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Generalized Fuzzy Bonferroni Harmonic Mean Operators and Their Applications in Group Decision Making

by

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Graduate School of Education

Pukyong National University

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(일반화된 퍼지 Bonferroni 조화평균 연산자와 집단의사결정에의 응용)

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CONTENTS

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요 약

Bonferroni 평균 (BM) 연산자는 집계 인수의 상관관계를 반영하는 중요한 집성 기술이다. Sun과 Sun [24]이 BM과 조화평균 연산자에 기초하여 퍼지 Bonferroni 조화 평균 (FBHM) 및 퍼지 ordered Bonferroni 조화 평균 (FOBHM) 연산자를 소개하였다.

본 논문에서는 FBHM과 FOBHM 연산자들의 성질들을 조사하고, 어떤 두 가지 집계 인수 대신에 세 가지 집계 인수의 상관관계를 고려하여 일반화된 퍼지 weighted Bonferroni 조화평균 (GFWBHM) 연산자와 일반화된 퍼지 ordered wighted Bonferroni 조화평균 (GFOWBHM) 연산자를 개발하기 위해 기존의 연산자들을 확장한다. 특히, 이러한 모든 연산자는 집계 인수가 구간 또는 실수일 경우로 축소할 수 있다. 그리고 GFWBHM 및 GFOWBHM 연산자에 따라, 여러 속성을 가진 집단의 의사결정에 접근방법을 제시하고 실제 예제를 통하여 제안된 접근방법의 효율성을 보였다.

1 Introduction

Multiple attribute group decision making (MAGDM) is the common phenomenon in modern life, which is to select the optimal alternative(s) from several alternatives or to get their ranking by aggregating the performances of each alternative under several attributes, in which the aggregation techniques play an important role. Considering the relationships among the aggregated arguments, we can classify the aggregation techniques into two categories, the ones which consider the aggregated argument dependently. For the first category, the well-known ordered weighted averaging (OWA) operator [1,2] is the representative, on the basis of which a lot of generalizations have been developed, such as the ordered weighted geometric (OWG) operator [3,4,5], the ordered weighted harmonic mean (OWHM) operator [6], the continuous ordered weighted averaging (C-OWA) operator [7], the continuous ordered weighted geometric(C-OWG)operator [8], and so on. The second category can reduce to two subcategories: the first subcategory focuses on changing the weight vector of the aggregation operators, such as the Choquet integral-based aggregation operators [9], in which the correlations of the aggregated arguments are measured subjectively by the decision makers, and the representatives of another subcategory are the power averaging (PA) operator [11] and the power geometric (PG) operator [12], both of which allow the aggregated arguments to support each other in aggregation process, on the basis of which the weighted vector is determined. The second subcategory focuses on the aggregated arguments such as the Bonferroni meas (BM) operator [13]. Yager [14] provided an interpretation of BM operator as involving a product of each argument with the average of the other arguments, a combined averaging and "anding" operator. Beliakov et al. [15] presented a composed aggregation technique called the generalized Bonferroni mean (GBM) operator, which models the average of the conjunctive expressions and the average of remaining. In fact, they extended the BM operator by considering the correlations of any three aggregated arguments instead of any two. However, both the BM operator and the GBM operator ignore some aggregation information and the weight vector of the aggregated arguments. To overcome this drawback, Xia et al. [16] developed the generalized weighted Bonferroni mean (GWBM) operator as the weighted version of the GBM operator. Based on the GBM operator and geometric mean operator, they also developed the generalized Bonferroni geometric mean (GBGM) operator. The fundamental characteristic of the GWBM operator is that it focuses on the group opinions, while the GBGM operator gives more importance to the individual opinions. Because of the usefulness of the aggregation techniques, which reflect the correlations of arguments, most of them have been extended to fuzzy, intuitionistic fuzzy or hesitant fuzzy environment [17-21].

Harmonic mean is the reciprocal of arithmetic mean of reciprocal, which is a conservative average to be used to provide for aggregation lying between the max and min operators, and is widely used as a tool to aggregate central tendency data [22]. In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data information. However, in many situations, the input arguments take the form of triangular fuzzy numbers because of time pressure, lack of knowledge and people's limited expertise related with problem domain. Therefore, "How to aggregate fuzzy data by using the harmonic mean?" is an interesting research topic and is worth paying attention to. So Xu [22] developed the fuzzy harmonic mean operators such as fuzzy weighted harmonic mean (FWHM) operator, fuzzy ordered weighted harmonic mean(FOWHM) operator and fuzzy hybrid harmonic mean (FHHM) operator, and applied them to MAGDM, Wei [23] developed fuzzy induced ordered weighted harmonic mean (FIOWHM) operator and then, based on the FWHM and FIOWHM operators, presented the approach to MAGDM. Sun and Sun [24] further applied the BM operator to fuzzy environment, and introduced the fuzzy Bonferroni harmonic mean (FBHM) operator and the fuzzy ordered Bonferroni harmonic mean (FOBHM) operator, and applied the FOBHM operator to multiple attribute decision making. In this thesis, we will develop some new harmonic aggregation operators, including the generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator and generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator, and apply them to MAGDM.

In order to do this, the remainder of this thesis is arranged in following sections. Chapter 2 first review some aggregation operators, including the BM, GBM, and GWBM operators. The some basis concepts related to triangular fuzzy numbers and some operational laws of triangular fuzzy numbers are introduced. The desirable properties of the FBHM and FOBHM operators are discussed. We extend them, by considering the correlations of any three aggregated arguments instead of any two, to develop generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator and generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator. Especially, all these operators can be reduced to aggregate interval or real numbers. Chapter 3 presents an approach to MAGDM based on the GFWBHM and GFOWBHM operators. Chapter 4 illustrates the presented approach with a practical example, and verify and show the advantages of the presented approach. Chapter 5 ends the paper with some concluding remarks.

2 Generalized fuzzy Bonferroni harmonic mean operators

The Bonferroni mean operator was initially proposed by Bonferroni [13] and was also investigated intensively by Yager [14]:

Definition 2.1 Let $p, q \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{BM}^{p,q}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}}, \tag{1}$$

then $\mathrm{BM}^{p,q}$ is called the Bonferroni mean (BM) operator.

Beliakov et al. [15] further extended the BM operator by considering the correlations of any three aggregated arguments instead of any two.

Definition 2.2 Let $p, q, r \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers. If

$$\mathrm{GBM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n(n-1)(n-2)} \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{n} a_i^p a_j^q a_k^r \right)^{\frac{1}{p+q+r}}, \tag{2}$$

then $\mathrm{GBM}^{p,q,r}$ is called the generalized Bonferroni mean (GBM) operator.

Especially, if $r = 0$, then the GBM operator reduces to the BM operator. However, it is noted that both the BM operator and the GBM operator do not consider the situation that $i = j$ or $j = k$ or $i = k$, and the weight vector of the aggregated arguments is not also considered. To overcome this drawback, Xia et al. [16] defined the weighted version of the GBM operator.

Definition 2.3 Let $p, q, r \geq 0$ and a_i $(i = 1, 2, \ldots, n)$ be a collection of nonnegative numbers with the weight vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_i > 0,\ i =$

$1, 2, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{GWBM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \left(\sum_{i,j,k=1}^{n} w_i w_j w_k a_i^p a_j^q a_k^r \right)^{\frac{1}{p+q+r}}, \tag{3}$$

then $\mathrm{GWBM}^{p,q,r}$ is called the generalized weighted Bonferroni mean (GWBM) operator.

Some special cases can be obtained as the change of the parameters:

(1) If $r = 0$, then the GWBM operator is reduced to the following:

$$\begin{aligned} \mathrm{GWBM}^{p,q,0}(a_1, a_2, \ldots, a_n) &= \left(\sum_{i,j,k=1}^{n} w_i w_j w_k a_i^p a_j^q \right)^{\frac{1}{p+q}} \\ &= \left(\sum_{i=1}^{n} w_i w_j a_i^p a_j^q \sum_{k=1}^{n} w_k \right)^{\frac{1}{p+q}} \\ &= \left(\sum_{i=1}^{n} w_i w_j a_i^p a_j^q \right)^{\frac{1}{p+q}}, \end{aligned} \tag{4}$$

which is the weight Bonferroni mean (WBM) operator.

(2) If $q = 0$ and $r = 0$, then the GWBM operator is reduced to the following:

$$\begin{aligned} \mathrm{GWBM}^{p,0,0}(a_1, a_2, \ldots, a_n) &= \left(\sum_{i,j,k=1}^{n} w_i w_j w_k a_i^p \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^{n} w_i a_i^p \sum_{j=1}^{n} w_j \sum_{k=1}^{n} w_k \right)^{\frac{1}{p}} \\ &= \left(\sum_{i=1}^{n} w_i a_i^p \right)^{\frac{1}{p}}, \end{aligned} \tag{5}$$

which is the generalized weighted averaging operator. Furthermore, in this case, let us look at the GWBM operator for some special cases of p.

