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On Implications And Characteristics for Nonlinear Mappings

by

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On Implications And Characteristics for Nonlinear Mappings
(비선형사상들의 포함관계와 특성)

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비선형사상들의 포함관계와 특성

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요　　약

X는 바나흐(Banach) 공간이고 X^*는 X의 쌍대공간(dual space)이라 하자. $J: X \rightarrow X^*$가 정규화된 쌍대사상(the normalized duality)이라 함은 각 $x \in X$에 대하여 집합

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \qquad (1)$$

로 대응시키는 다가함수를 뜻한다. 특히, X가 매끄러운(smooth) 바나흐 공간이면 (1)처럼 정의된 정규화된 쌍대사상(normalized duality mapping) J는 단가(single-valued)이므로

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in X \qquad (2)$$

로 정의된 Lyapunov 범함수 $\phi : X \times X \rightarrow \mathbb{R}$가 잘 정의된다.

본 논문의 2절에서는 여러 가지 비선형사상들의 포함관계를 체계적으로 분류하여 상호 비교・분석하였으며, 더욱 그 역 포함관계가 성립하지 않는 예제들을 제시하였다. 3절에서는 (2)처럼 정의된 ϕ-범함수에 대한 여러 가지 특성들을 조사하였으며, 4절에서는 ϕ-비선형 사상들의 상호포함관계를 조사하고 그 역의 포함관계에 대한 예제들을 제시하였다.

1 Introduction

Let X be a real Banach space with norm $\|\cdot\|$ and let X^* be the dual of X. Denote by $\langle\cdot,\cdot\rangle$ the duality product. The normalized duality mapping from X to X^* is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in X$. When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$.

Recall that a Banach space X is said to be *strictly convex* [38] if $\|(x+y)/2\| < 1$ for all x, $y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\|x_n - y_n\| \to 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\|(x_n + y_n)/2\| \to 1$.

Let $S(X) = \{x \in X : \|x\| = 1\}$ be the unit sphere of X. Then the Banach space X is said to be *smooth* provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists for each x, $y \in S(X)$. It is also said to be *uniformly smooth* if the limit in (1.1) is attained uniformly for $x, y \in S(X)$. It is well known that if X is smooth, then the duality mapping J is single-valued and norm-to-weak* continuous. Furthermore, if X is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of X. Some properties of the duality mapping have been given in [12, 34, 38]. A Banach space X is said to have the *Kadec-Klee* property if a sequence $\{x_n\}$ of X satisfying that $x_n \rightharpoonup x \in X$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if X is uniformly convex, then X has the Kadec-Klee property; see [12, 16, 38] for more details.

Let X be a smooth Banach space and let $\phi : X \times X \to \mathbb{R}$ be defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in X.$$

The purpose of this paper is systematically to classify the implications concerning to several nonlinear mappings. In section 2, we suggest several examples of nonlinear mappings which are comparable each other and we finally raise an open question; see Question 2.20. In section 3, after introducing three projections on Banach spaces and their characteristics, we prove an equivalent condition between norm convergence and ϕ-convergence; see Proposition 3.6. Finally, in section 4, we classify several ϕ-nonlinear mappings recently studied by many authors and give some examples which are comparable each other.

2 Several nonlinear mappings

Let C be a nonempty closed convex subset of a real Banach space X and let $T : C \to C$ be a mapping.

Definition 2.1. The mapping T is said to be *Lipschitzian* if

$$\|Tx - Ty\| \le L\|x - y\|, \quad x, y \in C,$$

where $L := L_T$ denotes the *Lipschitz constant* of T. Obviously, it is equivalent to the following property: for each $n \in \mathbb{N}$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad x, y \in C. \tag{2.1}$$

For a Lipschitzian mapping T, we say:

- T is *uniformly k-Lipschiztain* if $k_n = k$ for all $n \in N$;

- T is *nonexpansive* if $k_n = 1$ for all $n \geq 1$;

- T is *asymptotically nonexpansive* [13] if $\lim_{n\to\infty} k_n = 1$.

The first non-Lipschitzian mapping was introduce by Kirk [27]; we say that T is a mapping of *asymptotically nonexpansive type* if

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x-y\|) \leq 0 \tag{2.2}$$

for every $x \in C$, and T^N is continuous for some $N \geq 1$. In 1993, Bruck et al [7] introduced the stronger definition than (ANT) as follows:

Definition 2.2. ([7]) T is said to be *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \leq 0. \tag{2.3}$$

Note that if we define

$$c_n := \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x-y\|) \vee 0, \tag{2.4}$$

where $a \vee b := \max\{a, b\}$, then (2.3) ensures that $c_n \to 0$ and

$$\|T^n x - T^n y\| \leq \|x-y\| + c_n \tag{2.5}$$

for all $x, y \in C$ and $n \geq 1$. Obviously, (2.5) implies (2.3) in case $c_n \to 0$. Therefore, we conclude:

Proposition 2.3. *T satisfies (2.3) $\Leftrightarrow$ (2.5) holds for some sequence $\{c_n\}$ with $c_n \to 0$.*

For the purpose of unifying nonlinear mappings mentioned above, Alber et al [5] introduced the following definition:

Definition 2.4. ([5]) T is said to be *total asymptotically nonexpansive* if there exist two nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n, \beta_n \to 0$, $\tau \in \Gamma(R_+)$ and $n_0 \in \mathbb{N}$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + \alpha_n \tau(\|x - y\|) + \beta_n, \quad x, y \in C,\ n \geq n_0, \tag{2.6}$$

where $\tau \in \Gamma(R_+)$ if and only if τ is strictly increasing, continuous on R_+ and $\tau(0) = 0$.

