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# Subordination and superordination for analytic function associated with a linear operator

by

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# (선형 연산자를 포함한 해석함수들의 종속과 초종속 성질)

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선형연산자를 포함한 해석함수들의 종속과 초종속 성질

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요 약

Miller와 Mocanu [12]는 미분 종속원리를 이용하여 해석함수들의 다양한 기하학적 성질들을 조사하였다. 최근에는 미분종속의 상대개념을 소개하고 어떤 비선형 적분연산자에 대하여 종속 문제들과 그 쌍대이론인 초종속 문제들을 연구하였다. 또한 Bulboacă[2,3]와 Srivastava[16]는 일반화된 비선형 적분들에 대한 종속 및 그 쌍대이론인 초종속을 연구하여 Miller와 Mocanu [12]의 여러 결과들을 확장하였다.

본 논문에서는 Carlson와 Shaffer[4]에 의해서 소개된 선형 연산자에 대하여 종속 및 초종속 문제들을 연구하고 sandwich 형태의 새로운 결과들을 조사하였다. 또한 어떤 선형 적분연산자에 대하여 여러 가지 종속성질들을 연구하였다. 더욱이, 논문에서 소개된 주 결과들을 가우스 초기하 함수에 응용하여 여러 종속 성질들을 조사하였다.

## 1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let $f$ and $F$ be members of $\mathcal{H}$. The function $f$ is said to be subordinate to $F$, or $F$ is said to be superordinate to $f$, if there exists a function $w$ analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function $F$ is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$ (cf. [11,20]).

**Definition 1 [12]** Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let $h$ be univalent in $\mathbb{U}$. If $p$ is analytic in $\mathbb{U}$ and satisfies the differential subordination

$$\phi(p(z), zp'(z)) \prec h(z) \ \ (z \in \mathbb{U}), \tag{1.1}$$

then $p$ is called a solution of the differential subordination. The univalent function $q$ is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all $p$ satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (1.1) is said to be the best dominant.

**Definition 2 [13]** Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let $h$ be analytic in $\mathbb{U}$. If $p$ and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)) \ \ (z \in \mathbb{U}), \tag{1.2}$$

then $p$ is called a solution of the differential superordination. An analytic function $q$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all $p$ satisfying (1.2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (1.2) is said to be the best subordinant.

**Definition 3 [13]** We denote by $\mathcal{Q}$ the class of functions $f$ that are analytic and injective on $\overline{\mathbb{U}}\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic and in the open unit disk $\mathbb{U}$ with $f'(z) \neq 0$. Now we define $\phi(a, c; z)$ by

$$\phi(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \ \ (c \neq 0, -1, -2, \cdots),$$

where $(x)_k$ is the Pochhammer symbol(or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1)\cdots(x+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases}$$

Let $f \in \mathcal{A}$. Denote by $L(a, c) : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$L(a, c)f(z) = \phi(a, c; z) * f(z) \ \ (z \in \mathbb{U}), \tag{1.3}$$

where the symbol $(*)$ stands for the Hadamard product (or convolution). We observe that

$$L(2, 1)f(z) = zf'(z) \text{ and } L(n+1, 1)f(z) = D^n f(z),$$

where $n$ is any real number greater than $-1$, and the symbol $D^n$ is the Ruscheweyh derivative [19](also, see [7]) for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Furthermore,it is easily verified from the definition of the operator $L(a, c)$ that

$$z(L(a,c)f(z))' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z) \tag{1.4}$$

The operator $L(a,c)$ was introduced and studied by Saitoh [20]. This operator is an extension of the familiar Carlson-Shaffer operator $L(a,c)$ which has been used widely on the space of analytic and univalent functions in $\mathbb{U}$ ( see, for details [4]; see also [21,22]).

