Thesis for the Degree of Doctor of Philosophy

Some Aggregation Operators in Group Decision Making

by

Jung Mi Park

Department of Applied Mathematics

The Graduate School

Pukyong National University

August 2013

Some Aggregation Operators in Group Decision Making

집단의사결정에서의 집성연산자

Advisor : Prof. Jin Han Park

by
Jung Mi Park

A thesis submitted in partial fulfillment of the requirement
for the degree of

Doctor of Philosophy

in the Department of Applied Mathematics, The Graduate School,
Pukyong National University

August 2013

Some Aggregation Operators in Group Decision Making

A Dissertation

by

Jung Mi Park

Approved by :

(Chairman) Young Chel Kwun, Ph. D.

(Member) Yong-Soo Pyo, Ph. D.

(Member) Sung-Jin Cho, Ph. D.

(Member) Jong Jin Seo, Ph. D.

(Member) Jin Han Park, Ph. D.

August 23, 2013

CONTENTS

집단의사결정에서의 집성연산자

박 정 미

부경대학교 대학원 응용수학과

요 약

본 논문에서는 불확실한 환경 혹은 언어적 환경에서의 집단의사결정문제를 해결하기 위한 집성연산자를 소개하고 그 해결방안을 제안한 것으로 그 내용은 다음과 같이 요약된다.

첫째로, 멱조화(PH)연산자와 멱순위가중조화(POWH)연산자를 소개하고 그들의 성질을 찾고 PH와 POWH연산자들을 구간수를 포함하는 초기 변수들을 집성할 연산자들을 제안할 수 있는 불확실한 환경으로 확장한다. 더욱이 위의 연산자들을 이용하여 집단의사결정문제의 해결방안을 제시하고, 수치적인 예를 통해 해결방안의 적용을 설명한다.

둘째로, 언어적 변수 형태를 택하는 선호정보를 집성하기 위해 2-튜플언어적조화(2TLH)연산자, 2-튜플언어적가중조화(2TLWH)연산자 등과 같은 새로운 언어적집성연산자를 소개하고, 2TLWH연산자와 2TLHH연산자를 기반으로 2-튜플언어적 정보를 갖는 다속성의사결정에 대한 해결방안을 제안하고, 예를 통해 기존의 해결방안과 비교함으로서 제안된 방안의 실제성과 효용성을 보인다.

셋째로, 초기 변수에 의존하는 가중벡터를 갖는 비선형가중평균집성 도구인 몇가지 2-튜플언어적선순위집성연산자를 소개하고, 우선순위가 다른 속성을 갖는 언어적 정보를 위하는 다속성집단의사결정의 해결방안을 제안하기 위해 그들을 사용하며 수치적인 예를 통하여 해결방안의 적용을 설명한다.

Chapter 1

Introduction

Information aggregation is an essential process of gathering relevant information from multiple sources by using a proper aggregation technique. Many techniques, such as the weighted average operator [12], the weighted geometric mean operator [1], harmonic mean operator [4], weighted harmonic mean (WHM) operator [4], ordered weighted average (OWA) operator [56], ordered weighted geometric operator [5, 51], weighted OWA operator [31], induced OWA operator [63], induced ordered weighted geometric operator [53], uncertain OWA operator [52], hybrid aggregation operator [41], linguistic aggregation operators [59, 16, 19, 42, 26, 27] and so on have been developed to aggregate data information. However, yet most of existing aggregation operators do not take into account the information about the relationship between the values being fused. Yager [58] introduced a tool to provide more versatility in the information aggregation process, i.e., developed a power average (PA) operator and a power OWA (POWA) operator. In some situations, however, these two operators are unsuitable to deal with the arguments taking the forms of multiplicative variables because of lack of knowledge, or data, and decision makers' limited expertise related to the problem domain. So, based on this tool, Xu and Yager [55] developed additional new geometric aggregation operators, including the power geometric (PG) operator, weighted PG operator and power ordered weighted geometric (POWG) operator, whose weighting vectors depend upon the input arguments and allow values being aggregated to support and reinforce each other, and applied them to group decision making based on multiplicative preference relations.

Group decision making (i.e., multi-expert) is a typical decision making activity where utilizing several experts alleviate some of the decision making difficulties due to the problem's complexity and uncertainty. In the real world, the uncertainty, constraints, and even unclear knowledge of the experts imply that decision makers cannot provide exact numbers to express their opinions. Linguistic variables are a very useful tool to express a decision maker's preference information over objects in process of decision making under uncertain or vague environments [64, 65]. In order to get a decision result, an important step is the aggregation of linguistic variables. Over the last decades, various linguistic aggregation operators have been proposed. We classify these operators into the following categories: (1) the linguistic aggregation operators are based on the semantic model, such as the linguistic approximation operator [7], linguistic OWA operator [3, 14, 16, 18, 8, 10], linguistic weighted OWA operator [31] and inverse-LOWA operator [13], these operators use linguistic terms as labels for fuzzy numbers while the computations over them are done directly over those fuzzy numbers; (2) the linguistic aggregation operators based on the symbolic model [14, 9, 28], which make computations on the indexes of the linguistic labels; (3) the linguistic aggregation operators, which compute with words directly, such as the linguistic weighted averaging (LWA) operator [48], extended ordered weighted averaging (EOWA) operator [44], extended ordered weighted geometric (EOWG) operator [44], linguistic weighted arithmetic averaging (LWAA) operator [34, 37], linguistic weighted geometric averaging (LWGA) operator [42], linguistic ordered weighted geometric averaging (LOWGA) operator [42], linguistic hybrid geometric averaging (LHGA) operator [42], uncertain LWA (ULWA) operator [43, 38, 36], uncertain linguistic hybrid aggregation (ULHA) operator [43], induced uncertain LOWA (IULOWA) operator [47], uncertain linguistic ordered weighted geometric (ULOWG) operator [46], induced uncertain linguistic ordered weighted geometric (IULOWG) operator [46], induced linguistic generalized ordered weighted averaging (ILGOWA) operator [25], induced linguistic generalized hybrid averaging (ILGHA) operator [25], induced linguistic quasi-arithmetic OWA (Quasi-ILOWA) operator [25] and linguistic power aggregation operator [66]; (4) the linguistic aggregation operators based on the 2-tuple linguistic representation model [19, 20], which represent the linguistic information with a pair of values called 2-tuple, composed by a linguistic term and number, including 2-tuple weighted averaging operator [19], 2-tuple OWA operator [19], 2-tuple weighted geometric averaging (TWGA) operator [39], 2-tuple ordered weighted geometric averaging (TOWGA) operator [39] and 2-tuple hybrid geometric averaging (THGA) operator [39]. The operators in (1) and (2) develop some approximation processes to express the results in initial expression domain, which produce the consequent loss of information and hence bring about the lack of precision, while the operators in (3) and (4) allow a continuous representation of the linguistic information on its domain, and thus they can represent any counting of information obtained in an aggregation process without any loss of information.

In this thesis, we consider and study three methods to solve multiple attribute group decision making (MAGDM) problems under uncertain or linguistic environment. We briefly summarize the contents of each chapter as follows.

Harmonic mean is a conservative average, which is widely used to aggregate central tendency data. In Chapter 2, we develop some new harmonic aggregation operators, including the power-harmonic (PH) operator, weighted PH operator, and power-ordered weighted harmonic (POWH) operator, and apply them to group decision making. In order to do this, we first review some aggregation operators, including the PA, PG, POWA and POWG operators. Then, we develop a PH operator and its weighted form based on the PA (or PG) operator and the harmonic mean, and a POWH operator based on the POWA (POWG) operator and the harmonic mean, and investigate some of their properties, such as commutativity, idempotency and boundedness. The relationship among the PA, PG and PH operators and the relationship the POWA, POWG and POWH operators are also discussed. We utilize the weighted PH and POWH operators, respectively, to develop an approach to group decision making. Furthermore, we extend the developed operators to the uncertain environment and develop an approach to group decision making based on uncertain preference relations.

In the existing literature, the harmonic mean is generally considered as a fusion technique of numerical data information. However, in many situations, the input arguments take the form of 2-tuple linguistic variables because of time pressure, lack of knowledge and people's limited expertise related with problem domain. Therefore, "how to aggregate 2-tuple linguistic variables by using the harmonic mean?" is an interesting research topic and is worth paying attention to. In Chapter 3, we focus our attention on developing some 2-tuple linguistic harmonic (2TLH) operators. To do so, we present the basic concept related to 2-tuple linguistic representation, and develops some 2TLH operators, such as 2-tuple linguistic weighted harmonic (2TLWH) operator, 2-tuple linguistic ordered weighted harmonic (2TLOWH) operator and 2-tuple linguistic hybrid harmonic (2TLHH) operator, and investigate some of their properties. An approach to MAGDM based on the developed operators is presented. We illustrate the presented approach with a practical example, and verify and show the advantages of the presented approach and makes a comparative study to the existing approach.

More and more multiple attribute decision making theories and methods under linguistic environment have been developed. Current methods are under the assumption that the attributes are at the same priority level. However, in real and practical multiple attribute decision making problem, the attributes generally have different priority levels. To overcome this drawback, motivated by the idea of prioritized aggregation operators [61], in Chapter 4, we develop some 2-tuple linguistic prioritized aggregation operators such as 2-tuple linguistic prioritized weighted harmonic (2TLPWH) operator and 2-tuple linguistic prioritized ordered weighted harmonic (2TLPOWH) operator. The prominent characteristic of these operators is that they take into account prioritization among the attributes. Then, we apply them to group decision making, with linguistic information, in which the attributes are in different priority levels. Finally, an example is used to illustrate the applicability of the developed approach.

Chapter 2

Power harmonic operators and their applications in group decision making

The power average (PA) operator, power geometric (PG) operator, power ordered weighted average (POWA) operator and power ordered weighted geometric (POWG) operator are the nonlinear weighted aggregation tools whose weighting vectors depend on input arguments. In this chapter, we develop a power harmonic (PH) operator and a power ordered weighted harmonic (POWH) operator, and study some properties of these operators. Then we extends the PH and POWH operators to uncertain environments, i.e, develop an uncertain PH (UPH) operator and its weighted form, and uncertain POWH (UPOWH) operator to aggregate the input arguments taking the form of interval numbers. Moreover, we utilize the weighted PH and POWH operators, respectively, to develop an approach to group decision making based on preference relations and utilize the weighted UPH and UPOWH operators, respectively, to develop an approach to group decision making based on uncertain preference relations. Finally, an example is used to illustrate the applicability of both the developed approaches.

2.1 Power harmonic operators

Yager [58] introduced a nonlinear weighted average aggregation operation tool, which is called PA operator, and can be defined as follows:

$$\mathrm{PA}(a_1, a_2, \ldots, a_n) = \frac{\sum_{i=1}^{n}(1+T(a_i))a_i}{\sum_{i=1}^{n}(1+T(a_i))}, \tag{2.1}$$

where

$$T(a_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(a_i, a_j) \tag{2.2}$$

and $\mathrm{Sup}(a, b)$ is the support for a from b, which satisfies the following three properties: 1) $\mathrm{Sup}(a, b) \in [0, 1]$, 2) $\mathrm{Sup}(a, b) = \mathrm{Sup}(b, a)$, 3) $\mathrm{Sup}(a, b) \geq \mathrm{Sup}(x, y)$ if $|a - b| < |x - y|$.

Yager [58], based on the OWA operator [56] and PA operator, also defined a POWA operator as follows:

$$\mathrm{POWA}(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} u_i a_{\mathrm{index}(i)}, \tag{2.3}$$

where index is an indexing function such that $\mathrm{index}(i)$ is the index of the ith largest of the arguments a_j $(j = 1, 2, \ldots, n)$, and thus $a_{\mathrm{index}(i)}$ is the ith largest argument of a_j $(j = 1, 2, \ldots, n)$, and u_i $(i = 1, 2, \ldots, n)$ are a collection of weights such that

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right),\ R_i = \sum_{j=1}^{i} V_{\mathrm{index}(j)},\ TV = \sum_{i=1}^{n} V_{\mathrm{index}(i)},$$
$$V_{\mathrm{index}(i)} = 1 + T(a_{\mathrm{index}(i)}) \tag{2.4}$$

where $g : [0, 1] \to [0, 1]$ is a basic unit-interval monotone (BUM) function having the following properties: 1) $g(0) = 0$, 2) $g(1) = 1$, 3) $g(x) \geq g(y)$ if $x > y$, and $T(a_{\mathrm{index}(i)})$ denotes the support of the ith largest argument by all the other arguments, i.e.,

$$T(a_{\mathrm{index}(i)}) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(a_{\mathrm{index}(i)}, a_{\mathrm{index}(j)}), \tag{2.5}$$

where $\mathrm{Sup}(a_{\mathrm{index}(i)}, a_{\mathrm{index}(j)})$ indicates the support of the jth largest argument for the ith largest argument.

Based on the PA operator and the geometric mean, in the following, Xu and Yager [55] defined the PG operator:

$$\mathrm{PG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}} \tag{2.6}$$

where a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, and $T(a_i)$ satisfies the condition (2.2). Based on the POWA operator and the geometric mean, Xu and Yager [55] also defined the power ordered weighted geometric (POWG) operator as follows:

$$\mathrm{POWG}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_{\mathrm{index}(i)}^{u_i} \tag{2.7}$$

which satisfies the conditions (2.4) and (2.5), and $a_{\mathrm{index}(i)}$ is the ith largest argument of a_j $(j = 1, 2, \ldots, n)$.

Based on PA operator and the harmonic mean, in the following, we define a PH operator:

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))a_i}} \tag{2.8}$$

where a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, and $T(a_i)$ satisfies the condition (2.2). Clearly, the PH operator is a nonlinear weighted harmonic aggregation operator, and the weight $\frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))}$ of the argument a_i depends on all the input arguments a_j $(j = 1, 2, \ldots, n)$ and allows the argument values to support each other in the harmonic aggregation process.

Lemma 2.1.1 [21, 22, 67] *Letting* $x_i > 0$, $\alpha_i > 0$, $i = 1, 2, \ldots, n$, *and* $\sum_{i=1}^{n} \alpha_i = 1$, *then*

$$\frac{1}{\sum_{i=1}^{n} \frac{\alpha_i}{x_i}} \leq \prod_{i=1}^{n} (x_i)^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i x_i \tag{2.9}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

By Lemma 2.1.1, we have the following theorem.

Theorem 2.1.2 *Assuming that* a_j $(j = 1, 2, \ldots, n)$ *are a collection of arguments, then we have*

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) \le \mathrm{PG}(a_1, a_2, \ldots, a_n) \le \mathrm{PA}(a_1, a_2, \ldots, a_n). \tag{2.10}$$

Now, we discuss some properties of the PH operator.

Theorem 2.1.3 *Letting* $\mathrm{Sup}(a_i, a_j) = k$*, for all* $i \ne j$*, then*

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \tag{2.11}$$

which indicates that when all supports are the same, the PH operator is simply the harmonic mean.

Especially, if $\mathrm{Sup}(a_i, a_j) = 0$ for all $i \ne j$, i.e., all the supports are zero, then there is no support in the harmonic aggregation process, and in this case, by the condition (2.2), we have $T(a_i) = 0$, $i = 1, 2, \ldots, n$, then

$$\frac{1 + T(a_i)}{\sum_{i=1}^{n}(1 + T(a_i))} = \frac{1}{n},\ i = 1, 2, \ldots, n \tag{2.12}$$

and thus, by (2.8) and (2.12), it is clear that the PH operator reduces to the harmonic mean.

Theorem 2.1.4 *Let* a_j $(j = 1, 2, \ldots, n)$ *be a collection of arguments, then we have the following properties.*

1) (Commutativity): If $(a'_1, a'_2, \ldots, a'_n)$ *is any permutation of* $(a_1, a_2, \ldots, a_n)$*, then*

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = \mathrm{PH}(a'_1, a'_2, \ldots, a'_n). \tag{2.13}$$

2) (Idempotency): If $a_j = a$ *for all* j*, then*

$$\mathrm{PH}(a_1, a_2, \ldots, a_n) = a. \tag{2.14}$$

3) (Boundedness):

$$\min_i a_i \le \mathrm{PH}(a_1, a_2, \ldots, a_n) \le \max_i a_i. \tag{2.15}$$

In (2.8), all the objects that are being aggregated are of equal importance. In many situations, the weights of the objects should be taken into account, for example, in group decision making, the decision makers usually have different importance and thus, need to be assigned different weights. Suppose that each object that is being aggregated has a weight indicating its importance, then we define the weighted form of (2.8) as follows:

$$\mathrm{PH}_w(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{w_i(1+T'(a_i))}{\sum_{i=1}^{n} w_i(1+T'(a_i))a_i}} \tag{2.16}$$

where

$$T'(a_i) = \sum_{j=1, j\neq i}^{n} w_j \mathrm{Sup}(a_i, a_j) \tag{2.17}$$

with the condition

$$w_i \in [0, 1],\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} w_i = 1. \tag{2.18}$$

Obviously, the weighted PH operator has the properties, as described in Theorem 2.1.2, as well as 2) and 3) of Theorem 2.1.4. However, Theorem 2.1.3 and 1) of Theorem 2.1.4 do not hold for the weighted PH operator.

Based on the POWA operator and the harmonic mean, we define a power ordered weighted harmonic (POWH) operator as follows:

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{u_i}{a_{\mathrm{index}(i)}}} \tag{2.19}$$

which satisfies the conditions (2.4) and (2.5), and $a_{\mathrm{index}(i)}$ is the ith largest argument of a_j $(j = 1, 2, \ldots, n)$.

Especially, if $g(x) = x$, then the POWH operator reduces to the PH operator, In fact, by (2.4), we have

$$\mathrm{POWH}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{i=1}^{n} \frac{u_i}{a_{\mathrm{index}(i)}}} = \frac{1}{\sum_{i=1}^{n} \frac{g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right)}{a_{\mathrm{index}(i)}}}$$

$$= \frac{1}{\sum_{i=1}^{n} \frac{\frac{R_i}{TV} - \frac{R_{i-1}}{TV}}{a_{\text{index}(i)}}} = \frac{1}{\sum_{i=1}^{n} \frac{\frac{V_{\text{index}(i)}}{TV}}{a_{\text{index}(i)}}}$$

$$= \frac{1}{\sum_{i=1}^{n} \frac{1+T(a_i)}{\sum_{i=1}^{n}(1+T(a_i))a_i}}$$

$$= \text{PH}(a_1, a_2, \ldots, a_n). \quad (2.20)$$

By Lemma 2.1.1, we the following theorem.

Theorem 2.1.5 *Assuming that a_j $(j = 1, 2, \ldots, n)$ are a collection of arguments, then we have*

$$\text{POWH}(a_1, a_2, \ldots, a_n) \leq \text{POWG}(a_1, a_2, \ldots, a_n) \leq \text{POWA}(a_1, a_2, \ldots, a_n). \quad (2.21)$$

From Theorem 2.1.3 and (2.20), we have the following corollary.

Corollary 2.1.6 *Letting $\text{Sup}(a_i, a_j) = k$ for all $i \neq j$, and $g(x) = x$, then we have*

$$\text{POWH}(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} \quad (2.22)$$

which indicates that when all supports are the same, the POWH operator is simply the harmonic mean.

Similar to Theorem 2.1.4, we have the following theorem.

Theorem 2.1.7 *Let a_j $(j = 1, 2, \ldots, n)$ be a collection of arguments, then we have the following properties.*

1) (Commutativity): If $(a'_1, a'_2, \ldots, a'_n)$ is any permutation of $(a_1, a_2, \ldots, a_n)$, then

$$\text{POWH}(a_1, a_2, \ldots, a_n) = \text{POWH}(a'_1, a'_2, \ldots, a'_n). \quad (2.23)$$

2) (Idempotency): If $a_j = a$ for all j, then

$$\text{POWH}(a_1, a_2, \ldots, a_n) = a. \quad (2.24)$$

3) (Boundedness):

$$\min_i a_i \leq \text{POWH}(a_1, a_2, \ldots, a_n) \leq \max_i a_i. \tag{2.25}$$

From the above-mentioned theoretical analysis, the difference between the weighted PH and POWH operators is that the weighted PH operator emphasizes the importance of each argument, while the POWH operator weights the importance of the ordered position of each argument.

2.2 Approach to group decision making

Let us consider a group decision making problem. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives and let $D = \{d_1, d_2, \ldots, d_m\}$ be a set of decision makers, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, with $w_k \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^m w_k = 1$. The decision maker d_k compare each pair of alternatives (x_i, x_j) and provides his/her preference value $a_{ij}^{(k)}$ over them and constructs the preference relation A_k on the set X, which is defined as a matrix $A_k = (a_{ij}^{(k)})_{n\times n}$ under the following condition:

$$a_{ij}^{(k)} \geq 0, \quad a_{ij}^{(k)} + a_{ji}^{(k)} = 1, \quad a_{ii}^{(k)} = \frac{1}{2}, \quad \text{for all } i, j = 1, 2, \ldots, n. \tag{2.26}$$

Then, we utilize the weighted PH operator to develop an approach to group decision making based on preference relations, which involves the following steps.

Approach I.
Step 1: Calculate the supports

$$\text{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)}) = 1 - \frac{|a_{ij}^{(k)} - a_{ij}^{(l)}|}{\sum_{l=1, l\neq k}^m |a_{ij}^{(k)} - a_{ij}^{(l)}|}, \; l = 1, 2, \ldots, m \tag{2.27}$$

which satisfy the support condition 1)-3) in Section 2.1.