1) If $p = 1$, the GWBM operator is reduced to the weighted averaging (WA) operator.

2) If $p \to 0$, then the GWBM operator is reduced to the weighted geometric (WG) operator.

3) If $p \to +\infty$, then the GWBM operator is reduced to the max operator.

The above aggregation techniques can only deal with the situation that the arguments are represented by the exact numerical values, but are invalid if the aggregation information is given in other forms, such as triangular fuzzy number [25], which is a widely used tool to deal with uncertainty and fuzzyness, described as follows:

Definition 2.4 [25] A triangular fuzzy number $\hat{a}$ can be defined by a triplet $[a^L, a^M, a^U]$. The membership function $\mu_{\hat{a}}(x)$ is defined as:

$$\mu_{\hat{a}}(x) = \begin{cases} 0, & \text{x} < a^L; \\ \frac{x-a^L}{a^M-a^L}, & a^L \leq x \leq a^M; \\ \frac{x-a^U}{a^M-a^U}, & a^M \leq x \leq a^U; \\ 0, & \text{x} > a^U, \end{cases}$$

where $a^U \geq a^M \geq a^L \geq 0$, a^L and a^U stand for the lower and upper values of $\hat{a}$, respectively, and a^M stands for the modal value [25]. Especially, if any two of a^L, a^M and a^U are equal, then $\hat{a}$ is reduced to an interval number; if all a^L, a^M and a^U are equal, then $\hat{a}$ is reduced to a real number. For convenience, we let Ω be the set of all triangular fuzzy numbers. Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then some operational laws defined as follows [25]:

1) $\hat{a} + \hat{b} = [a^L, a^M, a^U] + [b^L, b^M, b^U] = [a^L + b^L, a^M + b^M, a^U + b^U]$;

2) $\lambda\hat{a} = \lambda[a^L, a^M, a^U] = [\lambda a^L, \lambda a^M, \lambda a^U]$;

3) $\hat{a} \times \hat{b} = [a^L, a^M, a^U] \times [b^L, b^M, b^U] = [a^L b^L, a^M b^M, a^U b^U]$;

4) $\frac{1}{\hat{a}} = \frac{1}{[a^L, a^M, a^U]} = [\frac{1}{a^U} \frac{1}{a^M} \frac{1}{a^L}]$.

In order to compare two triangular fuzzy numbers, Xu [22] provided the following definition:

Definition 2.5 Let $\hat{a} = [a^L, a^M, a^U]$ and $\hat{b} = [b^L, b^M, b^U]$ be two triangular fuzzy numbers, then the degree of possibility of $\hat{a} \geq \hat{b}$ is defined as follows:

$$p(\hat{a} \geq \hat{b}) = \delta \max\left\{1 - \max\left(\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0\right), 0\right\}$$
$$+(1-\delta)\max\left\{1 - \max\left(\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0\right), 0\right\}, \;\; \delta \in [0,1] \quad (6)$$

which satisfies the following properties:

$$0 \leq p(\hat{a} \geq \hat{b}) \leq 1, \;\; p(\hat{a} \geq \hat{a}) = 0.5, \;\; p(\hat{a} \geq \hat{b}) + p(\hat{b} \geq \hat{a}) = 1. \quad (7)$$

Here, δ reflects the decision maker's risk-bearing attitude. If $\delta > 0.5$, then the decision maker is risk lover; If $\delta = 0.5$, then the decision maker is neutral to risk; If $\delta < 0.5$, then the decision maker is risk avertor.

In the following, we shall give a simple procedure for ranking of the triangular fuzzy numbers $\hat{a}_i$ $(i = 1, 2, \ldots, n)$. First, by using (6), we compare each $\hat{a}_i$ with all the $\hat{a}_j$ $(j = 1, 2, \ldots, n)$, for simplicity, let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$, then we develop a possibility matrix [26,27] as

$$\mathrm{P} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ & & \vdots & \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}, \quad (8)$$

where $p_{ij} \geq 0, \;\; p_{ij} + p_{ji} = 1, \;\; p_{ii} = \frac{1}{2}, \;\; i, j = 1, 2, \ldots, n.$

Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$. Then, in accordance with the values of p_i $(i = 1, 2, \ldots, n)$, we rank the $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ in descending order.

To aggregate the triangular fuzzy correlated information, based on the BM and weighted harmonic mean operators, Sun and Sun [24] developed the fuzzy Bonferroni harmonic mean operator. Because this operator consider the weight vector of the aggregated arguments, we redefine this operator as fuzzy weighted Bonferroni harmonic mean operator:

Definition 2.6 [24] Let $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{a}_i$ $(i = 1, 2, \ldots, n)$, where w_i indicates the importance degree of $\hat{a}_i$, satisfying $w_i > 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n w_i = 1$. If

$$\mathrm{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j=1}^n \frac{w_i w_j}{\hat{a}_i^p \hat{a}_j^q}\right)^{\frac{1}{p+q}}}$$

$$= \left[\frac{1}{\left(\sum_{i,j=1}^n \frac{w_i w_j}{(a_i^L)^p (a_j^L)^q}\right)^{\frac{1}{p+q}}} + \frac{1}{\left(\sum_{i,j=1}^n \frac{w_i w_j}{(a_i^M)^p (a_j^M)^q}\right)^{\frac{1}{p+q}}} + \frac{1}{\left(\sum_{i,j=1}^n \frac{w_i w_j}{(a_i^U)^p (a_j^U)^q}\right)^{\frac{1}{p+q}}}\right] \tag{9}$$

where $p, q \geq 0$, then $\mathrm{FWBHM}^{p,q}$ is called the fuzzy weighted Bonferroni harmonic mean (FWBHM) operator.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FWBHM operator is reduced to the following:

$$\mathrm{FBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\frac{1}{n^2}\sum_{i,j=1}^n \frac{1}{\hat{a}_i^p \hat{a}_j^q}\right)^{\frac{1}{p+q}}}, \tag{10}$$

which we call the fuzzy Bonferroni harmonic mean (FBHM) operator.

In addition, a special case can obtained as the change of parameter: If $q = 0$, then the FWBHM operator is reduced to the following:

$$\mathrm{FWBHM}^{p,0}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j=1}^n \frac{w_i w_j}{\hat{a}_i^p}\right)^{\frac{1}{p}}} = \frac{1}{\left(\sum_{i=1}^n \frac{w_i}{\hat{a}_i^p} \sum_{j=1}^n w_j\right)^{\frac{1}{p}}}$$

$$= \frac{1}{\left(\sum_{i=1}^n \frac{w_i}{\hat{a}_i^p}\right)^{\frac{1}{p}}}$$

$$= \left[\frac{1}{\left(\sum_{i=1}^{n} \frac{w_i}{(a_i^L)^p}\right)^{\frac{1}{p}}}, \frac{1}{\left(\sum_{i=1}^{n} \frac{w_i}{(a_i^M)^p}\right)^{\frac{1}{p}}}, \frac{1}{\left(\sum_{i=1}^{n} \frac{w_i}{(a_i^U)^p}\right)^{\frac{1}{p}}} \right], \tag{11}$$

which we call the fuzzy weighted generalized harmonic mean (FWGHM) operator.

On the bases of the operational laws of triangular fuzzy numbers, the FWBHM operator has the following properties:

Theorem 2.7 Let $p, q \geq 0$, and $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency**: If all $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ are equal, i.e., $\hat{a}_i = \hat{a}$, for all i, then

$$\text{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness**:

$$\hat{a}^- \leq \text{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \hat{a}^+,$$

where $\hat{a}^- = [\min_i(a_i^L), \min_i(a_i^M), \min_i(a_i^U)]$, $\hat{a}^+ = [\max_i(a_i^L), \max_i(a_i^M), \max_i(a_i^U)]$.

(3) **Monotonicity**: Let $\hat{a}_i^* = [a_i^{L*}, a_i^{M*}, a_i^{U*}]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then if $a_i^L \leq a_i^{L*}$, $a_i^M \leq a_i^{M*}$ and $a_I^M \leq a_i^{U*}$ for all i, then

$$\text{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \text{FWBHM}^{p,q}(\hat{a}_1^*, \hat{a}_2^*, \ldots, \hat{a}_n^*).$$

(4) **Commutativity**: Let $\hat{a}_i' = [a_i^{L\prime}, a_i^{M\prime}, a_i^{U\prime}]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then

$$\text{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \text{FWBHM}^{p,q}(\hat{a}_1', \hat{a}_2', \ldots, \hat{a}_n'),$$

where $(\hat{a}_1', \hat{a}_2', \ldots, \hat{a}_n')$ is any permutation of $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$.

