Almost δ-Almost Continuous Functions

거의 δ-거의 연속함수

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by

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A Dissertation

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거의 δ-거의 연속함수

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요약

본 논문의 목적은 Nasef와 Noiri 등에 의해 소개된 거의 전연속성(almost precontinuity)과 Raychaudhuri와 Mukherjee 등에 의해서 소개된 δ-거의 연속성 (δ-almost continuiuty)의 일반화 개념인 거의 δ-거의 연속함수(almost δ-almost continuous function)를 소개하고, 이 함수에 대한 여러 가지 성질들과 특성들을 조사하였다. 그리고, 위에서 언급한 연속함수 (즉, 거의 전연속성와 δ-거의 연속성)뿐만 아니라 연속함수의 여러 가지 다른 형태와 거의 δ-거의 연속함수(almost δ-almost continuous function)와의 관계에 대하여 조사하였다.

1 Introduction

Two types of almost continuous functions were introduced by Singal and Singal [21] and Husain [8], respectively. Long and Carnahan [10] pointed out that these notions of almost continuity are independent of each other. Mashhour et al. [11] introduced and investigated preopen sets and precontinuity in topological spaces. Noiri [16] showed that precontinuity is equivalent to almost continuity in the sense of Husain and obtained some characterizations of almost continuity in the sense of Singal and Singal and almost continuity in the sense of Husain, respectively.

In 1983, Abd El-Monsef et al.[1] introduced and investigated β-open sets and β-continuity in topological spaces. Nasef and Noiri [12] defined almost precontinuous functions and almost β-continuous functions as a generalization of precontinuity and β-continuity, respectively, and used almost precontinuity to obtain a decomposition of almost continuity in the sense of Singal and Singal. Noiri [17] obtained further properties and some characterizations of almost β-continuous functions.

In 1993, Raychaudhuri and Mukherjee [19] defined δ-almost continuous functions as a generalization of almost continuity in the sense of Husain and introduced the notion of δ-preopen sets and used it to characterize δ-almost continuity.

The purpose of this paper is to define almost δ-almost continuous functions and to obtain several characterizations and properties of such functions. The notion of almost δ-almost continuity is a generalization of each almost precontinuity and δ-almost continuity. Moreover, the relationships between almost δ-almost continuous functions and some known concepts are also discussed.

2 Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated.

Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ represent the closure of $A$and the interior of A with respect to τ, respectively. A subset A is said to be *regular open* (resp. *regular closed*) if $A = \mathrm{int}(\mathrm{cl}(A))$ (resp. $A = \mathrm{cl}(\mathrm{int}(A))$). The *$\delta$-interior* [22] of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-$\mathrm{int}(A)$. A subset A is called *δ-open* [22] if $A = \delta$-$\mathrm{int}(A)$, i.e., a set is δ-open if it is the union of regular open sets. The complement of δ-open set is called *δ-closed*. Alternatively, a set A of (X, τ) is called *δ-closed* [22] if $A = \delta$-$\mathrm{cl}(A)$, where δ-$\mathrm{cl}(A) = \{x \in X : A \cap \mathrm{int}(\mathrm{cl}(U)) \neq \phi,\ U \in \tau \text{ and } x \in U\}$.

A subset A is said to be *α-open* [13] (resp. *semi-open* [9], *preopen* [11], *β-open* [1] or *semi-preopen* [2], *δ-preopen* [19]) if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$ (resp. $A \subset \mathrm{cl}(\mathrm{int}(A))$, $A \subset \mathrm{int}(\mathrm{cl}(A))$, $A \subset \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$, $A \subset \mathrm{int}(\delta\text{-}\mathrm{cl}(A))$). The complement of a α-open (resp. semi-open, preopen, β-open, δ-preopen) set is said to be *α-closed* (resp. *semi-closed, preclosed, β-closed, δ-preclosed*). The family of all δ-preopen (resp. δ-preclosed) sets of X is denoted by δ-$PO(X)$ (resp. δ-$PC(X)$) and the family $\{U \in \delta\text{-}PO(X) : x \in U\}$ is denoted by δ-$PO(X, x)$, where x is a point of X. The intersection of all δ-preclosed sets of X containing A is called the *δ-preclosure* [19] of A and is denoted by δ-$\mathrm{pcl}(A)$. The α-closure, semi-closure, preclosure and β-closure are similarly defined and denoted by α-$\mathrm{cl}(A)$, $\mathrm{scl}(A)$, $\mathrm{pcl}(A)$ and β-$\mathrm{cl}(A)$. The union of all δ-preopen sets of X contained in A is called *δ-preinterior* [19] and is denoted by δ-pint(A).

The following lemmas are useful in sequel:

Lemma 2.1 [19] *Let A be a subset of a space X. Then the following properties hold:*

(a) $\delta\text{-pcl}(X - A) = X - \delta\text{-pint}(A)$,

(b) $x \in \delta\text{-pcl}(A)$ *if and only if* $A \cap U \neq \phi$ *for each* $U \in \delta\text{-}PO(X, x)$,

(c) A *is δ-preclosed in* X *if and only if* $A = \delta\text{-pcl}(A)$,

(d) $\delta\text{-pcl}(A)$ *is δ-preclosed in* X,

(e) $\delta\text{-pcl}(A) = A \cup \text{cl}(\delta\text{-int}(A))$ *and* $\delta\text{-pint}(A) = A \cap \text{int}(\delta\text{-cl}(A))$.