Making use of the principle of subordination, Miller et al. [14] obtained some subordination theorems involving certain integral operators for analytic functions in $\mathbb{U}$. Also Owa and Srivastava [17] investigated the subordination properties of certain integral operators (see also [2]). Moreover, Miller and Mocanu [13] considered differential superordinations, as the dual problem of differential subordinations (see also [3]).We also remark in passing that some more interesting results related to subordination and superordination may be founded in [5]and[6]. In the present paper, we investigate the subordination and superordination preserving properties of the linear operator $L(a,c)$ defined by (1.3) with the sandwich-type theorems.

## 2. A set of Lemmas

The following lemmas will be required in our present investigation.

**Lemma 1 [10].** *Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the condition:*

$$\text{Re}\{H(is,t)\} \le 0,$$

*for all real $s$ and $t \le -n(1+s^2)/2$, where $n$ is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and*

$$\text{Re}\{H(p(z), zp'(z))\} > 0 \ \ (z \in \mathbb{U}),$$

*then* $\mathrm{Re}\{p(z)\} > 0$ *in* $\mathbb{U}$.

**Lemma 2 [11].** *Let* $\beta, \gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in \mathcal{H}(\mathbb{U})$ *with* $h(0) = c$. *If* $\mathrm{Re}\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, *then the solution of the differential equation*:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \ \ (z \in \mathbb{U})$$

*with* $q(0) = c$ *is analytic in* $\mathbb{U}$ *and satisfies* $\mathrm{Re}\{\beta q(z) + \gamma\} > 0$ *for* $z \in \mathbb{U}$.

**Lemma 3 [12].** *Let* $p \in \mathcal{Q}$ *with* $p(0) = a$ *and let* $q(z) = a + a_n z^n + \cdots$ *be analytic in* $\mathbb{U}$ *with* $q(z) \not\equiv a$ *and* $n \geq 1$. *If* $q$ *is not subordinate to* $p$, *then there exist points* $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ *and* $\zeta_0 \in \partial\mathbb{U} \setminus E(f)$, *for which* $q(\mathbb{U}_{r_0}) \subset p(\mathbb{U})$,

$$q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \ \ (m \geq n).$$

A function $L(z,t)$ defined on $\mathbb{U} \times [0, \infty)$ is the subordination chain (or Löwner chain) if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0, \infty)$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for $z \in \mathbb{U}$ and $0 \leq s < t$.

**Lemma 4 [13].** *Let* $q \in \mathcal{H}[a, 1]$, *let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\varphi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z,t) = \varphi(q(z), tzq'(z))$ *is a subordination chain and* $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \varphi(p(z), zp'(z)) \ \ (z \in \mathbb{U})$$

*implies that*

$$q(z) \prec p(z) \ \ (z \in \mathbb{U}).$$

*Furthermore, if* $\varphi(q(z), zp'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$, *then* $q$ *is the best subordinant.*

**Lemma 5 [18].** *The function* $L(z,t) = a_1(t)z + \cdots$ *with* $a_1(t) \neq 0$ *and* $\lim_{t\to\infty} |a_1(t)| = \infty$. *Suppose that* $L(\cdot;t)$ *is analytic in* $\mathbb{U}$ *for all* $t \geq 0$, $L(z;\cdot)$ *is continuously differentiable on* $[0,\infty)$ *for all* $z \in \mathbb{U}$. *If* $L(z;t)$ *satisfies*

$$|L(z;t)| \leq K_0|a_1(t)| \quad (|z| < r_0 < 1;\ 0 \leq t < \infty))$$

*for some positive constants* $K_0$ *and* $r_0$ *and*

$$\mathrm{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty),$$

*then* $L(z;t)$ *is a subordination chain.*

## 3. Main Results

Firstly, we begin by proving the following subordination theorem involving the multiplier transformation $L(a,c)$ defined by (1.3).

