right]$) using the following formulas:

$$\hat{r}_{ij}^{(k)} = \frac{\hat{a}_{ij}^{(k)}}{\sum_{j=1}^{n} \hat{a}_{ij}^{(k)}} = \left[\frac{{a_{ij}}^{L(k)}}{\sum_{j=1}^{n} {a_{ij}}^{U(k)}}, \frac{{a_{ij}}^{M(k)}}{\sum_{j=1}^{n} {a_{ij}}^{M(k)}}, \frac{{a_{ij}}^{U(k)}}{\sum_{j=1}^{n} {a_{ij}}^{L(k)}}\right],$$
$$\text{for benefit attribute } G_i, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots, s, \quad (44)$$

$$\hat{r}_{ij}^{(k)} = \frac{1/\hat{a}_{ij}^{(k)}}{\sum_{j=1}^{n}(1/\hat{a}_{ij}^{(k)})} = \left[\frac{1/{a_{ij}}^{U(k)}}{\sum_{j=1}^{n}(1/{a_{ij}}^{L(k)})}, \frac{1/{a_{ij}}^{M(k)}}{\sum_{j=1}^{n}(1/{a_{ij}}^{M(k)})}, \frac{1/{a_{ij}}^{L(k)}}{\sum_{j=1}^{n}(1/{a_{ij}}^{U(k)})}\right],$$
$$\text{for cost attribute } G_i, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots, s. \quad (45)$$

Step 2. Utilize the FWQM operator:

$$\hat{r}_{j}^{(k)} = \text{FWQM}(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)}) = \left(\sum_{i=1}^{m} w_i(\hat{r}_{ij}^{(k)})^2\right)^{\frac{1}{2}}$$
$$= \left[\left(\sum_{i=1}^{m} w_i(\hat{r}_{ij}^{L(k)})^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{m} w_i(\hat{r}_{ij}^{M(k)})^2\right)^{\frac{1}{2}}, \left(\sum_{i=1}^{m} w_i(\hat{r}_{ij}^{U(k)})^2\right)^{\frac{1}{2}}\right] \quad (46)$$

or the FWCHM operator:

$$\hat{r}_j^{(k)} = \text{FWCHM}(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)}) = \frac{\sum_{i=1}^{m} w_i (\hat{r}_{ij}^{(k)})^2}{\sum_{i=1}^{m} w_i \hat{r}_{ij}^{(k)}}$$
$$= \left[\frac{\sum_{i=1}^{m} w_i (\hat{r}_{ij}^{L(k)})^2}{\sum_{i=1}^{m} w_i \hat{r}_{ij}^{U(k)}}, \frac{\sum_{i=1}^{m} w_i (\hat{r}_{ij}^{M(k)})^2}{\sum_{i=1}^{m} w_i \hat{r}_{ij}^{M(k)}}, \frac{\sum_{i=1}^{m} w_i (\hat{r}_{ij}^{L(k)})^2}{\sum_{i=1}^{m} w_i \hat{r}_{ij}^{U(k)}} \right] \tag{47}$$

to aggregate all the elements in the jth column of $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 3. Utilize the FHQM operator:

$$\hat{r}_j = \text{FHQM}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(s)}) = \left(\sum_{k=1}^{s} \omega_k (\dot{\hat{r}}_j^{(\sigma(k))})^2 \right)^{\frac{1}{2}}$$
$$= \left[\left(\sum_{k=1}^{s} \omega_k (\dot{r}_j^{L(\sigma(k))})^2 \right)^{\frac{1}{2}}, \left(\sum_{k=1}^{s} \omega_k (\dot{r}_j^{M(\sigma(k))})^2 \right)^{\frac{1}{2}}, \left(\sum_{k=1}^{s} \omega_k (\dot{r}_j^{U(\sigma(k))})^2 \right)^{\frac{1}{2}} \right] \tag{48}$$

or the FHCHM operator:

$$\hat{r}_j = \text{FHCHM}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(s)}) = \frac{\sum_{k=1}^{s} \omega_k (\dot{\hat{r}}_j^{(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{\hat{r}}_j^{(\sigma(k))}}$$
$$= \left[\frac{\sum_{k=1}^{s} \omega_k (\dot{r}_j^{L(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{r}_j^{U(\sigma(k))}}, \frac{\sum_{k=1}^{s} \omega_k (\dot{r}_j^{M(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{r}_j^{M(\sigma(k))}}, \frac{\sum_{k=1}^{s} \omega_k (\dot{r}_j^{L(\sigma(k))})^2}{\sum_{k=1}^{s} \omega_k \dot{r}_j^{U(\sigma(k))}} \right] \tag{49}$$

to aggregate the overall attribute values $\hat{r}_j^{(k)}$ $(k = 1, 2, \ldots, s)$ corresponding to the decision maker d_k $(k = 1, 2, \ldots, s)$ and get the collective overall attribute value $\hat{r}_j$, where $\dot{\hat{r}}_j^{(\sigma(k))} = [\dot{r}_j^{L(\sigma(k))}, \dot{r}_j^{M(\sigma(k))}, \dot{r}_j^{U(\sigma(k))}]$ is the kth largest of the weighted data $\dot{\hat{r}}_j^{(k)}$ $(\dot{\hat{r}}_j^{(k)} = s v_k \hat{r}_j^{(k)}$, $k = 1, 2, \ldots, s)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_s)^T$ is the weighting vector of the FHQM (or FHCHM) operator, with $\omega_k \geq 0$ and $\sum_{k=1}^{s} \omega_k = 1$.

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, \ldots, n)$ by using Equation (11), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix $P = (p_{ij})_{n \times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$, and then reorder

$\hat{r}_j$ $(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_j $(j = 1, 2, \ldots, n)$.

Step 5. Rank all the alternatives x_j $(j = 1, 2, \ldots, n)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 6. End.

5 Illustrative example

In this chapter, we use a multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library(adopted from [20, 13]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives x_j $(j = 1, 2, 3, 4, 5)$ are to be evaluated using triangular fuzzy numbers by the three decision makers d_k $(k = 1, 2, 3)$ (whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under three major impacts: economic, functional, and operational. Two monetary attributes and six nonmonetary attributes (that is, G_1: owning cost ($\$/\text{ft}^2$), G_2: operating cost ($\$/\text{ft}^2$), G_3: performance $(^*)$, G_4: noise level (Db), G_5: maintainability $(^*)$, G_6: reliability (%), G_7: flexibility $(^*)$, G_8: safety $(^*)$, where * unit is from $0 - 1$ scale, three attributes G_1, G_2, and G_4 are cost attributes, and the other five attributes are benefit attributes, suppose that the weight vector of the attributes G_i $(i = 1, 2, \ldots, 8)$ is $w = (0.05, 0.08, 0.14, 0.12, 0.18, 0.21, 0.05, 0.17)^T$) emerged from three impacts is Tables 1-3.

To select the best air-conditioning system, we first utilize the approach based on the FWQM and FHQM operators, the main steps are as follows:

Step 1. By using Equations (44) and (45), we normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrices $A^{(k)}$ $(k = 1, 2, 3)$ into the corresponding element in the matrices $R^{(k)} = (\hat{r}_{ij})_{8\times 5}$ $(k = 1, 2, 3)$ (Tables 4-6):