Now it is natural to consider more stronger one than the concept of mappings which are total asymptotically nonexpansive.

Definition 2.5. T is said to be *square total asymptotically nonexpansive* if (2.6) in Definition 2.4 can be replaced by

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + \widetilde{\alpha}_n \widetilde{\tau}(\|x - y\|^2) + \widetilde{\beta}_n, \tag{2.7}$$

for all $x, y \in C$ and $n \geq m_0$, where $m_0 \in \mathbb{N}$, $\widetilde{\alpha}_n, \widetilde{\beta}_n \to 0$ and $\widetilde{\tau} \in \Gamma(R^+)$.

Remark 2.6. Note that the property (2.6) with $\alpha_n = 0$ for all $n \geq 1$ reduces to (2.5) with $\beta_n = c_n$; moreover, if we take $\tau(t) = t$ for all $t \geq 0$ and $\beta_n = 0$ for all $n \geq 1$ in (2.6), it is reduced to (2.1) with $k_n = 1 + \alpha_n$.

Now we summarize the connection between the classes of nonlinear mappings considered above. We use the following notation:

$$
\begin{aligned}
\text{(N)} &= \text{the class of nonexpansive mappings} \\
\text{(L)} &= \text{the class of Lipschiztian mappings} \\
\text{(UC)} &= \text{the class of uniformly continuous mappings} \\
\text{(UL)} &= \text{the class of uniformly Lipschiztian mappings} \\
\text{(AN)} &= \text{the class of asymptotically nonexpansive mappings} \\
\text{(TAN)} &= \text{the class of totally asymptotically nonexpansive mappings} \\
({}^2\,\text{TAN}) &= \text{the class of square totally asymptotically nonexpansive mappings} \\
\text{(ANIS)} &= \text{the class of asymptotically nonexpansive mappings in the intermediate sense}
\end{aligned}
$$

We say that T is AN, ANIS and TAN, in abbreviated forms, for T belonging to the classes (AN), (ANIS) and (TAN).

Remark 2.7. The following implications holds.

(i) (N) $\subset$ (AN) $\subset$ (UL) $\subset$ (L) $\subset$ (UC).

(ii) (ANIS) $\cup$ (AN) $\subset$ (TAN).

(iii) Assume $\delta := \operatorname{diam}(C) < \infty$; then

$$
\begin{aligned}
\|T^n x - T^n y\| &\leq \|x-y\| + \alpha_n \sup_{x,y\in C} \tau(\|x-y\|) + \beta_n \\
&\leq \|x-y\| + \alpha_n \tau(\delta) + \beta_n \\
&= \|x-y\| + c_n, \quad x,y \in C,\ n \geq 1,
\end{aligned}
$$

where $c_n := \alpha_n \tau(\delta) + \beta_n \to 0$. Hence we conclude: if C is bounded and T is TAN, then T satisfies (2.5); in particular, (AN) $\subset$ (ANIS) by (i).

(iv) If C is bounded, then (TAN) $=$ (^{2} TAN).

(v) $Tx = \sqrt{x}$ is uniformly continuous but not Lipschitzian on $[0, \infty)$. Also, $Tx = 2x$ is Lipschitzian but not uniformly k-Lipschitzian.

Here we introduce an example of a Lipschiztian mapping which is AN but not nonexpansive. The following example is originally due to [13] in ℓ^2 spaces.

Example 2.8. ([23]; see Example 3.13). *Let B denote the unit ball in the space $X = \ell^p$, where $1 < p < \infty$. Obviously, X is uniformly convex and uniformly smooth. Let $T : B \to B$ be defined by*

$$Tx = (0, x_1^2, \lambda_1 x_2, \lambda_2 x_3, \ldots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in B$, where $0 < \lambda_n < 1$ for all $n \geq 1$ and $\prod_{n=1}^{\infty} \lambda_n = \frac{1}{2}$. Then:

(a) *T is Lipschitzian, i.e., $\|Tx - Ty\| \leq 2\|x - y\|, \quad x, y \in C$;*

(b) *T is AN, i.e., $\|T^{n+1}x - T^{n+1}y\| \leq 2\prod_{i=1}^{n-1} \lambda_i \|x - y\|, \quad x, y \in C,\ n \in \mathbb{N}$;*

(c) *T is not nonexpansive.*

Proof. Noticing that, for $x = (x_1, x_2, \ldots) \in B$,

$$T^n x = \Big(\overbrace{0, \ldots, 0}^{n}, \prod_{i=1}^{n-1} \lambda_i\, x_1^2, \prod_{i=1}^{n} \lambda_i\, x_2, \prod_{i=2}^{n+1} \lambda_i\, x_3, \ldots\Big).$$