**Theorem 1.** *Let* $f, g \in \mathcal{A}$. *Suppose that*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

$$\left(\phi(z) := (1-\alpha)\frac{L(a+1,c)g(z)}{z} + \alpha\frac{L(a,c)g(z)}{z};\ a > 0;\ 0 \leq \alpha < 1;\ z \in \mathbb{U}\right), \tag{3.1}$$

*where*

$$\delta = \frac{(1-\alpha)^2 + a^2 - |(1-\alpha)^2 - a^2|}{4a(1-\alpha)}. \tag{3.2}$$

*If* $f$ *and* $g$ *satisfy the following subordination condition* :

$$(1-\alpha)\frac{L(a+1,c)f(z)}{z} + \alpha\frac{L(a,c)f(z)}{z} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{3.3}$$

*then*

$$\frac{L(a,c)f(z)}{z} \prec \frac{L(a,c)g(z)}{z} \quad (z \in \mathbb{U}).$$

*Moreover, the function $L(a,c)g(z)/z$ is the best dominant.*

*Proof.* Let us define the functions $F$ and $G$ by

$$F(z) := \frac{L(a,c)f(z)}{z} \quad \text{and} \quad G(z) := \frac{L(a,c)g(z)}{z}, \tag{3.4}$$

respectively. Without loss of generality, we can assume that $G$ is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. We first show that, if the function $q$ is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{3.5}$$

then

$$\text{Re}\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Taking the logarithmic differentiation on both sides of the second equation in (3.4) and using (1.4) for $g \in \mathcal{A}$, we obtain

$$a\phi(z) = aG(z) + (1-\alpha)zG'(z) \tag{3.6}$$

Now, by differentiating both sides of (3.6), we obtain

$$az\phi'(z) = (1-\alpha)zG'(z)\left(q(z) + \frac{a}{1-\alpha}\right),$$

which, in conjunction with (3.6), yields the relationship:

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + a/(1-\alpha)} \\ &= q(z) + \frac{zq'(z)}{q(z) + a/(1-\alpha)} \equiv h(z). \end{aligned} \tag{3.7}$$

From (3.1), we have

$$\mathrm{Re}\left\{h(z)+\frac{a}{1-\alpha}\right\}>0 \quad (z\in\mathbb{U}),$$

and by using Lemma 2, we conclude that the differential equation (3.7) has a solution $q\in\mathcal{H}(\mathbb{U})$ with $q(0)=h(0)=1$. Let us put

$$H(u,v)=u+\frac{v}{u+a/(1-\alpha)}+\delta, \tag{3.8}$$

where $\delta$ is given by (3.2). From (3.1), (3.7) and (3.8), we obtain

$$\mathrm{Re}\{H(q(z),zq'(z))\}>0 \quad (z\in\mathbb{U}).$$

Now we proceed to show that $\mathrm{Re}\{H(is,t)\}\leq 0$ for all real $s$ and $t\leq -(1+s^2)/2$. From (3.8), we have

$$\begin{aligned}\mathrm{Re}\{H(is,t)\} &= \mathrm{Re}\left\{is+\frac{t}{is+a/(1-\alpha)}+\delta\right\}\\ &= \frac{ta/(1-\alpha)}{|a/(1-\alpha)+is|^2}+\delta \\ &\leq -\frac{E_\delta(s)}{2|a/(1-\alpha)+is|^2},\end{aligned} \tag{3.9}$$

where

$$E_\delta(s):=\left(\frac{a}{1-\alpha}-2\delta\right)s^2-\frac{a}{1-\alpha}\left(2\delta\frac{a}{1-\alpha}-1\right). \tag{3.10}$$

For $\delta$ given by (3.2), we can prove easily that the expression $E_\delta(s)$ given by (3.10) is positive or equal to zero. Hence from (3.9), we see that $\mathrm{Re}\{H(is,t)\}\leq 0$ for all real $s$ and $t\leq -(1+s^2)/2$. Thus, by using Lemma 1.1, we conclude that $\mathrm{Re}\{q(z)\}>0$ for all $z\in\mathbb{U}$. That is, $q$ is convex in $\mathbb{U}$.