Table 1: Triangular fuzzy number decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[3.5, 4.0, 4.7]$	$[1.7, 2.0, 2.3]$	$[3.5, 3.8, 4.2]$	$[3.5, 3.8, 4.5]$	$[3.3, 3.8, 4.0]$
G_2	$[5.5, 6.0, 6.5]$	$[4.8, 5.1, 5.5]$	$[4.5, 5.2, 5.5]$	$[4.5, 4.7, 5.0]$	$[5.5, 5.7, 6.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.5, 0.56, 0.6]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.85, 0.9]$	$[0.6, 0.7, 0.8]$
G_4	$[35, 40, 45]$	$[70, 73, 75]$	$[65, 68, 70]$	$[40, 42, 45]$	$[50, 55, 60]$
G_5	$[0.4, 0.45, 0.5]$	$[0.4, 0.44, 0.6]$	$[0.7, 0.76, 0.8]$	$[0.9, 0.97, 1.0]$	$[0.5, 0.54, 0.6]$
G_6	$[95, 98, 100]$	$[70, 73, 75]$	$[80, 83, 90]$	$[90, 93, 95]$	$[85, 90, 95]$
G_7	$[0.3, 0.35, 0.5]$	$[0.7, 0.75, 0.8]$	$[0.8, 0.9, 1.0]$	$[0.6, 0.75, 0.8]$	$[0.4, 0.5, 0.6]$
G_8	$[0.7, 0.74, 0.8]$	$[0.5, 0.53, 0.6]$	$[0.6, .68, 0.7]$	$[0.7, 0.8, 0.9]$	$[0.8, .85, 0.9]$

Step 2. Utilize the FWQM operator (46) to aggregate all elements in the jth column $R^{(K)}$ and get the overall attribute value $\hat{r}_j^{(k)}$:

$$\begin{aligned}
\hat{r}_1^{(1)} &= [0.1736, 0.2029, 0.2436]\,, \hat{r}_2^{(1)} = [0.1473, 0.1751, 0.2167]\,,\\
\hat{r}_3^{(1)} &= [0.1689, 0.1985, 0.2354]\,, \hat{r}_4^{(1)} = [0.2043, 0.2422, 0.2759]\,,\\
\hat{r}_5^{(1)} &= [0.1687, 0.1991, 0.2370]\,,\\
\hat{r}_1^{(2)} &= [0.1770, 0.2044, 0.2417]\,, \hat{r}_2^{(2)} = [0.1622, 0.1878, 0.2191]\,,\\
\hat{r}_3^{(2)} &= [0.1744, 0.1974, 0.2314]\,, \hat{r}_4^{(2)} = [0.1977, 0.2342, 0.2676]\,,\\
\hat{r}_5^{(2)} &= [0.1717, 0.1979, 0.2333]\,,\\
\hat{r}_1^{(3)} &= [0.0714, 0.0795, 0.0892]\,, \hat{r}_2^{(3)} = [0.0573, 0.0638, 0.0772]\,,\\
\hat{r}_3^{(3)} &= [0.0699, 0.0831, 0.0959]\,, \hat{r}_4^{(3)} = [0.0782, 0.0879, 0.1004]\,,\\
\hat{r}_5^{(3)} &= [0.0704, 0.0781, 0.0890]\,.
\end{aligned}$$

Step 3. Utilize the FHQM operator (48) (suppose that its weight vector is $\omega = (0.243, 0514, 0.243)^T$ determined by using the normal distribution based method [17], let $\sigma = 0.5$) to aggregate the overall attribute value $\hat{r}_j^{(k)}$ $(k = 1, 2, 3)$ corresponding to the decision maker d_k $(k = 1, 2, 3)$, and get the collective overall

Table 2: Triangular fuzzy number decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.0, 4.3, 4.5]$	$[2.1, 2.2, 2.4]$	$[5.0, 5.1, 5.2]$	$[4.3, 4.4, 4.5]$	$[3.0, 3.3, 3.5]$
G_2	$[6.0, 6.3, 6.5]$	$[5.0, 5.1, 5.2]$	$[4.5, 4.7, 5.0]$	$[5.0, 5.1, 5.3]$	$[7.0, 7.5, 8.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.4, 0.5, 0.6]$	$[0.5, .55, 0.6]$	$[0.7, 0.75, 0.8]$	$[0.7, 0.8, 0.9]$
G_4	$[37, 38, 39]$	$[70, 73, 75]$	$[65, 66, 67]$	$[40, 42, 45]$	$[50, 52, 55]$
G_5	$[0.4, 0.5, 0.6]$	$[0.5, 0.55, 0.6]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.95, 1.0]$	$[0.4, 0.44, 0.5]$
G_6	$[92, 93, 95]$	$[70, 75, 80]$	$[83, 84, 85]$	$[90, 91, 92]$	$[90, 93, 95]$
G_7	$[0.4, 0.45, 0.5]$	$[0.8, 0.85, 0.9]$	$[0.7, 0.73, 0.8]$	$[0.7, 0.85, 0.9]$	$[0.4, 0.45, 0.5]$
G_8	$[0.6, 0.7, 0.8]$	$[0.6, 0.65, 0.7]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.76, 0.8]$	$[0.7, 0.8, 0.9]$