Especially, if $\sum_{l=1, l\neq k}^m |a_{ij}^{(k)} - a_{ij}^{(l)}| = 0$, then we stipulate $\text{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)}) = 1$.

Step 2: Utilize the weights w_k $(k = 1, 2, \ldots, m)$ of the decision makers d_k $(k = 1, 2, \ldots, m)$ to calculate the weighted support $T'(a_{ij}^{(k)})$ of the preference value $a_{ij}^{(k)}$ by the other preference values $a_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T'(a_{ij}^{(k)}) = \sum_{l=1, l\neq k}^{m} w_l \mathrm{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)}) \tag{2.28}$$

and calculate the weights $v_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$ associated with the preference values $a_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$

$$v_{ij}^{(k)} = \frac{w_k \left(1 + T'(a_{ij}^{(k)})\right)}{\sum_{k=1}^{m} w_k \left(1 + T'(a_{ij}^{(k)})\right)}, \ k = 1, 2, \ldots, m \tag{2.29}$$

where $v_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} v_{ij}^{(k)} = 1$.

Step 3: Utilize the weighted PH operator to aggregate all the individual preference relations $A_k = (a_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective preference relation $A = (a_{ij})_{n\times n}$, where

$$a_{ij} = \mathrm{PH}_w(a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(m)}) = \frac{1}{\sum_{k=1}^{m} \frac{v_{ij}^{(k)}}{a_{ij}^{(k)}}}, \quad i, j = 1, 2, \ldots, n. \tag{2.30}$$

Step 4: Utilize the normalizing rank aggregation method (NRAM) [54] given by

$$v_i = \frac{\sum_{j=1}^{n} a_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}}, \ i = 1, 2, \ldots, n \tag{2.31}$$

to derive the priority vector $v = (v_1, v_2, \ldots, v_n)^T$ of $A = (a_{ij})_{n\times n}$, where $v_i > 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} v_i = 1$.

Step 5: Rank all alternatives x_i $(i = 1, 2, \ldots, n)$ in accordance with the priority weights v_i $(i = 1, 2, \ldots, n)$. The more the wight v_i, the better the alternative x_i will be.

In the case where the information about the weights of decision makers is unknown, then we utilize the POWH operator to develop an approach to group decision making based on preference relations, which can be described as follows.

Approach II.

Step 1: Calculate the supports

$$\mathrm{Sup}(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}) = 1 - \frac{|a_{ij}^{\mathrm{index}(k)} - a_{ij}^{\mathrm{index}(l)}|}{\sum_{l=1,l\neq k}^{m} |a_{ij}^{\mathrm{index}(k)} - a_{ij}^{\mathrm{index}(l)}|},\ l = 1, 2, \ldots, m \quad (2.32)$$

which indicates the support of the lth largest preference value $a_{ij}^{\mathrm{index}(l)}$ for the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$ of $a_{ij}^{(s)}$ $(s = 1, 2, \ldots, m)$. Especially, if $\sum_{l=1,l\neq k}^{m} |a_{ij}^{\mathrm{index}(k)} - a_{ij}^{\mathrm{index}(l)}| = 0$, then we stipulate $\mathrm{Sup}(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}) = 1$. It is necessary to point out that the support measure is a similarity measure, which can be used to measure the degree that a preference value provided by a decision maker is close to another one provided by other decision maker in a group decision making problem. Thus, $\mathrm{Sup}(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)})$ denotes the similarity degree between the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$ and the lth largest preference value $a_{ij}^{\mathrm{index}(l)}$.

Step 2: Calculate the support $T(a_{ij}^{\mathrm{index}(k)})$ of the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$ by the other preference values $a_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T(a_{ij}^{\mathrm{index}(k)}) = \sum_{l=1,l\neq k}^{m} \mathrm{Sup}(a_{ij}^{\mathrm{index}(k)}, a_{ij}^{\mathrm{index}(l)}) \quad (2.33)$$

and by (2.4), calculate the weight $u_{ij}^{(k)}$ associated with the kth largest preference value $a_{ij}^{\mathrm{index}(k)}$, where

$$u_{ij}^{(k)} = g\left(\frac{R_{ij}^{(k)}}{TV_{ij}}\right) - g\left(\frac{R_{ij}^{(k-1)}}{TV_{ij}}\right),\ R_{ij}^{(k)} = \sum_{l=1}^{k} V_{ij}^{\mathrm{index}(l)},$$
$$TV_{ij} = \sum_{l=1}^{m} V_{ij}^{\mathrm{index}(l)}, V_{ij}^{\mathrm{index}(l)} = 1 + T(a_{ij}^{\mathrm{index}(l)}) \quad (2.34)$$

where $u_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} u_{ij}^{(k)} = 1$, and g is the BUM function described in Section 2.1.

Step 3: Utilize the POWH operator to aggregate all the individual preference relations $A_k = (a_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective preference relation

$A = (a_{ij})_{n\times n}$, where

$$a_{ij} = \mathrm{POWH}(a_{ij}^{(1)}, a_{ij}^{(2)}, \ldots, a_{ij}^{(m)}) = \frac{1}{\sum_{k=1}^{m} \frac{u_{ij}^{(k)}}{a_{ij}^{\mathrm{index}(k)}}}, \quad i, j = 1, 2, \ldots, n. \quad (2.35)$$

Step 4: For this step, see Approach I.

Step 5: For this step, see Approach I.

In the above-mentioned two approaches, Approach I considers the situations where the weighted PH operator to aggregate all the individual preference relations into the collective preference relation and then the NRAM method to derive its priority vector, and using this, we can rank and select the given alternatives. While Approach II considers the situations where the information about the weights of decision makers is unknown and utilizes the POWH operator to aggregate all the individual preference relations into collective preference relation, then it also uses the NRAM method to find the final decision result.

2.3 Uncertain power harmonic operators

In this section, we consider the situations where the input arguments cannot be expressed in exact numerical values, but value range (i.e., interval numbers) can be obtained. We first review some operational laws, which are related to interval numbers [35, 2].

Let $\tilde{a} = [a^L, a^U]$ and $\tilde{b} = [b^L, b^U]$ be two interval numbers, where $a^U \geq a^L > 0$ and $b^U \geq b^L > 0$, then we have the following operational laws.

1) $\tilde{a} + \tilde{b} = [a^L, a^U] + [b^L, b^U] = [a^L + b^L, a^U + b^U]$.

2) $\tilde{a}\tilde{b} = [a^L, a^U] \cdot [b^L, b^U] = [a^l b^L, a^U, b^U]$.

3) $\lambda\tilde{a} = \lambda[a^L, a^U] = [\lambda a^L, \lambda a^U]$, where $\lambda > 0$.

4) $\frac{1}{\tilde{a}} = \frac{1}{[a^L, a^U]} = [\frac{1}{a^U}, \frac{1}{a^L}]$.

5) $\frac{\tilde{a}}{\tilde{b}} = \frac{[a^L, a^U]}{[b^L, b^U]} = [\frac{a^L}{b^U}, \frac{a^U}{b^L}]$.

In order to rank interval numbers, we now introduce a possibility degree formula [11] for the comparison between the interval numbers $\tilde{a} = [a^L, a^U]$ and

$\tilde{b} = [b^L, b^U]$:

$$p(\tilde{a} \geq \tilde{b}) = \min\left\{\max\left(\frac{a^U - b^L}{a^U - a^L + b^U - b^L}, 0\right), 1\right\}, \tag{2.36}$$

where $p(\tilde{a} \geq \tilde{b})$ is called the possibility degree of $\tilde{a} \geq \tilde{b}$, which satisfies

$$0 \leq p(\tilde{a} \geq \tilde{b}) \leq 1,\ p(\tilde{a} \geq \tilde{b}) + p(\tilde{b} \geq \tilde{a}) = 1,\ p(\tilde{a} \geq \tilde{a}) = 0.5. \tag{2.37}$$

Let $\tilde{a}_j = [a_j^L, a_j^U]$ $(j = 1, 2, \ldots, n)$ be a collection of interval numbers, then based on the previous operational laws of interval numbers, we extend the PH operator to uncertain environments and define an UPH operator as follows:

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{i=1}^{n} \frac{1+T(\tilde{a}_i)}{\sum_{i=1}^{n}(1+T(\tilde{a}_i))\tilde{a}_i}} \tag{2.38}$$

where

$$T(\tilde{a}_i) = \sum_{j=1, j\neq i}^{n} \mathrm{Sup}(\tilde{a}_i, \tilde{a}_j) \tag{2.39}$$

and $\mathrm{Sup}(\tilde{a}, \tilde{b})$ is the support for $\tilde{a}$ from $\tilde{b}$, which satisfies the following three properties: 1) $\mathrm{Sup}(\tilde{a}, \tilde{b}) \in [0, 1]$, 2) $\mathrm{Sup}(\tilde{a}, \tilde{b}) = \mathrm{Sup}(\tilde{b}, \tilde{a})$, 3) $\mathrm{Sup}(\tilde{a}, \tilde{b}) \geq \mathrm{Sup}(\tilde{x}, \tilde{y})$ if $d(\tilde{a}, \tilde{b}) < d(\tilde{x}, \tilde{y})$, where d is a distance measure.

Similar to the PH operator, the UPH operator has the following properties.

Theorem 2.3.1 *Letting* $\mathrm{Sup}(\tilde{a}_i, \tilde{a}_j) = k$ *for all* $i \neq j$, *then*

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{\tilde{a}_i}} \tag{2.40}$$

which indicates that when all the supports are the same, the UPH operator is simply the uncertain harmonic mean.

Theorem 2.3.2 *Let* $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ *be a collection of interval numbers, then we have the following properties.*

1) (Commutativity): If $(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n')$ *is any permutation of* $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, *then*

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \mathrm{UPH}(\tilde{a}_1', \tilde{a}_2', \ldots, \tilde{a}_n'). \tag{2.41}$$

2) (Idempotency): If $\tilde{a}_j = \tilde{a}$ *for all* j, *then*

$$\mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{2.42}$$

3) (Boundedness):

$$\min_i \tilde{a}_i \leq \mathrm{UPH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max_i \tilde{a}_i. \tag{2.43}$$

If the weights of the objects are taken into account, then we define the weighted form of (2.38) as follows:

$$\mathrm{UPH}_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{i=1}^n \frac{w_i(1+T'(\tilde{a}_i))}{\sum_{i=1}^n w_i(1+T'(\tilde{a}_i))\tilde{a}_i}} \tag{2.44}$$

where

$$T'(\tilde{a}_i) = \sum_{j=1, j\neq i}^{n} w_j \mathrm{Sup}(\tilde{a}_i, \tilde{a}_j) \tag{2.45}$$

with the condition

$$w_i \in [0,1], \; i = 1, 2, \ldots, n, \; \sum_{i=1}^n w_i = 1. \tag{2.46}$$

Obviously, the weighted UPH operator has the properties of 2) and 3) in Theorem 2.3.2. However, Theorem 2.3.1 and 1) of Theorem 2.3.2 do not hold for the weighted UPH operator.

Based on the POWH operator and the possibility degree formula, we define a UPOWH operator as follows:

$$\mathrm{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{1}{\sum_{i=1}^n \frac{u_i}{\tilde{a}_{\mathrm{index}(i)}}} \tag{2.47}$$

where $\tilde{a}_{\text{index}(i)}$ is the ith largest interval number of $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$, and

$$u_i = g\left(\frac{R_i}{TV}\right) - g\left(\frac{R_{i-1}}{TV}\right),\ R_i = \sum_{j=1}^{i} V_{\text{index}(j)},$$
$$TV = \sum_{i=1}^{n} V_{\text{index}(i)},\ V_{\text{index}(j)} = 1 + T(\tilde{a}_{\text{index}(i)}) \tag{2.48}$$

and $T(\tilde{a}_{\text{index}(i)})$ denotes the support of the ith largest interval number by all the other interval numbers, i.e.,

$$T(\tilde{a}_{\text{index}(i)}) = \sum_{j=1}^{n} \text{Sup}(\tilde{a}_{\text{index}(i)}, \tilde{a}_{\text{index}(j)}) \tag{2.49}$$

where $\text{Sup}(\tilde{a}_{\text{index}(i)}, \tilde{a}_{\text{index}(j)})$ indicates the support of the jth largest interval number for the ith largest interval number (here, we can use the possibility degree formula (2.36) to rank interval numbers).

Especially, if $g(x) = x$, then the UPOWH operator reduces to the UPH operator.

From Theorem 2.3.1, we have the following corollary.

Corollary 2.3.3 *Letting* $\text{Sup}(\tilde{a}_{\text{index}(i)}, \tilde{a}_{\text{index}(j)}) = k$ *for all* $i \neq j$, *and* $g(x) = x$, *then*

$$\text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \frac{n}{\sum_{i=1}^{n} \frac{1}{\tilde{a}_i}} \tag{2.50}$$

which indicates that when the supports are the same, the UPOWH operator is simply the uncertain harmonic mean.

Similar to Theorem 2.3.2, we have the following theorem.

Theorem 2.3.4 *Let* $\tilde{a}_j$ $(j = 1, 2, \ldots, n)$ *be a collection of interval numbers, then we have the following properties.*

1) (Commutativity): If $(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$ *is any permutation of* $(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n)$, *then*

$$\text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \text{UPOWH}(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n). \tag{2.51}$$

2) (Idempotency): If $\tilde{a}_j = \tilde{a}$ for all j, then

$$\text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \tilde{a}. \tag{2.52}$$

3) (Boundedness):

$$\min_i \tilde{a}_i \leq \text{UPOWH}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \leq \max_i \tilde{a}_i. \tag{2.53}$$

2.4 Approach to group decision making based on uncertain preference relations

As mentioned in Section 2.2, in this section, we will apply the weighted UPH and UPOWH operators to group decision making based on uncertain preference relations. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives and let $D = \{d_1, d_2, \ldots, d_m\}$ be a set of decision makers, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, with $w_k \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^m w_k = 1$. The decision maker d_k compare each pair of alternatives (x_i, x_j) and provides his/her preference value range $\tilde{a}_{ij}^{(k)} = [a_{ij}^{L(k)}, a_{ij}^{U(k)}]$ over them and constructs the uncertain preference relation $\tilde{A}_k$ on the set X, which is defined as a matrix $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{n \times n}$ under the following condition:

$$\begin{aligned} &a_{ij}^{U(k)} \geq a_{ij}^{L(k)} > 0, \quad a_{ij}^{L(k)} + a_{ji}^{U(k)} = 1, \quad a_{ji}^{L(k)} + a_{ij}^{U(k)} = 1, \\ &a_{ii}^{L(k)} = a_{ii}^{U(k)} = \frac{1}{2}, \quad i, j = 1, 2, \ldots, n. \end{aligned} \tag{2.54}$$

Then, we utilize the weighted UPH operator to develop an approach to group decision making based on uncertain preference relations, which involves the following steps.

Approach III.

Step 1: Calculate the supports

$$\text{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = 1 - \frac{d(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)})}{\sum_{l=1, l \neq k}^m d(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)})}, \ l = 1, 2, \ldots, m \tag{2.55}$$

which satisfy the support condition 1)-3) in Section 2.3. Here, without loss of generality, we let

$$d(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = \frac{1}{2}(|a_{ij}^{L(l)} - a_{ij}^{L(k)}| + |a_{ij}^{U(l)} - a_{ij}^{U(k)}|). \tag{2.56}$$

Especially, if $\sum_{l=1,l\neq k}^{m} d(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = 0$, then we stipulate $\text{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) = 1$.

Step 2: Utilize the weights w_k $(k = 1, 2, \ldots, m)$ of the decision makers d_k $(k = 1, 2, \ldots, m)$ to calculate the weighted support $T'(\tilde{a}_{ij}^{(k)})$ of the uncertain preference value $\tilde{a}_{ij}^{(k)}$ by the other uncertain preference values $\tilde{a}_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T'(\tilde{a}_{ij}^{(k)}) = \sum_{l=1,l\neq k}^{m} w_l \text{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}) \tag{2.57}$$

and calculate the weights $\dot{v}_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$ associated with the uncertain preference values $\tilde{a}_{ij}^{(k)}$ $(k = 1, 2, \ldots, m)$

$$\dot{v}_{ij}^{(k)} = \frac{w_k\left(1 + T'(\tilde{a}_{ij}^{(k)})\right)}{\sum_{k=1}^{m} w_k\left(1 + T'(\tilde{a}_{ij}^{(k)})\right)},\ k = 1, 2, \ldots, m \tag{2.58}$$

where $\dot{v}_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} \dot{v}_{ij}^{(k)} = 1$.

Step 3: Utilize the weighted UPH operator to aggregate all the individual uncertain preference relations $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective uncertain preference relation $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where

$$\begin{aligned} \tilde{a}_{ij} &= [a_{ij}^{l}, a_{ij}^{U}] = \text{UPH}_w(\tilde{a}_{ij}^{(1)}, \tilde{a}_{ij}^{(2)}, \ldots, \tilde{a}_{ij}^{(m)}) \\ &= \frac{1}{\sum_{k=1}^{m} \frac{\dot{v}_{ij}^{(k)}}{\tilde{a}_{ij}^{(k)}}},\ i, j = 1, 2, \ldots, n. \end{aligned} \tag{2.59}$$

Step 4: Utilize the uncertain NRAM (UNRAM) given by

$$\tilde{v}_i = \frac{\sum_{j=1}^{n} \tilde{a}_{ij}}{\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_{ij}},\ i = 1, 2, \ldots, n \tag{2.60}$$

to derive the uncertain priority vector $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n)^T$ of $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$.

Step 5: Compare each pair of the uncertain priority weights $\tilde{v}_i$ $(i = 1, 2, \ldots, n)$ by using the possibility degree formula (2.36) and construct a possibility degree matrix $P = (p_{ij})_{n\times n}$, where $p_{ij} = p(\tilde{v}_i \geq \tilde{v}_j)$, $i, j = 1, 2, \ldots, n$, which satisfy $p_{ij} \geq 0$ $p_{ij} + p_{ji} = 1$, $p_{ii} = 0.5$, $i, j = 1, 2, \ldots, n$. Summing all the elements in each line of the matrix P, we get

$$p_i = \sum_{j=1}^{n} p_{ij}, \quad i = 1, 2, \ldots, n. \tag{2.61}$$

Then we rank the uncertain priority weights $\tilde{v}_i$ $(i = 1, 2, \ldots, n)$ in descending order in accordance with p_i $(i = 1, 2, \ldots, n)$.

Step 6: Rank all alternatives x_i $(i = 1, 2, \ldots, n)$ in accordance with the descending order of the uncertain priority weights $\tilde{v}_i$ $(i = 1, 2, \ldots, n)$.

In the case where the information about the weights of decision makers is unknown, then we utilize the UPOWH operator to develop an approach to group decision making based on uncertain preference relations, which can be described as follows.

Approach IV.

Step 1: Calculate the supports

$$\mathrm{Sup}(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) = 1 - \frac{d(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)})}{\sum_{l=1, l\neq k}^{m} d(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)})}, \ l = 1, 2, \ldots, m \tag{2.62}$$

which indicates the support of lth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(l)}$ for the kth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(k)}$ of $\tilde{a}_{ij}^{(s)}$ $(s = 1, 2, \ldots, m)$ (here, we can use Step 5 of Approach III to rank uncertain preference values). Especially, if $\sum_{l=1, l\neq k}^{m} d(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) = 0$, then we stipulate $\mathrm{Sup}(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) = 1$.

Step 2: Calculate the support $T(\tilde{a}_{ij}^{\mathrm{index}(k)})$ of the kth largest uncertain preference value $\tilde{a}_{ij}^{\mathrm{index}(k)}$ by the other uncertain preference values $\tilde{a}_{ij}^{(l)}$ $(l = 1, 2, \ldots, m$, and $l \neq k)$

$$T(\tilde{a}_{ij}^{\mathrm{index}(k)}) = \sum_{l=1, l\neq k}^{m} \mathrm{Sup}(\tilde{a}_{ij}^{\mathrm{index}(k)}, \tilde{a}_{ij}^{\mathrm{index}(l)}) \tag{2.63}$$

and by (2.48), calculate the weight $\dot{u}_{ij}^{(k)}$ associated with the kth largest uncertain preference value $\tilde{a}_{ij}^{\text{index}(k)}$, where

$$\dot{u}_{ij}^{(k)} = g\left(\frac{\dot{R}_{ij}^{(k)}}{TV'_{ij}}\right) - g\left(\frac{\dot{R}_{ij}^{(k-1)}}{TV'_{ij}}\right),\ \dot{R}_{ij}^{(k)} = \sum_{l=1}^{k} V_{ij}^{\text{index}(l)},$$
$$TV'_{ij} = \sum_{l=1}^{m} V_{ij}^{\text{index}(l)},\ V_{ij}^{\text{index}(l)} = 1 + T(\tilde{a}_{ij}^{\text{index}(l)}) \tag{2.64}$$

where $\dot{u}_{ij}^{(k)} \geq 0,\ k = 1, 2, \ldots, m$, and $\sum_{k=1}^{m} \dot{u}_{ij}^{(k)} = 1$, and g is the BUM function described in Section 2.1.

Step 3: Utilize the UPOWH operator to aggregate all the individual uncertain preference relations $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ into the collective uncertain preference relation $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$, where

$$\begin{aligned}\tilde{a}_{ij} &= [a_{ij}^L, a_{ij}^U] = \text{UPOWH}(\tilde{a}_{ij}^{(1)}, \tilde{a}_{ij}^{(1)}, \ldots, \tilde{a}_{ij}^{(m)}) \\ &= \frac{1}{\sum_{k=1}^{m} \frac{\dot{u}_{ij}^{(k)}}{\tilde{a}_{ij}^{\text{index}(k)}}},\ i, j = 1, 2, \ldots, n.\end{aligned} \tag{2.65}$$

Step 4: For this step, see Approach III.
Step 5: For this step, see Approach III.
Step 6: For this step, see Approach III.