Thus we have $\|T^n x - T^n y\| \leq 2\prod_{i=1}^{n-1} \lambda_i \|x - y\|$ for all $n \geq 2$. Obviously, since $2\prod_{i=1}^{n-1} \lambda_i \downarrow 1$, T is AN. On the other hand, since $\|Tx - Ty\| = \frac{3}{4} > \frac{1}{2} = \|x - y\|$ for $x = (1, 0, 0, \ldots)$ and $y = (1/2, 0, 0, \ldots)$, T is not nonexpansive. $\square$

Remark 2.9. Consider either $\lambda_n := 1 - \frac{1}{(n+1)^2}$ or $\lambda_n := \exp\left(\frac{1}{2n} - \frac{1}{2n-1}\right)$ to get a sequence satisfying that $0 < \lambda_n < 1$ and $\prod_{n=1}^{\infty} \lambda_n = \frac{1}{2}$. Indeed, for the second case, since $0 < \exp(-x) < 1$ for all $x > 0$, we must find a sequence $\{\alpha_n\}$ such that

$$\prod_{n=1}^{\infty} \lambda_n = \prod_{n=1}^{\infty} \exp(-\alpha_n) = \exp(-\sum_{n=1}^{\infty} \alpha_n) = \frac{1}{2}.$$

This is equivalent to

$$\sum_{n=1}^{\infty} \alpha_n = \ln 2 = \sum_{n=1}^{\infty} \Big(\frac{1}{2n-1} - \frac{1}{2n}\Big)$$

because

$$\begin{aligned} s_n &:= \sum_{k=1}^{n} \Big(\frac{1}{2k-1} - \frac{1}{2k}\Big) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \\ &= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+k/n} \to \int_0^1 \frac{1}{1+x} dx = \ln 2. \end{aligned}$$

Furthermore, since $1 - x \le \exp(-x)$ for all $x \in (0,1)$, we observe

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \Rightarrow \quad \prod_{n=1}^{\infty} (1 - \alpha_n) = 0$$

for all $\alpha_n \in (0,1)$ since

$$0 \le \prod_{n=1}^{\infty} (1 - \alpha_n) \le \prod_{n=1}^{\infty} \exp(-\alpha_n) = \exp(-\sum_{n=1}^{\infty} \alpha_n) = 0.$$

Recall that $\{f_n\}$ converges uniformly to f on C if

$$\|f_n - f\| := \sup_{x \in C} \|f_n(x) - f(x)\| \to 0$$

as $n \to \infty$, where $f_n,\, f : C\,(\subset X) \to C$. We say that $T^n x$ converges uniformly to p on C whenever $\{f_n := T^n\}$ converges uniformly to a point $p \in C$ on C, that is,

$$\sup_{x \in C} \|T^n x - p\| \to 0 \quad \text{as } n \to \infty.$$

Proposition 2.10. *If $T^n x$ converges uniformly to some point $p \in C$ on C, then T satisfies (2.5).*

Proof. Let $c_n := \sup_{x,y\in C} \|T^n x - T^n y\|$. Then $c_n \to 0$ since

$$0 \le c_n = \sup_{x,y\in C} \|T^n x - T^n y\| \le \sup_{x\in C} \|T^n x - p\| + \sup_{y\in C} \|p - T^n y\| \to 0$$

as $n \to \infty$. From the construction of c_n it readily follows that

$$\|T^n x - T^n y\| \le c_n, \quad x, y \in C,\ n \ge 1,$$

which immediately implies (2.5). $\square$

Remark 2.11. However, the converse of Proposition 2.10 does not hold in general; see Example 2.18.

As a direct consequence of Proposition 2.10, we introduce non-Lipschitzian mappings which are asymptotically nonexpansive in the intermediate sense.

Example 2.12. ([20]). *Let $C := \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$ and $0 < |k| < 1$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx := \begin{cases} kx \sin\frac{1}{x}, & \text{if } x \ne 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Then $F(T) = \{0\}$ and $T^n x$ converges uniformly to 0 on C. Since T is clearly uniformly continuous, it follows from Proposition 2.10 that T is ANIS. However, T is not Lipschitzian; see Example 4.3 of [20] for the proof.

Example 2.13. ([25]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} \alpha, & x \in [0, \alpha]; \\ \frac{\alpha}{\sqrt{1-\alpha}}\sqrt{1-x}, & x \in [\alpha, 1], \end{cases}$$

where $\alpha \in (0,1)$. Then $F(T)=\{\alpha\}$ and $T^n x = \alpha$ for all $x \in C$, $n \geq 2$; hence T is ANIS as in Example 2.12. However, T is not Lipschitzian; see Example 3.9 of [25] for the proof.

Example 2.14. ([17]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} (\sqrt{2}-1)\sqrt{\frac{1}{2}-x} + \frac{1}{\sqrt{2}}, & \text{if } 0 \leq x \leq 1/2; \\ \sqrt{x}, & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

Then $F(T) = \{1\}$ and $T^n x$ converges uniformly to 1 on C; hence T is ANIS as in Example 2.12. However, T is not Lipschitzian; see Example 1.2 of [17] for more details.

A mapping satisfying the property (2.3) do not always guarantee its *non-Lipschitz*. The following two examples are *uniformly Lipschitzian* ANIS mappings.