Proof. (2) and (4) are can be proven easily, and we only prove (1) and (3).

(1) Since $\hat{a}_i = \hat{a}$, we have

$$\begin{aligned}
\mathrm{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\hat{a}^p \hat{a}^q}\right)^{\frac{1}{p+q}}} = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\hat{a}^{p+q}}\right)^{\frac{1}{p+q}}} \\
&= \frac{\hat{a}}{\left(\sum_{i=1}^{n} w_i \sum_{j=1}^{n} w_j\right)^{\frac{1}{p+q}}} \\
&= \hat{a}.
\end{aligned}$$

(3) Since $a_i^L \le a_i^{L*}$, $a_i^M \le a_i^{M*}$ and $a_I^M \le a_i^{U*}$ for all i, then $(a_i^L)^p (a_j^L)^q \le (a_i^{L*})^p (a_j^{L*})^q$, $(a_i^M)^p (a_j^M)^q \le (a_i^{M*})^p (a_j^{M*})^q$ and $(a_i^U)^p (a_j^U)^q \le (a_i^{U*})^p (a_j^{U*})^q$. Hence we have

$$\begin{aligned}
&\mathrm{FWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\hat{a}_i^p \hat{a}_j^q}\right)^{\frac{1}{p+q}}} \\
&= \left[\frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^L)^p (a_j^L)^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^M)^p (a_j^M)^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^U)^p (a_j^U)^q}\right)^{\frac{1}{p+q}}} \right] \\
&\le \left[\frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^{L*})^p (s_j^{L*})^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^{M*})^p (a_j^{M*})^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^{U*})^p (a_j^{U*})^q}\right)^{\frac{1}{p+q}}} \right] \\
&= \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(\hat{a}_i^*)^p (\hat{a}_j^*)^q}\right)^{\frac{1}{p+q}}} = \mathrm{FWBHM}^{p,q}(\hat{a}_1^*, \hat{a}_2^*, \ldots, \hat{a}_n^*).
\end{aligned}$$

Especially, if the triangular fuzzy numbers $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_i = [a_i^L, a_i^U]$ $(i = 1, 2, \ldots, n)$, then the FWBHM operator (9) is reduced to the uncertain weighted Bonferroni harmonic mean (UWBHM) operator:

$$\mathrm{UWBHM}^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\tilde{a}_i^p \tilde{a}_j^q}\right)^{\frac{1}{p+q}}}$$

$$= \left[\frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^L)^p (a_j^L)^q} \right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{(a_i^U)^p (a_j^U)^q} \right)^{\frac{1}{p+q}}} \right]. \tag{12}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the UWBHM operator is reduced to the uncertain Bonferroni harmonic mean (UBHM) operator:

$$\text{UBHM}^{p,q}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\left(\frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{\tilde{a}_i^p \tilde{a}_j^q} \right)^{\frac{1}{p+q}}}$$

$$= \left[\frac{1}{\left(\frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{(a_i^L)^p (a_j^L)^q} \right)^{\frac{1}{p+q}}}, \frac{1}{\left(\frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{(a_i^U)^p (a_j^U)^q} \right)^{\frac{1}{p+q}}} \right]. \tag{13}$$

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \ldots, n$, then the UWBHM operator (12) is reduced to the weighted Bonferroni harmonic mean (WBHM) operator:

$$\text{WBHM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{a_i^p a_j^q} \right)^{\frac{1}{p+q}}}. \tag{14}$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WBHM operator is reduced to the Bonferroni harmonic mean (BHM) operator:

$$\text{BHM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\frac{1}{n^2} \sum_{i,j=1}^{n} \frac{1}{a_i^p a_j^q} \right)^{\frac{1}{p+q}}}. \tag{15}$$

Example 2.8 Given a Collection of triangular fuzzy numbers: $\hat{a}_1 = [2, 3, 4]$, $\hat{a}_2 = [1, 2, 4]$, $\hat{a}_3 = [2, 4, 6]$, $\hat{a}_4 = [1, 3, 5]$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of $\hat{a}_i$ $(i = 1, 2, 3, 4)$, then by FWBHM operator (9) (let $p = q = 2$), we have

$$\text{FWBHM}^{2,2}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4)$$

$$= \left[\frac{1}{\left(\sum_{i,j=1}^{4} \frac{w_i w_j}{(a_i^L)^2 (a_j^L)^2}\right)^{\frac{1}{4}}}, \frac{1}{\left(\sum_{i,j=1}^{4} \frac{w_i w_j}{(a_i^M)^2 (a_j^M)^2}\right)^{\frac{1}{4}}}, \frac{1}{\left(\sum_{i,j=1}^{4} \frac{w_i w_j}{(a_i^U)^2 (a_j^U)^2}\right)^{\frac{1}{4}}} \right]$$
$$= [1.27, 2.95, 4.64].$$

Based on the OWA and FWBHM operators and Definition 2.5, we define fuzzy ordered weight Bonferroni harmonic mean (FOWBHM) operator as below:

Definition 2.9 Let $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. For $p, q \geq 0$, a fuzzy ordered weighted Bonferroni harmonic mean (FOWBHM) operator of dimension n is a mapping $\text{FOWBHM}^{p,q} : \Omega^n \to \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Furthermore,

$$\text{FOWBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{\hat{a}_{\sigma(i)}^p \hat{a}_{\sigma(j)}^q}\right)^{\frac{1}{p+q}}}$$
$$= \left[\frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{(a_{\sigma(i)}^L)^p (a_{\sigma(j)}^L)^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{(a_{\sigma(i)}^M)^p (a_{\sigma(j)}^M)^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{(a_{\sigma(i)}^U)^p (a_{\sigma(j)}^U)^q}\right)^{\frac{1}{p+q}}} \right], \quad (16)$$

where $\hat{a}_{\sigma(i)} = [a_{\sigma(i)}^L, a_{\sigma(i)}^M, a_{\sigma(i)}^U]$ $(i = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{a}_{\sigma(i-1)} \geq \hat{a}_{\sigma(i)}$ for all i.

However, if there is a tie between $\hat{a}_i$ and $\hat{a}_j$, then we replace each of $\hat{a}_i$ and $\hat{a}_j$ by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these k replicas of their average. The weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ can be determined by using some weight determining methods like the normal distribution based method, see Refs [28,29,30] for more details.

If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$ then the FOWBHM operator is reduced to the FBHM operator; in addition, if $q = 0.$ then the FOWBHM operator is reduced to the following:

$$\begin{aligned} \text{FOWBHM}^{p,0}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) &= \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{\hat{a}^p_{\sigma(i)}}\right)^{\frac{1}{p}}} \\ &= \frac{1}{\left(\sum_{i=1}^{n} \frac{\omega_i}{\hat{a}^p_{\sigma(i)}} \sum_{j=1}^{n} \omega_j\right)^{\frac{1}{p}}} \\ &= \frac{1}{\left(\sum_{i=1}^{n} \frac{\omega_i}{\hat{a}^p_{\sigma(i)}}\right)^{\frac{1}{p}}} \\ &= \left[\frac{1}{\left(\sum_{i=1}^{n} \frac{\omega_i}{(a^L_{\sigma(i)})^p}\right)^{\frac{1}{p}}}, \frac{1}{\left(\sum_{i=1}^{n} \frac{\omega_i}{(a^M_{\sigma(i)})^p}\right)^{\frac{1}{p}}}, \frac{1}{\left(\sum_{i=1}^{n} \frac{\omega_i}{(a^U_{\sigma(i)})^p}\right)^{\frac{1}{p}}}\right] \quad (17) \end{aligned}$$

which we call the fuzzy ordered weighted generalized harmonic mean (FOWGHM) operator.