2.5 Illustrative example

Four university students share a house, where they intend to have broadband Internet connection installed (adapted from [55, 33]). There are four options available to choose from, which are provided by three Internet service providers.

1) Option 1 (x_1): 1 Mbps broadband;
2) Option 2 (x_2): 2 Mbps broadband;
3) Option 3 (x_3): 3 Mbps broadband;
4) Option 4 (x_4): 8 Mbps broadband.

Since the Internet service and its monthly bill will be shared among the four students d_k $(k = 1, 2, 3, 4)$ (whose weight vector $w = (0.3, 0.3, 0.2, 0.2)^T$), they decide to perform a group decision analysis. Suppose that the students reveal their preference relations for the options independently and anonymously and construct the following preference relations, respectively:

$$A_1 = \begin{pmatrix} 0.5 & 0.4 & 0.5 & 0.8 \\ 0.6 & 0.5 & 0.8 & 0.9 \\ 0.5 & 0.2 & 0.5 & 0.6 \\ 0.2 & 0.1 & 0.4 & 0.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.5 & 0.8 & 0.7 & 0.4 \\ 0.2 & 0.5 & 0.6 & 0.6 \\ 0.3 & 0.4 & 0.5 & 0.8 \\ 0.6 & 0.4 & 0.2 & 0.5 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0.5 & 0.4 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.3 & 0.7 \\ 0.3 & 0.7 & 0.5 & 0.6 \\ 0.4 & 0.3 & 0.4 & 0.5 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0.5 & 0.7 & 0.7 & 0.5 \\ 0.3 & 0.5 & 0.4 & 0.4 \\ 0.3 & 0.6 & 0.5 & 0.9 \\ 0.5 & 0.6 & 0.1 & 0.5 \end{pmatrix}.$$

Since the weights of students are given, we then utilize Approach I to find the decision result.

We first utilize (2.27) to calculate the supports $\mathrm{Sup}(a_{ij}^{(k)}, a_{ij}^{(l)})$ $(i, j, k, l = 1, 2, 3, 4, k \neq l)$, which are contained in the matrices $S^{kl} = (S^{kl}(a_{ij}^{(k)}, a_{ij}^{(l)}))_{4\times 4}$ $(k = 1, 2, 3, 4)$, respectively

$$S^{12} = \begin{pmatrix} 1 & 0.429 & 0.667 & 0.556 \\ 0.429 & 1 & 0.818 & 0.700 \\ 0.667 & 0.818 & 1 & 0.600 \\ 0.556 & 0.700 & 0.600 & 1 \end{pmatrix}, \quad S^{13} = \begin{pmatrix} 1 & 1 & 0.667 & 0.778 \\ 1 & 1 & 0.545 & 0.800 \\ 0.667 & 0.545 & 1 & 1 \\ 0.778 & 0.800 & 1 & 1 \end{pmatrix}$$

$$S^{14} = \begin{pmatrix} 1 & 0.571 & 0.667 & 0.667 \\ 0.571 & 1 & 0.636 & 0.500 \\ 0.667 & 0.636 & 1 & 0.400 \\ 0.667 & 0.500 & 0.400 & 1 \end{pmatrix}, \quad S^{21} = \begin{pmatrix} 1 & 0.556 & 0 & 0.429 \\ 0.556 & 1 & 0.714 & 0.500 \\ 0 & 0.714 & 1 & 0.600 \\ 0.429 & 0.500 & 0.600 & 1 \end{pmatrix}$$

$$S^{23} = \begin{pmatrix} 1 & 0.556 & 1 & 0.714 \\ 0.556 & 1 & 0.571 & 0.833 \\ 1 & 0.571 & 1 & 0.600 \\ 0.714 & 0.833 & 0.600 & 1 \end{pmatrix}, \quad S^{24} = \begin{pmatrix} 1 & 0.889 & 1 & 0.857 \\ 0.889 & 1 & 0.714 & 0.667 \\ 1 & 0.714 & 1 & 0.800 \\ 0.857 & 0.667 & 0.800 & 1 \end{pmatrix}$$

$$S^{31} = \begin{pmatrix} 1 & 1 & 0 & 0.600 \\ 1 & 1 & 0.444 & 0.667 \\ 0 & 0.444 & 1 & 1 \\ 0.600 & 0.667 & 1 & 1 \end{pmatrix}, \quad S^{32} = \begin{pmatrix} 1 & 0.429 & 1 & 0.600 \\ 0.429 & 1 & 0.667 & 0.833 \\ 1 & 0.667 & 1 & 0.600 \\ 0.600 & 0.833 & 0.600 & 1 \end{pmatrix}$$

$$S^{34} = \begin{pmatrix} 1 & 0.571 & 1 & 0.800 \\ 0.571 & 1 & 0.889 & 0.500 \\ 1 & 0.889 & 1 & 0.400 \\ 0.800 & 0.500 & 0.400 & 1 \end{pmatrix}, \quad S^{41} = \begin{pmatrix} 1 & 0.571 & 0 & 0.400 \\ 0.571 & 1 & 0.429 & 0.500 \\ 0 & 0.429 & 1 & 0.571 \\ 0.400 & 0.500 & 0.571 & 1 \end{pmatrix}$$

$$S^{42} = \begin{pmatrix} 1 & 0.857 & 1 & 0.800 \\ 0.857 & 1 & 0.714 & 0.800 \\ 1 & 0.714 & 1 & 0.857 \\ 0.800 & 0.800 & 0.857 & 1 \end{pmatrix}, \quad S^{43} = \begin{pmatrix} 1 & 0.571 & 1 & 0.800 \\ 0.571 & 1 & 0.857 & 0.700 \\ 1 & 0.857 & 1 & 0.571 \\ 0.800 & 0.70 & 0.571 & 1 \end{pmatrix}.$$

Then, we utilize the weight vector $w = (0.3, 0.3, 0.2, 0.2)^T$ of the students d_k $(k = 1, 2, 3, 4)$ and (2.28) to calculate the weighted supports $T'(a_{ij}^{(k)})$ $(i, j, k = 1, 2, 3, 4)$ of the preference values $a_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$, which are contained in the matrices $T'_k = (T'(a_{ij}^{(k)}))_{4\times4}$ $(k = 1, 2, 3, 4)$, respectively

$$T'_1 = \begin{pmatrix} 0.700 & 0.443 & 0.467 & 0.456 \\ 0.443 & 0.700 & 0.482 & 0.470 \\ 0.467 & 0.482 & 0.700 & 0.460 \\ 0.456 & 0.470 & 0.460 & 0.700 \end{pmatrix}, \quad T'_2 = \begin{pmatrix} 0.700 & 0.456 & 0.400 & 0.443 \\ 0.456 & 0.700 & 0.471 & 0.450 \\ 0.400 & 0.471 & 0.700 & 0.460 \\ 0.443 & 0.450 & 0.460 & 0.700 \end{pmatrix}$$

$$T'_3 = \begin{pmatrix} 0.800 & 0.543 & 0.500 & 0.520 \\ 0.543 & 0.800 & 0.511 & 0.550 \\ 0.500 & 0.511 & 0.800 & 0.560 \\ 0.520 & 0.550 & 0.560 & 0.800 \end{pmatrix}, \quad T'_4 = \begin{pmatrix} 0.800 & 0.543 & 0.500 & 0.520 \\ 0.543 & 0.800 & 0.514 & 0.530 \\ 0.500 & 0.514 & 0.800 & 0.543 \\ 0.520 & 0.530 & 0.543 & 0.800 \end{pmatrix}$$

and then utilize (2.29) to calculate the weights $v_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$ associated with the preference values $a_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$, which are contained in the matrices $V_k = (v_{ij}^{(k)})_{4\times4}$ $(k = 1, 2, 3, 4)$, respectively

$$V_1 = \begin{pmatrix} 0.293 & 0.291 & 0.301 & 0.296 \\ 0.291 & 0.293 & 0.298 & 0.295 \\ 0.301 & 0.298 & 0.293 & 0.293 \\ 0.295 & 0.295 & 0.293 & 0.293 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0.293 & 0.294 & 0.288 & 0.293 \\ 0.293 & 0.293 & 0.296 & 0.292 \\ 0.287 & 0.296 & 0.293 & 0.293 \\ 0.293 & 0.292 & 0.293 & 0.293 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0.207 & 0.207 & 0.205 & 0.206 \\ 0.208 & 0.207 & 0.203 & 0.208 \\ 0.206 & 0.203 & 0.207 & 0.208 \\ 0.206 & 0.208 & 0.208 & 0.207 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0.207 & 0.208 & 0.206 & 0.206 \\ 0.208 & 0.207 & 0.203 & 0.205 \\ 0.206 & 0.203 & 0.207 & 0.206 \\ 0.206 & 0.205 & 0.206 & 0.207 \end{pmatrix}.$$

Based on this, we utilize the weighted PH operator (2.30) to aggregate all the individual preference relations $A_k = (a_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3, 4)$ into the collective preference relation

$$A = \begin{pmatrix} 0.5000 & 0.5237 & 0.6248 & 0.5383 \\ 0.3344 & 0.5000 & 0.4878 & 0.6157 \\ 0.3411 & 0.3499 & 0.5000 & 0.6992 \\ 0.3460 & 0.2121 & 0.2093 & 0.5000 \end{pmatrix}.$$

After this, we utilize the NRAM (2.31) to derive the priority vector of A

$$v = (0.3003, 0.2661, 0.2596, 0.1740)^T.$$

Using this, we get the ranking of the options as follows:

$$x_1 \succ x_2 \succ x_3 \succ x_4.$$

In this case where the preferences constructed by students d_k $(k = 1, 2, 3, 4)$ are uncertain preference relations, for example

$$\tilde{A}_1 = \begin{pmatrix} [0.5, 0.5] & [0.3, 0.5] & [0.4, 0.5] & [0.7, 0.8] \\ [0.5, 0.7] & [0.5, 0.5] & [0.7, 0.8] & [0.7, 0.9] \\ [0.5, 0.6] & [0.2, 0.3] & [0.5, 0.5] & [0.5, 0.6] \\ [0.2, 0.3] & [0.1, 0.3] & [0.4, 0.5] & [0.5, 0.5] \end{pmatrix},$$

$$\tilde{A}_2 = \begin{pmatrix} [0.5, 0.5] & [0.7, 0.8] & [0.5, 0.7] & [0.4, 0.6] \\ [0.2, 0.3] & [0.5, 0.5] & [0.5, 0.6] & [0.6, 0.7] \\ [0.3, 0.5] & [0.4, 0.5] & [0.5, 0.5] & [0.7, 0.9] \\ [0.4, 0.6] & [0.3, 0.4] & [0.1, 0.3] & [0.5, 0.5] \end{pmatrix},$$

$$\tilde{A}_3 = \begin{pmatrix} [0.5, 0.5] & [0.4, 0.5] & [0.5, 0.7] & [0.4, 0.6] \\ [0.5, 0.6] & [0.5, 0.5] & [0.3, 0.4] & [0.7, 0.8] \\ [0.3, 0.5] & [0.6, 0.7] & [0.5, 0.5] & [0.5, 0.7] \\ [0.4, 0.6] & [0.2, 0.3] & [0.3, 0.5] & [0.5, 0.5] \end{pmatrix},$$

$$\tilde{A}_4 = \begin{pmatrix} [0.5, 0.5] & [0.6, 0.8] & [0.6, 0.7] & [0.3, 0.5] \\ [0.2, 0.4] & [0.5, 0.5] & [0.3, 0.4] & [0.4, 0.5] \\ [0.3, 0.4] & [0.6, 0.7] & [0.5, 0.5] & [0.7, 0.9] \\ [0.5, 0.7] & [0.5, 0.6] & [0.1, 0.3] & [0.5, 0.5] \end{pmatrix}.$$

Then we can utilize Approach III to derive the ranking of the four options, and the following decision steps are need.

Step 1: Utilize (2.55) and (2.56) to calculate the supports $\mathrm{Sup}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)})$ $(i, j, k, l = 1, 2, 3, 4)$, which are contained in the matrices $S^{kl} = (S^{kl}(\tilde{a}_{ij}^{(k)}, \tilde{a}_{ij}^{(l)}))_{4\times 4}$ $(k = 1, 2, 3, 4)$, respectively

$$S^{12} = \begin{pmatrix} 1 & 0.500 & 0.700 & 0.706 \\ 0.500 & 1 & 0.800 & 0.727 \\ 0.700 & 0.800 & 1 & 0.546 \\ 0.706 & 0.727 & 0.546 & 1 \end{pmatrix}, \quad S^{13} = \begin{pmatrix} 1 & 0.929 & 0.700 & 0.706 \\ 0.929 & 1 & 0.600 & 0.909 \\ 0.700 & 0.600 & 1 & 0.909 \\ 0.706 & 0.909 & 0.909 & 1 \end{pmatrix}$$

$$S^{14} = \begin{pmatrix} 1 & 0.571 & 0.600 & 0.588 \\ 0.571 & 1 & 0.600 & 0.364 \\ 0.600 & 0.600 & 1 & 0.546 \\ 0.588 & 0.364 & 0.546 & 1 \end{pmatrix}, \quad S^{21} = \begin{pmatrix} 1 & 0.500 & 0.250 & 0.286 \\ 0.500 & 1 & 0.667 & 0.667 \\ 0.250 & 0.667 & 1 & 0.444 \\ 0.286 & 0.667 & 0.444 & 1 \end{pmatrix}$$

$$S^{23} = \begin{pmatrix} 1 & 0.571 & 1 & 1 \\ 0.571 & 1 & 0.667 & 0.778 \\ 1 & 0.667 & 1 & 0.556 \\ 1 & 0.778 & 0.556 & 1 \end{pmatrix}, \quad S^{24} = \begin{pmatrix} 1 & 0.929 & 0.750 & 0.714 \\ 0.929 & 1 & 0.666 & 0.555 \\ 0.750 & 0.666 & 1 & 1 \\ 0.714 & 0.555 & 1 & 1 \end{pmatrix}$$

$$S^{31} = \begin{pmatrix} 1 & 0.917 & 0.250 & 0.286 \\ 0.917 & 1 & 0.333 & 0.889 \\ 0.250 & 0.333 & 1 & 0.889 \\ 0.286 & 0.889 & 0.889 & 1 \end{pmatrix}, \quad S^{32} = \begin{pmatrix} 1 & 0.500 & 1 & 1 \\ 0.500 & 1 & 0.667 & 0.778 \\ 1 & 0.667 & 1 & 0.556 \\ 1 & 0.778 & 0.556 & 1 \end{pmatrix}$$

$$S^{34} = \begin{pmatrix} 1 & 0.583 & 0.750 & 0.714 \\ 0.583 & 1 & 1 & 0.333 \\ 0.750 & 1 & 1 & 0.556 \\ 0.714 & 0.333 & 0.556 & 1 \end{pmatrix}, \quad S^{41} = \begin{pmatrix} 1 & 0.500 & 0.334 & 0.364 \\ 0.500 & 1 & 0.333 & 0.588 \\ 0.334 & 0.333 & 1 & 0.444 \\ 0.364 & 0.588 & 0.444 & 1 \end{pmatrix}$$

$$S^{42} = \begin{pmatrix} 1 & 0.917 & 0.833 & 0.818 \\ 0.917 & 1 & 0.667 & 0.765 \\ 0.833 & 0.667 & 1 & 1 \\ 0.818 & 0.765 & 1 & 1 \end{pmatrix}, \quad S^{43} = \begin{pmatrix} 1 & 0.583 & 0.833 & 0.818 \\ 0.583 & 1 & 1 & 647 \\ 0.833 & 1 & 1 & 0.556 \\ 0.818 & 0.647 & 0.556 & 1 \end{pmatrix}.$$

Step 2: Utilize the weight vector $w = (0.3, 0.3, 0.2, 0.2)^T$ of the students d_k $(k = 1, 2, 3, 4)$ and (2.57) to calculate the weighted supports $T'(\tilde{a}_{ij}^{(k)})$ $(i, j, k = 1, 2, 3, 4)$ of the uncertain preference values $\tilde{a}_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$, which are contained in the matrices $T'_k = (T'(\tilde{a}_{ij}^{(k)}))_{4\times 4}$ $(k = 1, 2, 3, 4)$, respectively

$$T'_1 = \begin{pmatrix} 0.700 & 0.450 & 0.470 & 0.471 \\ 0.450 & 0.700 & 0.480 & 0.473 \\ 0.470 & 0.480 & 0.700 & 0.455 \\ 0.471 & 0.473 & 0.455 & 0.700 \end{pmatrix}, \quad T'_2 = \begin{pmatrix} 0.700 & 0.450 & 0.425 & 0.429 \\ 0.450 & 0.700 & 0.467 & 0.467 \\ 0.425 & 0.467 & 0.700 & 0.444 \\ 0.429 & 0.467 & 0.444 & 0.700 \end{pmatrix}$$

$$T'_3 = \begin{pmatrix} 0.800 & 0.542 & 0.525 & 0.529 \\ 0.542 & 0.800 & 0.500 & 0.567 \\ 0.525 & 0.500 & 0.800 & 0.544 \\ 0.529 & 0.567 & 0.544 & 0.800 \end{pmatrix}, \quad T'_4 = \begin{pmatrix} 0.800 & 0.548 & 0.517 & 0.518 \\ 0.542 & 0.800 & 0.500 & 0.535 \\ 0.517 & 0.500 & 0.800 & 0.544 \\ 0.518 & 0.535 & 0.544 & 0.800 \end{pmatrix}$$

and then utilize (2.58) to calculate the weights $\dot{v}_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$ associated with the uncertain preference values $\tilde{a}_{ij}^{(k)}$ $(i, j, k = 1, 2, 3, 4)$, which are contained in the matrices $V_k = (\dot{v}_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3, 4)$, respectively

$$V_1 = \begin{pmatrix} 0.293 & 0.293 & 0.299 & 0.298 \\ 0.293 & 0.293 & 0.299 & 0.294 \\ 0.299 & 0.299 & 0.293 & 0.293 \\ 0.298 & 0.294 & 0.291 & 0.293 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0.293 & 0.293 & 0.289 & 0.290 \\ 0.293 & 0.293 & 0.297 & 0.293 \\ 0.289 & 0.297 & 0.293 & 0.291 \\ 0.290 & 0.293 & 0.291 & 0.293 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0.207 & 0.207 & 0.207 & 0.207 \\ 0.207 & 0.207 & 0.202 & 0.209 \\ 0.207 & 0.202 & 0.207 & 0.208 \\ 0.207 & 0.209 & 0.208 & 0.207 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0.207 & 0.207 & 0.205 & 0.205 \\ 0.207 & 0.207 & 0.202 & 0.204 \\ 0.205 & 0.202 & 0.207 & 0.208 \\ 0.205 & 0.204 & 0.208 & 0.207 \end{pmatrix}.$$

Step 3: Utilize the weighted UPH operator to aggregate all the individual uncertain preference relations $\tilde{A}_k = (\tilde{a}_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3, 4)$ into the collective uncertain preference relation

$$\tilde{A} = \begin{pmatrix} [0.5000, 0.5000] & [0.4430, 0.6154] & [0.4806, 0.6253] & [0.4253, 0.6208] \\ [0.2857, 0.4430] & [0.5000, 0.5000] & [0.4223, 0.5322] & [0.5823, 0.7068] \\ [0.3407, 0.4992] & [0.3435, 0.4613] & [0.5000, 0.5000] & [0.5831, 0.7463] \\ [0.3182, 0.4728] & [0.1862, 0.3638] & [0.1559, 0.3752] & [0.5000, 0.5000] \end{pmatrix}.$$

Step 4: Utilize the UNRAM (2.60) to derive the uncertain priority vector $\tilde{v} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4)^T$ of $\tilde{A}$

$$\tilde{v}_1 = [0.2185, 0.3596], \tilde{v}_2 = [0.2116, 0.3323],$$
$$\tilde{v}_3 = [0.2089, 0.3361], \tilde{v}_4 = [0.1371, 0.2607].$$

Step 5: In order to rank $\tilde{v}_i$ $(i = 1, 2, 3, 4)$, by (2.36), we construct the possibility degree matrix

$$P = \begin{pmatrix} 0.5000 & 0.5654 & 0.5618 & 0.8406 \\ 0.4346 & 0.5000 & 0.4979 & 0.7989 \\ 0.4382 & 0.5021 & 0.5000 & 0.7933 \\ 0.1594 & 0.2011 & 0.2067 & 0.5000 \end{pmatrix}$$

and by (2.61), we have

$$p_1 = 2.4678, p_2 = 2.2314, p_3 = 2.2336, p_4 = 1.0672.$$

Then, $\tilde{v}_1 > \tilde{v}_3 > \tilde{v}_2 > \tilde{v}_4$, and thus, we rank the options x_i $(i = 1, 2, 3, 4)$ in accordance with the descending order of $\tilde{v}_i$ $(i = 1, 2, 3, 4)$

$$x_1 \succ x_3 \succ x_2 \succ x_4.$$

From the previous numerical results, it can be known that the ranking of the options in the latter case are slightly different from the former case due to change of the input arguments.