Example 2.15. ([24]). *Let $X = \mathbb{R}$ and $C = [0,1]$. For each $x \in C$, let $T : C \to C$ be defined by*

$$Tx = \begin{cases} kx, & \text{if } 0 \leq x \leq 1/2; \\ \frac{k}{2k-1}(k-x), & \text{if } 1/2 \leq x \leq k; \\ 0, & \text{if } k \leq x \leq 1, \end{cases}$$

where $1/2 < k < 1$. Then $F(T) = \{0\}$ and $T^n x$ converges uniformly to 0 on C. Obviously, T is uniformly continuous. By Proposition 2.10, T is ANIS. Furthermore, T is uniformly Lipschitzian. Indeed, if $0 \leq x \leq 1/2$ and $1/2 \leq y \leq k$, then $T^n x = k^n x$ and $T^n y = \frac{k^n}{2k-1}(k - x)$. Therefore, we see that

$$\begin{aligned}
|T^n x - T^n y| &= \Big|k^n x - \frac{k^n}{2} + \frac{k^n}{2} - \frac{k^n}{2k-1}(k-y)\Big| \\
&= \Big|k^n\Big(x - \frac{1}{2}\Big) + \frac{k^n}{2k-1}\Big[\Big(k - \frac{1}{2}\Big) - (k-y)\Big]\Big| \\
&\le k^n\Big|x - \frac{1}{2}\Big| + \frac{k^n}{2k-1}\Big|y - \frac{1}{2}\Big| \le \frac{k}{2k-1}|x-y|.
\end{aligned}$$

The remaining cases are obvious. Hence T is uniformly $\frac{k}{2k-1}$-Lipschitzian.

Example 2.16. ([17]; see Example 1.3). *For any $k > 0$, let $\{a_n\}$ be a sequence of positive numbers such that $a_n \downarrow 0$ and $\prod_{n=1}^{\infty}(1 + a_n) = k$. Set*

$$b_n := \frac{1}{2^{n+1}(1 + a_n)}, \quad n \ge 1.$$

Let $T : C \to C$ be defined by

$$Tx = \begin{cases} (1 + a_1)x + 1/2, & \text{if } x \in [0, b_1]; \\ 1/2 + 1/4, & \text{if } x \in [b_1, 1/2] \end{cases}$$

and

$$Tx = \begin{cases} (1 + a_n)\Big(x - \sum\limits_{i=1}^{n-1} \frac{1}{2^i}\Big) + \sum\limits_{i=1}^{n} \frac{1}{2^i}, & \text{if } x \in \Big[\sum\limits_{i=1}^{n-1} \frac{1}{2^i}, \sum\limits_{i=1}^{n-1} \frac{1}{2^i} + b_n\Big]; \\ \sum\limits_{i=1}^{n+1} \frac{1}{2^i}, & \text{if } x \in \Big[\sum\limits_{i=1}^{n-1} \frac{1}{2^i} + b_n, \sum\limits_{i=1}^{n} \frac{1}{2^i}\Big], \quad n \ge 2 \end{cases}$$

and $T1 = 1$. Then $F(T) = \{1\}$ and $T^n x$ converges uniformly to 1 on C. Since T is continuous on C, T is also uniformly continuous on C. By Proposition 2.10, T is ANIS. Furthermore, T is uniformly k-Lipschitzian.

Remark 2.17. (i) Since $1 + x \le e^x$ for all $x \in \mathbb{R}$, we easily find a sequence $\{a_n\}$ of positive numbers such that $a_n \downarrow 0$ and $\prod_{n=1}^{\infty}(1 + a_n) = k\,(> 1)$. Indeed, since the sequence $\{s_n\}$, $s_n := \prod_{k=1}^{n}(1+a_k)$, of its nth partial sums is strictly increasing and

$$\prod_{n=1}^{\infty}(1+a_n) \le \prod_{n=1}^{\infty} e^{a_n} = e^{\sum_{n=1}^{\infty} a_n} = e^{\ln k} = k,$$

it suffices to find a (convergent) geometric series replaced with $a_n := r^n,\ 0 \le r < 1$ such that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} r^n = \frac{r}{1-r} = \ln k$$

$$\Leftrightarrow \quad r = \frac{\ln k}{1+\ln k}.$$

(ii) Note that if we take $a_n \equiv 0$ in Example 2.16, then T is clearly nonexpansive.

As a slight modification of Example 2.16, we shall give an example of a uniformly Lipschitzian ANIS mapping defined a (unbounded) closed convex subset C on which is not converges uniformly.

Example 2.18. *Consider $C := [0,\infty) \subset \mathbb{R}$. Let T be defined on $[0,1]$ as in Example 2.16 and define $Tx = x$ on $[1,\infty)$. Since $T^n x$ converges uniformly to 1 on $[0,1]$, setting $c_n := \sup\{\|T^n x - T^n y\| : x, y \in [0,1]\} \to 0$, then T satisfies (2.5), i.e.,*

$$\|T^n x - T^n y\| \le \|x-y\| + c_n, \quad x, y \in C,\ n \ge 1.$$

In view of Example 2.16, T is uniformly k-Lipschitzian. Therefore, $T : C \to C$ is ANIS.

Now we introduce an example of a mapping which is k-lipschitzian involution but not ANT.