Next, we prove that the subordination condition (3.3) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{3.11}$$

for the functions $F$ and $G$ defined by (3.4). For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(1-\alpha)(1+t)}{a} zG'(z) \ \ (z \in \mathbb{U};\ 0 \leq t < \infty).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(\frac{a+(1-\alpha)(1+t)}{a}\right) \neq 0 \quad (0 \leq t < \infty).$$

This shows that the function

$$L(z,t) = a_1(t)z + \cdots$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0, \infty)$. By using the well-known growth and distortion theorems for convex functions, it is easy to check that the first part of Lemma 5 is satisfied. Furthermore, we have

$$\operatorname{Re}\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \operatorname{Re}\left\{\frac{a}{1-a} + (1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right\} > 0,$$

since $G$ is convex and $a > 0$. Therefore, by virtue of Lemma 4, $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\phi(z) = G(z) + \frac{1-\alpha}{a} zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

This implies that

$$L(\zeta, t) \notin L(\mathbb{U}, 0) = \phi(\mathbb{U})$$

for $\zeta \in \partial\mathbb{U}$ and $t \in [0, \infty)$.

Now suppose that $F$ is not subordinate to $G$, then by Lemma 3, there exists points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F(z_0) = (1+t)\zeta_0 G'(\zeta_0) \ \ (0 \leq t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{(1-\alpha)(1+t)}{a} \zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1-\alpha}{a} z_0 F'(z_0) \\ &= \frac{L(a+1, c) f(z_0)}{z_0} \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (3.3). This contrdicts the above observation that $L(\zeta_0, t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (3.3) must imply the subordination given by (3.11). Considering $F(z) = G(z)$, we see that the function $G$ is the best dominant. This evidently completes the proof of Theorem 1.

We next prove a dual problem of Theorem 1, in the sense that the subordinations are replaced by superordinations.

**Theorem 2.** *Let $f, g \in \mathcal{A}$. Suppose that*

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta$$

$$\left(\phi(z) := (1-\alpha)\frac{L(a+1,c)g(z)}{z} + \alpha\frac{L(a,c)g(z)}{z};\ a > 0;\ 0 \leq \alpha < 1;\ z \in \mathbb{U}\right),$$

*where $\delta$ is given by* (3.2)*. If $L(a+1,c)f(z)/z$ is univalent in $\mathbb{U}$ and $L(a,c)f(z)/z$ $\in \mathcal{H}[1,1] \cap \mathcal{Q}$, then*

$$\phi(z) \prec (1-\alpha)\frac{L(a+1,c)f(z)}{z} + \alpha\frac{L(a,c)f(z)}{z} \quad (z \in \mathbb{U}) \tag{3.12}$$

*implies that*

$$\frac{L(a,c)g(z)}{z} \prec \frac{L(a,c)f(z)}{z} \quad (z \in \mathbb{U}).$$

*Moreover, the function $L(a,c)g(z)/z$ is the best subordinant.*

*Proof.* Now let us define the functions $F$ and $G$, respectively, by (3.4). We first note that, if the function $q$ is defined by (3.5), by using (3.6), then we obtain

$$\begin{aligned} \phi(z) &= G(z) + \frac{1-\alpha}{a} zG'(z) \\ &=: \varphi(G(z), zG'(z)). \end{aligned} \tag{3.13}$$

After a simple calculation, the equation (3.12) yields the relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + a/(1-\alpha)}.$$

Then by using the same method as in the proof of Theorem 1, we can prove that $\mathrm{Re}\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, $G$ defined by (3.4) is convex(univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (3.12) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \tag{3.14}$$

for the functions $F$ and $G$ defined by (3.4). Now consider the function $L(z,t)$ defined by

$$L(z,t) := G(z) + \frac{(1-\alpha)t}{a} zG'(z) \;\; (z \in \mathbb{U};\; 0 \leq t < \infty).$$

Since $G$ is convex and $(1-\alpha)/a > 0$, we can prove easily that $L(z,t)$ is a subordination chain as in the proof of Theorem 1. Therefore according to Lemma

4, we conclude that the superordination condition (3.12) must imply the superordination given by (3.14). Furthermore, since the differential equation (3.13) has the univalent solution $G$, it is the best subordinant of the given differential superordination. Therefore we complete the proof of Theorem 2.

If we combine this Theorem 1 and Theorem 2, then we obtain the following sandwich-type theorem.