Lemma 2.2 [19] *Let A be a subset of a space (X, τ). Then $A \in \delta\text{-}PO(X)$ if and only if $A \cap U \in \delta\text{-}PO(X)$ for each regular open (δ-open) set of X.*

Definition 2.3 A function $f : X \to Y$ is said to be:

(a) *R-map* [5] if $f^{-1}(V)$ is regular open in X for each regular open set V of Y;

(b) *δ-continuous* [15] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(\text{int}(\text{cl}(U)) \subset \text{int}(\text{cl}(V))$;

(c) *almost continuous* [21] (briefly, *a.c.S.*) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \text{int}(\text{cl}(V))$;

(d) *almost continuous* [8] (briefly, *a.c.H.*) if for each $x \in X$ and each open set V of Y containing $f(x)$, $\text{cl}(f^{-1}(V))$ is a neighborhood of x.

Definition 2.4 A function $f : X \to Y$ is said to be:

(a) *precontinuous* [11] (resp. *β-continuous* [1], *δ-almost continuous* [19]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a preopen (resp. β-open, δ-preopen) set U containing x such that $f(U) \subset V$;

(b) *almost precontinuous* [12] (resp. *almost β-continuous* [12]) if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists a preopen (resp. β-open) set U containing x such that $f(U) \subset V$.

3 Characterizations

Definition 3.1 A function $f : X \to Y$ is said to be *almost δ-almost continuous* (briefly, a.δ.a.c.) if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \delta\text{-}PO(X, x)$ such that $f(U) \subset (V)$.

Theorem 3.2 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is a.δ.a.c.;*

(b) *for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \delta\text{-}PO(X, x)$ such that $f(U) \subset \text{int}(\text{cl}(V))$;*

(c) *$f^{-1}(V) \in \delta\text{-}PO(X)$ for each regular open set V of Y;*

(d) *$f^{-1}(F) \in \delta\text{-}PC(X)$ for each regular closed set F of Y.*

Proof The proof is obvious and is thus omitted. □

Theorem 3.3 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is a.δ.a.c.;*

(b) *$f(\delta\text{-pcl}(A)) \subset \delta\text{-cl}(f(A))$ for every subset A of X;*

(c) *$\delta\text{-pcl}(f^{-1}(B)) \subset f^{-1}(\delta\text{-cl}(B))$ for every subset B of Y;*

(d) *$f^{-1}(F) \in \delta\text{-}PC(X)$ for every δ-closed set F of Y;*

(e) *$f^{-1}(V) \in \delta\text{-}PO(X)$ for every δ-open set V of Y.*

Proof (a)⇒(b): Let A be any subset of X. Since $\delta\text{-cl}(f(A))$ is δ-closed in Y, it is denoted by $\cap\{F_\alpha : F_\alpha \text{ is regular closed}, \alpha \in \nabla\}$, where ∇ is an index set. By Theorem 3.2, we have $A \subset f^{-1}(\delta\text{-cl}(f(A))) = \cap\{f^{-1}(F_\alpha : \alpha \in \nabla\} \in \delta\text{-}PC(X)$ and hence $\delta\text{-pcl}(A) \subset f^{-1}(\delta\text{-cl}(f(A)))$. Therefore, we obtain $f(\delta\text{-pcl}(A)) \subset \delta\text{-cl}(f(A))$.

(b)⇒(c): Let B be any subset of X. Then we have $f(\delta\text{-pcl}(f^{-1}(B))) \subset \delta\text{-cl}(f(f^{-1}(B))) \subset \delta\text{-cl}(f^{-1}(B))$ and hence $\delta\text{-pcl}(f^{-1}(B)) \subset f^{-1}(\delta\text{-cl}(B))$.

(c)⇒(d): Let F be any δ-closed set of Y. Then we have $\delta\text{-pcl}f^{-1}(F)) \subset f^{-1}(\delta\text{-cl}(F)) = f^{-1}(F)$ and hence $f^{-1}(F) \in \delta\text{-}PC(X)$.

(d)⇒(e): Let V be any δ-open set of Y. Then we have $f^{-1}(Y - V) = X - f^{-1}(V) \in \delta\text{-}PC(X)$ and hence $V \in \delta\text{-}PO(X)$.

(e)⇒(a): Let V be any regular open set of Y. Since V is δ-open in Y, $f^{-1}(V) \in \delta\text{-}PO(X)$ and hence by Theorem 3.2, f is a.δ.a.c. □

Lemma 3.4 [2] scl(V) =int(cl(V)) *for every preopen set* V *of a space* X.