Table 3: Triangular fuzzy number decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.3, 4.4, 4.6]$	$[2.2, 2.4, 2.5]$	$[4.5, 4.8, 5.0]$	$[4.7, 4.9, 5.0]$	$[3.1, 3.2, 3.4]$
G_2	$[6.4, 6.7, 7.0]$	$[5.0, 5.2, 5.5]$	$[4.7, 4.8, 4.9]$	$[5.5, 5.7, 6.0]$	$[6.0, 6.5, 7.0]$
G_3	$[0.8, 0.85, 0.9]$	$[0.5, 0.6, 0.7]$	$[0.6, 0.7, 0.8]$	$[0.7, 0.8, 0.9]$	$[0.7, 0.75, 0.8]$
G_4	$[36, 38, 40]$	$[72, 73, 75]$	$[67, 68, 70]$	$[45, 48, 50]$	$[55, 57, 60]$
G_5	$[0.4, 0.46, 0.5]$	$[0.4, 0.45, 0.6]$	$[0.8, 0.95, 1.0]$	$[0.8, 0.85, 0.9]$	$[0.5, 0.55, 0.6]$
G_6	$[93, 94, 95]$	$[77, 78, 80]$	$[85, 87, 90]$	$[90, 94, 95]$	$[90, 96, 100]$
G_7	$[0.4, 0.5, 0.6]$	$[0.8, 0.9, 1.0]$	$[0.8, 0.86, 0.9]$	$[0.6, 0.7, 0.8]$	$[0.5, 0.57, 0.6]$
G_8	$[0.7, 0.78, 0.8]$	$[0.5, 0.55, 0.6]$	$[0.6, 0.68, 0.7]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.85, 0.9]$

Table 4: Normalized triangular fuzzy number decision matrix $R^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.12, 0.16, 0.21]$	$[0.25, 0.32, 0.43]$	$[0.14, 0.17, 0.21]$	$[0.13, 0.17, 0.21]$	$[0.14, 0.17, 0.22]$
G_2	$[0.15, 0.18, 0.21]$	$[0.18, 0.21, 0.24]$	$[0.18, 0.20, 0.25]$	$[0.20, 0.23, 0.25]$	$[0.16, 0.19, 0.21]$
G_3	$[0.18, 0.23, 0.30]$	$[0.13, 0.16, 0.20]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.24, 0.30]$	$[0.15, 0.20, 0.27]$
G_4	$[0.22, 0.26, 0.32]$	$[0.13, 0.14, 0.16]$	$[0.14, 0.15, 0.17]$	$[0.22, 0.25, 0.28]$	$[0.16, 0.19, 0.23]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.20, 0.24, 0.28]$	$[0.26, 0.31, 0.34]$	$[0.14, 0.17, 0.21]$
G_6	$[0.21, 0.22, 0.24]$	$[0.15, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.23]$	$[0.19, 0.21, 0.23]$
G_7	$[0.08, 0.11, 0.18]$	$[0.19, 0.23, 0.29]$	$[0.22, 0.28, 0.36]$	$[0.16, 0.23, 0.29]$	$[0.11, 0.15, 0.21]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.19, 0.21]$	$[0.18, 0.22, 0.27]$	$[0.21, 0.24, 0.27]$

attribute value $\hat{r}_j$:

$$\hat{r}_1 = [0.1568, 0.1818, 0.2160], \hat{r}_2 = [0.1385, 0.1619, 0.1939],$$
$$\hat{r}_3 = [0.1536, 0.1771, 0.2086], \hat{r}_4 = [0.1791, 0.2119, 0.2417],$$
$$\hat{r}_5 = [0.1523, 0.1771, 0.2095].$$