**Theorem 3.** *Let $f, g_k \in \mathcal{A}(k=1,2)$. Suppose that*

$$\mathrm{Re}\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \tag{3.15}$$

$$\left(\phi_k(z) := (1-\alpha)\frac{L(a+1,c)g_k(z)}{z} + \alpha\frac{L(a,c)g_k(z)}{z};\ a>0;\ 0\le\alpha<1;\ z\in\mathbb{U}\right),$$

*where $\delta$ is given by* (3.2). *If $L(a+1,c)f(z)/z$ is univalent in $\mathbb{U}$ and $L(a,c)f(z)/z \in \mathcal{H}[1,1]\cap\mathcal{Q}$, then*

$$\phi_1(z) \prec (1-\alpha)\frac{L(a+1,c)f(z)}{z} + \alpha\frac{L(a,c)f(z)}{z} \prec \phi_2(z) \quad (z\in\mathbb{U})$$

*implies that*

$$\frac{L(a,c)g_1(z)}{z} \prec \frac{L(a,c)f(z)}{z} \prec \frac{L(a,c)g_2(z)}{z} \quad (z\in\mathbb{U}).$$

*Moreover, the functions $L(a,c)g_1(z)/z$ and $L(a,c)g_2(z)/z$ are the best subordinant and the best dominant, respectively.*

The assumption of Theorem 3, that the functions $L(a+1,c)f(z)/z$ and $L(a,c)f(z)/z$ need to be univalent in $\mathbb{U}$, may be replaced by another conditions in the following result.

**Corollary 1.** *Let $f, g_k \in \mathcal{A}(k=1,2)$. Suppose also that the condition* (3.15) *is satisfied and*

$$\operatorname{Re}\left\{1+\frac{z\psi''(z)}{\psi'(z)}\right\} > -\delta \tag{3.16}$$

$$\left(\psi(z) := (1-\alpha)\frac{L(a+1,c)f(z)}{z} + \alpha\frac{L(a,c)f(z)}{z};\ a>0;\ 0\leq\alpha<1;\ z\in\mathbb{U}\right),$$

*where* $\delta$ *is given by* (3.2). *Then*

$$\phi(z) \prec \frac{L(a+1,c)f(z)}{z} \prec \phi(z) \quad (z\in\mathbb{U})$$

*implies that*

$$\frac{L(a,c)g_1(z)}{z} \prec \frac{L(a,c)f(z)}{z} \prec \frac{L(a,c)g_2(z)}{z} \quad (z\in\mathbb{U}).$$

*Moreover, the functions* $L(a,c)g_1(z)/z$ *and* $L(a,c)g_2(z)/z$ *are the best subordinant and the best dominant, respectively.*

*Proof.* In order to prove Corollary 1, we have to show that the condition (3.16) implies the univalence of $\psi(z)$ and $F(z) := L(a,c)f(z)/z$. Since $0<\delta\leq 1/2$ from (3.2) in Theorem 1 , the condition (3.16) means that $\psi$ is a close-to-convex function in $\mathbb{U}$ (see [6]) and hence $\psi$ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 1, we can prove the convexity(univalence) of $F$ and so the details may be omitted. Therefore, from Theorem 3, we obtain Corollary 1.

Taking $a=c=1$ and $\alpha=0$ in Theorem 3, we have the following result.

**Corollary 2.** *Let* $f, g_k \in \mathcal{A}$. *Suppose that*

$$\operatorname{Re}\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{1}{2} \quad (\phi_k(z) := g_k'(z);\ k=1,2;\ z\in\mathbb{U}).$$

*If* $f'(z)$ *is univalent in* $\mathbb{U}$ *and* $f(z)/z \in \mathcal{H}[1,1]\cap\mathcal{Q}$, *then*

$$g_1'(z) \prec f'(z) \prec g_2'(z) \quad (z\in\mathbb{U})$$

*implies that*

$$\frac{g_1(z)}{z} \prec \frac{f(z)}{z} \prec \frac{g_2(z)}{z} \quad (z \in \mathbb{U}).$$

*Moreover, the functions* $g_1(z)/z$ *and* $g_2(z)/z$ *are the best subordinant and the best dominant, respectively.*

Next, we consider the generalized Libera integral operator $F_\mu$ $(\mu > -1)$ defined by (cf. [1,7,9,15])

$$F_\mu(f)(z) := \frac{\mu+1}{z^\mu} \int_0^z t^{\mu-1} f(t)dt \ \ (f \in \mathcal{A};\ \mu > -1) \tag{3.17}$$

Now, we obtain the following sandwich-type result involving the integral operator defined by (3.17).