Theorem 3.5 *For a function* $f : X \to Y$, *the following are equivalent:*

(a) f *is a.*δ*.a.c.;*

(b) *for each* $x \in X$ *and each open set* V *of* Y *containing* $f(x)$, *there exists* $U \in \delta\text{-}PO(X, x)$ *such that* $f(U) \subset \text{scl}(V)$*;*

(c) $\delta\text{-pcl}(f^{-1}(\text{cl}(\text{int}(\text{cl}(B))))) \subset f^{-1}(\text{cl}(B))$ *for each subset* B *of* Y*;*

(d) $\delta\text{-pcl}(f^{-1}(\text{cl}(\text{int}(F)))) \subset f^{-1}(F)$ *for each closed set* F *of* Y*;*

(e) $\delta\text{-pcl}(f^{-1}(\text{cl}(V))) \subset f^{-1}(\text{cl}(V))$ *for each open set* V *of* Y*;*

(f) $f^{-1}(V) \subset \delta\text{-pint}(f^{-1}(\text{scl}(V)))$ *for each open set* V *of* Y*;*

(g) $f^{-1}(V) \subset \text{int}(\delta\text{-cl}(f^{-1}(\text{scl}(V))))$ *for each open set* V *of* Y.

Proof (a)⇔(b): It follows from Lemma 3.4.

(a)⇒(c): Let B be any subset of X. Assume that $x \in X - f^{-1}(\text{cl}(B))$. Then $f(x) \in Y - \text{cl}(B)$ and there exists an open set V containing $f(x)$ such that $V \cap B = \emptyset$; hence $\text{int}(\text{cl}(V)) \cap \text{icl}(\text{int}(\text{cl}(B))) = \emptyset$. Since f is a.δ.a.c., there exists $U \in \delta\text{-}PO(X, x)$ such that $f(U) \subset \text{int}(\text{cl}(V))$. Therefore, we have

$U \cap f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B)))) = \emptyset$ and hence $x \in X - \delta$-pcl$(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B)))))$. Thus we obtain δ-pcl$(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(B))))) \subset f^{-1}(\mathrm{cl}(B))$.

(c)⇒(d): Let F be any closed set of Y. Then we have

$$\delta\text{-pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(F)))) = \delta\text{-pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(F))))) \subset f^{-1}(\mathrm{cl}(\mathrm{int}(F))) \subset f^{-1}(F).$$

(d)⇒(e): For any open set V of Y, $\mathrm{cl}(V)$ is regular closed in Y and we have

$$\delta\text{-pcl}(f^{-1}(\mathrm{cl}(V))) = \delta\text{-pcl}(f^{-1}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}(V))))) \subset f^{-1}(\mathrm{cl}(V)).$$

(e)⇒(f): Let V be any open set of Y. Then $Y - \mathrm{cl}(V)$ is open in Y and by Lemmas 2.1 (a) and 3.4 we have

$$\begin{aligned} X - \delta\text{-pint}(f^{-1}(\mathrm{scl}(V))) &= \delta\text{-pcl}(f^{-1}(Y - \mathrm{int}(\mathrm{cl}(V)))) \\ &\subset f^{-1}(\mathrm{cl}(Y - \mathrm{cl}(V))) \subset X - f^{-1}(V). \end{aligned}$$

Therefore, we obtain $f^{-1}(V) \subset \delta$-pint$(f^{-1}(\mathrm{scl}(V)))$.

(f)⇒(g): Let V be any open set of Y. By using Theorem 2.1 in [19], we obtain

$$f^{-1}(V) \subset \delta\text{-pint}(f^{-1}(\mathrm{scl}(V))) \subset \mathrm{int}(\delta\text{-cl}(f^{-1}(\mathrm{scl}(V)))).$$

(g)⇒(b): Let x be any point of X and V be any open set of Y containing $f(x)$. By Lemma 2.1, we have

$$x \in f^{-1}(V) \subset \mathrm{int}(\delta\text{-cl}(f^{-1}(\mathrm{scl}(V)))) = \delta\text{-pint}(f^{-1}(\mathrm{scl}(V))).$$

Then there exists $U \in \delta\text{-}PO(X, x)$ such that $U \subset \delta$-pint$(f^{-1}(\mathrm{scl}(V)))$ and hence $f(U) \subset \mathrm{int}(\mathrm{cl}(V))$. □

Lemma 3.6 [16] *For a subset of a space X, then following hold:*

(a) α-cl$(V) = \mathrm{cl}(V)$ *for each β-open set V of X.*

(b) $\mathrm{pcl}(V) = \mathrm{cl}(V)$ *for each semi-open set V of X.*

Theorem 3.7 *For a function $f : X \to Y$, the following are equivalent:*

(a) *f is a.δ.a.c.;*

(b) *δ-pcl$(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}(V))$ for each β-open set V of Y;*

(c) *δ-pcl$(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}(V))$ for each semi-open set V of Y;*

(d) *$f^{-1}(V) \subset \delta$-pint$(f^{-1}(\mathrm{int}(\mathrm{cl}(V))))$ for each preopen set V of Y;*

(e) *δ-pcl$(f^{-1}(V)) \subset f^{-1}(\alpha\text{-cl}(V))$ for each β-open set V of Y;*

(f) *δ-pcl$(f^{-1}(V)) \subset f^{-1}(\mathrm{pcl}(V))$ for each semi-open set V of Y;*

(g) *$f^{-1}(V) \subset \delta$-pint$(f^{-1}(\mathrm{scl}(V)))$ for each preopen set V of Y.*

Proof (a)$\Rightarrow$(b): Let V be any β-open set of Y. Then by Theorem 2.4 in [2] cl(V) is regular open in Y. Since f is a.δ.a.c., $f^{-1}(\mathrm{cl}(V)$ is δ-preclosed in X and hence δ-pcl$(f^{-1}(V)) \subset f^{-1}(\mathrm{cl}(V))$.