Especially, if the triangular fuzzy numbers $\hat{a}_i = [a^L_i, a^M_i, a^U_i]$ $(i = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_i = [a^L_i, a^U_i]$ $(i = 1, 2, \ldots, n)$, then the FOWBHM operator is reduced to the uncertain ordered weighted Bonferroni harmonic mean (UOWBHM) operator:

$$\begin{aligned} \text{UOWBHM}^{p,q}(\tilde{a}_i, \tilde{a}_2, \ldots, \tilde{a}_n) &= \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{\tilde{a}^p_{\sigma(i)} \tilde{a}^q_{\sigma(j)}}\right)^{\frac{1}{p+q}}} \\ &= \left[\frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{(a^L_{\sigma(i)})^p (a^L_{\sigma(j)})^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{(a^U_{\sigma(i)})^p (a^U_{\sigma(j)})^q}\right)^{\frac{1}{p+q}}}\right], \quad (18) \end{aligned}$$

where $\tilde{a}_{\sigma(i)} = [a^L_{\sigma(i)}, a^U_{\sigma(i)}]$ $(i = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\sigma(i-1)} \geq \tilde{a}_{\sigma(i)}$ for all i. If there is a tie between $\hat{a}_i$ and $\hat{a}_j$, then we replace each of $\hat{a}_i$ and $\hat{a}_j$ by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $a^L_i = a^U_i = a_i$, for all $i = 1, 2, \ldots, n$, then the UOWBHM operator is reduced to the ordered weighted Bonferroni harmonic mean (OWBHM) operator:

$$\mathrm{OWBHM}^{p,q}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{\omega_i \omega_j}{b_i^p b_j^q}\right)^{\frac{1}{p+q}}}, \tag{19}$$

where b_i is the ith largest of a_i $(i = 1, 2, \ldots, n)$. The OWBHM operator (19) has some special cases:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\mathrm{OWBHM}^{p,q}(a_1, a_2, \ldots, a_n) = \max\{a_i\} = b_1. \tag{20}$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\mathrm{OWBHM}^{p,q}(a_1, a_2, \ldots, a_n) = \min\{a_i\} = b_n. \tag{21}$$

(3) If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\begin{aligned}\mathrm{OWBHM}^{p,q}(a_1, a_2, \ldots, a_n) &= \frac{1}{\left(\frac{1}{n^2}\sum_{i,j=1}^{n} \frac{1}{b_i^p b_j^q}\right)^{\frac{1}{p+q}}} = \frac{1}{\left(\frac{1}{n^2}\sum_{i,j=1}^{n} \frac{1}{a_i^p a_j^q}\right)^{\frac{1}{p+q}}} \\ &= \mathrm{BHM}^{p,q}(a_1, a_2, \ldots, a_n).\end{aligned} \tag{22}$$

Example 2.10 Given a collection of triangular fuzzy numbers: $\hat{a}_1$=[3,4,6], and $\hat{a}_2$=[1,2,4], $\hat{a}_3$=[2,4,5], $\hat{a}_4$=[3,5,6], and $\hat{a}_5$=[2,5,7]. To rank these triangular fuzzy number, we first compare each triangular fuzzy number $\hat{a}_i$ with all triangular fuzzy numbers $\hat{a}_j$ $(j = 1, 2, 3, 4, 5)$ by using (6) (without loss of generality, set

$\delta = 0.5$), let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$ $(i, j = 1, 2, 3, 4, 5)$, then we utilize these possibility degrees to construct the following matrix $P = (p_{ij})_{5\times5}$:

$$P = \begin{pmatrix} 0.500 & 1 & 0.667 & 0.333 & 0.375 \\ 0 & 0.500 & 0 & 0 & 0 \\ 0.333 & 1 & 0.500 & 0.125 & 0.200 \\ 0.667 & 1 & 0.875 & 0.500 & 0.467 \\ 0.625 & 1 & 0.800 & 0.533 & 0.500 \end{pmatrix},$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.875, p_2 = 0.500, p_3 = 2.158, p_4 = 3.509, p_5 = 3.458.$$

and then we rank the triangular fuzzy numbers $\hat{a}_i$ $(i = 1, 2, 3, 4, 5)$ in descending order in accordance with the values of p_i $(i = 1, 2, 3, 4, 5)$:

$$\hat{a}_{\sigma(1)} = \hat{a}_4,\ \hat{a}_{\sigma(2)} = \hat{a}_5,\ \hat{a}_{\sigma(3)} = \hat{a}_1,\ \hat{a}_{\sigma(4)} = \hat{a}_3,\ \hat{a}_{\sigma(5)} = \hat{a}_2.$$

Suppose that the weight vector of the FOWBHM operator is $\omega = (0.1117, 0.2365, 0.3036, 0.2365, 0.1117)^T$ (derived by the normal distribution based method [28]), then by (16) (let $p = q = 2$), we get

$$\begin{aligned} &\text{FOWBHM}^{2,2}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) \\ &= \left[\frac{1}{\left(\sum_{i,j=1}^{5} \frac{\omega_i\omega_j}{(a_i^L)^2(a_j^L)^2}\right)^{\frac{1}{4}}}, \frac{1}{\left(\sum_{i,j=1}^{5} \frac{\omega_i\omega_j}{(a_i^M)^2(a_j^M)^2}\right)^{\frac{1}{4}}}, \frac{1}{\left(\sum_{i,j=1}^{5} \frac{\omega_i\omega_j}{(a_i^U)^2(a_j^U)^2}\right)^{\frac{1}{4}}}\right] \\ &= [1.901, 3.632, 5.509] \end{aligned}$$

Both the FWBHM and FOWBHM operators, however, can only deal with the situation that there are correlations between any two aggregated arguments, but not the situation that there exist connections among any three aggregated arguments.

To solve this issue, and motivated by Definition 2.3, we define the following:

Definition 2.11 Let $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{a}_i$ $(i = 1, 2, \ldots, n)$, where w_i indicates the importance degree of $\hat{a}_i$, satisfying $w_i > 0$, $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n w_i = 1$. For $p, q, r \geq 0$, if

$$\mathrm{GFWBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j,k=1}^n \frac{w_i w_j w_k}{\hat{a}_i^p \hat{a}_j^q \hat{a}_k^r}\right)^{\frac{1}{p+q+r}}}$$

$$= \left[\frac{1}{\left(\sum_{i,j,k=1}^n \frac{w_i w_j w_k}{(a_i^L)^p (a_j^L)^q (a_k^L)^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1}^n \frac{w_i w_j w_k}{(a_i^M)^p (a_j^M)^q (a_k^M)^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1}^n \frac{w_i w_j w_k}{(a_i^U)^p (a_j^U)^q (a_k^U)^r}\right)^{\frac{1}{p+q+r}}}\right]. \quad (23)$$

then $\mathrm{GFWBHM}^{p,q,r}$ is called generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the GFWBHM operator is reduced to the following:

$$\mathrm{GFBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\frac{1}{n^3} \sum_{i,j,k=1}^n \frac{1}{\hat{a}_i^p \hat{a}_j^q \hat{a}_k^r}\right)^{\frac{1}{p+q+r}}}, \quad (24)$$

which we call the generalized fuzzy Bonferroni harmonic mean (GFBHM) operator.

In addition, some special cases can be obtained as the change of parameters:
(1) If $r = 0$, then the GFWBHM operator is reduced to

$$\mathrm{GFWBHM}^{p,q,0}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j,k=1}^n \frac{w_i w_j w_k}{\hat{a}_i^p \hat{a}_j^q}\right)^{\frac{1}{p+q}}}$$

$$= \frac{1}{\left((\sum_{k=1}^{n} w_k) \sum_{i,j=1}^{n} \frac{w_i w_j}{\hat{a}_i^p \hat{a}_j^q} \right)^{\frac{1}{p+q}}}$$

$$= \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\hat{a}_i^p \hat{a}_j^q} \right)^{\frac{1}{p+q}}} \quad (25)$$

which is the FWBHM operator.

(2) If $q = 0$ and $r = 0$, then the GFWBHM operator is reduced to

$$\text{GFWBHM}^{p,0,0}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{\hat{a}_i^p} \right)^{\frac{1}{p}}}$$

$$= \frac{1}{\left((\sum_{j=1}^{n} w_j)(\sum_{k=1}^{n} w_k) \sum_{i,j=1}^{n} \frac{w_i}{\hat{a}_i^p} \right)^{\frac{1}{p}}}$$

$$= \frac{1}{\left(\sum_{i=1}^{n} \frac{w_i}{\hat{a}_i^p} \right)^{\frac{1}{p}}} \quad (26)$$

which is FWGHM operator. In this case, if $p = 1$, then FWGHM operator is reduced to FWHM operator.

Similar to the FWBHM operator, the GFWBHM operator has the following properties:

Theorem 2.12 Let $p, q, r \geq 0$, and $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, the following are valid:

(1) **Idempotency**: If all $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ are equal, i.e., $\hat{a}_i = \hat{a}$, for all i, then

$$\text{GFWBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \hat{a}.$$

(2) **Boundedness**:

$$\hat{a}^- \leq \text{GFWBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \hat{a}^+,$$

where $\hat{a}^- = [\min_i(a_i^L), \min_i(a_i^M), \min_i(a_i^U)]$, $\hat{a}^+ = [\max_i(a_i^L), \max_i(a_i^M), \max_i(a_i^U)]$.