Example 2.19. ([22]). *Let* $X := \mathbb{R}$, $C := [-\frac{1}{k}, 1]$, *where* $1 < k < 2$. *Define a mapping* $T : C \to C$ *by*

$$Tx := \begin{cases} -kx, & \textit{if } -\frac{1}{k} \le x \le 0; \\ -\frac{1}{k}x, & \textit{if } 0 \le x \le 1. \end{cases}$$

Then:

(a) $T^2x = x$ *for all* $x \in C$ *(hence,* $T^{2n-1} = T$ *for all* $n \ge 1$*);*

(b) T *is uniformly* k*-lipschitzian;*

(c) T *does not satisfy (2.2); hence it is not ANT.*

Indeed, it suffice to show: T *is not ANT. To this end, for each* $x \in C$,

$$\begin{aligned} & \limsup_{n\to\infty} \sup_{y\in C} \{|T^n x - T^n y| - |x - y|\} \\ \ge \quad & \sup\{|Ty| - |y| : y \in [-1/k, 1]\} \\ = \quad & \sup\{(k-1)|y| : -1/k \le y \le 0\} \\ = \quad & (k-1)\frac{1}{k} = 1 - \frac{1}{k} > 0. \end{aligned}$$

Finally, we raise a question as follows.

Question 2.20. *Find either uniformly Lipschitzian or non-Lipschitzian mappings which are TAN but not ANIS.*

3 Projections on Banach spaces

Let X be a real normed space with its dual X^*. In 1994, Alber [2, 3, 4] introduced the Lyapunov functional $V : X \times X^* \to \mathbb{R}$ defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \quad x \in X,\, x^* \in X^*. \tag{3.1}$$

Here we list below some properties of the Lyapunov functional V from [4]:

(i) $V(x, x^*)$ is continuous;

(ii) $V(x, x^*)$ is convex relative to x when x^* is fixed and relative to x^* when x is fixed;

(iii) $(\|x\| - \|x^*\|)^2 \leq V(x, x^*) \leq (\|x\| + \|x^*\|)^2$;

(iv) $V(x, x^*) = 0$ if and only if $x^* \in J(x)$.

In fact, if $V(x, x^*) = 0$, by (iii) we see $||x|| = \|x^*\|$. Also, $\langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2$. From the definition of J, it follows that $x^* \in Jx$. The converse is obvious.

Let X be a smooth Banach space. Since the normalized duality mapping J from X to X^* is single-valued, the Lyapunov functional $\phi : X \times X \to \mathbb{R}$ is well defined as

$$\phi(x, y) = V(x, Jy) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad x, y \in X. \tag{3.2}$$

Also, we shall list some basic properties pertaining to the Lyapunov functional ϕ:

Proposition 3.1. (a) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad x, y \in X$.

(b) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad x, y, z \in X$.

(c) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\| \, \|Jx - Jy\| + \|y - x\| \, \|y\|$.

(d) *If X is strictly convex, then $\phi(x, y) = 0$ if and only if $x = y$ for any $x, y \in X$; see [30].*

(e) *$\phi(\cdot, y)$ is weakly lower semicontinuous on X; see Lemma 2.3.2 in [28]. Moreover, it is continuous and convex on X while $\phi(x, \cdot)$ is only continuous.*

(f) $\phi(\cdot, y)$ *is strictly convex if and only if* X *is strictly convex.*

(g) *If* X *is uniformly convex and smooth and* $\{x_n\}, \{z_n\}$ *are two sequences of* X *such that either* $\{x_n\}$ *or* $\{z_n\}$ *is bounded, then* $\phi(x_n, z_n) \to 0 \Leftrightarrow \|x_n - z_n\| \to 0$*; see Proposition 2.9 of [19] and Propposition 2.9 of [23].*

(h) *If* X *is reflexive, smooth, strictly convex Banach space, then* X *has the Kadec-Klee property if and only if* X *satisfies the following property (KT):*

(KT) *Given a sequence* $\{x_n\}$ *in a* $X \in (S)$ *and* $x(\neq 0) \in X$*,*
$\phi(x_n, x) \to 0$ *if and only if* $x_n \to x$*;*

see Proposition 2.10 of [23].

Theorem 3.2. ([39]). *Let* $r > 0$ *and let* X *be a Banach space. Then* X *is uniformly convex if and only if there exists* $g \in \Gamma[0, \infty)$ *such that*

$$\|x + y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle + g(\|y\|), \quad x, y \in B_r,\ j(x) \in Jx, \tag{3.3}$$

where $B_r := \{x \in X : \|x\| \leq r\}$ *and* $g \in \Gamma[0, \infty) \Leftrightarrow g : [0, \infty) \to [0, \infty)$ *is a continuous, strictly increasing, and convex function on* $[0, \infty)$ *with* $g(0) = 0$*.*

Using Theorem 3.2, Kamimura and Takahashi [19] proved the following result to deduce strong convergence of a proximal type algorithm in Banach spaces.

Proposition 3.3. ([19]) *Let* X *be a uniformly convex and smooth Banach space and let* $\{x_n\}, \{z_n\}$ *be two sequences of* X*. If* $\phi(x_n, z_n) \to 0$ *and either* $\{x_n\}$ *or* $\{z_n\}$ *is bounded, then* $\|x_n - z_n\| \to 0$*.*

Remark 3.4. Note that the converse of Proposition 3.3 remains true; see Proposition 2.7 of [23].