(b)$\Rightarrow$(c): This is obvious since every semi-open set is β-open.

(c)$\Rightarrow$(a): Let F be any regular closed set of Y. Then F is semi-open in Y and hence δ-pcl$(f^{-1}(F)) \subset f^{-1}(\mathrm{cl}(F)) = f^{-1}(F)$. This shows that $f^{-1}(F)$ is δ-preclosed in X. Therefore, f is a.δ.a.c.

(a)$\Rightarrow$(d): Let V be any preopen set of Y. Then $V \subset \mathrm{int}(\mathrm{cl}(V))$ and $\mathrm{int}(\mathrm{cl}(V))$ is regular open in Y. Since f is a.δ.a.c., $f^{-1}(\mathrm{int}(\mathrm{cl}(V)))$ is δ-preopen in X and hence we obtain $f^{-1}(V) \subset f^{-1}(\mathrm{int}(\mathrm{cl}(V))) \subset \delta$-pint$(f^{-1}(\mathrm{int}(\mathrm{cl}(V))))$.

(d)$\Rightarrow$(a): Let V be any regular open set of Y. Then V is preopen and $f^{-1}(V) \subset \delta$-pint$(f^{-1}(\mathrm{int}(\mathrm{cl}(V)))) = \delta$-pint$(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is δ-preopen in X and hence f is a.δ.a.c.

(b)$\Leftrightarrow$(e): It follows from Lemma 3.6 (a).

(c)$\Leftrightarrow$(f): It follows from Lemma 3.6 (b).

(d)$\Leftrightarrow$(d): It follows from Lemma 3.4. □

Theorem 3.8 *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ be the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. Then g is a.δ.a.c. if and only if f is a.δ.a.c.*

Proof (*Necessity*). Let $x \in X$ and V be aany regular open set of Y containing $f(x)$. Then $X \times V$ is regular open in $X \times Y$ and $g(x) = (x, f(x)) \in X \times Y$. Since g is a.δ.a.c., there exists $U \in \delta\text{-}PO(X, x)$ such that $g(U) \subset X \times V$. Therefore, we obtain $f(U) \subset V$ and hence f is a.δ.a.c.

(*Sufficiency*). Let $x \in X$ and W be any regular open set of $X \times Y$ containing $g(x)$. There exist regular open sets U_1 and V of X and Y, respectively, such that $(x, f(x)) \in U_1 \times V \subset W$. Since f is a.δ.a.c., there exists $U_2 \in \delta\text{-}PO(X, x)$ such that $f(U_2) \subset V$. Put $U = U_1 \cap U_2$, then by Lemma 2.2 we obtain $U \in \delta\text{-}PO(X, x)$ and $g(U) \subset U_1 \times V \subset W$. This shows that g is a.δ.a.c. □

Lemma 3.9 [19] *Let A and X_0 be subsets of a space X.*

(a) *If $A \in \delta\text{-}PO(X)$ and X_0 is δ-open in X, then $A \cap X_0 \in \delta\text{-}PO(X_0)$.*

(b) *If $A \in \delta\text{-}PO(X_0)$ and X_0 is δ-open in X, then $A \in \delta\text{-}PO(X)$.*

Theorem 3.10 *If $f : X \to Y$ is a.δ.a.c. and A is δ-open in X, then the restriction $f/A : A \to Y$ is a.δ.a.c.*

Proof Let V be any regular open set of Y. By Theorem 3.2, we have $f^{-1}(V) \in \delta\text{-}PO(X)$ and hence by Lemma 3.9 (a) $(f/A)^{-1}(V) = f^{-1}(V) \cap A \in \delta\text{-}PO(A)$. Thus, it follows from Theorem 3.2 that f/A is a.δ.a.c. □

Theorem 3.11 *Let $f : X \to Y$ be a function and $\{U_\alpha : \alpha \in \nabla\}$ be a cover of X by δ-open sets of X. If $f/U_\alpha : U_\alpha \to Y$ is a.δ.a.c. for each $\alpha \in \nabla$, then f is a.δ.a.c.*

Proof Let V be any regular open set of Y. Then, we have

$$f^{-1}(V) = X \cap f^{-1}(V) = \cup\{U_\alpha \cap f^{-1}(V) : \alpha \in \nabla\} = \cup\{(f/U_\alpha)^{-1}(V) : \alpha \in \Delta\}.$$

Since f/U_α is a.δ.a.c., $(f/U_\alpha)^{-1}(V) \in \delta\text{-}PO(U_\alpha)$ for each $\alpha \in \nabla$. By Lemma 3.9 (b), $(f/U_\alpha)^{-1}(V) \in \delta\text{-}PO(X)$ for each $\alpha \in \nabla$ and hence $f^{-1}(V) \in \delta\text{-}PO(X)$. Therefore, f is a.δ.a.c. □