(3) **Monotonicity**: Let $\hat{a}_i^* = [a_i^{L*}, a_i^{M*}, a_i^{U*}]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then if $a_i^L \leq a_i^{L*}$, $a_i^M \leq a_i^{M*}$ and $a_I^U \leq a_i^{U*}$ for all i, then

$$\mathrm{GFWBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \leq \mathrm{GFWBHM}^{p,q,r}(\hat{a}_1^*, \hat{a}_2^*, \ldots, \hat{a}_n^*).$$

(4) **Commutativity**: Let $\hat{a}_i' = [a_i^{L\prime}, a_i^{M\prime}, a_i^{U\prime}]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers, then

$$\mathrm{GFWBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \mathrm{GFWBHM}^{p,q,r}(\hat{a}_1', \hat{a}_2', \ldots, \hat{a}_n'),$$

where $(\hat{a}_1', \hat{a}_2', \ldots, \hat{a}_n')$ is any permutation of $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n)$.

Especially, if the triangular fuzzy numbers $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ are reduced to the interval numbers $\tilde{a}_i = [a_i^L, a_i^U]$ $(i = 1, 2, \ldots, n)$, then the GFWBHM operator (18) is reduced to the generalized uncertain weighted Bonferroni harmonic mean (GUWBHM) operator:

$$\mathrm{GUWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{\tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r}\right)^{\frac{1}{p+q+r}}}$$
$$= \left[\frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{(a_i^L)^p (a_j^L)^q (a_k^L)^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{(a_i^U)^p (a_j^U)^q (a_k^U)^r}\right)^{\frac{1}{p+q+r}}}\right]. \quad (27)$$

If $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the GUWBHM operator is reduced to the generalized uncertain Bonferroni harmonic mean (GUBHM):

$$\mathrm{GUBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\left(\frac{1}{n^3} \sum_{i,j,k=1}^{n} \frac{1}{\tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r}\right)^{\frac{1}{p+q+r}}}$$
$$= \left[\frac{1}{\left(\frac{1}{n^3} \sum_{i,j,k=1}^{n} \frac{1}{(a_i^L)^p (a_j^L)^q (a_k^L)^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\frac{1}{n^3} \sum_{i,j,k=1}^{n} \frac{1}{(a_i^U)^p (a_j^U)^q (a_k^U)^r}\right)^{\frac{1}{p+q+r}}}\right] \quad (28)$$

If $a_i^L = a_i^U = a_i$, for all $i = 1, 2, \ldots, n$, then the GUWBHM operator is reduced to the generalized weighted Bonferroni harmonic mean (GWBHM) operator:

$$\mathrm{GWBHM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r}\right)^{\frac{1}{p+q+r}}}. \tag{29}$$

In this case, if $p = 1$ and $q = r = 0$, the GWBHM operator is reduced to the weighted harmonic mean (WHM) operator.

Example 2.13 Consider the four triangular fuzzy numbers $\hat{a}_i$ and their weight vector w given in Example 2.8. Then by the GFWBHM operator (23) (without of generalization, let $p = q = r = 3$), we have

$$\begin{aligned}
&\mathrm{GFWBHM}^{3,3,3}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4) \\
&= \left[\frac{1}{\left(\sum_{i,j,k=1}^{4} \frac{w_i w_j w_k}{(a_i^L)^3 (a_j^L)^3 (a_k^L)^3}\right)^{\frac{1}{9}}}, \frac{1}{\left(\sum_{i,j,k=1}^{4} \frac{w_i w_j w_k}{(a_i^M)^3 (a_j^M)^3 (a_k^M)^3}\right)^{\frac{1}{9}}}, \right. \\
&\qquad \left. \frac{1}{\left(\sum_{i,j,k=1}^{4} \frac{w_i w_j w_k}{(a_i^U)^3 (a_j^U)^3 (a_k^U)^3}\right)^{\frac{1}{9}}}\right] \\
&= [1.21, 2.89, 4.59].
\end{aligned}$$

Definition 2.14 Let $\hat{a}_i = [a_i^L, a_i^M, a_i^U]$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. For $p, q, r \geq 0$, a generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator of dimension n is a mapping $\mathrm{GFOWBHM}^{p,q,r} : \Omega^n \to \Omega$, that has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Furthermore,

$$\mathrm{GFOWBHM}^{p,q,r}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{\hat{a}_{\sigma(i)}^p \hat{a}_{\sigma(j)}^q \hat{a}_{\sigma(k)}^r}\right)^{\frac{1}{p+q+r}}}$$

$$= \left[\frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{(a^L_{\sigma(i)})^p (a^L_{\sigma(j)})^q (a^L_{\sigma(k)})^r} \right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{(a^M_{\sigma(i)})^p (a^M_{\sigma(j)})^q (a^M_{\sigma(k)})^r} \right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{(a^U_{\sigma(i)})^p (a^U_{\sigma(j)})^q (a^U_{\sigma(k)})^r} \right)^{\frac{1}{p+q+r}}} \right], \tag{30}$$

where $\hat{a}_{\sigma(i)} = [a^L_{\sigma(i)}, a^M_{\sigma(i)}, a^U_{\sigma(i)}]$ $(i = 1, 2, \dots, n)$, and $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $\hat{a}_{\sigma(i-1)} \geq \hat{a}_{\sigma(i)}$ for all i.

However, if there is a tie between $\hat{a}_i$ and $\hat{a}_j$, then we replace each of $\hat{a}_i$ and $\hat{a}_j$ by their average $(\hat{a}_i + \hat{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $\omega = (\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})^T$ then the GFOWBHM operator is reduced to the GF-BHM operator. Moreover, some special cases can be obtained as the change of parameters: If $r = 0$, then the GFOWBHM operator is reduced to FOWBHM operator; if $r = 0$ and $q = 0$, then GFOWBHM operator is reduced to FOWGHM operator.

Especially, if the triangular fuzzy numbers $\hat{a}_i = [a^L_i, a^M_i, a^U_i]$ $(i = 1, 2, \dots, n)$ are reduced to the interval numbers $\tilde{a}_i = [a^L_i, a^U_i]$ $(i = 1, 2, \dots, n)$, then the GFOWBHM operator is reduced to the generalized uncertain ordered weighted Bonferroni harmonic mean (GUOWBHM) operator:

$$\text{GUOWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_n) = \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{\tilde{a}^p_{\sigma(i)} \tilde{a}^q_{\sigma(j)} \tilde{a}^r_{\sigma(k)}} \right)^{\frac{1}{p+q+r}}}$$

$$= \left[\frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{(a^L_{\sigma(i)})^p (a^L_{\sigma(j)})^q (a^L_{\sigma(k)})^r} \right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{(a^U_{\sigma(i)})^p (a^U_{\sigma(j)})^q (a^U_{\sigma(k)})^r} \right)^{\frac{1}{p+q+r}}} \right], \tag{31}$$

where $\tilde{a}_{\sigma(i)} = [a^L_{\sigma(i)}, a^U_{\sigma(i)}]$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\tilde{a}_{\sigma(i-1)} \geq \tilde{a}_{\sigma(i)}$ for all i. If there is a tie between $\tilde{a}_i$ and $\tilde{a}_j$, then we replace each of $\tilde{a}_i$ and $\tilde{a}_j$ by their average $(\tilde{a}_i + \tilde{a}_j)/2$ in process of aggregation. If k items are tied, then we replace these by k replicas of their average.

If $a^L_i = a^U_i = a_i$, for all $i = 1, 2, \ldots, n$, then the GUOWBHM operator is reduced to the generalized ordered weighted Bonferroni harmonic mean (GOWBHM) operator:

$$\mathrm{GOWBHM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\sum_{i,j,k=1}^{n} \frac{\omega_i \omega_j \omega_k}{b_i^p b_j^q b_k^r} \right)^{\frac{1}{p+q+r}}}, \tag{32}$$

where b_i is the ith largest of a_i $(i = 1, 2, \ldots, n)$. In this case, if $p = 1$ and $q = r = 0$, then the GOWBHM operator is reduced to the ordered weighted harmonic mean (OWHM) operator.

The GOWBHM operator (32) has some special cases:

(1) If $\omega = (1, 0, \ldots, 0)^T$, then

$$\mathrm{GOWBHM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \max\{a_i\} = b_1. \tag{33}$$

(2) If $\omega = (0, 0, \ldots, 1)^T$, then

$$\mathrm{GOWBHM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \min\{a_i\} = b_n. \tag{34}$$

(3) If $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{GOWBHM}^{p,q,r}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\frac{1}{n^3} \sum_{i,j,k=1}^{n} \frac{1}{b_i^p b_j^q b_k^r} \right)^{\frac{1}{p+q+r}}}$$

$$= \frac{1}{\left(\frac{1}{n^3}\sum_{i,j,k=1}^{n}\frac{1}{a_i^p a_j^q a_k^r}\right)^{\frac{1}{p+q+r}}}, \qquad (35)$$

which we call the generalized Bonferroni harmonic mean (GBHM) operator.