**Theorem 4** *Let* $f, g \in \mathcal{A}$. *Suppose also that*

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \tag{3.18}$$

$$\left(\phi_k(z) := \frac{L(a,c)g_k(z)}{z};\ k = 1, 2;\ z \in \mathbb{U}\right)$$

*where*

$$\delta = \frac{1 + (\mu+1)^2 - |1 - (\mu+1)^2|}{4(\mu+1)} \ \ (\mu > -1). \tag{3.19}$$

*If* $L(a,c)f(z)/z$ *is univalent in* $\mathbb{U}$ *and* $L(a,c)F_\mu(f)(z)/z \in \mathcal{H}[1,1] \cap \mathcal{Q}$, *then the following subordination condition :*

$$\phi_1(z) \prec \frac{L(a,c)f(z)}{z} \prec \phi_2(z) \quad (z \in \mathbb{U}),$$

*implies that*

$$\frac{L(a,c)F_\mu(g_1)(z)}{z} \prec \frac{L(a,c)F_\mu(f)(z)}{z} \prec \frac{L(a,c)F_\mu(g_2)(z)}{z} \quad (z \in \mathbb{U}).$$

*Moreover, the function* $L(a,c)F_\mu(g_1)(z)/z$ *and* $L(a,c)F_\mu(g_2)(z)/z$ *are the best subordinant and the best dominant, respectively.*

*Proof.* Let us define the functions $F$ and $G_k (k=1,2)$ by

$$F(z) := \frac{L(a,c)F_\mu(f)(z)}{z} \quad \text{and} \quad G_k(z) := \frac{L(a,c)F_\mu(g_k)(z)}{z},$$

respectively. Without loss of generality, as in the proof of Theorem 1, we can assume that $G$ is analytic and univalent on $\overline{\mathbb{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$.

From the definition of the integral operator $F_\mu$ defined by (3.17), we obtain

$$z(L(a,c)F_\mu(f)(z))' = (\mu+1)L(a,c)f(z) - \mu L(a,c)F_\mu(f)(z) \tag{3.20}$$

Then from (3.18) and (3.20), we have

$$(\mu+1)\phi(z) = (\mu+1)G_k(z) + zG_k'(z). \tag{3.21}$$

Setting

$$q_k(z) = 1 + \frac{zG_k''(z)}{G_k'(z)} \quad (z \in \mathbb{U}),$$

and differentiating both sides of (3.21), we obtain

$$1 + \frac{z\phi_k''(z)}{\phi_k'(z)} = q_k(z) + \frac{zq_k'(z)}{q_k(z)+\mu+1}.$$

The remaining part of the proof is similar to that of Theorem 1 and so we may omit for the proof involved.

By using the same methods as in the proof of Corollary 1, we have the following result.

**Corollary 3.** *Let* $f, g_k \in \mathcal{A} (k=1,2)$. *Suppose that the condition* (3.18) *is satisfied and*

$$\text{Re}\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \left(z\in\mathbb{U};\ \phi_k(z) := \frac{L(a,c)f(z)}{z}\right),$$

*where* $\delta$ *is given by* (3.19). *Then*

$$\phi_1(z) \prec \frac{L(a,c)f(z)}{z} \prec \phi_2(z) \quad (z\in\mathbb{U})$$

*implies that*

$$\frac{L(a,c)F_\mu(g_1)(z)}{z} \prec \frac{L(a,c)F_\mu(f)(z)}{z} \prec \frac{L(a,c)F_\mu(g_2)(z)}{z} \quad (z\in\mathbb{U}).$$