Theorem 3.12 *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Then the composition $g \circ f : X \to Z$ is a.δ.a.c. if f and g satisfy one of the following conditions:*

(a) *f is a.δ.a.c. and g is R-map,*

(b) *f is δ-almost continuous and g is a.c.S.*

Let $\{X_\alpha : \alpha \in \nabla\}$ and $\{Y_\alpha : \alpha \in \nabla\}$ be any two families of spaces with the same index set ∇. For each $\alpha \in \nabla$, let $f_\alpha : X_\alpha \to Y_\alpha$ be a function. The product space $\prod\{X_\alpha : \alpha \in \nabla\}$ will be denoted by $\prod X_\alpha$ and the product function $\prod f_\alpha : \prod X_\alpha \to \prod Y_\alpha$ defined by $\prod f(x) = \prod\{f_\alpha(x_\alpha) : \alpha \in \nabla\}$ for each $x = \{x_\alpha\} \in \prod X_\alpha$, is simply denoted by $f : \prod X_\alpha \to \prod Y_\alpha$.

Lemma 3.13 *Let A_α be a nonempty subset of X_α for each $\alpha \in \nabla$ and*

$$A = \prod\{A_{\alpha(i)} : i = 1, 2, \ldots, n\} \times \prod\{X_\alpha : \alpha \neq \alpha(i),\ \alpha \in \nabla\}$$

be a nonempty subset of $\prod X_\alpha$, where n is a positive integer. Then $A \in \delta\text{-}PO(\prod X_\alpha)$ if and only if $A_{\alpha(i)} \in \delta\text{-}PO(X_{\alpha(i)})$ for each $i = 1, 2, \ldots, n$.

Proof (*Necessity*). The natural projection is open and continuous. Thus, the necessity follows from the fact that the open continuous image of a δ-preopen set is δ-preopen.

(*Sufficiency*). Since $A_{\alpha(i)} \in \delta\text{-}PO(X_{\alpha(i)})$ for each i, we have

$$\prod_{i=1}^{n} A_{\alpha(i)} \times \prod_{\alpha \neq \alpha(i)} X_\alpha \subset \prod_{i=1}^{n} \mathrm{int}(\delta\text{-cl}(A_{\alpha(i)})) \times \prod_{\alpha \neq \alpha(i)} X_\alpha = \mathrm{int}\left(\delta\text{-cl}\left(\prod_{i=1}^{n} A_{\alpha(i)} \times \prod_{\alpha \neq \alpha(i)} X_\alpha\right)\right)$$

This shows that $A \in \delta\text{-}PO(\prod X_\alpha)$. □

Theorem 3.14 *If a function $f : X \to \prod Y_\alpha$ is a.δ.a.c., then $P_\alpha \circ f : X \to Y_\alpha$ is a.δ.a.c. for each $\alpha \in \nabla$, where P_α is the projection of $\prod Y_\alpha$ onto Y_α.*

Proof Let V_α be any regular open set of Y_α. Since P_α is continuous open, it is an R-map and hence $P_\alpha^{-1}(V_\alpha)$ is regular open in $\prod Y_\alpha$. By Theorem 3.2, $f^{-1}(P_\alpha^{-1}(V_\alpha)) = (P_\alpha \circ f)^{-1}(V_\alpha) \in \delta\text{-}PO(X)$. This shows that $P_\alpha \circ f$ is a.δ.a.c. □

Theorem 3.15 *The product function $f : \prod X_\alpha \to \prod Y_\alpha$ is a.δ.a.c. if and only if $f_\alpha : X_\alpha \to Y_\alpha$ is a.δ.a.c. for each $\alpha \in \nabla$.*

Proof (*Necessity*). Let γ be an arbitrary fixed index and V_γ be any regular open set of Y_γ. Then $\prod Y_\beta \times V_\gamma$ is regular open in $\prod Y_\alpha$, where $\beta \in \nabla$ and $\beta \neq \gamma$, and hence $f^{-1}(\prod Y_\beta \times V_\gamma) = \prod X_\beta \times f_\gamma{}^{-1}(V_\gamma) \in \delta\text{-}PO(\prod X_\alpha)$. Since every cotinuous open function preserves δ-open sets, $f_\gamma{}^{-1}(V_\gamma) \in \delta\text{-}PO(X_\gamma)$ and hence f_γ is a.δ.a.c.