2.6 Conclusions

In this chapter, based on the PA operator, we have developed several new nonlinear weighted harmonic aggregation operators including the PH operator, weighted PH operator, POWH operator, UPH operator, weighted UPH operator and UPOWH operator. We have studied some desired properties of the developed operators, such as commutativity, idempotency and boundedness. The fundamental idea of these operators is that the weight of each input argument depends on the other input arguments and allows argument values to support each other in the harmonic aggregation process. Moreover, we have applied the developed operators to aggregate all individual preference (or uncertain preference) relations into collective preference (or uncertain preference) under various group decision making environment and then developed some group decision making approaches. The merit of the developed approaches is that they can take all the decision arguments and their relationships into account. In the future, we will develop several applications of the developed aggregation operators in other fields, such as pattern recognition, supply chain management and image processing.

Chapter 3

2-tuple linguistic harmonic operators and their applications in group decision making

Harmonic mean is reciprocal of arithmetic mean of reciprocal, which is a conservative average to be used to provide for aggregation lying between max and min operators. In this chapter, we develop some new linguistic aggregation operators such as 2-tuple linguistic harmonic (2TLH) operator, 2-tuple linguistic weighted harmonic (2TLWH) operator, 2-tuple linguistic ordered weighted harmonic (2TLOWH) operator, and 2-tuple linguistic hybrid harmonic (2TLHH) operator, which can be utilized to aggregate preference information taking the form of linguistic variables, and then study some desirable properties of the operators. Based on the 2TLWH and 2TLHH operators, we present an approach to multiple attribute decision making with 2-tuple linguistic information. Finally, illustrative example is given to verify the developed approach and to demonstrate its practicality and effectiveness by comparing with the existing approaches.

3.1 Preliminaries

The linguistic variables are used in processes of computing with words that imply their fusion, aggregation and comparison, etc. The most often used models dealing with linguistic information are: (1) the semantic model that uses the linguistic terms just as labels for fuzzy numbers, while the computations over them are done directly over those fuzzy numbers, (2) the second one is the symbolic model that uses the order index of the linguistic terms to make direct computation on labels, and (3) the third model is based on the linguistic 2-tuple.

The 2-tuple linguistic representation model, proposed by Herrera and Martínez [19, 20], is based on the symbolic model and in addition the concept called *Symbolic Translation*. In this section, we first review some concept of the 2-tuple.

Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, where s_i represents a possible value for a linguistic variable, and it must have the following characteristics [15, 17]:

1) The set is ordered: $s_i \geq s_j$ if $i \geq j$;

2) There is the negation operator: $\text{neg}(s_i) = s_j$ such that $j = t - i$;

3) Max operator: $\max(s_i, s_j) = s_i$ if $s_i \geq s_j$;

4) Min operator: $\min(s_i, s_j) = s_i$ if $s_i \leq s_j$.

For example, S can be defined so as its elements are uniformly distributed on a scale on which a total order is defined [42]:

$$\begin{aligned} S = \{ & s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, \\ & s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, \\ & s_8 = \text{extremely good}\}. \end{aligned}$$

Definition 3.1.1 [19] Let β be the result of an aggregation of the indices of a set of labels assessed in linguistic term set S, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g + 1$ the cardinality of S. Let $i = \text{round}(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [0.5, -0.5)$ then α is called a *symbolic translation*.

From this concept, Herrera and Martínez [19] developed the linguistic representation model which represents the linguistic information by means of 2-tuple (s_i, α_i), $s_i \in S$ and $\alpha_i \in [-0.5, 0.5)$:

• s_i represents the linguistic label center of the information;

• α_i is a numerical value expressing the value of the translation from the original result β to the closest index label i in the linguistic term set S, i.e., the symbolic translation.

This model defines a set of transformation functions between linguistic terms and 2-tuples and between numeric values and 2-tuples.

Definition 3.1.2 [19] Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation, then the 2-tuple that expresses the equivalent information to β is obtained with the function $\triangle : [0, g] \to S \times [-0.5, 0.5)$ defined by

$$\begin{aligned} &\triangle : [0, g] \to S \times [-0.5, 0.5) \\ &\triangle(\beta) = (s_i, \alpha_i), \text{ with } \begin{cases} s_i, & i = \text{round}(\beta), \\ \alpha_i = \beta - i, & \alpha_i \in [-0.5, 0.5), \end{cases} \end{aligned} \tag{3.1}$$

where round$(\cdot)$ is the usual round operation, s_i has the closest index label to β and α_i is the value of the symbolic translation.

Contrarily, let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and (s_i, α_i) be a 2-tuple. There is always a $\triangle^{-1}$ function:

$$\begin{aligned} &\triangle^{-1} : S \times [-0.5, 0.5) \to [0, g] \\ &\triangle^{-1}(s_i, \alpha_i) = i + \alpha_i = \beta \end{aligned} \tag{3.2}$$

such that from a 2-tuple it returns its equivalent numerical value $\beta \in [0, g]$.

From Definitions 3.1.1 and 3.1.2, it is obvious that the conversion of a linguistic term into a linguistic 2-tuple consists of adding a value zero as symbolic translation:

$$s_i \in S \Longrightarrow (s_i, 0). \tag{3.3}$$

By ordinary lexicographic order, Herrera and Martínez [19] defined the comparison of linguistic information represented by 2-tuples.

Definition 3.1.3 [19] Let (s_k, α_k) and (s_l, α_l) be two 2-tuples, with each one representing a counting of information, then:

- if $k < l$ then (s_k, α_k) is smaller than (s_l, α_l), denoted by $(s_k, \alpha_k) < (s_l, \alpha_l)$;
- if $k = l$ then

1) if $\alpha_k = \alpha_l$ then (s_k, α_k) and (s_l, α_l) represent the same information, denoted by $(s_k, \alpha_k) \sim (s_l, \alpha_l)$;

2) if $\alpha_k < \alpha_l$ then (s_k, α_k) is smaller than (s_l, α_l), denoted by $(s_k, \alpha_k) < (s_l, \alpha_l)$;

3) if $\alpha_k > \alpha_l$ then (s_k, α_k) is bigger than (s_l, α_l), denoted by $(s_k, \alpha_k) > (s_l, \alpha_l)$.

3.2 2-tuple linguistic harmonic operators

Definition 3.2.1 [12] Let WAA : $R^n \to R$, if

$$\mathrm{WAA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j a_j, \tag{3.4}$$

where R is the set of real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$, then WAA is called the weighted arithmetic averaging (WAA) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WAA}(a_1, a_2, \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WAA operator is reduced to the arithmetic averaging (AA) operator, i.e.,

$$\mathrm{AA}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{j=1}^{n} a_j. \tag{3.5}$$

Definition 3.2.2 [4] Let WHM : $(R^+)^n \to R^+$, if

$$\mathrm{WHM}(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{a_j}}, \tag{3.6}$$

where R^+ is the set of all positive real numbers, a_j $(j = 1, 2, \ldots, n)$ is a collection of positive real numbers, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of a_j $(j = 1, 2, \ldots, n)$, with $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, then WHM is called the weighted harmonic mean (WHM) operator. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $\mathrm{WHM}(a_1, a_2, \ldots, a_n) = a_i$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the WHM operator is reduced to the harmonic mean (HM) operator, i.e.,

$$\mathrm{HM}(a_1, a_2, \ldots, a_n) = \frac{n}{\sum_{j=1}^n \frac{1}{a_j}}. \tag{3.7}$$

The WAA and WHM operators first weight all the given data, and then aggregate all these weighted data into a collective one. Yager [56] introduced and studied the OWA operator that weights the ordered positions of the data instead of weighting the data themselves.

Definition 3.2.3 [56] An OWA operator of dimension n is a mapping OWA : $R^n \to R$ that has an associated vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$. Furthermore,

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^n w_j b_j, \tag{3.8}$$

where b_j is the jth largest of a_i $(i = 1, 2, \ldots, n)$. Especially, if $w_i = 1$, $w_j = 0$, $j \neq i$, then $b_n \leq \mathrm{OWA}(a_1, a_2, \ldots, a_n) = b_i \leq b_1$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then

$$\mathrm{OWA}(a_1, a_2, \ldots, a_n) = \frac{1}{n}\sum_{j=1}^n b_j = \frac{1}{n}\sum_{j=1}^n a_j = \mathrm{AA}(a_1, a_2, \ldots, a_n). \tag{3.9}$$

The WAA, WHM and OWA operators have only been used in situation in which the input arguments are the exact values. However, judgements of people depend on personal psychological aspects such as experience, learning, situation, state of mind, and so forth. It is more suitable to provide their preferences by means of linguistic variables rather than numerical ones.

For convenience, let $\tilde{S}$ be the set of all linguistic 2-tuples. In the following, we extend the WHM operator (3.6) to linguistic 2-tuple environment:

Definition 3.2.4 Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples, and let 2TLWH : $\tilde{S}^n \to \tilde{S}$, if

$$\text{2TLWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = \triangle \left(\frac{1}{\sum_{i=1}^{n} \frac{w_i}{\triangle^{-1}(r_i, \alpha_i)}} \right) \tag{3.10}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of (r_i, α_i) $(i = 1, 2, \ldots, n)$, with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, then 2TLWH is called the 2-tuple linguistic weighted harmonic (2TLWH) operator. Especially, if $w_j = 1$, $w_i = 0$, $i \neq j$, then $\text{2TLWH}((r_1, \alpha_1), (r_2, \alpha_2) \ldots, (r_n, \alpha_n)) = (r_j, \alpha_j)$; if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then 2TLWH operator is reduced to the 2-tuple linguistic harmonic (2TLH) operator:

$$\text{2TLH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = \triangle \left(\frac{n}{\sum_{i=1}^{n} \frac{1}{\triangle^{-1}(r_i, \alpha_i)}} \right). \tag{3.11}$$

Example 3.2.5 Given a collection of linguistic 2-tuples: $(r_1, \alpha_1) = (s_2, 0.2)$, $(r_2, \alpha_2) = (s_5, -0.3)$, $(r_3, \alpha_3) = (s_6, 0.3)$ and $(r_4, \alpha_4) = (s_3, -0.4)$, let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of (r_i, α_i) $(i = 1, 2, 3, 4)$, then by (3.11), we have

$$\begin{aligned} \text{2TLWH}((r_1, \alpha_1), (r_2, \alpha_2), (r_3, \alpha_3), (r_4, \alpha_4)) &= \triangle \left(\frac{1}{\frac{0.3}{2.2} + \frac{0.1}{4.7} + \frac{0.2}{6.3} + \frac{0.4}{2.6}} \right) \\ &= (s_3, 0.01). \end{aligned}$$

Theorem 3.2.6 *Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples, then we have:*

$$\min_i (r_i, \alpha_i) \leq \text{2TLWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \leq \max_i (r_i, \alpha_i). \tag{3.12}$$

Proof Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of (r_i, α_i) $(i = 1, 2, \ldots, n)$, $\triangle^{-1}(r_i, \alpha_i) = \beta_i$ for any i, $\min_i (r_i, \alpha_i) = (r_k, \alpha_k)$ and $\max_i (r_i, \alpha_i) = (r_l, \alpha_l)$, then

$$\text{2TLWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = \triangle \left(\frac{1}{\sum_{i=1}^{n} \frac{w_i}{\triangle^{-1}(r_i, \alpha_i)}} \right) = \triangle \left(\frac{1}{\sum_{i=1}^{n} \frac{w_i}{\beta_i}} \right)$$

$$\geq \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{w_i}{\beta_k}}\right) = \triangle(\beta_k) = (r_k, \alpha_k),$$

$$2\text{TLWH}((r_1,\alpha_1),(r_2,\alpha_2),\ldots,(r_n,\alpha_n)) = \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{w_i}{\triangle^{-1}(r_i,\alpha_i)}}\right) = \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{w_i}{\beta_i}}\right)$$

$$\leq \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{w_i}{\beta_l}}\right) = \triangle(\beta_l) = (r_l, \alpha_l).$$

Hence

$$\min_i (r_i, \alpha_i) \leq 2\text{TLWH}((r_1,\alpha_1),(r_2,\alpha_2),\ldots,(r_n,\alpha_n)) \leq \max_i (r_i, \alpha_i).$$

Based on the OWA and 2TLWH operators, we define a 2-tuple linguistic ordered weighted harmonic (2TLOWH) operators as follows:

Definition 3.2.7 Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples. A 2TLOWH operator of dimension n is a mapping 2TLOWH : $\tilde{S}^n \to \tilde{S}$, that has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$. Furthermore,

$$2\text{TLOWH}((r_1,\alpha_1),(r_2,\alpha_2),\ldots,(r_n,\alpha_n)) = \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\triangle^{-1}(r_{\sigma(i)},\alpha_{\sigma(i)})}}\right), \quad (3.13)$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ and $(r_{\sigma(i)}, \alpha_{\sigma(i)}) \geq (r_{\sigma(i+1)}, \alpha_{\sigma(i+1)})$ for all i.

Especially, if there is a tie between (r_i, α_i) and (r_j, α_j), then we replace each of (r_i, α_i) and (r_j, α_j) by their average $((r_i, \alpha_i) + (r_j, \alpha_j))/2$ in the process of aggregation. If k items are tied, then we replace these by k replicas of their average. The weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ can be determined by using some weight determining methods like the normal distribution based method [32-35].

In the following, we shall look at some desirable properties associated with the 2TLOWH operator.

Theorem 3.2.8 *Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples, then we have the following properties:*

1) (Commutativity): If $((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n))$ is any permutation of $((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n))$, then

$$\begin{aligned} &\mathrm{2TLOWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad = \mathrm{2TLOWH}((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n)). \end{aligned} \tag{3.14}$$

2) (Idempotency): If $(r_i, \alpha_i) = (r, \alpha)$ for all i, then

$$\mathrm{2TLOWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = (r, \alpha). \tag{3.15}$$

3) (Monotonicity): Let (r'_i, α'_i) $(i = 1, 2, \ldots, n,\ r'_i \in S,\ \alpha'_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples, if $(r_i, \alpha_i) \leq (r'_i, \alpha'_i)$, for all i, then

$$\begin{aligned} &\mathrm{2TLOWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad \leq \mathrm{2TLOWH}((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n)). \end{aligned} \tag{3.16}$$

4) (Boundedness):

$$\min_i (r_i, \alpha_i) \leq \mathrm{2TLOWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \leq \max_i (r_i, \alpha_i). \tag{3.17}$$

Proof Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be an associated vector of 2TLOWH operator such that $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$. Let $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ be a permutation of $(1, 2, \ldots, n)$ such that $(r_{\sigma(i-1)}, \alpha_{\sigma(i-1)}) \geq (r_{\sigma(i)}, \alpha_{\sigma(i)})$ for all i, and let $\triangle^{-1}(r_{\sigma(i)}, \alpha_{\sigma(i)}) = \beta_{\sigma(i)}$ for any i.

(1) Since $((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n))$ is a permutation of $((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n))$, $\triangle^{-1}(r'_{\sigma(i)}, \alpha'_{\sigma(i)}) = \beta_{\sigma(i)}$ for any i. Hence

$$\begin{aligned} \mathrm{2TLOWH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) &= \triangle\left(\frac{1}{\sum_{i=1}^n \frac{\omega_i}{\triangle^{-1}(r_{\sigma(i)}, \alpha_{\sigma(i)})}}\right) \\ &= \triangle\left(\frac{1}{\sum_{i=1}^n \frac{\omega_i}{\beta_{\sigma(i)}}}\right) = \triangle\left(\frac{1}{\sum_{i=1}^n \frac{\omega_i}{\triangle^{-1}(r'_{\sigma(i)}, \alpha'_{\sigma(i)})}}\right) \\ &= \mathrm{2TLOWH}((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n)). \end{aligned}$$

(2) Let $\triangle^{-1}(r,\alpha)=\beta$. Since $(r_i,\alpha_i)=(r,\alpha)$ for all i, $\triangle^{-1}(r_{\sigma(i)},\alpha_{\sigma(i)})=\beta_{\sigma(i)}=\beta$ for any i. Hence

$$\begin{aligned}\text{2TLOWH}((r_1,\alpha_1),(r_2,\alpha_2),\ldots,(r_n,\alpha_n)) &= \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\triangle^{-1}(r_{\sigma(i)},\alpha_{\sigma(i)})}}\right)\\ &= \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\beta_{\sigma(i)}}}\right)=\triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\beta}}\right)\\ &= \triangle(\beta)=(r,\alpha).\end{aligned}$$

(3) Let $\triangle^{-1}(r'_{\sigma(i)},\alpha'_{\sigma(i)})=\beta'_{\sigma(i)}$ for any i. Since $(r_i,\alpha_i)\leq(r'_i,\alpha'_i)$ for all i, $\triangle^{-1}(r_{\sigma(i)},\alpha_{\sigma(i)})=\beta_{\sigma(i)}\leq\beta'_{\sigma(i)}=\triangle^{-1}(r'_{\sigma(i)},\alpha'_{\sigma(i)})$ for all i. Hence

$$\begin{aligned}\text{2TLOWH}((r_1,\alpha_1),(r_2,\alpha_2),\ldots,(r_n,\alpha_n)) &= \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\triangle^{-1}(r_{\sigma(i)},\alpha_{\sigma(i)})}}\right)\\ &= \triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\beta_{\sigma(i)}}}\right)\leq\triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\beta'_{\sigma(i)}}}\right)=\triangle\left(\frac{1}{\sum_{i=1}^{n}\frac{\omega_i}{\triangle^{-1}(r'_{\sigma(i)},\alpha'_{\sigma(i)})}}\right)\\ &= \text{2TLOWH}((r'_1,\alpha'_1),(r'_2,\alpha'_2),\ldots,(r'_n,\alpha'_n)).\end{aligned}$$

(4) Similar to the proof of Theorem 3.2.6.

Especially, if $\omega=(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})^T$, then the 2TLOWH operator is reduced to the 2TLH operator. In fact,

$$\begin{aligned}\text{2TLOWH}((r_1,\alpha_1),(r_2,\alpha_2),\ldots,(r_n,\alpha_n)) &= \triangle\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\triangle^{-1}(r_{\sigma(i)},\alpha_{\sigma(i)})}}\right)\\ &= \triangle\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\triangle^{-1}(r_i,\alpha_i)}}\right)=\text{2TLH}((r_1,\alpha_1),(r_2,\alpha_2)\ldots,(r_n,\alpha_n)).\end{aligned}$$

Clearly, the fundamental characteristic of the 2TWH operator is that it consider the importance of each given linguistic argument, whereas the fundamental characteristic of the 2TLOWH operator is the reordering step, and it weights all the ordered positions of the linguistic arguments instead of weighting the given linguistic arguments themselves. By combining the advantages of the 2TLWH and 2TLOWH operators, in following, we develop a 2-tuple linguistic hybrid harmonic (2TLHH) operator that weights both the given linguistic arguments and their ordered positions.

Definition 3.2.9 Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples. A 2TLHH operator of dimension n is a mapping $\mathrm{2TLHH} : \tilde{S}^n \rightarrow \tilde{S}$, which has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$, such that

$$\mathrm{2TLHH}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = \triangle\left(\frac{1}{\sum_{i=1}^{n} \frac{\omega_i}{\triangle^{-1}(\dot{r}_{\sigma(i)}, \dot{\alpha}_{\sigma(i)})}}\right) \tag{3.18}$$

where $(\dot{r}_{\sigma(i)}, \dot{\alpha}_{\sigma(i)})$ is the ith largest of the weighted linguistic 2-tuples $(\dot{r}_i, \dot{\alpha}_i)$ $((\dot{r}_i, \dot{\alpha}_i) = \triangle(n w_i \triangle^{-1}(r_i, \alpha_i)),\ i = 1, 2, \ldots, n)$, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of (r_i, α_i) $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, and n is the balancing coefficient.

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $(\dot{r}_i, \dot{\alpha}_i) = (r_i, \alpha_i)$, $i = 1, 2, \ldots, n$, in this case, the 2TLHH operator is reduced to the 2TLOWH operator. Moreover, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the 2TLHH operator is reduced to the 2TLWH operator.

Example 3.2.10 Given a collection of linguistic 2-tuples: $(r_1, \alpha_1) = (s_2, 0.2)$, $(r_2, \alpha_2) = (s_5, -0.3)$, $(r_3, \alpha_3) = (s_6, 0.3)$ and $(r_4, \alpha_4) = (s_3, -0.4)$, and let $w = (0.3, 0.1, 0.2, 0.4)^T$ be the weight vector of (r_i, α_i) $(i = 1, 2, 3, 4)$. Then we get the weighted linguistic 2-tuples:

$$\begin{aligned}
(\dot{r}_1, \dot{\alpha}_1) &= \triangle(4 \times 0.3 \times \triangle^{-1}(s_2, 0.2)) = \triangle(2.64) = (s_3, -0.36),\\
(\dot{r}_2, \dot{\alpha}_2) &= \triangle(4 \times 0.1 \times \triangle^{-1}(s_5, -0.3)) = \triangle(1.88) = (s_2, -0.12),\\
(\dot{r}_3, \dot{\alpha}_3) &= \triangle(4 \times 0.2 \times \triangle^{-1}(s_6, 0.3)) = \triangle(5.04) = (s_5, 0.04),\\
(\dot{r}_4, \dot{\alpha}_4) &= \triangle(4 \times 0.4 \times \triangle^{-1}(s_3, -0.4)) = \triangle(4.16) = (s_4, 0.16).
\end{aligned}$$

By using Definition 3.1.3, we rank the linguistic 2-tuples (r_i, α_i) $(i = 1, 2, 3, 4)$:

$$(\dot{r}_{\sigma(1)}, \dot{\alpha}_{\sigma(1)}) = (s_5, 0.04),\ (\dot{r}_{\sigma(2)}, \dot{\alpha}_{\sigma(2)}) = (s_4, 0.16),$$
$$(\dot{r}_{\sigma(3)}, \dot{\alpha}_{\sigma(3)}) = (s_3, -0.36),\ (\dot{r}_{\sigma(4)}, \dot{\alpha}_{\sigma(4)}) = (s_2, -0.12).$$

Suppose that the weighting vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$ of the 2TLHH operator is $\omega = (0.2, 0.3, 0.4, 0.1)^T$, then by (3.18), we get

$$\begin{aligned} \text{2TLHH}((r_1, \alpha_1), (r_2, \alpha_2), (r_3, \alpha_3), (r_4, \alpha_4)) &= \triangle\left(\frac{1}{\frac{0.2}{5.04} + \frac{0.3}{4.16} + \frac{0.4}{2.64} + \frac{0.1}{1.88}}\right) \\ &= (s_3, 0.16). \end{aligned}$$

3.2.1 Generalizations of 2TLOWH operators

In the following, generalizations of the 2TLOWH operator are presented by using generalized and quasi-arithmetic means.