Proposition 3.5. ([23]; see Proposition 2.9). *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_n) \to 0$ if and only if $\|x_n - z_n\| \to 0$.*

More generally, we can prove

Proposition 3.6. *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}, \{z_n\}$ be two sequences of X. If either $\{x_n\}$ or $\{z_n\}$ is bounded, then $\phi(x_n, z_m) \to 0$ as $n, m \to \infty$ if and only if $\|x_n - z_m\| \to 0$ as $n, m \to \infty$.*

Proof. ($\Rightarrow$) Since $\phi(x_n, z_m) \to 0$ as $n, m \to \infty$, $\{\phi(x_n, z_m)\}$ is bounded. If one of $\{x_n\}$ and $\{z_m\}$ is bounded, so is the other by (a) of Proposition 3.1. By Theorem 3.2, there exists $g \in \Gamma_c(\mathbb{R}_+)$ such that

$$\begin{aligned} g(\|y_n - z_m\|) &\leq \|z_m - (y_n - z_m)\|^2 - \|z_m\|^2 - 2\langle y_n - z_m, Jz_m\rangle \\ &= \|y_n\|^2 - \|z_m\|^2 - 2\langle y_n, Jz_m\rangle + 2\|z_m\|^2 \\ &= \phi(y_n, z_m). \end{aligned}$$

It follows from $\phi(x_n, z_m) \to 0$ as $n, m \to \infty$ that $g(\|x_n - z_m\|) \to 0$ as $n, m \to \infty$. Then the properties of g yield that $\|x_n - z_m\| \to 0$ as $n, m \to \infty$.

($\Leftarrow$) Since $x_n - z_m \to 0$ as $n, m \to \infty$, it is not hard to see that if either $\{x_n\}$ or $\{z_m\}$ is bounded, then the other is also bounded. Now let $x \in X$ be fixed. Then noticing that

$$\begin{aligned} |\phi(x_n, x) - \phi(z_m, x)| &= |\,\|x_n\|^2 - \|z_m\|^2 + 2\langle z_m - x_n, Jx\rangle\,| \\ &\leq |\|x_n\| - \|z_m\||(\|x_n\| + \|z_m\|) + 2\|z_m - x_n\|\,\|x\|| \\ &\leq \|x_n - z_m\|(\|x_n\| + \|z_m\| + 2\|x\|) \to 0 \end{aligned}$$

as $n, m \to \infty$ and using the identity equation (b) of Proposition 3.1, we have

$$\begin{aligned} 0 \leq \phi(x_n, z_m) &= \phi(x_n, x) - \phi(z_m, x) + 2\langle x_n - z_m, Jx - Jz_m \rangle \\ &\leq |\, \phi(x_n, x) - \phi(z_m, x) \,| + 2\|x_n - z_m\|(\|x\| + \|z_m\|) \to 0 \end{aligned}$$

as $n, m \to \infty$. The proof is complete. □

Remark 3.7. In particular, taking $z_n = x_n$ for all $n \in \mathbb{N}$, it follows from Proposition 3.6 that $\phi(x_n, x_m) \to 0$ as $n, m \to \infty$ if and only if $\{x_n\}$ is a Cauchy sequence in X; hence it converges.

Let C be a nonempty subset of a real Banach space X. We say that C is said to be a *Chebyshev set* if to each $x \in X$ there exists a unique $x_0 \in C$ such that

$$\|x - x_0\| = d(x, C) = \inf_{y \in C} \|x - y\|.$$

In this case, we may define the *nearest point projection* (or called *metric projection*) $P_C : X \to C$ by assigning x_0 to x.

Proposition 3.8. ([15]; see Proposition 3.4; pp.13).

Let C be a convex Chebyshev set in X and $x \in X$. Then, $x_0 = P_C x$ if and only if there exists $j \in J(x - x_0)$ such that $\langle y - x_0, j \rangle \leq 0$ for all $y \in C$.

Let X be a reflexive and strictly convex Banach space and let C be a nonempty closed convex subset of X. For every (fixed) $x \in X$, consider $f(y) = \|x - y\|^2$ for $y \in X$. Then $f : X \to [0, \infty)$ is a proper strictly convex and continuous function and $f(y) \to \infty$ as $\|y\| \to \infty$. By Theorem 1.2 of [6], there exists a unique $x_0 \in C$ such that

$$\begin{aligned} & f(x_0) = \inf_{y \in C} f(y) \\ \Leftrightarrow \quad & \|x - x_0\| = \inf_{y \in C} \|x - y\| = d(x, C). \end{aligned} \tag{3.4}$$

Therefore, the closed convex subset C of a reflexive and strictly convex Banach space X is a Chebyshev set and hence $P_C : X \to C$ is a *nearest point projection* (or *metric projection*) on X. Combined with Proposition 3.8, we have the following

Proposition 3.9. *Let C be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space X. Then*

$$x_0 = P_C x \quad \Leftrightarrow \quad \langle y - x_0, J(x - x_0)\rangle \leq 0, \quad y \in C. \tag{3.5}$$

The following definition is originally due to Definition 6.2 in [4] based on uniformly convex and uniformly smooth Banach spaces.