Example 2.15 Consider the four triangular fuzzy numbers $\hat{a}_i$ and their weight vector w given in Example 2.10. Then by the GFOWBHM operator (30) (let $p = q = r = 3$), we have

$$\begin{aligned}
&\mathrm{GFOWBHM}^{3,3,3}(\hat{a}_1, \hat{a}_2, \hat{a}_3, \hat{a}_4, \hat{a}_5) \\
&= \left[\frac{1}{\left(\sum_{i,j,k=1}^{5}\frac{w_i w_j w_k}{(a_{\sigma(i)}^L)^3(a_{\sigma(j)}^L)^3(a_{\sigma(k)}^L)^3}\right)^{\frac{1}{9}}}, \frac{1}{\left(\sum_{i,j,k=1}^{5}\frac{w_i w_j w_k}{(a_{\sigma(i)}^M)^3(a_{\sigma(j)}^M)^3(a_{\sigma(k)}^M)^3}\right)^{\frac{1}{9}}},\right. \\
&\qquad \left.\frac{1}{\left(\sum_{i,j,k=1}^{5}\frac{w_i w_j w_k}{(a_{\sigma(i)}^U)^3(a_{\sigma(j)}^U)^3(a_{\sigma(k)}^U)^3}\right)^{\frac{1}{9}}},\right] \\
&= [1.751, 3.410, 5.422]
\end{aligned}$$

In the following chapter, we will apply the developed operators to multiple attribute group decision making.

3 An approach to multiple attribute group decision making with triangular fuzzy information

For a group decision making with triangular fuzzy information, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives, and $G = \{G_1, G_2, \ldots, G_m\}$ be the set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_n)^T$ with $w_i \geq 0$ and $\sum_{i=2}^{m} w_i = 1$, and let $D = \{d_1, d_2, \ldots, d_s\}$ be the set of decision makers, whose weight vector is $v = (v_1, v_2, \ldots, v_s)^T$, where $v_k \geq 0$ and $\sum_{k=1}^{s} v_k = 1$. Suppose that $A^{(k)} = (\hat{a}_{ij}^{(k)})_{m\times n}$ is the triangular fuzzy decision matrix, where $\hat{a}_{ij}^{(k)} = [a_{ij}^{L(k)}, a_{ij}^{M(k)}, a_{ij}^{U(k)}]$ is an attribute value, which takes the form of triangular fuzzy number, given by the decision maker $d_k \in D$, for the alternative $x_j \in X$ with respect to the attribute $G_i \in G$.

In the following, we apply the GFWBHM and GFOWBHM operators to group decision making with triangular fuzzy information.

Step 1. Normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrix $A^{(k)}$ into a corresponding element in the matrix $R^{(k)} = (\hat{r}_{ij}^{(k)})_{m\times n}$ ($\hat{r}_{ij}^{(k)} = [\hat{r}_{ij}^{L(k)}, \hat{r}_{ij}^{M(k)}, \hat{r}_{ij}^{U(k)}]$) using the following formulas:

$$\begin{aligned}\hat{r}_{ij}^{(k)} &= \frac{\hat{a}_{ij}^{(k)}}{\sum_{j=1}^{n} \hat{a}_{ij}^{(k)}} \\ &= \left[\frac{a_{ij}^{L(k)}}{\sum_{j=1}^{n} a_{ij}^{U(k)}}, \frac{a_{ij}^{M(k)}}{\sum_{j=1}^{n} a_{ij}^{M(k)}}, \frac{a_{ij}^{U(k)}}{\sum_{j=1}^{n} a_{ij}^{L(k)}}\right], \text{ for benefit attribute } G_i, \\ & i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots, s. \end{aligned} \tag{36}$$

$$\begin{aligned}\hat{r}_{ij}^{(k)} &= \frac{1/\hat{a}_{ij}^{(k)}}{\sum_{j=1}^{n} (1/\hat{a}_{ij}^{(k)})} \\ &= \left[\frac{1/a_{ij}^{U(k)}}{\sum_{j=1}^{n} (1/a_{ij}^{L(k)})}, \frac{1/a_{ij}^{M(k)}}{\sum_{j=1}^{n} (1/a_{ij}^{M(k)})}, \frac{1/a_{ij}^{L(k)}}{\sum_{j=1}^{n} (1/a_{ij}^{U(k)})}\right], \text{for cost attribute } G_i, \\ & i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, k = 1, 2, \ldots, s. \end{aligned} \tag{37}$$

Step 2. Utilize the GFWBHM operator (23):

$$\hat{r}_j^{(k)} = \mathrm{GFWBHM}^{p,q,r}(\hat{r}_{1j}^{(k)}, \hat{r}_{2j}^{(k)}, \ldots, \hat{r}_{mj}^{(k)}) = \frac{1}{\left(\sum_{i,h,l=1}^{m} \frac{w_i w_h w_l}{(\hat{r}_{ij}^{(k)})^p,(\hat{r}_{ij}^{(k)})^q,(\hat{r}_{ij}^{(k)})^r}\right)^{\frac{1}{p+q+r}}}$$

$$= \left[\frac{1}{\left(\sum_{i,h,l=1}^{m} \frac{w_i w_h w_l}{(r_{ij}^{L(k)})^p,(r_{ij}^{L(k)})^q,(r_{ij}^{L(k)})^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,h,l=1}^{m} \frac{w_i w_h w_l}{(r_{ij}^{M(k)})^p,(r_{ij}^{M(k)})^q,(r_{ij}^{M(k)})^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,h,l=1}^{m} \frac{w_i w_h w_l}{(r_{ij}^{U(k)})^p,(r_{ij}^{U(k)})^q,(r_{ij}^{U(k)})^r}\right)^{\frac{1}{p+q+r}}} \right], \tag{38}$$

to aggregate all the elements in the jth column of $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(k)}$ of the alternative x_j corresponding to the decision maker d_k.

Step 3. Utilize the GFOWBHM operator (30):

$$\hat{r}_j = \mathrm{GFOWBHM}^{p,q,r}(\hat{r}_j^{(1)}, \hat{r}_j^{(2)}, \ldots, \hat{r}_j^{(s)}) = \frac{1}{\left(\sum_{k,h,l=1}^{s} \frac{\omega_k \omega_h \omega_l}{(\dot{\hat{r}}_j^{\sigma(k)})^p(\dot{\hat{r}}_j^{\sigma(h)})^q(\dot{\hat{r}}_j^{\sigma(l)})^r}\right)^{\frac{1}{p+q+r}}}$$

$$= \left[\frac{1}{\left(\sum_{k,h,l=1}^{s} \frac{\omega_k \omega_h \omega_l}{(\dot{r}_j^{L(\sigma(k))})^p(\dot{r}_j^{L(\sigma(h))})^q(\dot{r}_j^{L(\sigma(l))})^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{k,h,l=1}^{s} \frac{\omega_k \omega_h \omega_l}{(\dot{r}_j^{M(\sigma(k))})^p(\dot{r}_j^{M(\sigma(h))})^q(\dot{r}_j^{M(\sigma(l))})^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{k,h,l=1}^{s} \frac{\omega_k \omega_h \omega_l}{(\dot{r}_j^{U(\sigma(k))})^p(\dot{r}_j^{U(\sigma(h))})^q(\dot{r}_j^{U(\sigma(l))})^r}\right)^{\frac{1}{p+q+r}}} \right] \tag{39}$$

to aggregate the overall attribute values $\hat{r}_j^{(k)}$ $(k = 1, 2, \ldots, s)$ corresponding to the decision maker d_k $(k = 1, 2, \ldots, s)$ and get the collective overall attribute value $\hat{r}_j$, where $\dot{\hat{r}}_j^{(\sigma(k))} = [\dot{r}_j^{L(\sigma(k))}, \dot{r}_j^{M(\sigma(k))}, \dot{r}_j^{(U\sigma(k))}]$ is kth largest of the weighted data $\dot{\hat{r}}_j^{(k)}$ $(\dot{\hat{r}}_j^{(k)} = sv_k \hat{r}_j^{(k)}, k = 1, 2, \ldots, s)$, $\omega = (\omega_1, \omega_2, \ldots, \omega_s)^T$ is the weighting vector of the GFOWBHM operator, with $\omega_k \geq 0$ and $\sum_{k=1}^{s} \omega_k = 1$.

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1, 2, \ldots, n)$ by using (6), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$, and then construct the possibility matrix $P = (p_{ij})_{n \times n}$, where $p_{ij} \geq 0$, $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all elements in each line of matrix P, we have $p_i = \sum_{j=1}^{n} p_{ij}$, $i = 1, 2, \ldots, n$, and then reorder $\hat{r}_j$ $(j = 1, 2, \ldots, n)$ in descending order in accordance with the values of p_j $(j = 1, 2, \ldots, n)$.