Definition 3.2.11 Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples. A 2-tuple linguistic generalized ordered weighted averaging (2TLGOWA) operator of dimension n is a mapping 2TLGOWA : $\tilde{S}^n \to \tilde{S}$ that has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ such that $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$. Furthermore,

$$\begin{aligned} &\text{2TLGOWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad = \triangle\left(\left(\sum_{i=1}^n \omega_i \left(\triangle^{-1}(r_{\sigma(i)}, \alpha_{\sigma(i)})\right)^\lambda\right)^{\frac{1}{\lambda}}\right), \end{aligned} \tag{3.19}$$

where λ is a parameter such that $\lambda \in (-\infty, \infty) - \{0\}$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $(r_{\sigma(i)}, \alpha_{\sigma(i)}) \geq (r_{\sigma(i+1)}, \alpha_{\sigma(i+1)})$ for all i.

The 2TLGOWA operators have many desirable properties: commutativity, monotonicity, boundedness and idempotency. Especially, if there are ties between linguistic 2-tuples, as in the case of 2TLOWH operator, we replace each of the tied arguments by their 2-tuple linguistic generalized mean in the process of aggregation. The 2TLGOWA operator provides a parameterized family of aggregation operators. In order to study this family, we can analyze the weighting vector w or the parameter ω. Especially, if $\omega_i = \frac{1}{n}$ for all i, then the 2TLGOWA operator is reduced to the 2-tuple linguistic generalized mean (2TLGM) operator; if $\omega_j = 1$, $\omega_i = 0$ for all $i \neq j$, and $(r_j, \alpha_j) = \min_i(r_i, \alpha_i)$ (resp. $(r_j, \alpha_j) = \max_i(r_i, \alpha_i)$), then the 2TLGOWA operator is reduced to the 2-tuple linguistic minimum (resp. maximum) operator; if $(r_i, \alpha_i) \geq (r_{i+1}, \alpha_{i+1})$ for all i, then the 2TLGOWA operator is reduced to the 2-tuple linguistic weighted generalized mean (2TLWGM) operator. Some special cases can be obtained as the change of the parameters:

1) If $\lambda = 1$, then the 2TLGOWA operator is reduced to the 2TLOWA operator.

2) If $\lambda \to 0$, then the 2TLGOWA operator is reduced to the 2TLOWG operator.

3) If $\lambda = -1$, then the 2TLGOWA operator is reduced to the 2TLOWH operator.

4) If $\lambda = 2$, then the 2TLGOWA operator is reduced to the 2-tuple linguistic ordered weighted quadric averaging (2TLOWQA) operator.

5) If $\lambda = -\infty$, then the 2TLGOWA operator is reduced to the 2-tuple linguistic minimum operator.

6) If $\lambda = \infty$, then the 2TLGOWA operator is reduced to the 2-tuple linguistic maximum operator.

Definition 3.2.12 Let (r_i, α_i) $(i = 1, 2, \ldots, n,\ r_i \in S,\ \alpha_i \in [-0.5, 0.5))$ be a collection of linguistic 2-tuples. A 2-tuple linguistic ordered weighted quasi-arithmetic averaging (Quasi-2TLOWA) operator of dimension n is a mapping Quasi-2TLOWA : $\tilde{S}^n \to \tilde{S}$ that has an associated weighting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$, such that

$$\begin{aligned}&\text{Quasi-2TLOWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n))\\ &\qquad = \triangle\left(g^{-1}\left(\sum_{i=1}^{n} \omega_i g\left(\triangle^{-1}(r_{\sigma(i)}, \alpha_{\sigma(i)})\right)\right)\right) \qquad (3.20)\end{aligned}$$

where g is a continuous strictly monotone function, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $(r_{\sigma(i)}, \alpha_{\sigma(i)}) \geq (r_{\sigma(i+1)}, \alpha_{\sigma(i+1)})$ for all i.

As we can see, the 2TLGOWA operator is a particular case of the Quasi-2THOWA operator when $g(x) = x^{\lambda}$. Note that all properties and particular cases commented in 2TLGOWA operator are also discussed in this generalization.

3.3 An approach to group decision making

Now we consider a multiple attribute group decision making (MAGDM) problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n feasible alternatives and $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_i \geq 0$, $i = 1, 2, \ldots, m$, $\sum_{i=1}^{m} w_i = 1$. Let $D = \{d_1, d_2, \ldots, d_l\}$ be the set of l decision makers, and $v = (v_1, v_2, \ldots, v_l)^T$ be the weight vector of decision makers, where $v_k \geq 0$, $k = 1, 2, \ldots, l$, $\sum_{k=1}^{l} v_k = 1$. Suppose that $A^{(k)} = (a_{ij}^{(k)})_{m \times n}$ is the linguistic decision matrix, where $a_{ij}^{(k)} \in S$ is preference value, which takes the form of linguistic variables, given by the decision maker $d_k \in D$, for alternative $x_j \in X$ with respect to attribute $G_i \in G$. Group decision making problems follow a common resolution scheme [19, 29] composed by the following two phases:

• *Aggregation phase*: It combines the individual preferences to obtain a collective preference value for each alternative.

• *Exploitation phase*: It orders the collective preference values to obtain the best alternative(s).

In the following we shall utilize the 2TLWH and 2TLHH operators to propose an approach to MAGDM with linguistic information.

Step 1: Transform the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{m \times n}$ into 2-tuple linguistic decision matrix $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{m \times n}$, $k = 1, 2, \ldots, l$.

Step 2: Utilize the decision information given in matrix $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{m \times n}$, and the 2TLWH operator:

$$(a_j^{(k)}, \alpha_j^{(k)}) = \text{2TLWH}((a_{1j}^{(k)}, 0), (a_{2j}^{(k)}, 0), \ldots, (a_{mj}^{(k)}, 0))$$

$$= \triangle\left(\frac{1}{\sum_{i=1}^{m}\frac{w_i}{\triangle^{-1}(a_{ij}^{(k)},0)}}\right),\ j = 1,2,\ldots,n \tag{3.21}$$

to aggregate all the elements in the jth column of $\tilde{A}^{(k)}$ and get the overall attribute value $(a_j^{(k)}, \alpha_j^{(k)})$ of the alternative x_j corresponding to the decision maker d_k.

Step 3: Utilize the 2TLHH operator:

$$\begin{aligned}(a_j, \alpha_j) &= \text{2TLHH}((a_j^{(1)}, \alpha_j^{(1)}), (a_j^{(2)}, \alpha_j^{(2)}), \ldots, (a_j^{(l)}, \alpha_j^{(l)})) \\ &= \triangle\left(\frac{1}{\sum_{k=1}^{l}\frac{\omega_k}{\triangle^{-1}(\dot{a}_j^{\sigma(k)}, \dot{\alpha}_j^{\sigma(k)})}}\right), j = 1,2,\ldots,n \end{aligned}\tag{3.22}$$

to aggregate the overall attribute values $(a_j^{(k)}, \alpha_j^{(k)})$ $(k = 1,2,\ldots,l)$ corresponding to the decision maker d_k $(k = 1,2,\ldots,l)$ and get the collective overall attribute value (a_j, α_j), where $(\dot{a}_j^{(\sigma(k))}, \dot{\alpha}_j^{(\sigma(k))})$ is the kth largest of the weighted data $(\dot{a}_j^{(k)}, \dot{\alpha}_j^{(k)})$ $((\dot{a}_j^{(k)}, \dot{\alpha}_j^{(k)}) = \triangle(lv_k \triangle^{-1}(a_j^{(k)}, \alpha_j^{(k)}))$, $k = 1,2,\ldots,l)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_l)^T$ is the weighting vector of the 2TLHH operator, with $\omega_k \geq 0$ and $\sum_{k=1}^{l}\omega_k = 1$.

Step 4: Utilize the collective overall attribute value (a_j, α_j) $(j = 1,2,\ldots,n)$ to rank the alternatives $x_j (j = 1,2,\ldots,n)$, and then select the most desirable one.

Step 5: End.

3.4 Illustrative examples

Example 3.4.1 Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted from [14]). There is a panel with five possible alternatives in which to invest the money: 1) x_1 is a car industry; 2) x_2 is a food company; 3) x_3 is a computer company; 4) x_4 is an arms company; 5) x_5 is a TV company.

The investment company must take a decision according to the following four attributes (suppose that the weighting vector of four attributes is $w =$

$(0.35, 0.15, 0.20, 0.30)^T$): 1) G_1 is the risk analysis; 2) G_2 is the growth analysis; G_3 is the social-political impact analysis; 4) G_4 is the environmental impact analysis.

The five possible alternatives x_j ($j = 1, 2, 3, 4, 5$) are evaluated using the linguistic term set S:

$$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor}, \\ s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good}, \\ s_8 = \text{extremely good}\}$$

by three decision makers d_k ($k = 1, 2, 3$) (whose weighting vector is $v = (0.35, 0.25, 0.40)^T$) under the above three attributes G_i ($i = 1, 2, 3, 4$), and construct, respectively, the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{4\times5}$ ($k = 1, 2, 3$) as listed in Tables 3.1-3.3.

To get the best alternative(s), the following steps are involved:

Step 1: Transform the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{4\times5}$ ($k = 1, 2, 3$) into 2-tuple linguistic decision matrix $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times5}$ ($k = 1, 2, 3$) (see Tables 3.4-3.6):

Step 2: Utilize the 2TLWH operator (3.21) to aggregate all the elements in the jth column of $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times5}$ and get the overall attribute value $(a_j^{(k)}, \alpha_j^{(k)})$ of the alternative x_j:

$$(a_1^{(1)}, \alpha_1^{(1)}) = (s_3, -0.01), (a_2^{(1)}, \alpha_2^{(1)}) = (s_6, -0.07), (a_3^{(1)}, \alpha_3^{(1)}) = (s_5, -0.18), \\ (a_4^{(1)}, \alpha_4^{(1)}) = (s_4, -0.50), (a_5^{(1)}, \alpha_5^{(1)}) = (s_5, 0.42), (a_1^{(2)}, \alpha_1^{(2)}) = (s_3, -0.28), \\ (a_2^{(2)}, \alpha_2^{(2)}) = (s_5, -0.38), (a_3^{(2)}, \alpha_3^{(2)}) = (s_5, -0.40), (a_4^{(2)}, \alpha_4^{(2)}) = (s_6, 0.32), \\ (a_5^{(2)}, \alpha_5^{(2)}) = (s_4, 0.38), (a_1^{(3)}, \alpha_1^{(3)}) = (s_3, -0.24), (a_2^{(3)}, \alpha_2^{(3)}) = (s_5, 0.38), \\ (a_3^{(3)}, \alpha_3^{(3)}) = (s_6, -0.40), (a_4^{(3)}, \alpha_4^{(3)}) = (s_5, -0.31), (a_5^{(3)}, \alpha_5^{(3)}) = (s_6, -0.28).$$

Step 3: Utilize the 2TLHH operator (3.22) (whose weight vector is $\omega = (0.243, 0.514, 0.243)^T$ determined by using normal distribution based method) to aggregate the overall attribute values $(a_j^{(k)}, \alpha_j^{(k)})$ ($k = 1, 2, 3$) to the decision

Table 3.1: Linguistic decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_4	s_8	s_4	s_8	s_6
G_2	s_5	s_7	s_5	s_6	s_4
G_3	s_3	s_7	s_5	s_3	s_7
G_4	s_2	s_4	s_6	s_2	s_5

Table 3.2: Linguistic decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_4	s_6	s_5	s_7	s_5
G_2	s_5	s_6	s_4	s_6	s_3
G_3	s_2	s_6	s_4	s_2	s_6
G_4	s_2	s_3	s_5	s_4	s_4

Table 3.3: Linguistic decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_4	s_6	s_5	s_7	s_5
G_2	s_6	s_4	s_5	s_5	s_7
G_3	s_2	s_5	s_7	s_6	s_6
G_4	s_2	s_6	s_6	s_3	s_6

Table 3.4: 2-tuple linguistic decision matrix $\tilde{A}^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_8, 0)$	$(s_4, 0)$	$(s_8, 0)$	$(s_6, 0)$
G_2	$(s_5, 0)$	$(s_7, 0)$	$(s_5, 0)$	$(s_6, 0)$	$(s_4, 0)$
G_3	$(s_3, 0)$	$(s_7, 0)$	$(s_5, 0)$	$(s_3, 0)$	$(s_7, 0)$
G_4	$(s_2, 0)$	$(s_4, 0)$	$(s_6, 0)$	$(s_2, 0)$	$(s_5, 0)$

Table 3.5: 2-tuple linguistic decision matrix $\tilde{A}^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_5, 0)$
G_2	$(s_5, 0)$	$(s_6, 0)$	$(s_4, 0)$	$(s_6, 0)$	$(s_3, 0)$
G_3	$(s_2, 0)$	$(s_6, 0)$	$(s_4, 0)$	$(s_2, 0)$	$(s_6, 0)$
G_4	$(s_2, 0)$	$(s_3, 0)$	$(s_5, 0)$	$(s_4, 0)$	$(s_4, 0)$

Table 3.6: 2-tuple linguistic decision matrix $\tilde{A}^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_5, 0)$
G_2	$(s_6, 0)$	$(s_4, 0)$	$(s_5, 0)$	$(s_5, 0)$	$(s_7, 0)$
G_3	$(s_2, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_6, 0)$	$(s_6, 0)$
G_4	$(s_2, 0)$	$(s_6, 0)$	$(s_6, 0)$	$(s_3, 0)$	$(s_6, 0)$

maker d_k $(k = 1, 2, , 3)$ and get the collective overall attribute value (a_j, α_j):

$$(a_1, \alpha_1) = (s_3, -0.193), (a_2, \alpha_2) = (s_5, 0.255), (a_3, \alpha_3) = (s_5, -0.196),$$
$$(a_4, \alpha_4) = (s_5, -0.407), (a_5, \alpha_5) = (s_5, 0.008).$$

Step 4: Utilize the collective overall attribute value (a_j, α_j) $(j = 1, 2, 3, 4, 5)$ to rank the alternatives x_j $(j = 1, 2, 3, 4, 5)$:

$$x_2 \succ x_5 \succ x_3 \succ x_4 \succ x_1$$

and thus the most desirable alternative is x_2.

In order to compare performance with the existing method [50], in following, the semantic model, the fuzzy weighted harmonic mean (FWHM) and fuzzy hybrid harmonic mean (FHHM) operators [50] are used to computing the overall attribute values. The semantics of the linguistic terms are given by triangular fuzzy numbers defined in the interval $[0, 1]$, which are usually described by the membership functions. For example, we may assign the following semantics to the set of nine linguistic terms:

$$s_0 = (0, 0, 0.12), s_1 = (0, 0.12, 0.25), s_2 = (0.12, 0.25, 0.37),$$
$$s_3 = (0.25, 0.37, 0.5), s_4 = (0.37, 0.5, 0.62), s_5 = (0.5, 0.62, 0.75),$$
$$s_6 = (0.62, 0.75, 0.87), s_7 = (0.75, 0.87, 1), s_8 = (0.87, 1, 1).$$

In order to get the most desirable alternative(s), the following steps are involved:

Step 1; Transform the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{4\times5}$ $(k = 1, 2, 3)$ into triangular fuzzy number decision matrix $\bar{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{4\times5}$ $(k = 1, 2, 3)$, where $\hat{a}_{ij}^{(k)} = (\hat{a}_{ij}^{L(k)}, \hat{a}_{ij}^{M(k)}, \hat{a}_{ij}^{U(k)})$ (see Tables 3.7-3.9):

Step 2: Utilize the FWHM operator (see Definition C):

$$\hat{a}_j^{(k)} = (\hat{a}_j^{L(k)}, \hat{a}_j^{M(k)}, \hat{a}_j^{U(k)}) = \mathrm{FWHM}(\hat{a}_{1j}^{(k)}, \hat{a}_{2j}^{(k)}, \hat{a}_{3j}^{(k)}, \hat{a}_{4j}^{(k)})$$
$$= \frac{1}{\sum_{i=1}^{4} \frac{w_i}{\hat{a}_{ij}^{(k)}}} = \left(\frac{1}{\sum_{i=1}^{4} \frac{w_i}{\hat{a}_{ij}^{L(k)}}}, \frac{1}{\sum_{i=1}^{4} \frac{w_i}{\hat{a}_{ij}^{M(k)}}}, \frac{1}{\sum_{i=1}^{4} \frac{w_i}{\hat{a}_{ij}^{U(k)}}} \right), \ k = 1, 2, 3$$

Table 3.7: Triangular fuzzy number decision matrix $\bar{A}^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(0.37, 0.5, 0.62)$	$(0.87, 1, 1)$	$(0.37, 0.5, 0.62)$	$(0.87, 1, 1)$	$(0.62, 0.75, 0.87)$
G_2	$(0.5, 0.62, 0.75)$	$(0.75, 0.87, 1)$	$(0.5, 0.62, 0.75)$	$(0.62, 0.75, 0.87)$	$(0.37, 0.5, 0.62)$
G_3	$(0.25, 0.37, 0.5)$	$(0.75, 0.87, 1)$	$(0.5, 0.62, 0.75)$	$(0.25, 0.37, 0.5)$	$(0.75, 0.87, 1)$
G_4	$(0.12, 0.25, 0.37)$	$(0.37, 0.5, 0.62)$	$(0.62, 0.75, 0.87)$	$(0.12, 0.25, 0.37)$	$(0.5, 0.62, 0.75)$

Table 3.8: Triangular fuzzy number decision matrix $\bar{A}^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(0.37, 0.5, 0.62)$	$(0.62, 0.75, 0.87)$	$(0.5, 0.62, 0.75)$	$(0.75, 0.87, 1)$	$(0.5, 0.62, 0.75)$
G_2	$(0.5, 0.62, 0.75)$	$(0.62, 0.75, 0.87)$	$(0.37, 0.5, 0.62)$	$(0.62, 0.75, 0.87)$	$(0.25, 0.37, 0.5)$
G_3	$(0.12, 0.25, 0.37)$	$(0.62, 0.75, 0.87)$	$(0.37, 0.5, 0.62)$	$(0.12, 0.25, 0.37)$	$(0.62, 0.75, 0.87)$
G_4	$(0.12, 0.25, 0.37)$	$(0.25, 0.37, 0.5)$	$(0.5, 0.62, 0.75)$	$(0.37, 0.5, 0.62)$	$(0.37, 0.5, 0.62)$

Table 3.9: Triangular fuzzy number decision matrix $\bar{A}^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(0.37, 0.5, 0.62)$	$(0.62, 0.75, 0.87)$	$(0.5, 0.62, 0.75)$	$(0.75, 0.87, 1)$	$(0.5, 0.62, 0.75)$
G_2	$(0.62, 0.75, 0.87)$	$(0.37, 0.5, 0.62)$	$(0.5, 0.62, 0.75)$	$(0.5, 0.62, 0.75)$	$(0.75, 0.87, 1)$
G_3	$(0.12, 0.25, 0.37)$	$(0.5, 0.62, 0.75)$	$(0.75, 0.87, 1)$	$(0.62, 0.75, 0.87)$	$(0.62, 0.75, 0.87)$
G_4	$(0.12, 0.25, 0.37)$	$(0.62, 0.75, 0.87)$	$(0.62, 0.75, 0.87)$	$(0.25, 0.37, 0.5)$	$(0.62, 0.75, 0.87)$

to aggregate all the elements in the jth column of $\bar{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{4\times5}$ and get the overall attribute value $\hat{a}_j^{(k)}$ of the alternative x_j:

$$\hat{a}_1^{(1)} = (0.220, 0.373, 0.506),\ \hat{a}_2^{(1)} = (0.595, 0.739, 0.845),$$
$$\hat{a}_3^{(1)} = (0.470, 0.601, 0.727),\ \hat{a}_4^{(1)} = (0.254, 0.437, 0.577),$$
$$\hat{a}_5^{(1)} = (0.544, 0.675, 0.804),\ \hat{a}_1^{(2)} = (0.185, 0.340, 0.473),$$
$$\hat{a}_2^{(2)} = (0.429, 0.573, 0.712),\ \hat{a}_3^{(2)} = (0.445, 0.572, 0.699),$$
$$\hat{a}_4^{(2)} = (0.314, 0.499, 0.646),\ \hat{a}_5^{(2)} = (0.411, 0.544, 0.675),$$
$$\hat{a}_1^{(3)} = (0.187, 0.345, 0.479),\ \hat{a}_2^{(3)} = (0.539, 0.671, 0.796),$$
$$\hat{a}_3^{(3)} = (0.571, 0.696, 0.825),\ \hat{a}_4^{(3)} = (0.437, 0.581, 0.725),$$
$$\hat{a}_5^{(3)} = (0.586, 0.712, 0.839).$$