*Moreover, the functions* $L(a,c)F_\mu(g_1)(z)/z$ *and* $L(a,c)F_\mu(g_2)(z)/z$ *are the best subordinant and the best dominant, respectively.*

## 4. An Application to the Gauss Hypergeometric Function

We begin by recalling that the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by (see, for details, [16] and [23, Chapter 14])

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$(z\in\mathbb{U};\ b\in\mathbb{C};\ c\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0,-1,-2,\cdots\}),$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol (or the shifted factorial) defined (for $\lambda,\nu\in\mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\lambda(a)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+\nu-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}). \end{cases}$$

For this useful special function, the following Eulerian integral representation is fairly well-known [21, p. 293]:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt \tag{4.1} $$

$$ (\mathrm{Re}\{c\} > \mathrm{Re}\{a\} > 0;\ |\arg(1-z)| \leq \pi - \epsilon;\ 0 < \epsilon < \pi). $$

In view of (4.1), we set

$$ g(z) = \frac{z}{1-z}, $$

so that the definition (3.17) yields

$$ \begin{aligned} F_\mu(g)(z) &= \frac{\mu+1}{z^\mu}\int_0^z t^\mu (1-t)^{-1}dt \\ &= (\mu+1)z\int_0^1 u^\mu(1-zu)^{-1}du \\ &= z[{}_2F_1(\mu+1,1;\mu+2;z)] \quad (\mu > -1). \end{aligned} $$

Moreover, we note that

$$ \frac{g(z)}{z} = \frac{1}{1-z} \neq 0 \quad (z \in \mathbb{U}). $$

Thus, by applying to Theorem 4 with $g(z) = z/(1-z)$, we obtain the following result involving the Gauss hypergeometric function.

**Theorem 5.** *Let $f \in \mathcal{A}$ and $0 < \kappa \leq 1 + 2\delta$, where $\delta$ is defined by* (3.19). *If $f$ satisfies the subordination condition* :

$$ \frac{L(\kappa,1)f(z)}{z} \prec \frac{1}{(1-z)^\kappa} \quad (z \in \mathbb{U}), $$

*then*

$$ \frac{L(\kappa,1)F_\mu(f)(z)}{z} \prec {}_2F_1(\mu+1,\kappa;\mu+2;z) \quad (z \in \mathbb{U}). $$

By taking $\kappa = 1$ in Theorem 5, we are led to the following Corollary 4.

**Corollary 4.** *Let $f \in \mathcal{A}$. Then the subordination condition :*

$$\frac{f(z)}{z} \prec \frac{1}{1-z} \quad (z \in \mathbb{U}),$$

*implies that*

$$\frac{F_\mu(f)(z)}{z} \prec {}_2F_1(\mu+1, 1; \mu+2; z) \quad (z \in \mathbb{U}),$$

*where $F_\mu$ is given by* (3.17)*. Moreover, the function ${}_2F_1(\mu+1, 1; \mu+2; z)$ is the best dominant.*

Finally, we state the following Theorem 6 below as the dual result of Theorem 5, which can be obtained by applying Theorem 4.

**Theorem 6.** *Under the assumption of Theorem 5, suppose also that $L(\kappa, 1)f(z)/z$ is univalent in $\mathbb{U}$ and $L(\kappa, 1)F_\mu(f)(z)/z \in \mathcal{Q}$, where $F_\mu$ is given by* (3.17)*. If $f$ satisfies the superordination condition :*

$$\frac{1}{(1-z)^\kappa} \prec \frac{L(\kappa, 1)f(z)}{z} \quad (z \in \mathbb{U}),$$

*then*

$${}_2F_1(\mu+1, \kappa; \mu+2; z) \prec \frac{L(\kappa, 1)F_\mu(f)(z)}{z} \quad (z \in \mathbb{U}).$$

*Moreover, the function ${}_2F_1(\mu+1, \kappa; \mu+2; z)$ is the best subordinant.*

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