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, 3, 4, 5)$ by using Equation (11) (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix:

$$P = \begin{pmatrix} 0.5 & 0.8558 & 0.5869 & 0.0553 & 0.5882 \\ 0.1442 & 0.5 & 0.2209 & 0 & 0.2301 \\ 0.4131 & 0.7791 & 0.5 & 0 & 0.5031 \\ 0.9447 & 1 & 1 & 0.5 & 1 \\ 0.4118 & 0.7699 & 0.4969 & 0 & 0.5 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5861,\ p_2 = 1.0952,\ p_3 = 2.1953,\ p_4 = 4.4447,\ p_5 = 2.1786$$

and then we reorder $\hat{r}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_j $(j = 1, 2, 3, 4, 5)$:

$$\hat{r}_4 > \hat{r}_1 > \hat{r}_3 > \hat{r}_5 > \hat{r}_2.$$

Table 5: Normalized triangular fuzzy number decision matrix $R^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.16, 0.19]$	$[0.28, 0.32, 0.36]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.16, 0.17]$	$[0.19, 0.21, 0.25]$
G_2	$[0.17, 0.18, 0.19]$	$[0.21, 0.22, 0.23]$	$[0.21, 0.24, 0.26]$	$[0.20, 0.22, 0.23]$	$[0.13, 0.15, 0.17]$
G_3	$[0.18, 0.24, 0.30]$	$[0.11, 0.15, 0.20]$	$[0.13, 0.16, 0.20]$	$[0.18, 0.22, 0.27]$	$[0.18, 0.24, 0.30]$
G_4	$[0.25, 0.27, 0.29]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.15, 0.16]$	$[0.22, 0.24, 0.27]$	$[0.18, 0.20, 0.21]$
G_5	$[0.11, 0.15, 0.21]$	$[0.14, 0.17, 0.21]$	$[0.22, 0.26, 0.31]$	$[0.22, 0.29, 0.34]$	$[0.11, 0.13, 0.17]$
G_6	$[0.21, 0.21, 0.22]$	$[0.16, 0.17, 0.19]$	$[0.19, 0.19, 0.20]$	$[0.20, 0.21, 0.22]$	$[0.20, 0.21, 0.22]$
G_7	$[0.11, 0.14, 0.17]$	$[0.22, 0.26, 0.30]$	$[0.19, 0.22, 0.27]$	$[0.19, 0.26, 0.30]$	$[0.19, 0.14, 0.17]$
G_8	$[0.15, 0.20, 0.26]$	$[0.15, 0.19, 0.23]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.22, 0.26]$	$[0.18, 0.23, 0.29]$

Table 6: Normalized triangular fuzzy number decision matrix $R^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.17, 0.18]$	$[0.28, 0.30, 0.35]$	$[0.14, 0.15, 0.17]$	$[0.14, 0.15, 0.16]$	$[0.20, 0.23, 0.25]$
G_2	$[0.16, 0.17, 0.19]$	$[0.20, 0.22, 0.24]$	$[0.22, 0.24, 0.25]$	$[0.18, 0.20, 0.22]$	$[0.16, 0.17, 0.20]$
G_3	$[0.20, 0.23, 0.27]$	$[0.12, 0.16, 0.21]$	$[0.15, 0.19, 0.24]$	$[0.17, 0.22, 0.27]$	$[0.17, 0.20, 0.24]$
G_4	$[0.26, 0.28, 0.31]$	$[0.14, 0.15, 0.16]$	$[0.15, 0.16, 0.17]$	$[0.21, 0.22, 0.25]$	$[0.17, 0.19, 0.20]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.20, 0.24, 0.28]$	$[0.26, 0.31, 0.34]$	$[0.14, 0.17, 0.21]$
G_6	$[0.21, 0.22, 0.24]$	$[0.15, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.23]$	$[0.19, 0.21, 0.23]$
G_7	$[0.08, 0.11, 0.18]$	$[0.19, 0.23, 0.29]$	$[0.22, 0.28, 0.36]$	$[0.16, 0.23, 0.29]$	$[0.11, 0.15, 0.21]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.19, 0.21]$	$[0.18, 0.22, 0.27]$	$[0.21, 0.24, 0.27]$

Step 5. Rank all the alternatives x_j $(j = 1,2,3,4,5)$ by the ranking of $\hat{r}_j$ $(j = 1,2,3,4,5)$:

$$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$$

and thus the most desirable alternative is x_4.