(*Sufficiency*). Let $\{x_\alpha\}$ be any point in $\prod X_\alpha$ and W be any regular open set of $\prod Y_\alpha$ containing $f(\{x_\alpha\})$. There exists a finite subset ∇_0 of ∇ such that V_γ is regular open in Y_γ for each $\gamma \in \nabla_0$ and $\{f_\alpha(x_\alpha)\} \in \prod\{V_\gamma : \gamma \in \nabla_0\} \times \prod\{Y_\beta : \beta \in \nabla - \nabla_0\} \subset W$. For each $\gamma \in \nabla_0$, there exists $U_\gamma \in \delta\text{-}PO(X_\gamma, x_\gamma)$ such that $f_\gamma(U_\gamma) \subset V_\gamma)$. By Lemma 3.13, $U = \prod\{U_\gamma : \gamma \in \nabla_0\} \times \prod\{X_\beta : \beta \in \nabla - \nabla_0\}$ is δ-preopen set of $\prod X_\alpha$ containing $\{x_\alpha\}$ and $f(U) \subset W$. This shows that f is a.δ.a.c. □

4 Some properties

Definition 4.1 [18] The *δ-pre-frontier* of a subset A of a space X, denoted by δ_p-Fr(A), is defined by δ_p-Fr$(A) = \delta$-pcl$(A) \cap \delta$-pcl$(X \setminus A) = \delta$-pcl$(A) \setminus \delta$-pint(A).

Theorem 4.2 *The set of all points x of X at which a function $f : X \to Y$ is not a.δ.a.c. is identical with the union of the δ-pre-frontiers of the inverse images of regular open sets containing $f(x)$.*

Proof Let x be a point of X at which f is not a.δ.a.c. Then, there exists a regular open set V of Y containing $f(x)$ such that $U \cap (X - f^{-1}(V)) \neq \emptyset$ for every $U \in \delta$-$PO(X, x)$. Therefore, we have$x \in \delta$-pcl$(X - f^{-1}(V)) = X - \delta$-pint$(V)$ and $x \in f^{-1}(V)$. Thus, we obtain $x \in \delta_p$-Fr$(f^{-1}(V))$. Conversely, suppose that f is a.δ.a.c. at $x \in X$ and let V be a regular open set containing $f(x)$. Then there exists $U \in \delta$-$PO(X, x)$ such that $U \subset f^{-1}(V)$; hence $x \in \delta$-pint$(f^{-1}(V))$. Therefore, we obtain $x \in X - \delta_p$-Fr$(f^{-1}(V))$. This complete the proof. □

Theorem 4.3 *If $f : X \to Y$ is a.δ.a.c., $g : X \to Y$ is δ-continuous and Y is Hausdorff, then the set $\{x \in X : f(x) = g(x)\}$ is δ-preclosed in X.*

Proof Let $A = \{x \in X : f(x) = g(x)\}$ and $x \in X - A$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets V and W of Y such that $f(x) \in V$, $g(x) \in W$ and $V \cap W = \emptyset$; hence int(cl$V$)) $\cap$ int(cl$(W)) = \emptyset$. Since f is a.δ.a.c., there exists $G \in \delta$-$PO(X, x)$ such that $f(G) \subset$ int(cl(V)). Since g is δ-continuous, there exists an open set H of X containing x such that f(int(cl$(H)) \subset$ int(cl(W)). Now, put $U = G \cap$ int(cl(H)), then $U \in \delta$-$PO(X, x)$ and $f(U) \cap g(U) \subset$ int(cl$(V)) \cap$ int(cl$(W)) = \emptyset$. Therefore, we obtain $U \cap A = \emptyset$ and hence $x \in X_\delta$-cl(A). This shows that A is δ-preclosed in X. □

Theorem 4.4 *If $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ are a.δ.a.c. and Y is Hausdorff, then the set $\{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$ is δ-preclosed in $X_1 \times X_2$.*

Proof Let $A = \{(x_1, x_2) \in X_1 \times X_2 : f_1(x_1) = f_2(x_2)\}$ and $(x_1, x_2) \in X_1 \times X_2 - A$. Then $f_1(x_1) \neq f_2(x_2)$ and there exist open sets V_1 and V_2 such that $f_1(x_1) \in V_1$, $f_2(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$; hence $\text{int}(\text{cl}(V_1)) \cap \text{int}(\text{cl}(V_2)) = \emptyset$. Since f_1 and f_2 are a.δ.a.c., there exist $U_1 \in \delta\text{-}PO(X_1, x_1)$ and $U_2 \in \delta\text{-}PO(X_2, x_2)$ such that $f_1(U_1) \subset \text{int}(\text{cl}(V_1))$ and $f_2(U_2) \subset \text{int}(\text{cl}(V_2))$. Therefore, we obtain $(x_1, x_2) \in U_1 \times U_2 \subset X_1 \times X_2 - A$ and $U_1 \times U_2 \in \delta\text{-}PO(X_1 \times X_2)$. This shows that A is δ-preclosed in $X_1 \times X_2$. □

Theorem 4.5 *If $f : X \to Y$ is a.δ.a.c. and S is a δ-closed set of $X \times Y$, then $P_X(S \cap G(f))$ is δ-closed in X, where P_X represents the projection of $X \times Y$ onto X and $G(f)$ denotes the graph of f.*

Proof Let S be a δ-closed set of $X \times Y$ and $x \in \delta\text{-pcl}(P_X(S \cap G(f)))$. Let W be any regular open set of $X \times Y$ containing $(x, f(x))$. Then there exist regular open sets U and V of X and Y, respectively, such that $(x, f(x)) \in U \times V \subset W$. Since f is a.δ.a.c., by Theorem 3.5 and Lemma 2.2 we have

$$x \in f^{-1}(V) \subset \delta\text{-pint}(f^{-1}(V)) \quad \text{and} \quad U \cap \delta\text{-pint}(f^{-1}(V)) \in \delta\text{-}PO(X, x).$$