Step 3: Utilize the FHHM operator (see Definition E) (suppose that its weight vector is also $\omega = (0.243, 0.514, 0.243)^T$) and Definition B (without loss of generality, $\delta = 0.5$):

$$\hat{a}_j = (\hat{a}_j^L, \hat{a}_j^M, \hat{a}_j^U) = \text{FHHM}(\hat{a}_j^{(1)}, \hat{a}_j^{(2)}, \hat{a}_j^{(3)})$$
$$= \frac{1}{\sum_{k=1}^{3} \frac{\omega_k}{\dot{\hat{a}}_j^{(\sigma(k))}}} = \left(\frac{1}{\sum_{k=1}^{3} \frac{\omega_k}{\dot{a}_j^{L(\sigma(k))}}}, \frac{1}{\sum_{k=1}^{3} \frac{\omega_k}{\dot{a}_j^{M(\sigma(k))}}}, \frac{1}{\sum_{k=1}^{3} \frac{\omega_k}{\dot{a}_j^{U(\sigma(k))}}} \right), j = 1, 2, 3, 4, 5$$

to aggregate the overall attribute values $\hat{a}_j^{(k)}$ $(k = 1, 2, 3)$ corresponding to the decision maker d_k $(= 1, 2, 3)$ and get the collective overall attribute value $\hat{a}_j$, where $\dot{\hat{a}}_j^{(\sigma(k))} = (\dot{a}_j^{L(\sigma(k))}, \dot{a}_j^{M(\sigma(k))}, \dot{a}_j^{U(\sigma(k))})$ is the kth largest of the weighted data $\dot{\hat{a}}_j^{(k)}$ $(\dot{\hat{a}}_j^{(k)} = 3v_k\hat{a}_j^{(k)}$, $k = 1, 2, 3)$:

$$\hat{a}_1 = (0.1975, 0.3504, 0.4818),\ \hat{a}_2 = (0.5122, 0.6542, 0.7759),$$
$$\hat{a}_3 = (0.4706, 0.5980, 0.7233),\ \hat{a}_4 = (0.2918, 0.4720, 0.6139),$$
$$\hat{a}_5 = (0.4919, 0.6234, 0.7519).$$

Step 4: Compare each $\hat{a}_j$ with all $\hat{a}_i$ $(i = 1, 2, 3, 4, 5)$ by using Definition B (without loss of generality, set $\delta = 0.5$), and let $p_{ij} = p(\hat{a}_i \geq \hat{a}_j)$, and then construct a possibility matrix:

$$\mathbf{P} = \begin{pmatrix} 0.5000 & 0.0000 & 0.0000 & 0.1060 & 0.0000 \\ 1.0000 & 0.5000 & 0.7008 & 1.0000 & 0.6013 \\ 1.0000 & 0.2992 & 0.5000 & 0.9682 & 0.4017 \\ 0.8940 & 0.0000 & 0.0318 & 0.5000 & 0.0000 \\ 1.0000 & 0.3987 & 0.5983 & 1.0000 & 0.5000 \end{pmatrix}.$$

Summing all elements in each line of matrix $\mathbf{P}$, we have

$$p_1 = 0.6060,\ p_2 = 3.8021,\ p_3 = 3.1691,\ p_4 = 1.4258,\ p_5 = 3.4970$$

and then reorder $\hat{a}_j$ $(j = 1, 2, 3, 4, 5)$ in descending order in accordance with the values p_j $(j = 1, 2, 3, 4, 5)$:

$$\hat{a}_2 > \hat{a}_5 > \hat{a}_3 > \hat{a}_4 > \hat{a}_1.$$

Step 5: Rank all alternatives x_j $(j = 1, 2, 3, 4, 5)$ by the ranking of $\hat{a}_j$ $(j = 1, 2, 3, 4, 5)$:

$$x_2 \succ x_5 \succ x_3 \succ x_4 \succ x_1$$

and thus the most desirable alternative is x_2.

From the previous results, it can be known that the ranking of the options in the Xu's approach [50] is the same with the our approach. But there are some differences between them. Firstly, the FWHM operator transforms the linguistic assessment values into triangular fuzzy numbers and then the FHHM operator aggregates the overall attribute values, and thus the results do not exactly match any initial linguistic terms. They also cannot be translated into linguistic variables, the transformations only an approximate form of the initial expression, which would lost and distort the original information and hence may bring about lack of precision. On the contrary, our approach generates the overall attribute values which expressed by the linguistic values. Secondly, our results are meaningful and definite, for example, the result of x_2 is $(s_5, 0.255)$, which means that x_2 is more than "slightly good". But the result of FHHM operator is $(0.5122, 0.6542, 0.7759)$, we can not definitely know that the result is better than "slightly good" or "good", or inferior than "very good". Finally, in order to rank all the alternatives from the collective overall attribute values, the Xu's approach needs the formula of comparing triangular fuzzy numbers and the possibility matrix to rank the results, while our approach only uses the linguistic comparison rule to the results.

The following practical case was adapted from [6].

Example 4. Due to increasing customization, a leading Taiwan firm in the bicycle industry needs a flexible manufacturing system (FMS) to produce a customized bike, which is designing for customer's requirements. After performing task analysis, it has been identified that this system should be produce mountain bikes and road racing bikes for a customized order. After preliminary screening, three competing alternatives, x_1, x_2 and x_3 are identified that are capable of performing this production task. A committee of three decision makers, d_1, d_2 and d_3 has been formed to conduct further evaluation and to select the most suitable FMS. The attributes which are considered here in assessment of x_j $(j = 1, 2, 3)$ are: 1) G_1 is process flexibility; 2) G_2 is product quality; 3) G_3 is learning; 4) G_4 is exposure to labor unrest. The decision maker d_k $(k = 1, 2, 3)$ evaluates the performance of FMS x_j $(j = 1, 2, 3)$ according to the attributes G_i $(j = 1, 2, 3, 4)$ by using the linguistic terms in the set

$$\begin{aligned} S = \{&s_1 = \text{extremely low}, s_2 = \text{very low}, s_3 = \text{low}, s_4 = \text{slightly low},\\ &s_5 = \text{middle}, s_6 = \text{slightly high}, s_7 = \text{high}, s_8 = \text{very high},\\ &s_9 = \text{extremely high}\}. \end{aligned}$$

and constructs, respectively, the linguistic decision matrix $A^{(k)}$ $(k = 1, 2, 3)$ as listed in Tables 3.10-3.12. Let $v = (0.3, 0.4, 0.3)^T$ be the weight vector of the decision makers d_k $(k = 1, 2, 3)$, and $w = (0.35, 0.15, 0.20, 0.30)^T$ be the weight vector of the attributes G_i $(i = 1, 2, 3, 4)$.

Now we utilize the our approach to find the decision result. We first transform the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{4\times 3}$ $(k = 1, 2, 3)$ into 2-tuple linguistic

Table 3.10: Linguistic decision matrix $A^{(1)}$

	x_1	x_2	x_3
G_1	s_6	s_3	s_7
G_2	s_5	s_6	s_4
G_3	s_7	s_6	s_5
G_4	s_4	s_6	s_5

Table 3.11: Linguistic decision matrix $A^{(2)}$

	x_1	x_2	x_3
G_1	s_5	s_5	s_7
G_2	s_6	s_3	s_8
G_3	s_2	s_7	s_7
G_4	s_5	s_6	s_5

Table 3.12: Linguistic decision matrix $A^{(3)}$

	x_1	x_2	x_3
G_1	s_7	s_5	s_6
G_2	s_4	s_7	s_7
G_3	s_3	s_8	s_7
G_4	s_3	s_6	s_6

decision matrix $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times3}$ $(k = 1, 2, 3)$ and utilize the 2TLWH operator (3.21) to aggregate all the elements in the jth column of $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times3}$ and get the overall attribute value $(a_j^{(k)}, \alpha_j^{(k)})$ of the alternative x_j:

$$(a_1^{(1)}, \alpha_1^{(1)}) = (s_5, 0.21), (a_2^{(1)}, \alpha_2^{(1)}) = (s_4, 0.44), (a_3^{(1)}, \alpha_3^{(1)}) = (s_5, 0.33),$$
$$(a_1^{(2)}, \alpha_1^{(2)}) = (s_4, -0.08), (a_2^{(2)}, \alpha_2^{(2)}) = (s_5, 0.04), (a_3^{(2)}, \alpha_3^{(2)}) = (s_6, 0.36),$$
$$(a_1^{(3)}, \alpha_1^{(3)}) = (s_4, -0.07), (a_2^{(3)}, \alpha_2^{(3)}) = (s_6, 0.01), (a_3^{(3)}, \alpha_3^{(3)}) = (s_6, 0.32).$$

Next, we utilize the 2TLHH operator (3.22) (whose weight vector is $\omega = (0.243, 0.514, 0.243)^T$ determined by using normal distribution based method) to aggregate the overall attribute values $(a_j^{(k)}, \alpha_j^{(k)})$ $(k = 1, 2, 3)$ to the decision maker d_k $(k = 1, 2, , 3)$ and get the collective overall attribute value (a_j, α_j):

$$(a_1, \alpha_1) = (s_4, 0.174), (a_2, \alpha_2) = (s_5, 0.072), (a_3, \alpha_3) = (s_6, 0.066).$$

Then,we utilize the collective overall attribute value (a_j, α_j) $(j = 1, 2, 3)$ to rank the alternatives x_j $(j = 1, 2, 3)$:

$$x_3 \succ x_2 \succ x_1$$

and thus the most desirable alternative is x_3.

3.5 Conclusions

In this chapter, based on the harmonic mean operator, we have developed several new harmonic aggregation operators including the 2-tuple linguistic weighted harmonic (2TLWH), 2-tuple linguistic ordered weighted harmonic (2TLOWH) and 2-tuple linguistic hybrid harmonic (2TLHH) operators, which can be utilized to aggregate preference information taking the form of linguistic variables. We have studied some desired properties of the developed operators, such as commutativity, idempotency and boundedness. The 2TLHH operator generalizes both the 2TLWH operator and the 2TLOWH operator, and reflects the importance degrees of both the given linguistic arguments and their ordered positions. Based on the 2TLWH and the 2LHH operators, we have proposed an approach to multiple attribute group decision making with linguistic information. We have also applied the proposed approach to the problem of investing a sum of money in best option. The proposed approach are compared with Xu's approach [50] to show their advantages and effectiveness.

Appendix

Let $\hat{a} = (a^L, a^M, a^U)$, where $a^U \geq a^M \geq a^L \geq 0$, then $\hat{a}$ is called a triangular fuzzy number, where a^L and a^U stand for the lower and upper values of $\hat{a}$, respectively, and a^M stands for the modal value [32]. For convenience, we let Ω be the set of all triangular fuzzy numbers. Some operational laws of triangular fuzzy numbers as follows [32]:

Definition A. Let $\hat{a} = (a^L, a^M, a^U)$ and $\hat{b} = (b^L, b^M, b^U)$ be two triangular fuzzy numbers, then

1) $\hat{a} + \hat{b} = (a^L, a^M, a^U) + (b^L, b^M, b^U) = (a^L + b^L, a^M + b^M, a^U + b^U)$;

2) $\lambda\hat{a} = \lambda(a^L, a^M, a^U) = (\lambda a^L, \lambda a^M, \lambda a^U)$;

3) $\frac{1}{\hat{a}} = \frac{1}{(a^L, a^M, a^U)} = (\frac{1}{a^U}, \frac{1}{a^M}, \frac{1}{a^L})$.

In order to compare two triangular fuzzy numbers, Xu [50] provided the following definition:

Definition B. Let $\hat{a} = (a^L, a^M, a^U)$ and $\hat{b} = (b^L, b^M, b^U)$ be two triangular fuzzy numbers, then the degree of possibility of $\hat{a} \geq \hat{b}$ is defined as follows:

$$p(\hat{a} \geq \hat{b}) = \delta \max\left\{1 - \max\left(\frac{b^M - a^L}{a^M - a^L + b^M - b^L}, 0\right), 0\right\}$$
$$+(1-\delta)\max\left\{1 - \max\left(\frac{b^U - a^M}{a^U - a^M + b^U - b^M}, 0\right), 0\right\}, \ \delta \in [0,1]$$

which satisfies the following properties:

$$0 \leq p(\hat{a} \geq \hat{b}) \leq 1, \ p(\hat{a} \geq \hat{a}) = 0.5, \ p(\hat{a} \geq \hat{b}) + p(\hat{b} \geq \hat{a}) = 1.$$

Definition C. Let $\hat{a}_i = (a_i^L, a_i^M, a_i^U)$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A fuzzy weighted harmonic mean (FWHM) operator of dimension n is a mapping FWHM : $\Omega^n \to \Omega$, such that

$$\mathrm{FWHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\sum_{i=1}^n \frac{w_i}{\hat{a}_i}}$$
$$= \left(\frac{1}{\sum_{i=1}^n \frac{w_i}{a_i^L}}, \frac{1}{\sum_{i=1}^n \frac{w_i}{a_i^M}}, \frac{1}{\sum_{i=1}^n \frac{w_i}{a_i^U}} \right),$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\hat{a}_i$ $(i = 1, 2, \ldots, n)$ with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FWHM operator is reduced to the FHM operator:

$$\mathrm{FHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{n}{\sum_{i=1}^n \frac{1}{\hat{a}_i}}.$$

Definition D. Let $\hat{a}_i = (a_i^L, a_i^M, a_i^U)$ $(i = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A fuzzy ordered weighted harmonic mean (FOWHM) operator of dimension n is a mapping FOWHM : $\Omega^n \to \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$, such that

$$\mathrm{FOWHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\sum_{i=1}^n \frac{\omega_i}{\hat{a}_{\sigma(i)}}}$$
$$= \left(\frac{1}{\sum_{i=1}^n \frac{\omega_i}{a_{\sigma(i)}^L}}, \frac{1}{\sum_{i=1}^n \frac{\omega_i}{a_{\sigma(i)}^M}}, \frac{1}{\sum_{i=1}^n \frac{\omega_i}{a_{\sigma(i)}^U}} \right),$$

where $\hat{a}_{\sigma(i)} = [a_{\sigma(i)}^L, a_{\sigma(i)}^M, a_{\sigma(i)}^U]$ $(i = 1, 2, \ldots, n)$, and $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$ such that $\hat{a}_{\sigma(i-1)} \geq \hat{a}_{\sigma(i)}$ for all i.

Definition E. Let $\hat{a}_j = (a_j^L, a_j^M, a_j^U)$ $(j = 1, 2, \ldots, n)$ be a collection of triangular fuzzy numbers. A fuzzy hybrid harmonic mean (FHHM) operator of dimension n is a mapping FHHM : $\Omega^n \to \Omega$, which has an associated vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with $\omega_i \geq 0$ and $\sum_{i=1}^n \omega_i = 1$, such that

$$\mathrm{FHHM}(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) = \frac{1}{\sum_{i=1}^n \frac{\omega_i}{\dot{\hat{a}}_{\sigma(i)}}}$$

$$= \left(\frac{1}{\sum_{i=1}^{n} \frac{\omega_i}{\dot{a}^L_{\sigma(i)}}}, \frac{1}{\sum_{i=1}^{n} \frac{\omega_i}{\dot{a}^M_{\sigma(i)}}}, \frac{1}{\sum_{i=1}^{n} \frac{\omega_i}{\dot{a}^U_{\sigma(i)}}} \right),$$

where $\dot{\hat{a}}_{\sigma(i)} = [\dot{a}^L_{\sigma(i)}, \dot{a}^M_{\sigma(i)}, \dot{a}^U_{\sigma(i)}]$ is the ith largest of the weighted triangular fuzzy numbers $\dot{\hat{a}}_i$ ($\dot{\hat{a}}_i = nw_i\hat{a}_i$, $i = 1, 2, \ldots, n$), $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $\dot{\hat{a}}_i$ ($i = 1, 2, \ldots, n$) with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$, and n is the balancing coefficient. Especially, if $w = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then $\dot{\hat{a}}_i = \hat{a}_i$, $i = 1, 2, \ldots, n$, in this case, the FHHM operator is reduced to the FOWHM operator. Moreover, if $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, then the FHHH operator is reduced to the FWHM operator.

Chapter 4

2-tuple linguistic prioritized aggregation operators and their applications in group decision making

The prioritized average (PA) operator is a nonlinear weighted average aggregation tool whose weighting vectors depend on their input arguments. In this chapter, we develop some 2-tuple linguistic prioritized aggregation operators such as 2-tuple linguistic prioritized weighted average (2TLPWA) operator, 2-tuple linguistic prioritized weighted geometric (2TLPWG) operator, 2-tuple linguistic prioritized ordered weighted average (2TLPOWA) operator and 2-tuple linguistic prioritized ordered weighted geometric (2TLPOWG) operator. Then, we apply them to develop approaches to multiple attribute group decision making, with linguistic information, in which the attributes are in different priority levels. Finally, an example is used to illustrate the applicability of the developed approaches.

4.1 2-tuple linguistic prioritized aggregation operators

The prioritized average (PA) operator was originally introduced by Yager [61], which was defined as follows:

Definition 4.1.1 [61] Let $G = \{G_1, G_2, \ldots, G_n\}$ be a collection of attribute and that there is a prioritization between the attribute expressed by the linear ordering $G_1 \succ G_2 \succ G_3 \succ \cdots \succ G_n$, indicate attribute G_j has higher priority than G_k if $j < k$. The value $G_j(x)$ is the performance of any alternative x under attribute G_j, and satisfies $G_j(x) \in [0, 1]$. If

$$\mathrm{PA}(G(x)) = \sum_{j=1}^{n} w_j G_j(x), \tag{4.1}$$

where $w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}$, $T_j = \prod_{k=1}^{j-1} G_k(x)$ $(j = 2, 3, \ldots, n)$, $T_1 = 1$, then PA is called the prioritized average (PA) operator.

The prioritized average (PA) operators have usually been used in the situation where the input arguments are the exact values. In some situations, however, the input arguments take the form of 2-tuple linguistic variables rather than numerical ones because of time pressure, lack of knowledge, and the decision maker's limited attention and information processing capabilities.

Definition 4.1.2 Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, and (r_i, α_i) $(r_i \in S,\ \alpha_i \in [-0.5, 0.5),\ i = 1, 2, \ldots, n)$ be a collection of 2-tuple linguistic variables, then we define a 2-tuple linguistic prioritized weighted average (2TLPWA) operators as follows:

$$\mathrm{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = \triangle\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{j=1}^{n} T_j} \triangle^{-1}(r_i, \alpha_i)\right), \tag{4.2}$$

where $T_i = \prod_{k=1}^{i-1} \frac{1}{g} |\triangle^{-1}(r_k, \alpha_k)|$, $i = 2, 3, \ldots, n$, $T_1 = 1$ and $\frac{1}{g}|\triangle^{-1}(r_i, \alpha_i)|$ is the score values of (r_i, α_i) $(i = 2, \ldots, n)$.

Clearly, the 2TLPWA operator is a nonlinear weighted aggregation operator, and the weight $\frac{T_i}{\sum_{j=1}^{n} T_j}$ of the argument (r_i, α_i) depends on all the input arguments (r_j, α_j) $(j = 1, 2, \ldots, n)$ and allows the argument values to support each other in the prioritized aggregation process.

Example 4.1.3 Let $S = \{s_i : i = 0, 1, 2, \ldots, 8\}$ be a finite and totally ordered discrete linguistic term set. Assume that we have four 2-tuple linguistic variables: $(r_1, \alpha_1) = (s_2, 0.2)$, $(r_2, \alpha_2) = (s_5, -0.3)$, $(r_3, \alpha_3) = (s_6, 0.3)$ and $(r_4, \alpha_4) = (s_3, -0.4)$, then

$$T_1 = 1$$
$$T_2 = \frac{1}{8}| \triangle^{-1} (s_2, 0.2)| = \frac{2.2}{8} = 0.275$$
$$T_3 = (0.275)\frac{1}{8}| \triangle^{-1} (s_5, -0.3)| = (0.275)(0.588) = 0.162$$
$$T_4 = (0.162)\frac{1}{8}| \triangle^{-1} (s_6, 0.3)| = (0.162)(0.788) = 0.128.$$

We see $\sum_{i=1}^{4} T_i = 1.565$. From this we get the aggregated value

$$\begin{aligned} &\text{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), (r_3, \alpha_3), (r_4, \alpha_4)) \\ &= \triangle\Big(\frac{1}{1.565}(2.2) + \frac{0.275}{1.565}(4.7) + \frac{0.162}{1.565}(6.3) + \frac{0.128}{1.565}(2.6)\Big) \\ &= \triangle(3.096) = (s_3, 0.096). \end{aligned}$$

In the following we shall make an investigation on some desirable properties of the 2TLPWA operator.