Next, we utilize the approach based on the FWCHM and FHCHM operators to select best alternative(s), the main steps are as follows:

Step $1'$**.** See Step 1.

Step $2'$**.** Utilize the FWCHM operator (47) to aggregate all elements in the jth column $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$:

$$\hat{r}_1^{(1)} = [0.1263, 0.2076, 0.3530], \hat{r}_2^{(1)} = [0.1040, 0.1804, 0.3272],$$
$$\hat{r}_3^{(1)} = [0.1234, 0.2013, 0.3325], \hat{r}_4^{(1)} = [0.1528, 0.2452, 0.3777],$$
$$\hat{r}_5^{(1)} = [0.1208, 0.2007, 0.3376],$$
$$\hat{r}_1^{(2)} = [0.1316, 0.2084, 0.3413], \hat{r}_2^{(2)} = [0.1229, 0.1928, 0.3057],$$
$$\hat{r}_3^{(2)} = [0.1346, 0.2016, 0.3138], \hat{r}_4^{(2)} = [0.1483, 0.2365, 0.3638],$$
$$\hat{r}_5^{(2)} = [0.1293, 0.2021, 0.3229],$$
$$\hat{r}_1^{(3)} = [0.1479, 0.2093, 0.3008], \hat{r}_2^{(3)} = [0.1196, 0.1815, 0.2989],$$
$$\hat{r}_3^{(3)} = [0.1355, 0.2170, 0.3376], \hat{r}_4^{(3)} = [0.1530, 0.2235, 0.3340],$$
$$\hat{r}_5^{(3)} = [0.1409, 0.2004, 0.2921],$$

Step $3'$**.** Utilize the FHCHM operator (49) (suppose that its weight vector is $\omega = (0.243, 0.514, 0.243)^T$ determined by using the normal distribution based method [17], let $\delta = 0.5$) to aggregate the overall attribute value $\hat{r}_j^{(k)}(k = 1,2,3)$ corresponding to the decision maker $d_k(k = 1,2,3)$,and get the collective overall attribute value $\hat{r}_j$:

$$\hat{r}_1 = [0.0634, 0.2542, 1.0615], \hat{r}_2 = [0.0507, 0.2243, 1.0426],$$
$$\hat{r}_3 = [0.0631, 0.2484, 0.9940], \hat{r}_4 = [0.0774, 0.2948, 1.0861],$$
$$\hat{r}_5 = [0.0618, 0.2457, 1.0001].$$

Step $4'$**.** Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, 3, 4, 5)$ by using Equation (11) (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct a possibility matrix:

$$P = \begin{pmatrix} 0.5 & 0.5367 & 0.5158 & 0.4563 & 0.5179 \\ 0.4633 & 0.5 & 0.4785 & 0.4201 & 0.4806 \\ 0.4842 & 0.5215 & 0.5 & 0.4397 & 0.5021 \\ 0.5437 & 0.5799 & 0.5603 & 0.5 & 0.5622 \\ 0.4821 & 0.5194 & 0.4979 & 0.4378 & 0.5 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.5267, p_2 = 2.3426, p_3 = 2.4475, p_4 = 2.7460, p_5 = 2.4372$$

and then we reorder $\hat{r}_j (j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of $p_j (j = 1, 2, 3, 4, 5)$:

$$\hat{r}_4 > \hat{r}_1 > \hat{r}_3 > \hat{r}_5 > \hat{r}_2.$$

Step $5'$**.** Rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_4 \succ x_1 \succ x_3 \succ x_5 \succ x_2$$

and thus the most desirable alternative is x_4.

From the above analysis, the results obtained by the proposed approach are very similar to the ones obtained Xu's approach [13], but our approach is more flexible than that of Xu [13] because it can provide the decision makers more choices as parameters are assigned different values.

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