Since $x \in \delta\text{-pcl}(P_X(S \cap G(f)))$, by Lemma 2.1(b) $u \in [U \cap \delta\text{-pint}(f^{-1}(V))] \cap P_X(S \cap G(f))$ for some point u of X. This implies that $(u, f(u)) \in S$ and $f(u) \in V$. Thus, we have

$$\emptyset \neq (U \times V) \cap S \subset W \cap S \quad \text{and} \quad (x, f(x)) \in \delta\text{-cl}(S).$$

Since S is δ-closed, we have $(x, f(x)) \in S \cap G(f)$ and $x \in P_X(S \cap G(f))$. It follows from Lemma 2.1(d) that $P_X(S \cap G(f))$ is δ-preclosed in X. □

Corollary 4.6 *If $f : X \to Y$ has a δ-closed graph and $g : X \to Y$ is a.δ.a.c., then the set $\{x \in X : f(x) = g(x)\}$ is δ-preclosed in X.*

Proof Since $G(f)$ is δ-closed and $P_X(G(f) \cap G(g)) = \{x \in X : f(x) = g(x)\}$, it follows from above Theorem that $\{x \in X : f(x) = g(x)\}$ is δ-preclosed in X. □

Corollary 4.7 *If $f : X \to Y$ has a δ-closed graph and $g : X \to Y$ is δ-almost continuous, then the set $\{x \in X : f(x) = g(x)\}$ is δ-preclosed in X.*

Theorem 4.8 *If $f : X \to Y$ is an a.δ.a.c. injective function having δ-closed graph, then X is Hausdorff.*

Proof Let x, y be a pair of distinct points of X. Then $f(x) \neq f(y)$ and $(x, f(y)) \notin G(f)$. Since $G(f)$ is δ-closed, by Theorem 5.2 in [15] there exist regular open sets $U \subset X$ and $V \subset Y$ containing x and $f(y)$, respectively, such that $f(U) \cap V = \emptyset$. Since f is a.δ.a.c., $y \in f^{-1}(V) \subset \text{int}(\delta\text{-cl}(f^{-1}(V)))$. Therefore, we obtain $U \cap \text{int}(\delta\text{-cl}(f^{-1}(V))) = \emptyset$. This shows that X is Hausdorff. □

Corollary 4.9 [19] *If $f : X \to Y$ is a δ-almost continuous injective function having δ-closed graph, then X is Hausdorff.*

Theorem 4.10 *If for each pair of distinct points x_1 and x_2 in a space X, there exists a function f of X into a Hausdorff space Y such that* (a) $f(x_1) \neq f(x_2)$ *and* (b) *f is a.δ.a.c. at x_1 and x_2, then there exist $U_1 \in \delta$-$PO(X, x_1)$ and $U_2 \in \delta$-$PO(X, x_2)$ such that $U_1 \cap U_2 = \emptyset$.*

Proof Since Y is Hausdorff, there exist open sets V_1 and V_2 of Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$; hence $\text{int}(\text{cl}(V_1)) \cap \text{int}(\text{cl}(V_2)) = \emptyset$. Since

f is a.δ.a.c. at x_1 and x_2, there exist $U_1 \in \delta\text{-}PO(X, x_1)$ and $U_2 \in \delta\text{-}PO(X, x_2)$ such that $f(U_1) \subset \text{int}(\text{cl}(V_1))$ and $f(U_2) \subset \text{int}(\text{cl}(V_2))$. Therefore, we obtain $U_1 \cap U_2 = \emptyset$. □

Definition 4.11 A function $f : X \to Y$ has a *δ-preclosed graph* if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in \delta\text{-}PO(X, x)$ and an open set V of Y containing y such that $[U \times \text{cl}(V)] \cap G(f) = \emptyset$.

Lemma 4.12 *A function $f : X \to Y$ has a δ-preclosed graph if and only if for each $(x, y) \in X \times Y$ such that $y \neq f(x)$, there exist $U \in \delta\text{-}PO(X, x)$ and an open set V of Y containing y such that $f(U) \cap \text{cl}(V) = \emptyset$.*

Theorem 4.13 *If $f : X \to Y$ is an a.δ.a.c. function and Y is Hausdorff, then f has a δ-preclosed graph.*

Proof Let $(x, y) \in X \times Y$ such that $y \neq f(x)$. Then there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $V \cap W = \emptyset$; hence $V \cap \text{cl}(W) = \emptyset$. Then $f(x) \in Y - \text{cl}(W)$ and $Y - \text{cl}(W)$ is regular open in Y. There exists $U \in \delta\text{-}PO(X, x)$ such that $f(U) \subset Y - \text{cl}(W)$ and hence $f(U) \cap \text{cl}(W) = \emptyset$. Therefore, by Lemma 4.12 f has a δ-preclosed graph. □