Theorem 4.1.4 *Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, and (r_i, α_i) $(r_i \in S,\ \alpha_i \in [-0.5, 0.5),\ i = 1, 2, \ldots, n)$ be a collection of 2-tuple linguistic variables, then we have the following properties:*

1) (Idempotency): If $(r_i, \alpha_i) = (r, \alpha)$ for all i, then

$$\text{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = (r, \alpha). \tag{4.3}$$

2) (Monotonicity): Let (r'_i, α'_i) ($r'_i \in S$, $\alpha'_i \in [-0.5, 0.5)$, $i = 1, 2, \ldots, n$) be a collection of 2-tuple linguistic variables, if $(r_i, \alpha_i) \leq (r'_i, \alpha'_i)$, for all i, then

$$\begin{aligned} &\mathrm{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad \leq \mathrm{2TLPWA}((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n)). \end{aligned} \tag{4.4}$$

3) (Boundedness):

$$\min_i (r_i, \alpha_i) \leq \mathrm{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \leq \max_i (r_i, \alpha_i). \tag{4.5}$$

Based on the 2TLPWA operator and the geometric mean, in the following, we define a 2-tuple linguistic prioritized geometric (2TLPG) operator.

Definition 4.1.5 Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, and (r_i, α_i) ($r_i \in S$, $\alpha_i \in [-0.5, 0.5)$, $i = 1, 2, \ldots, n$) be a collection of 2-tuple linguistic variables, then we define a 2-tuple linguistic prioritized weighted geometric (2TLPWG) operators as follows:

$$\mathrm{2TLPWG}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = \triangle \left(\prod_{i=1}^{n} \left(\triangle^{-1}(r_i, \alpha_i) \right)^{\frac{T_i}{\sum_{j=1}^{n} T_j}} \right) \tag{4.6}$$

where $T_i = \prod_{k=1}^{i-1} \frac{1}{g} |\triangle^{-1}(r_i, \alpha_i)|$, $i = 2, 3, \ldots, n$, $T_1 = 1$ and $\frac{1}{g}|\triangle^{-1}(r_i, \alpha_i)|$ is the score values of (r_i, α_i) ($i = 2, \ldots, n$).

Example 4.1.6 Consider the 2-tuple linguistic variables (r_i, α_i) ($i = 1, 2, 3, 4$) given in Example 4.1.3. Then by using 2TLPG operator, the aggregated value is

$$\begin{aligned} &\mathrm{2TLPWG}((r_1, \alpha_1), (r_2, \alpha_2), (r_3, \alpha_3), (r_4, \alpha_4)) \\ &\qquad = \triangle \left((2.2)^{\frac{1}{1.565}} \times (4.7)^{\frac{0.275}{1.565}} \times (6.3)^{\frac{0.162}{1.565}} \times (2.6)^{\frac{0.128}{1.565}} \right) \\ &\qquad = \triangle(2.842) = (s_3, -0.158). \end{aligned}$$

By Lemma 2.1.1, we have the following theorem.

Theorem 4.1.7 *Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set. Assume that (r_i, α_i) ($r_i \in S$, $\alpha_i \in [-0.5, 0.5)$, $i = 1, 2, \ldots, n$) are a collection of 2-tuple linguistic variables, we then have*

$$\begin{aligned} &\text{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\quad \leq \text{2TLPWG}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)). \end{aligned} \tag{4.7}$$

Similar to Theorem 4.1.4, we have the following properties.

Theorem 4.1.8 *Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, and (r_i, α_i) ($r_i \in S$, $\alpha_i \in [-0.5, 0.5)$, $i = 1, 2, \ldots, n$) be a collection of 2-tuple linguistic variables, then we have the following properties:*

1) (Idempotency): If $(r_i, \alpha_i) = (r, \alpha)$ for all i, then

$$\text{2TLPWG}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = (r, \alpha). \tag{4.8}$$

2) (Monotonicity): Let (r'_i, α'_i) ($r'_i \in S$, $\alpha'_i \in [-0.5, 0.5)$, $i = 1, 2, \ldots, n$) be a collection of 2-tuple linguistic variables, if $(r_i, \alpha_i) \leq (r'_i, \alpha'_i)$, for all i, then

$$\begin{aligned} &\text{2TLPWG}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\quad \leq \text{2TLPWG}((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n)). \end{aligned} \tag{4.9}$$

3) (Boundedness):

$$\min_i (r_i, \alpha_i) \leq \text{2TLPWG}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \leq \max_i (r_i, \alpha_i). \tag{4.10}$$

The fundamental characteristic of both the 2TLPWA and 2TLPWG operators is that they weight all the given 2-tuple linguistic variables themselves, and the weighting vectors depend upon the input arguments. However, in many group decision making problems, we need to rearrange all the given arguments in descending (or ascending) order, and then weight the ordered positions of the input arguments so as relieve the influence of unfair arguments on decision result by assigning low weights. As a result, motivated by the idea of Yager's OWA operator [56, 45], in the following, we define 2-tuple linguistic prioritized ordered weighted aggregation operators.

Definition 4.1.9 Let $S = \{s_i : i = 0, 1, 2, \ldots, g\}$ be a finite and totally ordered discrete linguistic term set, and (r_i, α_i) ($r_i \in S$, $\alpha_i \in [-0.5, 0.5)$, $i = 1, 2, \ldots, n$) be a collection of 2-tuple linguistic variables, then we define a 2-tuple linguistic prioritized ordered weighted average (2TLPOWA) operator as follows:

$$\begin{aligned} &\mathrm{2TLPOWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad = \triangle\left(\sum_{i=1}^{n} u_i \triangle^{-1}(r_{\mathrm{index}(i)}, \alpha_{\mathrm{index}(i)})\right) \end{aligned} \tag{4.11}$$

where index is an indexing function such that index(i) is the index of the ith largest of the arguments (r_j, α_j) ($j = 1, 2, \ldots, n$), and thus $(r_{\mathrm{index}(i)}, \alpha_{\mathrm{index}(i)})$ is the ith largest argument of (r_j, α_j) ($j = 1, 2, \ldots, n$), and u_i ($i = 1, 2, \ldots, n$) are a collection of weights such that

$$u_i = f(R_i) - f(R_{i-1}),\ R_i = \sum_{k=1}^{i} V_{\mathrm{index}(k)},\ i = 1, 2, \ldots, n \tag{4.12}$$

where $f : [0, 1] \to [0, 1]$ is a basic unit-interval monotone (BUM) function having the following properties: 1) $f(0) = 0$; 2) $f(1) = 1$; 3) $f(x) \geq f(y)$ if $x > y$, and $V_{\mathrm{index}(k)}$ is the associated priority based importance weight of the kth largest argument $(r_{\mathrm{index}(k)}, \alpha_{\mathrm{index}(k)})$ of arguments (r_j, α_j) ($j = 1, 2, \ldots, n$) having the priority based importance weights V_j ($j = 1, 2, \ldots, n$) given by

$$V_j = \frac{T_j}{\sum_{k=1}^{n} T_k},\ T_j = \prod_{k=1}^{j-1} \frac{1}{g} |\triangle^{-1}(r_k, \alpha_k)|,\ j = 2, 3, \ldots, n,\ T_1 = 1, \tag{4.13}$$

where $\frac{1}{g}|\triangle^{-1}(r_j, \alpha_j)|$ is the score value of (r_j, α_j) ($j = 2, \ldots, n$).

Example 4.1.10 Consider the 2-tuple linguistic variables (r_i, α_i) ($i = 1, 2, 3, 4$) given in Example 4.1.3. We first order the arguments (r_i, α_i) as follows:

$$\begin{aligned} &(r_{\mathrm{index}(1)}, \alpha_{\mathrm{index}(1)}) = (s_6, 0.3),\ (r_{\mathrm{index}(2)}, \alpha_{\mathrm{index}(2)}) = (s_5, -0.3), \\ &(r_{\mathrm{index}(3)}, \alpha_{\mathrm{index}(3)}) = (s_3, -0.4),\ (r_{\mathrm{index}(4)}, \alpha_{\mathrm{index}(4)}) = (s_2, 0.2) \end{aligned}$$

and get the associated priority based importance weight $V_{\text{index}(i)}$ of $(r_{\text{index}(i)}, \alpha_{\text{index}(i)})$ $(i = 1, 2, 3, 4)$ as follows:

$$V_{\text{index}(1)} = \frac{0.162}{1.565} = 0.1035, V_{\text{index}(2)} = \frac{0.275}{1.565} = 0.1757,$$
$$V_{\text{index}(3)} = \frac{0.128}{1.565} = 0.0818, V_{\text{index}(4)} = \frac{1}{1.565} = 0.6390.$$

Assume that the BUM function f is given by $f(x) = x^2$. Using this and the normalized priority based weights, we obtain the weights u_i $(i = 1, 2, 3, 4)$:

$$u_1 = (0.1035)^2 = 0.0107,\ u_2 = (0.2792)^2 - (0.1035)^2 = 0.0672,$$
$$u_3 = (0.361)^2 - (0.2792)^2 = 0.0524,\ u_4 = (1)^2 - (0.361)^2 = 0.8697.$$

Using these values and 2TLOWA operator, the aggregated value is

$$\begin{aligned}
&\text{2TLPOWA}((r_1, \alpha_1), (r_2, \alpha_2), (r_3, \alpha_3), (r_4, \alpha_4)) \\
&\qquad = \triangle\left((0.0107)(6.3) + (0.0672)(4.7) + (0.0524)(2.6) + (0.8697)(2.2)\right) \\
&\qquad = \triangle(2.474) = (s_2, 0.474).
\end{aligned}$$

Especially, if $f(x) = x$, then the 2TLPOWA operator reduces to the 2TLPWA operator. In fact, by (4.13) and (4.14), we have

$$\begin{aligned}
\text{2TLPOWA}&((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\
&= \triangle\left(\sum_{i=1}^{n} u_i \triangle^{-1}(r_{\text{index}(i)}, \alpha_{\text{index}(i)})\right) \\
&= \triangle\left(\sum_{i=1}^{n} (f(R_i) - f(R_{i-1})) \triangle^{-1}(r_{\text{index}(i)}, \alpha_{\text{index}(i)})\right) \\
&= \triangle\left(\sum_{i=1}^{n} (R_i - R_{i-1}) \triangle^{-1}(r_{\text{index}(i)}, \alpha_{\text{index}(i)})\right) \\
&= \triangle\left(\sum_{i=1}^{n} V_{\text{index}(i)} \triangle^{-1}(r_{\text{index}(i)}, \alpha_{\text{index}(i)})\right) \\
&= \triangle\left(\sum_{i=1}^{n} \frac{T_i}{\sum_{j=1}^{n} T_j} \triangle^{-1}(r_i, \alpha_i)\right) \\
&= \text{2TLPWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)). \qquad (4.14)
\end{aligned}$$

Similar to Theorem 4.1.4, we have the following properties.

Theorem 4.1.11 *Let (r_i, α_i) $(r_i \in S,\ \alpha_i \in [-0.5, 0.5),\ i = 1, 2, \ldots, n)$ be a collection of 2-tuple linguistic variables, then we have the following properties:*

1) (Idempotency): If $(r_i, \alpha_i) = (r, \alpha)$ for all i, then

$$\text{2TLPOWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) = (r, \alpha). \tag{4.15}$$

2) (Monotonicity): Let (r'_i, α'_i) $(r'_i \in S,\ \alpha'_i \in [-0.5, 0.5),\ i = 1, 2, \ldots, n)$ be a collection of 2-tuple linguistic variables, if $(r_i, \alpha_i) \leq (r'_i, \alpha'_i)$, for all i, then

$$\begin{aligned} &\text{2TLPOWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad \leq \text{2TLPOWA}((r'_1, \alpha'_1), (r'_2, \alpha'_2), \ldots, (r'_n, \alpha'_n)). \end{aligned} \tag{4.16}$$

3) (Boundedness):

$$\min_i (r_i, \alpha_i) \leq \text{2TLPOWA}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \leq \max_i (r_i, \alpha_i). \tag{4.17}$$

Furthermore, based on the 2TLPOWA operator (4.12) and the geometric mean, now we define a 2-tuple linguistic prioritized ordered weighted geometric (2TLPOWG) operator.

Definition 4.1.12 Let (r_i, α_i) $(r_i \in S,\ \alpha_i \in [-0.5, 0.5),\ i = 1, 2, \ldots, n)$ be a collection of 2-tuple linguistic variables, then we define a 2-tuple linguistic prioritized ordered weighted geometric (2TLPOWG) operator as follows:

$$\begin{aligned} &\text{2TLPOWG}((r_1, \alpha_1), (r_2, \alpha_2), \ldots, (r_n, \alpha_n)) \\ &\qquad = \triangle\left(\prod_{i=1}^{n} \left(\triangle^{-1}(r_{\text{index}(i)}, \alpha_{\text{index}(i)})\right)^{u_i}\right), \end{aligned} \tag{4.18}$$

where u_i $(i = 1, 2, \ldots, n)$ is a collection of weights satisfying the conditions (4.13) and (4.14).

Especially, if $f(x) = x$, then the 2TLPOWG operator reduces to the 2TLPWG operator. Clearly, the weighting vectors of both the 2TLPOWA and 2TLPOWG operators not only depend upon the input arguments, but also emphasize the ordered positions of the given arguments. Furthermore, the 2TLPOWG operators have also the properties: commutativity, idempotency and boundedness.

4.2 Approaches to multiple attribute group decision making with linguistic information

In this section, we shall utilize the 2-tuple linguistic prioritized aggregation operators to multiple attribute decision making with linguistic information.

For a MAGDM problems with linguistic information, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n alternatives, $G = \{G_1, G_2, \ldots, G_m\}$ be a set of m attributes and that there is a prioritization between the attributes expressed by he linear ordering $G_1 \succ G_2 \succ \cdots \succ G_m$, indicate attribute G_j has a higher priority than G_s, if $j < s$. Let $E = \{e_1, e_2, \ldots, e_s\}$ be a set of s decision makers, whose weight vector is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)^T$, with $\lambda_k \geq 0$, $k = 1, 2, \ldots, s$, and $\sum_{k=1}^{s} \lambda_k = 1$. Suppose that $A^{(k)} = (a_{ij}^{(k)})_{m\times n}$ is the linguistic decision matrix, where $a_{ij}^{(k)} \in S$ is preference value, which takes the form of linguistic variables, given by the decision maker $e_k \in E$, for alternative $x_j \in X$ with respect to attribute $G_i \in G$.

Then, we utilize the 2TLPWA and 2TLPOWA (or 2TLPWG and 2TLPOWG) operators to propose an approach to multiple attribute group decision making with linguistic information, which involves the following steps:

Step 1: Transform the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{m\times n}$ into 2-tuple linguistic decision matrix $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{m\times n}$, $k = 1, 2, \ldots, s$.

Step 2: Calculate the values of $T_{ij}^{(k)}$ $(k = 1, 2, \ldots, s)$ as follows:

$$\begin{aligned} T_{ij}^{(k)} &= \prod_{l=1}^{k-1} \frac{1}{g} | \triangle^{-1} (a_{ij}^{(l)}, 0)|, \ k = 2, 3, \ldots, s \\ T_{ij}^{(1)} &= 1 \end{aligned} \tag{4.19}$$

and utilize (4.13) and (4.14) to calculate the weights $u_{ij}^{(k)}$ associated with the kth largest argument $(a_{ij}^{\mathrm{index}(k)}, 0)$, where

$$u_{ij}^{(k)} = f(R_{ij}^{(k)}) - f(R_{ij}^{(k-1)}), \ R_{ij}^{(k)} = \sum_{l=1}^{k} V_{ij}^{\mathrm{index}(k)}, \tag{4.20}$$

where $V_{\mathrm{index}(k)}$ is the associated priority based importance weight of the kth largest argument $(a_{ij}^{\text{index}(k)}, 0)$ of arguments $(a_{ij}^{(l)}, 0)$ $(l = 1, 2, \ldots, s)$ having $V_{ij}^{(l)} = \frac{T_{ij}^{(l)}}{\sum_{k=1}^{s} T_{ij}^{(k)}}$ $(l = 1, 2, \ldots, s)$, and $u_{ij}^{(k)} \geq 0$, $k = 1, 2, \ldots, s$, and $\sum_{k=1}^{s} u_{ij}^{(k)} = 1$.

Step 3: Utilize the 2TLPOWA operator (4.12):

$$\begin{aligned}(a_{ij}, \alpha_{ij}) &= \text{2TLPOWA}((a_{ij}^{(1)}, 0), (a_{ij}^{(2)}, 0), \ldots, (a_{ij}^{(s)}, 0)) \\ &= \triangle \left(\sum_{k=1}^{s} u_{ij}^{(k)} \triangle^{-1} (a_{ij}^{\text{index}(k)}, 0) \right), \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n\end{aligned} \tag{4.21}$$

or the 2TLPOWG operator (4.19):

$$\begin{aligned}(a_{ij}, \alpha_{ij}) &= \text{2TLPOWG}((a_{ij}^{(1)}, 0), (a_{ij}^{(2)}, 0), \ldots, (a_{ij}^{(s)}, 0)) \\ &= \triangle \left(\prod_{k=1}^{s} \left(\triangle^{-1}(a_{ij}^{\text{index}(k)}, 0) \right)^{u_{ij}^{(k)}} \right), \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n\end{aligned} \tag{4.22}$$

to aggregate all the 2-tuple linguistic decision matrices $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{m \times n}$ $(k = 1, 2, \ldots, s)$ into the collective 2-tuple linguistic decision matrix $\tilde{A} = ((a_{ij}, \alpha_{ij}))_{m \times n}$, where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Step 4: Calculate the values of T_{ij} $(i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n)$ as follows:

$$\begin{aligned}&T_{ij} = \prod_{k=1}^{i-1} \frac{1}{g} | \triangle^{-1} (a_{kj}, \alpha_{kj})|, \ i = 2, 3, \ldots, m, \ j = 1, 2, \ldots, n \\ &T_{1j} = 1, \ j = 1, 2, \ldots, n\end{aligned} \tag{4.23}$$

Step 5: To get the overall preference value (a_j, α_j) corresponding to the alternative x_j, we aggregate all the preference values (a_{ij}, α_{ij}) $(i = 1, 2, \ldots, m)$ in the jth column of $\tilde{A}$ by using the 2TLPWA operator (4.2):

$$\begin{aligned}(a_j, \alpha_j) &= \text{2TLPWA}((a_{1j}, \alpha_{1j}), (a_{2j}, \alpha_{2j}), \ldots, (a_{mj}, \alpha_{mj})) \\ &= \triangle \left(\sum_{i=1}^{m} \frac{T_{ij}}{\sum_{i=1}^{m} T_{ij}} \triangle^{-1} (a_{ij}, \alpha_{ij}) \right), \ j = 1, 2, \ldots, n,\end{aligned} \tag{4.24}$$

or the 2TLPWG operator (4.6):

$$(a_j, \alpha_j) = \text{2TLPWG}((a_{1j}, \alpha_{1j}), (a_{2j}, \alpha_{2j}), \ldots, (a_{mj}, \alpha_{mj}))$$
$$= \triangle\left(\prod_{i=1}^{m}\left(\triangle^{-1}(a_{ij}, \alpha_{ij})\right)^{\frac{T_{ij}}{\sum_{i=1}^{m} T_{ij}}}\right), \; j = 1, 2, \ldots, n \quad (4.25)$$

Step 6: Utilize the collective overall attribute values (a_j, α_j) $(j = 1, 2, \ldots, n)$ and Definition 3.1.3 to rank the alternatives x_j $(j = 1, 2, \ldots, n)$, and then select the most desirable one.

Step 7: End.

4.3 Illustrative example

Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted from [14]). There is a panel with five possible alternatives in which to invest the money: 1) x_1 is a car industry; 2) x_2 is a food company; 3) x_3 is a computer company; 4) x_4 is an arms company; 5) x_5 is a TV company.

The investment company must take a decision according to the following four attributes: 1) G_1 is the risk analysis; 2) G_2 is the growth analysis; G_3 is the social-political impact analysis; 4) G_4 is the environmental impact analysis.

The five possible alternatives x_j $(j = 1, 2, 3, 4, 5)$ are evaluated using the linguistic term set S

$$S = \{s_0 = \text{extremely poor}, s_1 = \text{very poor}, s_2 = \text{poor}, s_3 = \text{slightly poor},$$
$$s_4 = \text{fair}, s_5 = \text{slightly good}, s_6 = \text{good}, s_7 = \text{very good},$$
$$s_8 = \text{extremely good}\}$$

by three decision makers e_k $(k = 1, 2, 3)$ under the above four attributes G_i $(i = 1, 2, 3, 4)$, and construct, respectively, the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{4\times 5}$ $(k = 1, 2, 3)$ as listed in Tables 4.1-4.3.