Step 5. Rank all alternatives s_j $(j = 1, 2, \ldots, n)$ by the ranking of $\hat{r}_j$ $(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 6. End.

4 Example illustrations

In this chapter, we use a multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library (adopted from [32,22]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problem facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers five feasible alternatives, which might be adapted to the physical structure of the library. The alternatives x_j $(j = 1, 2, 3, 4, 5)$ are to be evaluated using triangular fuzzy numbers by the three decision makers d_k $(k = 1, 2, 3)$(whose weight vector is $v = (0.4, 0.3, 0.3)^T$) under three major impacts: economic, functional, and operational. Two monetary attributes and six nonmonetary attributes (that is, G_1: owning cost (\$/ft^2), G_2: operating cost (\$/ft^2), G_3: performance(*), G_4: noise level (Db), G_5: maintainability(*), G_6: reliability (%), G_7: flexibility(*), G_8: safety(*), where * unit is from 0-1 scale, three attributes G_1, G_2, and G_4 are cost attributes, and the other five attributes are benefit attributes, suppose that the weight vector od the attributes G_i $(i = 1, 2, \ldots, 8)$ is $w = (0.05, 0.08, 0.14, 0.12, 0.18, 0.21, 0.05, 0.17)^T$) emerged from three impacts is Tables 1-3.

Table 1: Triangular fuzzy number decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[3.5, 4.0, 4.7]$	$[1.7, 2.0, 2.3]$	$[3.5, 3.8, 4.2]$	$[3.5, 3.8, 4.5]$	$[3.3, 3.8, 4.0]$
G_2	$[5.5, 6.0, 6.5]$	$[4.8, 5.1, 5.5]$	$[4.5, 5.2, 5.5]$	$[4.5, 4.7, 5.0]$	$[5.5, 5.7, 6.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.5, 0.56, 0.6]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.85, 0.9]$	$[0.6, 0.7, 0.8]$
G_4	$[35, 40, 45]$	$[70, 73, 75]$	$[65, 68, 70]$	$[40, 42, 45]$	$[50, 55, 60]$
G_5	$[0.4, 0.45, 0.5]$	$[0.4, 0.44, 0.6]$	$[0.7, 0.76, 0.8]$	$[0.9, 0.97, 1.0]$	$[0.5, 0.54, 0.6]$
G_6	$[95, 98, 100]$	$[70, 73, 75]$	$[80, 83, 90]$	$[90, 93, 95]$	$[85, 90, 95]$
G_7	$[0.3, 0.35, 0.5]$	$[0.7, 0.75, 0.8]$	$[0.8, 0.9, 1.0]$	$[0.6, 0.75, 0.8]$	$[0.4, 0.5, 0.6]$
G_8	$[0.7, 0.74, 0.8]$	$[0.5, 0.53, 0.6]$	$[0.6, 0.68, 0.7]$	$[0.7, 0.8, 0.9]$	$[0.8, 0.85, 0.9]$

Table 2: Triangular fuzzy number decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.0, 4.3, 4.5]$	$[2.1, 2.2, 2.4]$	$[5.0, 5.1, 5.2]$	$[4.3, 4.4, 4.5]$	$[3.0, 3.3, 3.5]$
G_2	$[6.0, 6.3, 6.5]$	$[5.0, 5.1, 5.2]$	$[4.5, 4.7, 5.0]$	$[5.0, 5.1, 5.3]$	$[7.0, 7.5, 8.0]$
G_3	$[0.7, 0.8, 0.9]$	$[0.4, 0.5, 0.6]$	$[0.5, 0.55, 0.6]$	$[0.7, 0.75, 0.8]$	$[0.7, 0.8, 0.9]$
G_4	$[37, 38, 39]$	$[70, 73, 75]$	$[65, 66, 67]$	$[40, 42, 45]$	$[50, 52, 55]$
G_5	$[0.4, 0.5, 0.6]$	$[0.5, 0.55, 0.6]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.95, 1.0]$	$[0.4, 0.44, 0.5]$
G_6	$[92, 93, 95]$	$[70, 75, 80]$	$[83, 84, 85]$	$[90, 91, 92]$	$[90, 93, 95]$
G_7	$[0.4, 0.45, 0.5]$	$[0.8, 0.85, 0.9]$	$[0.7, 0.73, 0.8]$	$[[0.7, 0.85, 0.9]$	$[0.4, 0.45, 0.5]$
G_8	$[0.6, 0.7, 0.8]$	$[0.6, 0.65, 0.7]$	$[0.5, 0.6, 0.7]$	$[0.7, 0.76, 0.8]$	$[0.7, 0.8, 0.9]$

Table 3: Triangular fuzzy number decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[4.3, 4.4, 4.6]$	$[2.2, 2.4, 2.5]$	$[4.5, 4.8, 5.0]$	$[4.7, 4.9, 5.0]$	$[3.1, 3.2, 3.4]$
G_2	$[6.4, 6.7, 7.0]$	$[5.0, 5.2, 5.5]$	$[4.7, 4.8, 4.9]$	$[5.5, 5.7, 6.0]$	$[6.0, 6.5, 7.0]$
G_3	$[0.8, 0.85, 0.9]$	$[0.5, 0.6, 0.7]$	$[0.6, 0.7, 0.8]$	$[0.7, 0.8, 0.9]$	$[0.7, 0.75, 0.8]$
G_4	$[36, 38, 40]$	$[72, 73, 75]$	$[67, 68, 70]$	$[45, 48, 50]$	$[55, 57, 60]$
G_5	$[0.4, 0.46, 0.5]$	$[0.4, 0.45, 0.6]$	$[0.8, 0.95, 1.0]$	$[0.8, 0.85, 0.9]$	$[0.5, 0.55, 0.6]$
G_6	$[93, 94, 95]$	$[77, 78, 80]$	$[85, 87, 90]$	$[90, 94, 95]$	$[90, 96, 100]$
G_7	$[0.4, 0.5, 0.6]$	$[0.8, 0.9, 1.0]$	$[0.8, 0.86, 0.9]$	$[0.6, 0.7, 0.8]$	$[0.5, 0.57, 0.6]$
G_8	$[0.7, 0.78, 0.8]$	$[0.5, 0.55, 0.6]$	$[0.6, 0.68, 0.7]$	$[0.8, 0.85, 0.9]$	$[0.8, 0.85, 0.9]$

In the following, we utilize the decision procedure to select the best air-conditioning system:

Step 1. By using (36) and (37), we normalize each attribute value $\hat{a}_{ij}^{(k)}$ in the matrices $A^{(k)}$ $(k = 1, 2, 3)$ into the corresponding element in the matrices $R^{(k)} = (\hat{r}_{ij})_{8\times 5}$ $(k = 1, 2, 3)$ (Tables 4-6):

Table 4: Normalized triangular fuzzy number decision matrix $R^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	[0.12, 0.16, 0.21]	[0.25, 0.32, 0.43]	[0.14, 0.17, 0.21]	[0.13, 0.17, 0.21]	[0.14, 0.17, 0.22]
G_2	[0.15, 0.18, 0.21]	[0.18, 0.21, 0.24]	[0.18, 0.20, 0.25]	[0.20, 0.23, 0.25]	[0.16, 0.19, 0.21]
G_3	[0.18, 0.23, 0.30]	[0.13, 0.16, 0.20]	[0.13, 0.17, 0.23]	[0.18, 0.24, 0.30]	[0.15, 0.20, 0.27]
G_4	[0.22, 0.26, 0.32]	[0.13, 0.14, 0.16]	[0.14, 0.15, 0.17]	[0.22, 0.25, 0.28]	[0.16, 0.19, 0.23]
G_5	[0.11, 0.14, 0.17]	[0.11, 0.14, 0.21]	[0.20, 0.24, 0.28]	[0.26, 0.31, 0.34]	[0.14, 0.17, 0.21]
G_6	[0.21, 0.22, 0.24]	[0.15, 0.17, 0.18]	[0.18, 0.19, 0.21]	[0.20, 0.21, 0.23]	[0.19, 0.21, 0.23]
G_7	[0.08, 0.11, 0.18]	[0.19, 0.23, 0.29]	[0.22, 0.28, 0.36]	[0.16, 0.23, 0.29]	[0.11, 0.15, 0.21]
G_8	[0.18, 0.21, 0.24]	[0.13, 0.15, 0.18]	[0.15, 0.19, 0.21]	[0.18, 0.22, 0.27]	[0.21, 0.24, 0.27]