5 Comparisons and examples

We obtain the following diagram by Definitions 2.3 and 3.1:

$$\begin{array}{ccccc} & & \beta\text{-continuity} & \rightarrow & \text{almost } \beta\text{-continuity} \\ & \nearrow & & \nearrow & \\ \text{pre-continuity} & \rightarrow & \text{almost precontinuity} & & \\ & \searrow & & \searrow & \\ & & \delta\text{-almost continuity} & \rightarrow & \text{almost } \delta\text{-almost continuity} \end{array}$$

However, the converses are not true in general as shown by Example 1 in [19], Examples 4.4 and 4.5 in [12] and the following examples:

Example 5.1 Let $X = \{a, b, c\}$, $Y = \{p, q, r, s, t\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}\}$ and $\sigma = \{\emptyset, Y, \{p, q, r, t\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function defined by $f(a) = p$, $f(b) = q$ and $f(c) = s$. Then f is a.δ.a.c. and β-continuous but not δ-almost continuous, since $\{p, q, r, t\}$ is open in (Y, σ) while $f^{-1}(\{p, q, r, t\}) = \{a, b\}$ is not δ-preopen in (X, τ).

Example 5.2 Let X, Y and τ be as in Example 5.1. Define a topology $\sigma = \{\emptyset, Y, \{s\}, \{p, q, r, t\}\}$ on Y and a function $f : (X, \tau) \rightarrow (Y, \sigma)$ defined as in Example 5.1. One can easily shows that f is β-continuous and hence almost β-continuous but not a.δ.a.c. because there exists a regular open set $\{p, q, r, t\}$ in (Y, σ) such that $f^{-1}(\{p, q, r, t\})$ is not δ-preopen in (X, τ).

Example 5.3 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, c\}, \{a, c.d\}\}$ and $\sigma = \{\emptyset, X, \{b\}, \{a, c\}\}$. Then the identity function $f : (X, \tau) \rightarrow (X, \sigma)$ is a.δ.a.c. However, f is not almost β-continuous since $\{b\}$ is regular open in (X, σ) while $f^{-1}(\{b\})$ is not β-open in (X, τ). Therefore, f is not almost precontinuous.

Definition 5.4 A space X is said to be:

(a) *semi-regular* if for any open set U of X and each point $x \in U$ there exits a regular open set V of X such that $x \in V \subset U$;

(b) *almost regular* [20] if for any regular closed set F of X and any point $x \in X - F$ there exist disjoint open sets U and V such that $x \in U$ and $F \subset V$.

Recall that a space X is *submaximal* if every dense subset of X is open in X. Nasef and Noiri [12] that in submaximal space every preopen set is open.

Theorem 5.5 *For a function $f : X \to Y$, the following are true:*

(a) *If f is a.δ.a.c. and Y is semi-regular, then f is δ-almost continuous.*

(b) *If f is a.δ.a.c. (resp. δ-almost continuous) and X is semi-regular, then f is precontinuous (resp. almost precontinuous).*

(c) *If f is a.δ.a.c. and X is semi-regular and submaximal, then f is a.c.S.*

Proof (a): Let $x \in X$ and V be an open set of Y containing $f(x)$. By the semi-regularity of Y, there exists a regular open set G of Y such that $f(x) \in G \subset V$. Since f is a.δ.a.c., there exists $U \in \delta\text{-}PO(X, x)$ such that $f(U) \subset \mathrm{int}(\mathrm{cl}(G)) = G \subset V$ and hence f is δ-almost continuous.

(b): The proof is similar to (a).

(c): It follows from (b) and Theorem 4.4 in [12]. □

Theorem 5.6 *If $f : X \to Y$ is a.δ.a.c. and Y is almost regular locally connected space such that $\delta\text{-cl}(f^{-1}(C)) \subset f^{-1}(\mathrm{cl}(C))$ for each connected subset C of Y, then f is a.c.S.*

Proof Let $x \in X$ and V be any open set of Y containing Y. Since Y is almost regular and locally connected, by Theorem 2.2 in [20] there exists a connected regular open set C of Y such that $f(x) \in C \subset \mathrm{cl}(C) \subset \mathrm{int}(\mathrm{cl}(V))$. Since f is a.δ.a.c.,

$x \in f^{-1}(C) \subset \text{int}(\delta\text{-cl}(f^{-1}(C)))$ and hence $\text{int}(\delta\text{-cl}(f^{-1}(C))) \subset \text{int}(f^{-1}(\text{cl}(C))) \subset f^{-1}(\text{int}(\text{cl}(V)))$. This shows that f is a.c.S. □

Corollary 5.7 *If $f : X \to Y$ is a.δ.a.c. and Y is regular locally connected space such that $\delta\text{-cl}(f^{-1}(C)) \subset f^{-1}(\text{cl}(C))$ for each connected subset C of Y, then f is continuous.*

Proof It follows from Theorem 5.6 and Theorem 9 in [14]. □

Corollary 5.8 [19] *If $f : X \to Y$ is δ-almost continuous and Y is regular locally connected space such that $\delta\text{-cl}(f^{-1}(C)) \subset f^{-1}(\text{cl}(C))$ for each connected subset C of Y, then f is continuous.*

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