Definition 3.10. ([29]; see Definition 1.1). Let X be a Banach space with its dual X^*. Let C be a nonempty closed convex subset of X. An operator $\pi_C : X^* \to 2^C$ is called a generalized projection on X^* if it associates with an arbitrary fixed point $x^* \in X^*$ the set of all minimal points of $V(x, x^*)$ over C, namely,

$$x \in \pi_C x^* \;\Leftrightarrow\; V(x, x^*) = \inf_{v \in C} V(v, x^*).$$

$\pi_C x^* \subset C$ is then called a generalized projection of the point x^*.

We list below some properties of the generalized projection π_C on X^* from [29]:

Proposition 3.11. (a) *If X is reflexive, then $\pi_C x^* (\neq \emptyset)$ is bounded, closed, and convex for any $x^* \in X^*$.*

(b) *For each $x \in C$ and $j_x \in J(x)$, we have $x \in \pi_C j_x$.*

(c) *If X is smooth, then, for given $x^* \in X^*$ and $x \in C$,*

$$x \in \pi_C x^* \quad \Leftrightarrow \quad \langle x - v, x^* - Jx\rangle \geq 0, \quad v \in C.$$

(d) *If X is smooth, then for given $x^* \in X^*$ and $x \in \pi_C x^*$, the following inequality holds:*

$$V(v, Jx) \leq V(v, x^*) - V(x, x^*), \quad v \in C.$$

(e) *The operator $\pi_C : X^* \to C$ is single valued if and only if X is strictly convex.*

(f) *If X is reflexive, smooth, strictly convex, and has the Kadec-Klee property, then $\pi_C : X^* \to C$ is continuous.*

Recall that if X is a smooth Banach space, then the Lyapunov functional $\phi : X \times X \to \mathbb{R}$ is well defined by

$$\phi(x, y) = V(x, Jy), \quad x, y \in X.$$

Proposition 3.12. ([19]: see Proposition 3).
Let X be a reflexive, smooth, and strictly convex Banach space, let C be a nonempty closed convex subset of X, and let $x \in X$. Then there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \inf_{z \in C} \phi(z, x).$$

Definition 3.13. ([2, 4, 19]). Then a mapping $\Pi_C : X \to C$ defined by $\Pi_C x = x_0$ for each $x \in X$ is called the *generalized projection*; see [2, 4, 19].

Remark 3.14. In Hilbert spaces, notice that the generalized projection is clearly coincident with the metric projection.

Proposition 3.15. ([2, 4, 19]) *Let C be a nonempty closed convex subset of a real Banach space X and let $x \in X$.*

(a) *If* X *is smooth, then*

$$x_0 = \Pi_C x \quad \Leftrightarrow \quad \langle y - x_0, Jx - Jx_0 \rangle \leq 0, \quad y \in C. \tag{3.6}$$

(b) *If* X *is reflexive, smooth, and strictly convex, then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad y \in C. \tag{3.7}$$

4 Nonlinear mappings relative to ϕ

Let X be a real Banach space and let C be a nonempty closed convex subset of X.

Definition 4.1. A point p in C is said to be an *asymptotic fixed point* of T [35] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$.

Definition 4.2. A mapping $f : C \to X$ is said to be *demiclosed* on C if the graph of f,

$$G(f) := \{(x, y) : x \in C, y = f(x)\}$$

is closed in $C \times X$, where C is equipped with the weak topology and X the strong topology. In other words, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup x, \ f(x_n) \to y \quad \Rightarrow \quad x \in C, \ f(x) = y.$$

See the page 108 of [14] for more related results.

Remark 4.3. Obviously, $F(T) \subset \hat{F}(T)$. Furthermore, note that if $I - T$ is demiclosed on C, then $\hat{F}(T) \subset F(T)$.

Definition 4.4. Let X be smooth. A mapping $T : C \to C$ is said to be *relatively totally asymptotically nonexpansive* (RTAN) if $F_T \neq \emptyset$, $F_T = \hat{F}_T$ and for each $n \geq 1$, there exist two nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n, \beta_n \to 0$, $\tau \in \Gamma(R_+)$ and $n_0 \in \mathbb{N}$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \alpha_n \tau(\phi(p, x)) + \beta_n, \quad x \in C,\ p \in F_T,\ n \geq n_0, \tag{4.1}$$

where $\tau \in \Gamma(R_+)$ if and only if τ is strictly increasing, continuous on R_+ and $\tau(0) = 0$. In particular, we say that T is *relatively asymptotically nonexpansive in the intermediate sense* (RANIS) if $\alpha_n \equiv 0$; *relatively asymptotically nonexpansive* (RAN) [1, 31, 26] if $\tau(t) = t$ for all $t \geq 0$ and $\beta_n \equiv 0$; *relatively nonexpansive* (RN) [8, 9, 10, 30, 23] if $\alpha_n,\ \beta_n \equiv 0$ in (4.3), in turns.