Step 2. Utilize the GFWBHM operator (38) (let $p = q = r = 3$) to aggregate all elements in the jth column $R^{(k)}$ and get the overall attribute value $\hat{r}_j^{(K)}$:

$$\hat{r}_1^{(1)} = [0.1390, 0.1753, 0.2187], \hat{r}_2^{(1)} = [0.1347, 0.1586, 0.1927],$$
$$\hat{r}_3^{(1)} = [0.1581, 0.1852, 0.2178], \hat{r}_4^{(1)} = [0.1900, 0.2289, 0.2651],$$
$$\hat{r}_5^{(1)} = [0.1565, 0.1911, 0.2311],$$
$$\hat{r}_1^{(2)} = [0.1480, 0.1851, 0.2248], \hat{r}_2^{(2)} = [0.1434, 0.1706, 0.1992],$$
$$\hat{r}_3^{(2)} = [0.1561, 0.1792, 0.2057], \hat{r}_4^{(2)} = [0.1927, 0.2228, 0.2477],$$
$$\hat{r}_5^{(2)} = [0.1499, 0.1761, 0.2098],$$
$$\hat{r}_1^{(3)} = [0.1459, 0.1811, 0.2704], \hat{r}_2^{(3)} = [0.1370, 0.1607, 0.1938],$$
$$\hat{r}_3^{(3)} = [0.1679, 0.1921, 0.2173], \hat{r}_4^{(3)} = [0.1883, 0.2138, 0.2395],$$
$$\hat{r}_5^{(3)} = [0.1678, 0.1922, 0.2215],$$

Step 3. Utilize the GFOWBHM operator (39) (suppose that its weight vector is $\omega = (0.243.0.514.0.243)^T$ determined by using the normal distribution based

Table 5: Normalized triangular fuzzy number decision matrix $R^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.16, 0.19]$	$[0.28, 0.32, 0.36]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.16, 0.17]$	$[0.19, 0.21, 0.25]$
G_2	$[0.17, 0.18, 0.19]$	$[0.21, 0.22, 0.23]$	$[0.21, 0.24, 0.26]$	$[0.20, 0.22, 0.23]$	$[0.13, 0.15, 0.17]$
G_3	$[0.18, 0.24, 0.30]$	$[0.11, 0.15, 0.20]$	$[0.13, 0.16, 0.20]$	$[0.18, 0.22, 0.27]$	$[0.18, 0.24, 0.30]$
G_4	$[0.25, 0.27, 0.29]$	$[0.13, 0.14, 0.15]$	$[0.15, 0.15, 0.16]$	$[0.22, 0.24, 0.27]$	$[0.18, 0.20, 0.21]$
G_5	$[0.11, 0.15, 0.21]$	$[0.14, 0.17, 0.21]$	$[0.22, 0.26, 0.31]$	$[0.22, 0.29, 0.34]$	$[0.11, 0.13, 0.17]$
G_6	$[0.21, 0.21, 0.22]$	$[0.16, 0.17, 0.19]$	$[0.19, 0.19, 0.20]$	$[0.20, 0.21, 0.22]$	$[0.20, 0.21, 0.22]$
G_7	$[0.11, 0.14, 0.17]$	$[0.22, 0.26, 0.30]$	$[0.19, 0.22, 0.27]$	$[0.19, 0.26, 0.30]$	$[0.19, 0.14, 0.17]$
G_8	$[0.15, 0.20, 0.26]$	$[0.15, 0.19, 0.23]$	$[0.13, 0.17, 0.23]$	$[0.18, 0.22, 0.26]$	$[0.18, 0.23, 0.29]$

Table 6: Normalized triangular fuzzy number decision matrix $R^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$[0.15, 0.17, 0.18]$	$[0.28, 0.30, 0.35]$	$[0.14, 0.15, 0.17]$	$[0.14, 0.15, 0.16]$	$[0.20, 0.23, 0.25]$
G_2	$[0.16, 0.17, 0.19]$	$[0.20, 0.22, 0.24]$	$[0.22, 0.24, 0.25]$	$[0.18, 0.20, 0.22]$	$[0.16, 0.17, 0.20]$
G_3	$[0.20, 0.23, 0.27]$	$[0.12, 0.16, 0.21]$	$[0.15, 0.19, 0.24]$	$[0.17, 0.22, 0.27]$	$[0.17, 0.20, 0.24]$
G_4	$[0.26, 0.28, 0.31]$	$[0.14, 0.15, 0.16]$	$[0.15, 0.16, 0.17]$	$[0.21, 0.22, 0.25]$	$[0.17, 0.19, 0.20]$
G_5	$[0.11, 0.14, 0.17]$	$[0.11, 0.14, 0.21]$	$[0.20, 0.29, 0.34]$	$[0.22, 0.26, 0.31]$	$[0.14, 0.17, 0.21]$
G_6	$[0.20, 0.21, 0.22]$	$[0.17, 0.17, 0.18]$	$[0.18, 0.19, 0.21]$	$[0.20, 0.21, 0.22]$	$[0.20, 0.21, 0.23]$
G_7	$[0.10, 0.14, 0.19]$	$[0.21, 0.25, 0.32]$	$[0.21, 0.24, 0.29]$	$[0.15, 0.20, 0.26]$	$[0.13, 0.16, 0.19]$
G_8	$[0.18, 0.21, 0.24]$	$[0.13, 0.15, 0.18]$	$[0.15, 0.18, 0.21]$	$[0.21, 0.23, 0.26]$	$[0.21, 0.23, 0.26]$

method [28], let $\delta = 0.5$ and $p = q = r = 3$) to aggregate the overall attribute value $\hat{r}_j^{(k)}$ $(k = 1,2,3)$ corresponding to the decision maker d_k $(k = 1,2,3)$ and get the collective overall attribute value $\hat{r}_j$:

$$\hat{r}_1 = [0.1459, 0.1816, 0.2195], \hat{r}_2 = [0.1395, 0.1649, 0.1962],$$
$$\hat{r}_3 = [0.1592, 0.1837, 0.2111], \hat{r}_4 = [0.1909, 0.2218, 0.2493],$$
$$\hat{r}_5 = [0.1552, 0.1830, 0.2191].$$

Step 4. Compare each $\hat{r}_j$ with all $\hat{r}_i$ $(i = 1,2,3,4,5)$ by using (6) (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{r}_i \geq \hat{r}_j)$ and then construct a possibility matrix:

$$P = \begin{pmatrix} 0.5 & 0.7387 & 0.4598 & 0 & 0.4610 \\ 0.2613 & 0.5 & 0.1638 & 0 & 0.1921 \\ 0.5402 & 0.8365 & 0.5 & 0 & 0.5010 \\ 1 & 1 & 1 & 0.5 & 1 \\ 0.5390 & 0.8079 & 0.1990 & 0 & 0.5 \end{pmatrix}.$$

Summing all elements in each line of matrix P, we have

$$p_1 = 2.1595, p_2 = 1.1171, p_3 = 2.3774, p_4 = 4.5, p_5 = 2.3459$$

and then reorder $\hat{r}_j$ $(j = 1,2,3,4,5)$ in descending order in accordance with the values of p_j $(j = 1,2,3,4,5)$:

$$\hat{r}_4 > \hat{r}_3 > \hat{r}_5 > \hat{r}_1 > \hat{r}_2.$$

Step 5. Rank all the alternatives x_j $(j = 1,2,3,4,5)$ by the ranking of $\hat{r}_j$ $(j = 1,2,3,4,5)$:

$$x_4 > x_3 > x_5 > x_1 > x_2.$$

and thus the most desirable alternative is x_4.

From the above analysis, the results obtained by the proposed approach are very similar to the ones obtained Xu's approach [22], but our approach is more flexible than that of Xu [22] because it can provide the decision makers more choices as parameters are assigned different values.

5 Conclusions

In this thesis, we have extended the GWBM operator to the triangular fuzzy environment and developed the fuzzy harmonic aggregation operators including the FWBHM and GFWBHM operators. Based on the these operators and Yager's OWA operator, we have developed the FOWBHM operator and the GFOWBHM operator, respectively, and discussed their properties and special cases. It has been pointed out that if all the input fuzzy data are reduced to the interval or numerical data, then the GFWBHM operator is reduced to the GUWBHM operator and GWBHM operator, respectively; the GFOWBHM operator is reduced to the GUOWBHM operator and GOWBHM operator, respectively. In these situations, the WHM (resp. OWHM) operator is the special case of the GWBHM (res. GOWBHM) operator. Based on the GFWBHM and and GFOWBHM operators, we have developed an approach to multiple attribute group decision making with triangular fuzzy information and have also applied the proposed approach the problem of determining what kind of air-conditioning systems should be installed in the library. Furthermore, the comparison of the proposed approach with other existing approaches is presented. The merit of the proposed approach is that it is more flexible than the classical ones because it can provide the decision makers more choices as parameters are assigned different values. Apparently, the proposed aggregation techniques and decision making method can also extended to the interval-valued triangular fuzzy environment.

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