Table 4.1: Linguistic decision matrix $A^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_4	s_8	s_4	s_8	s_6
G_2	s_5	s_7	s_5	s_6	s_4
G_3	s_3	s_7	s_5	s_3	s_7
G_4	s_2	s_4	s_6	s_2	s_5

Table 4.2: Linguistic decision matrix $A^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_4	s_6	s_5	s_7	s_5
G_2	s_5	s_6	s_4	s_6	s_3
G_3	s_2	s_6	s_4	s_2	s_6
G_4	s_2	s_3	s_5	s_4	s_4

Table 4.3: Linguistic decision matrix $A^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	s_4	s_6	s_5	s_7	s_5
G_2	s_6	s_4	s_5	s_5	s_7
G_3	s_2	s_5	s_7	s_6	s_6
G_4	s_2	s_6	s_6	s_3	s_6

To get the best alternative(s), we first utilize the 2TLPWA and 2TLPOWA operators to develop an approach to multiple attribute group decision making problem with linguistic information, which can be described as following:

Step 1: Transform the linguistic decision matrix $A^{(k)} = (a_{ij}^{(k)})_{4\times 5}$ $(k = 1, 2, 3)$ into 2-tuple linguistic decision matrix $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times 5}$ $(k = 1, 2, 3)$ (see Tables 4.4-4.6):

Step 2: Utilize (4.20) to calculate the values of $T_{ij}^{(k)}$ $(k = 1, 2, 3)$ which are contained in the matrices $T^{(k)} = (T_{ij}^{(k)})_{4\times 5}$, respectively

$$T^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \; T^{(2)} = \begin{pmatrix} 0.500 & 0.750 & 0.625 & 0.875 & 0.625 \\ 0.625 & 0.750 & 0.500 & 0.750 & 0.375 \\ 0.250 & 0.750 & 0.500 & 0.250 & 0.750 \\ 0.250 & 0.375 & 0.625 & 0.500 & 0.500 \end{pmatrix},$$

$$T^{(3)} = \begin{pmatrix} 0.250 & 0.563 & 0.391 & 0.766 & 0.391 \\ 0.469 & 0.375 & 0.313 & 0.469 & 0.328 \\ 0.063 & 0.469 & 0.438 & 0.188 & 0.563 \\ 0.063 & 0.281 & 0.469 & 0.188 & 0.375 \end{pmatrix},$$

and utilize (4.21) (suppose that the BUM function f is defined by $f(x) = x^2$) to calculate the weights $u_{ij}^{(k)}$ $(k = 1, 2, 3)$ which are contained in the matrices $U^{(k)} = (u_{ij}^{(k)})_{4\times 5}$, respectively

$$U^{(1)} = \begin{pmatrix} 0.326 & 0.187 & 0.096 & 0.144 & 0.246 \\ 0.050 & 0.222 & 0.304 & 0.203 & 0.037 \\ 0.581 & 0.203 & 0.051 & 0.017 & 0.187 \\ 0.581 & 0.029 & 0.229 & 0.088 & 0.040 \end{pmatrix},$$

$$U^{(2)} = \begin{pmatrix} 0.408 & 0.385 & 0.158 & 0.360 & 0.404 \\ 0.441 & 0.457 & 0.221 & 0.419 & 0.571 \\ 0.326 & 0.419 & 0.500 & 0.665 & 0.386 \\ 0.326 & 0.570 & 0.264 & 0.078 & 0.497 \end{pmatrix},$$

Table 4.4: 2-tuple linguistic decision matrix $\tilde{A}^{(1)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_8, 0)$	$(s_4, 0)$	$(s_8, 0)$	$(s_6, 0)$
G_2	$(s_5, 0)$	$(s_7, 0)$	$(s_5, 0)$	$(s_6, 0)$	$(s_4, 0)$
G_3	$(s_3, 0)$	$(s_7, 0)$	$(s_5, 0)$	$(s_3, 0)$	$(s_7, 0)$
G_4	$(s_2, 0)$	$(s_4, 0)$	$(s_6, 0)$	$(s_2, 0)$	$(s_5, 0)$

Table 4.5: 2-tuple linguistic decision matrix $\tilde{A}^{(2)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_5, 0)$
G_2	$(s_5, 0)$	$(s_6, 0)$	$(s_4, 0)$	$(s_6, 0)$	$(s_3, 0)$
G_3	$(s_2, 0)$	$(s_6, 0)$	$(s_4, 0)$	$(s_2, 0)$	$(s_6, 0)$
G_4	$(s_2, 0)$	$(s_3, 0)$	$(s_5, 0)$	$(s_4, 0)$	$(s_4, 0)$

Table 4.6: 2-tuple linguistic decision matrix $\tilde{A}^{(3)}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_6, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_5, 0)$
G_2	$(s_6, 0)$	$(s_4, 0)$	$(s_5, 0)$	$(s_5, 0)$	$(s_7, 0)$
G_3	$(s_2, 0)$	$(s_5, 0)$	$(s_7, 0)$	$(s_6, 0)$	$(s_6, 0)$
G_4	$(s_2, 0)$	$(s_6, 0)$	$(s_6, 0)$	$(s_3, 0)$	$(s_6, 0)$

$$U^{(3)} = \begin{pmatrix} 0.266 & 0.428 & 0.746 & 0.496 & 0.350 \\ 0.509 & 0.321 & 0.475 & 0.378 & 0.392 \\ 0.093 & 0.378 & 0.449 & 0.318 & 0.427 \\ 0.093 & 0.401 & 0.507 & 0.834 & 0.463 \end{pmatrix}.$$

Step 3: Utilize the 2TLPOWA operator (4.22) to aggregate all the individual 2-tuple linguistic decision matrices $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times5}$ $(k = 1, 2, 3)$ into the collective 2-tuple linguistic decision matrix $\tilde{A} = ((a_{ij}, \alpha_{ij}))_{4\times5}$ (see Table 4.7):

Table 4.7: Collective 2-tuple linguistic decision matrix $\tilde{A}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_6, 0.374)$	$(s_4, 0.254)$	$(s_7, 0.144)$	$(s_5, 0.246)$
G_2	$(s_5, 0.050)$	$(s_6, -0.420)$	$(s_5, -0.475)$	$(s_6, -0.378)$	$(s_4, -0.281)$
G_3	$(s_3, -0.419)$	$(s_6, -0.185)$	$(s_5, -0.347)$	$(s_3, -0.267)$	$(s_6, 0.187)$
G_4	$(s_2, 0)$	$(s_4, -0.343)$	$(s_5, 0.493)$	$(s_2, 0.254)$	$(s_5, -0.423)$

Step 4: Utilize (4.24) to calculate the values of T_{ij} which are contained in the matrix $T = (T_{ij})_{4\times5}$:

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.500 & 0.797 & 0.532 & 0.893 & 0.656 \\ 0.316 & 0.556 & 0.301 & 0.628 & 0.305 \\ 0.102 & 0.404 & 0.175 & 0.214 & 0.236 \end{pmatrix}.$$

Step 5: Utilize the decision information given in matrix $\tilde{A}$ and the 2TLPWA operator (4.25) to derive the collective overall preference value (a_j, α_j) of the alternative x_j $(j = 1, 2, 3, 4, 5)$:

$$(a_1, \alpha_1) = \triangle(3.934) = (s_4, -0.066),\ (a_2, \alpha_2) = \triangle(5.634) = (s_6, -0.366),$$
$$(a_3, \alpha_3) = \triangle(4.481) = (s_4, 0.481),\ (a_4, \alpha_4) = \triangle(5.252) = (s_5, 0.252),$$
$$(a_5, \alpha_5) = \triangle(4.849) = (s_5, -0.151).$$

Step 6: Rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ in accordance with the collective overall attribute values (a_j, α_j) $(j = 1, 2, 3, 4, 5)$ and Definition 3.1.3:

$$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_1$$

and thus the best alternative is x_2.

Based on the 2TLPWG and 2TLPOWG operators (here we also adopt the BUM function f as $f(x) = x^2$), then, in order to select the most desirable alternative(s), we can develop an approach to multiple attribute group decision making problem with linguistic information, which can be described as following:

Step $1'$: See Step 1.

Step $2'$: See Step 2.

Step $3'$*:* Utilize the 2TLPOWG operator (4.23) to aggregate all the individual 2-tuple linguistic decision matrices $\tilde{A}^{(k)} = ((a_{ij}^{(k)}, 0))_{4\times 5}$ $(k = 1, 2, 3)$ into the collective 2-tuple linguistic decision matrix $\tilde{A} = ((a_{ij}, \alpha_{ij}))_{4\times 5}$ (see Table 4.8):

Table 4.8: Collective 2-tuple linguistic decision matrix $\tilde{A}$

	x_1	x_2	x_3	x_4	x_5
G_1	$(s_4, 0)$	$(s_6, 0.332)$	$(s_4, 0.233)$	$(s_7, 0.136)$	$(s_5, 0.229)$
G_2	$(s_5, 0.046)$	$(s_5, 0.451)$	$(s_4, 0.497)$	$(s_6, -0.400)$	$(s_4, -0.352)$
G_3	$(s_3, -0.469)$	$(s_6, -0.222)$	$(s_5, -0.398)$	$(s_3, -0.332)$	$(s_6, 0.175)$
G_4	$(s_2, 0)$	$(s_4, -0.394)$	$(s_5, 0.470)$	$(s_2, 0.194)$	$(s_5, -0.458)$

Step $4'$*:* Utilize (4.24) to calculate the values of T_{ij} which are contained in the matrix $T = (T_{ij})_{4\times 5}$:

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.500 & 0.792 & 0.529 & 0.892 & 0.654 \\ 0.315 & 0.539 & 0.297 & 0.624 & 0.298 \\ 0.100 & 0.390 & 0.171 & 0.208 & 0.230 \end{pmatrix}.$$

Step 5′*:* Utilize the decision information given in matrix $\tilde{A}$ and the 2TLPWG operator (4.26) to derive the collective overall preference value (a_j, α_j) of the alternative x_j $(j = 1, 2, 3, 4, 5)$:

$$(a_1, \alpha_1) = \triangle(3.802) = (s_4, -0.198),\ (a_2, \alpha_2) = \triangle(5.491) = (s_5, 0.491),$$
$$(a_3, \alpha_3) = \triangle(4.452) = (s_4, 0.452),\ (a_4, \alpha_4) = \triangle(4.808) = (s_5, -0.192),$$
$$(a_5, \alpha_5) = \triangle(4.731) = (s_5, -0.269).$$

Step 6′*:* Rank all the alternatives x_j $(j = 1, 2, 3, 4, 5)$ in accordance with the collective overall attribute values (a_j, α_j) $(j = 1, 2, 3, 4, 5)$ and Definition 3.1.3:

$$x_2 \succ x_4 \succ x_5 \succ x_3 \succ x_1$$

and thus the best alternative is x_2.

In this section, we have proposed two approaches to solve the multiple attribute group decision making, with linguistic information, in which the attributes are in different priority level. From the above analysis, we can see that main advantages of the proposed operators and approaches are not only the fact that our operators accommodate the linguistic environment but also due to the consideration of the prioritization among the attributes, which makes it more feasible and practical.

4.4 Conclusions

In this chapter, we investigate MAGDM problems, with linguistic information, in which the attributes are in different priority level. Motivated by the idea of prioritized aggregation operators [61], we have developed some 2-tuple linguistic prioritized aggregation operators for aggregating linguistic information: 2-tuple linguistic prioritized weighted average (2TLPWA) operator, 2-tuple linguistic prioritized weighted geometric (2TLPWG) operator, 2-tuple linguistic prioritized ordered weighted average (2TLPOWA) operator, 2-tuple linguistic prioritized ordered weighted geometric (2TLPOWG) operator. The prominent characteristic of these proposed operators is that they take into account prioritization among the attributes. Then, we have utilized them to develop some approaches to solve the multiple attribute group decision making problem, with linguistic information, in which the attributes are in different priority level. A practical example about investment selection is given to verify the developed approaches and to demonstrate their practicality and effectiveness. In the future, we will develop several applications of the developed aggregation operators in other fields, such as pattern recognition, supply chain management and image processing.

Bibliography

[1] J. Aczél and T.L. Saaty, Procedures for synthesizing ratio judgements, J. Math. Psychol. 27 (1983) 93-102.

[2] G. Bojadziev and M. Bojadziev, Fuzzy Sets, Fuzzy Logic, Applications, World Scientific, Singapore, 1995.

[3] G. Bordogna, M. Fedrizzi and G. Pasi, A linguistic modeling of consensus in group decision making based on OWA operators, IEEE Trans. Syst. Man Cybern. Part A 27 (1997) 126-132.

[4] P.S. Bullen, D.S. Mitrinovi and P.M. Vasi, Means and Their Inequalities, Dordrecht, The Netherlands: Reidel, 1988.

[5] F. Chiclana, F. Herrera and E. Herrera-Viedma, Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations, Fuzzy Sets Syst. 122 (2001) 277-291.

[6] S.J. Chuu, Selecting the advanced manufacturing technology using fuzzy multiple attributes group decision making with multiple fuzzy information. Comput Ind Eng 57 (2009) 1033-1042.

[7] R. Degani and G. Bortolan, The problem of linguistic approximation in clinical decision making, Int. J. Approx. Reasoning 2 (1988) 143-162.

[8] M. Delgado, F. Herrera, E. Herrera-Viedma and L. Martínez, Combining numerical and linguistic information in group decision making, Inf. Sci. 107 (1998) 177-194.

[9] M. Delgado, J.L. Verdegay and M.A. Vila, On aggregation operators of linguistic labels, Int. J. Intell. Syst. 8 (1993) 351-370.

[10] Y.C. Dong, Y.F. Xu, H.Y. Li and B. Feng, The OWA-based consensus operator under linguistic representation models using position indexes, Eur. J. Oper. Res. 203 (2010) 455-463.

[11] G. Faccinetti, R.G. Ricci and S. Muzzioli, Note on ranking fuzzy triangular numbers, Int. J. Intell. Syst. 13 (1998) 613-622.

[12] J.C. Harsanyi, Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility, J. Polit. Econ. 63 (1955) 309-321.

[13] F. Herrera and E. Herrera-Viedma, Aggregation operators for linguistic weighted information, IEEE Trans. Syst. Man Cybern. Part A: Syst. Humans 27 (1997) 646-656.

[14] F. Herrera and E. Herrera-Viedma, Linguistic decision analysis: steps for solving decision problems under linguistic information, Fuzzy Sets Syst. 115 (2000) 67-82.

[15] F. Herrera, E. Herrera-Viedma and L. Martínez, A fusion approach for managing multi-granularity linguistic term sets in decision making, Fuzzy Sets Syst. 114 (2000) 43-58.

[16] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A sequential selection process in group decision making with a linguistic assessment approach, Inf. Sci. 85 (1995) 223-239.

[17] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, A model of consensus in group decision making under linguistic assessments, Fuzzy Sets Syst. 78 (1996) 73-87.

[18] F. Herrera, E. Herrera-Viedma and J.L. Verdegay, Linguistic measures based on fuzzy coincidence for reaching consensus in group decision making, Int. J. Approx. Reasoning 16 (1997) 309-334.

[19] F. Herrera and L. Martínez, A 2-tuple fuzzy linguistic representation model for computing with words, IEEE Trans. Fuzzy Syst. 8 (2000) 746-752.

[20] F. Herrera and L. Martínez, A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making, IEEE Trans. Syst. Man Cybern. Part B: Cybern. 31 (2001) 227-234.

[21] O. Holder, Über einen Mittelwertsatz, Göttingen Nachrichten (1889) 38-47.

[22] J.L. Jensen, Sur les fonctions convexes et les inégualités entre les valeurs moyennes, Acta Math. 30 (1906) 175-193.

[23] X.W. Liu and S.L. Han, Orness and parameterized RIM quantifier aggregation with OWA operators: A summary, Int. J. Approx. Reasoning 48 (2008) 77-97.

[24] J.M. Merigó, A.M. Gil-Lafuente, L.G. Zhou and H.Y. Chen, Generalization of the linguistic aggregation and its application in decision making, J. Syst. Eng. Elec. 22 (2011) 593-603.

[25] J.M. Merigó, A.M. Gil-Lafuente, L.G. Zhou and H.Y. Chen, Induced and linguistic generalized aggregation operators and their application in linguistic group decision making, Group Decis. Negot. 21 (2012) 531-549.

[26] J.H. Park, M.G. Gwak and Y.C. Kwun, Linguistic harmonic mean operators and their applications to group decision making, Int. J. Adv. Manuf. Technol. 57 (2011) 411-419.

[27] J.H. Park, M.G. Gwak and Y.C. Kwun, Uncertain linguistic harmonic mean operators and their applications to multiple attribute group decision making, Computing 93 (2011) 47-64.

[28] J.I. Peláez and J.M. Doňa, LAMA: A linguistic aggregation of majority additive operator, Int. J. Intell. Syst. 18 (2003) 809-820.

[29] M. Roubens, Fuzzy sets and decision analysis, Fuzzy Sets Syst. 90 (1997) 199-206.

[30] R. Sadiq and S. Tesfamariam, Probability density functions based weights for ordered weighted averaging (OWA) operators: An example of water quality indices, Eur. J. Oper. Res., in press.

[31] V. Torra, The weighted OWA operators, Int. J. Intell. Syst. 12 (1997) 153-166.

[32] P.J.M. Van Laarhoven and W. Pedrycz, A fuzzy extension of Saaty's priority theory, Fuzzy Sets Syst. 11 (1983) 199-227.

[33] Y.M. Wang and C. Parkan, Optimal aggregation of fuzzy preference relations with an application to broadband internet service selection, Eur. J. Oper. Res. 187 (2008) 1476-1486.

[34] Z.B. Wu and Y.H. Chen, The maximizing deviation method for group multiple attribute decision making under linguistic environment, Fuzzy Sets Syst. 158 (2007) 1608-1617.

[35] R.N. Xu and X.Y. Zhai, Extensions of the analytic hierarchy process in fuzzy environment, Fuzzy Sets Syst. 52 (1992) 251-257.

[36] Y.J. Xu and Z.J. Cai, Standard deviation method for determining the weights of group multiple attribute decision making under uncertain linguistic environment, in: The 7th World Congress on Intelligent Control and Autmation, IEEE, Chongqing, China, 2008, pp. 8311-8316.

[37] Y.J. Xu and Q.L. Da, Standard and mean deviation methods for linguistic group decision making and their applications, Expert Syst. Appl. 37 (2010) 5905-5912.

[38] Y.J. Xu, Q.L. Da and C.X. Zhao, Interactive approachfor multiple attribute decision making with incomplete weight information under linguistic environment, Systems Engineering and Electronics 31 (2009) 597-601.

[39] Y.J. Xu and L. Huang, An approach to group decision making problems based on 2-tuple linguistic aggregation operators, in: ISECS International Colloquium on Computing, Communication, Control, and Management, IEEE Computer Society, Guangzhou, China, 2008, pp. 73-77.

[40] Y. Xu, J.M. Merigó and H. Wang, Linguistic power aggregation operators and their application to multiple attribute group decision making, Appl. Math. Modelling (2011), doi: 10.1016/j.amp.2011.12.002.

[41] Z.S. Xu, Uncertain Multiple Attribute Decision Making: Methods and Applications, Beijing, China: Tsinghua Univ. Press, 2004.

[42] Z.S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, Inf. Sci. 166 (2004) 19-30.

[43] Z.S. Xu, Uncertain linguistic aggregation operators based on approach to multiple attribute group decision making under uncertain linguistic environment, Inf. Sci. 168 (2004) 171-184.

[44] Z.S. Xu, EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations, Int. J. Fuzz. Knowl.-Based Syst. 12 (2004) 791-810.

[45] Z.S. Xu, An overview of methods for determining OWA weights, Int. J. Intell. Syst. 20 (2005) 843-865.

[46] Z.S. Xu, An approach based on the uncertain LOWG and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations, Decis. Support Syst. 41 (2006) 488-499.

[47] Z.S. Xu, Induced uncertain linguistic OWA operators applied to group decision making, Inf. Fusion 7 (2006) 231-238.

[48] Z.S. Xu, On generalized induced linguistic aggregation operators, International Journal of General Systems 35 (2006) 17-28.

[49] Z.S. Xu, Linguistic aggregation operators: An overview, in: H. Bustine (Ed.) Fuzzy Sets and Their Extensions: Representation, Aggregation and Models, Springer, 2008, pp. 163-181.

[50] Z.S. Xu, Fuzzy harmonic mean operators, Int. J. Intell. Syst. 24 (2009) 152-172.

[51] Z.S. Xu and Q.L. Da, The ordered weighted geometric averaging operators, Int. J. Intell. Syst. 17 (2002) 709-716.

[52] Z.S. Xu and Q.L. Da, The uncertain OWA operators, Int. J. Intell. Syst. 17 (2002) 569-575.

[53] Z.S. Xu and Q.L. Da, An overview of operators aggregating information, Int. J. Intell. Syst. 18 (2003) 953-969.

[54] Z.S. Xu, Q.L. Da and L.H. Liu, Normalizing rank aggregation method for priority of a fuzzy preference relation and its effectiveness, Int. J. Approx. Reasoning 50 (2009) 1287-1297.

[55] Z.S. Xu and R.R. Yager, Power-geometric operators and their use in group decision making, IEEE Trans. Fuzzy Syst. 18 (2010) 94-105.

[56] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybern. 18 (1988) 183-190.

[57] R.R. Yager, An approach to ordinal decision making, Int. J. Approx. Reasoning 12 (1995) 237-261.

[58] R.R. Yager, The power average operator, IEEE Trans. Syst. Man Cybern. A. Syst. Humans 31 (2001) 724-731.

[59] R.R. Yager, Generalized OWA aggregation operator, Fuzzy Optim. Decision Making 3 (2004) 93-107.

[60] R.R. Yager, Centered OWA operators, Soft Comput. 11 (2007) 631-639.

[61] R.R. Yager, Prioritized aggregation operators, Int. J. Approx. Reason. 48 (2008) 263-274.

[62] R.R. Yager, Prioritized OWA aggregation, Fuzzy Optim. Decis. Mak. 8 (2009) 245-262.

[63] R.R. Yager and D.P. Filev, Induced ordered weighted averaging operators, IEEE Trans. Syst. Man Cybern. 29 (1999) 141-150.

[64] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning. Part 1, 2, and 3, Inf. Sci. 8 (1975) 199-249, 301-357; 9 (1976) 43-80.

[65] L.A. Zadeh, A computational approach to fuzzy quantifiers in natural languages, Comput. Math. Appl. 9 (1983) 149-184.

[66] L.G. Zhou and H.Y. Chen, A generalization of the power aggregation operators for linguistic environment and its application in group decision making, Know.-Based Syst. 26 (2012) 216-224.

[67] Wikipedia, http://en.wikipedia.org/wiki/Generalizedmean.