As an analogue, we accept the following dual concepts:

Definition 4.5. Let X be smooth. A mapping $T : C \to C$ is said to be *generalized total asymptotically nonexpansive* (GTAN) if $F_T \neq \emptyset$, $F_T = \hat{F}_T$ and there exist two nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n, \beta_n \to 0$, $\tau \in \Gamma(R_+)$ and $n_0 \in \mathbb{N}$ such that

$$\phi(T^n x, q) \leq \phi(x, q) + \alpha_n \tau(\phi(x, q)) + \beta_n, \quad x \in C,\ q \in F_T,\ n \geq n_0. \tag{4.2}$$

In particular, we say that T is *generalized asymptotically nonexpansive in the intermediate sense* (GANIS) if $\alpha_n \equiv 0$; *generalized asymptotically nonexpansive* (GAN) if $\tau(t) = t$ for all $t \geq 0$ and $\beta_n \equiv 0$; *generalized nonexpansive* (GN) [18] if $\alpha_n,\ \beta_n \equiv 0$ in (4.2), in turns.

Removing the condition $F_T = \hat{F}_T$ in the above definitions, we can define quasi-ϕ-nonlinear mappings.

Definition 4.6. Let X be smooth. A mapping $T : C \to C$ is said to be *totally quasi-ϕ-asymptotically nonexpansive* (TQ-ϕ-AN) [11, 36] if $F_T \neq \emptyset$ and there exist two nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_n, \beta_n \to 0$, $\tau \in \Gamma(R_+)$ and $n_0 \in \mathbb{N}$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \alpha_n \tau(\phi(p, x)) + \beta_n, \quad x \in C,\ p \in F_T,\ n \geq n_0. \tag{4.3}$$

In particular, we say that T is *quasi-ϕ-asymptotically nonexpansive in the intermediate sense* (Q-ϕ-ANIS) if $\alpha_n \equiv 0$; *quasi-ϕ-asymptotically nonexpansive* (Q-ϕ-AN) [32, 40] if $\tau(t) = t$ for all $t \geq 0$ and $\beta_n \equiv 0$; *quasi-ϕ-nonexpansive* (Q-ϕ-N) [32, 40] if $\alpha_n,\ \beta_n \equiv 0$ in (4.3), in turns.

We first introduce examples of quasi-ϕ-nonexpansive mapping.

Example 4.7. *From (3.7) note that if $X \in (R) \cap (S) \cap (SC)$, then*

$$\phi(y, \Pi_C x) \leq \phi(y, x), \quad x \in X,\ y \in C, \tag{4.4}$$

that is, $\Pi_C : X \to C$ is quasi-ϕ nonexpansive because of $F_{\Pi_C} = C \neq \emptyset$.

Example 4.8. ([30, 32]). *Let X be a Banach space and let $X \in (R) \cap (S) \cap (SC)$. Let A be a maximal monotone operator of X into X^* and $J_r = (J + rA)^{-1} J$ be the resolvent for A with $r > 0$ and $F(J_r) = A^{-1}0$. Then*

(a) *$J_r = (J + rA)^{-1} J : X \to D(A)$ is quasi-ϕ-nonexpansive; see [32].*

(b) *Furthermore, if $X \in (US)$, then $\hat{F}(J_r) = A^{-1}0 = F(J_r)$ and J_r is relatively nonexpansive; see [30].*

We next introduce an example of a mapping which is quasi-ϕ-nonexpansive but not relatively nonexpansive, which is originally due to [37].

Example 4.9. ([37]) *Let X be any smooth Banach space, and $x_0 \neq 0$ be any element of X. Define a mapping $T : E \to E$ as*

$$Tx = \begin{cases} \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right)x_0, & \text{if } x = (1/2 + 1/2^n)x_0; \\ -x, & \text{if } x \neq (1/2 + 1/2^n)x_0, \quad n \geq 1. \end{cases}$$

Then $F(T) = \{0\}$ and T is quasi-ϕ-nonexpansive but not relatively nonexpansive.

Also, we introduce an example of a uniformly Lipschitzian mapping which is relatively asymptotically nonexpansive but not relatively nonexpansive.

Example 4.10. ([26]; see Example 3.7). *Let $X = \ell^p$, where $1 < p < \infty$, and $C = \{x = (x_1, x_2, \ldots) \in X; x_n \geq 0\}$. Then C is a closed convex subset of X. Note that C is not bounded. Obviously, X is uniformly convex and uniformly smooth. Let $\{\lambda_n\}$ and $\{\bar{\lambda}_n\}$ be sequences of real numbers satisfying the following properties:*

(i) $0 < \lambda_n < 1$, $\bar{\lambda}_n > 1$, $\lambda_n \uparrow 1$ *and* $\bar{\lambda}_n \downarrow 1$,

(ii) $\lambda_{n+1}\,\bar{\lambda}_n = 1$ *and* $\bar{\lambda}_{n+j}\,\lambda_{j+1} < 1$ *for all n and j.*

(for examples, consider either $\lambda_n = 1 - \frac{1}{n+1}$, $\bar{\lambda}_n = 1 + \frac{1}{n+1}$ or $\lambda_n = e^{-1/n}$, $\bar{\lambda}_n = e^{1/(n+1)}$). Then we define $T : C \to C$ by

$$Tx = (0, \bar{\lambda}_1 |\sin x_1|, \lambda_2 x_2, \bar{\lambda}_2 x_3, \lambda_3 x_4, \bar{\lambda}_3 x_5, \cdots)$$

for all $x = (x_1, x_2, x_3, \ldots) \in C$. Obviously, $F(T) = \{0\}$, where $0 = (0, 0, \ldots) \in C$, and T is both AN and RAN. However, T is not relatively nonexpansive.

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